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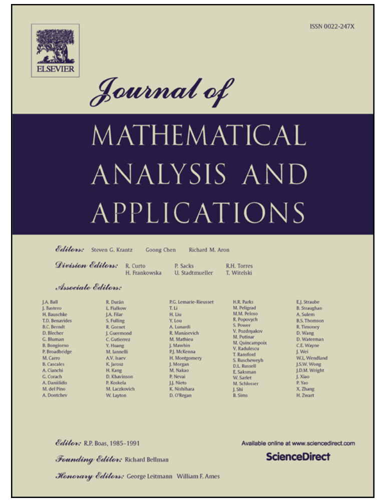

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Homogenization results for micro-contact elasticity problems

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Abstract

The asymptotic behavior of some elasticity problems, in a perforated domain, is analyzed. We address here the case of an ϵ-periodic perforated structure, with rigid inclusions of the same size as the period. The body occupying this domain is considered to be clamped along a part of its outer boundary and subjected to given tractions on the rest of the exterior boundary. Several nonlinear conditions on the boundary of the rigid inclusions are considered. The approach we follow is based on the periodic unfolding method, which allows us to deal with general materials.

Keywords: homogenization, frictional contact, the periodic unfolding method.

1. Introduction

The behavior of heterogeneous materials, with inhomogeneities at a length scale which is much smaller than the characteristic dimensions of the system, is of huge interest in the theory of composite materials. The homogenization theory was successfully applied for modeling the behavior of such materials, leading to appropriate macroscopic continuum models, obtained by averaging the rapid oscillations of the material properties. Besides, such effective models have the advantage of avoiding extensive numerical computations arising when dealing with the small scale behavior of the system.

This paper deals with the derivation of macroscopic models for some elasticity problems in periodically perforated domains with rigid inclusions of the same size as the period. This periodic structure is occupied by a linearly elastic body which is considered to be clamped along a part of its outer boundary. On the rest of the exterior boundary, surface tractions are given. The body is

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subjected to the action of given volume forces. Several nonlinear conditions on the boundary of the rigid inclusions are considered. More precisely, we study the case when a nonlinear Robin condition is imposed and, respectively, the case when unilateral contact with given friction is taken into consideration. By using the periodic unfolding method, introduced by Cioranescu, Damlamian & Griso [3] and by Cioranescu, Donato & Zaki [6], [7] (see, also, [4]), we obtain the corresponding macroscopic problems.

Similar problems have been addressed, using various tools and techniques, by many authors. The macroscopic behavior of a composite material with two elastic components is analyzed, in a formal way, by Léné & Leguillon in [17] and [18]. The homogenization of a contact problem of Signorini type in elasticity, by using the two-scale method, is, for the first time, addressed by Mikelić, Shillor & Tapiéro in [22]. In [14], Iosif'yan studies the asymptotic behavior of the solution of a classical problem in elasticity for a perforated body, clamped along its outer boundary and with a Signorini condition imposed on the surface of the cavities. A viscoelastic periodically perforated material with rigid inclusions, for which the contact and the friction are described by linear conditions, is considered by Gilbert, Panchenko & Xie in [12]. For an homogenized model for acoustic vibrations of composite materials with internal friction, the interested reader is referred to Gilbert, Panchenko & Xie [12]. A system of linear elasticity is considered in [14] for a periodically perforated domain in the case in which a nonlinear Robin condition is imposed on the boundary of the inclusions. Recently, Cioranescu, Damlamian & Orlik [5] perform the homogenization, via the periodic unfolding method, of a contact problem for an elastic body with closed and open cracks. Their problem involves the jump of the solution on the oscillating interface. For elasticity problems involving jumps at imperfect interfaces, see also Ene & Pasa [11], Lipton & Vernescu [19], [20], and Mei & Vernescu [21].

In this paper, for the Robin problem, we extend, via the periodic unfolding method, some of the results contained in [12] and [14], by considering general nonlinearities in the condition imposed on the boundary of the inclusions. Also, we establish an homogenization result for a Signorini problem with Tresca friction. The difficulties of this problem come from the fact that the unilateral condition generates a convex cone of admissible displacements, and, especially, from the fact that the friction condition involves a nonlinear functional containing the norm of the tangential displacement on the boundary of the rigid inclusions. As shown in Section 4, the macroscopic problem is different from the one addressed in [5]. In particular, for the frictionless contact case, we regain a result obtained, under more restrictive assumptions, in [14]. This frictionless problem was also addressed in [15], by the two-scale convergence method, for more general geometric structures of the inclusions on which the Signorini conditions act.

The structure of the paper is as follows: in Section 2, we formulate our microscopic problems, namely a nonlinear Robin problem and a Signorini-Tresca one. Section 3 is devoted to the homogenization of the Robin problem. In Section 4, we obtain the macroscopic behavior of the solution of the Signorini-

Tresca problem.

2. The microscopic problems

Let us consider a linearly elastic body occupying a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$ (the relevant physical cases are $n = 2$ or $n = 3$), with a Lipschitz boundary $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, where Γ_1, Γ_2 are open and disjoint parts of Γ, with $meas(\Gamma_1) > 0$.

The body is subjected to the action of a volume force of density $\boldsymbol{f}$ given in Ω and a surface traction of density $\boldsymbol{t}$ applied on Γ_2. The body is clamped on Γ_1 and, so, the displacement vector $\boldsymbol{u}$ vanishes here.

Let $Y = (0,1)^n$ be the representative cell and T, the rigid part, be an open subset of Y, with a Lipschitz boundary ∂T and such that $\bar{T} \subset Y$. Let $Y^* = Y \backslash \bar{T}$ be the elastic part. We assume that the set of all translated images of $\bar{T}$ of the form $\epsilon(\mathbf{l} + \bar{T})$, with $\mathbf{l} \in \mathbb{Z}^n$, does not intersect the boundary $\partial\Omega$. We denote by T_ϵ the set of the inclusions contained in Ω, i.e.

$$T_\epsilon = \bigcup_{\mathbf{l} \in \mathcal{K}_\epsilon} \epsilon(\mathbf{l} + T),$$

where $\mathcal{K}_\epsilon = \{\mathbf{l} \in \mathbb{Z}^n / \epsilon(\mathbf{l} + \bar{T}) \subset \Omega\}$.

We define the periodic perforated domain by

$$\Omega_\epsilon = \Omega \backslash \bar{T}_\epsilon.$$

Also, we denote by $\boldsymbol{e}$ and $\boldsymbol{\sigma}^\epsilon$ the strain and, respectively, the stress tensor related, in the framework of linear elasticity, by the constitutive law:

$$\sigma^\epsilon_{ij} = a^\epsilon_{ijkh} e_{kh}(\boldsymbol{u}^\epsilon),$$

where

$$e_{ij}(\boldsymbol{u}^\epsilon) = \frac{1}{2}\left(\frac{\partial u^\epsilon_i}{\partial x_j} + \frac{\partial u^\epsilon_j}{\partial x_i}\right) \quad 1 \leq i, j \leq n.$$

Here and below we adopt the usual summation convention. We suppose that the fourth order elasticity tensor $A^\epsilon = (a^\epsilon_{ijkh})$ is given by $a^\epsilon_{ijkh}(x) = a_{ijkh}\left(\dfrac{x}{\epsilon}\right)$, where the elasticity coefficients $\{a_{ijkh}\}_{1 \leq i,j,k,h \leq n}$ are Y-periodic functions and satisfy the usual symmetry and ellipticity conditions:

$$\begin{aligned} & a_{ijkh} = a_{jihk} = a_{khij}\,, \\ & \exists\, \alpha > 0 \text{ such that } a_{ijkh}\xi_{ij}\xi_{kh} \geq \alpha |\boldsymbol{\xi}|^2\,,\ \forall \boldsymbol{\xi} = (\xi_{ij}) \in \mathbb{R}^{n \times n}. \end{aligned} \tag{2.1}$$

We use a classical decomposition in the normal and the tangential components of the displacement vector and of the stress vector on $\partial\Omega_\epsilon = \partial\Omega \cup \partial T_\epsilon$:

$$\begin{aligned} & u_\nu = u_i \nu^\epsilon_i\,, && \boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \boldsymbol{\nu}^\epsilon\,, \\ & \sigma_\nu = \sigma_{ij}\nu^\epsilon_i \nu^\epsilon_j\,, && \sigma_{\tau_i} = \sigma_{ij}\nu_j - \sigma_\nu \nu^\epsilon_i\,, \end{aligned}$$

where $\boldsymbol{\nu}^\epsilon$ is the exterior unit normal to $\partial\Omega_\epsilon$.

Under the previous notation, we first consider the following elasticity problem with a nonlinear Robin condition on the boundary of the inclusions ∂T_ϵ:
Problem $(\boldsymbol{\mathcal{P}}_{1\epsilon})$: Find a displacement vector $\boldsymbol{u}^\epsilon : \Omega_\epsilon \to \mathbb{R}^n$ such that

$$\begin{cases} -\operatorname{div} \boldsymbol{\sigma}^\epsilon = \boldsymbol{f} \qquad \text{in } \ \Omega_\epsilon, \\ \boldsymbol{u}^\epsilon = 0 \qquad \text{on } \ \Gamma_1, \\ \boldsymbol{\sigma}^\epsilon \cdot \boldsymbol{\nu}^\epsilon = \boldsymbol{t} \qquad \text{on } \ \Gamma_2, \\ \boldsymbol{\sigma}^\epsilon \cdot \boldsymbol{\nu}^\epsilon = -\epsilon(\boldsymbol{k}^\epsilon + \boldsymbol{h}(\boldsymbol{u}^\epsilon)) \qquad \text{on } \ \partial T_\epsilon. \end{cases} \tag{2.2}$$

Here, $\boldsymbol{k}^\epsilon = (k_1^\epsilon, \dots, k_n^\epsilon)$ with $k_i^\epsilon(x) = k_i\left(\dfrac{x}{\epsilon}\right)$, for $i = 1, \dots, n$, k_i being a Y-periodic function.

Also, in (2.2), $\boldsymbol{h}(\boldsymbol{u}^\epsilon) = (h_1(u_1^\epsilon), h_2(u_2^\epsilon), \dots, h_n(u_n^\epsilon))$, where the function $h_i : \mathbb{R} \to \mathbb{R}$, for $i = 1, \dots, n$, has the following properties:

$$\begin{cases} h_i \text{ is continuously differentiable}, \\ h_i \text{ is non-decreasing}, \\ h_i(0) = 0, \\ \text{there exist a positive constant } C \text{ and a exponent } q, \\ \quad \text{with } 0 \le q \le \infty \text{ if } n = 2 \text{ and } 0 \le q \le \dfrac{n}{n-2} \text{ if } n > 2, \text{ such that} \\ \quad |h_i'(s)| \le C(1 + |s|^{q-1}) \quad \forall s \in \mathbb{R}. \end{cases} \tag{2.3}$$

The second problem we address involves frictional contact on the boundary of the inclusions. The contact is described by Signorini conditions and the friction by a Coulomb law with given friction (Tresca friction).
Problem $(\boldsymbol{\mathcal{P}}_{2\epsilon})$: Find a displacement vector $\boldsymbol{u}^\epsilon : \Omega_\epsilon \to \mathbb{R}^n$ such that

$$\begin{cases} -\operatorname{div} \boldsymbol{\sigma}^\epsilon = \boldsymbol{f} \qquad \text{in } \ \Omega_\epsilon, \\ \boldsymbol{u}^\epsilon = 0 \qquad \text{on } \ \Gamma_1, \\ \boldsymbol{\sigma}^\epsilon \cdot \boldsymbol{\nu}^\epsilon = \boldsymbol{t} \qquad \text{on } \ \Gamma_2, \\ \sigma_\nu^\epsilon \le 0 \ \ u_\nu^\epsilon \le 0\,, \ \sigma_\nu^\epsilon u_\nu^\epsilon = 0 \qquad \text{on } \ \partial T_\epsilon, \\ |\boldsymbol{\sigma}_\tau^\epsilon| \le \epsilon g^\epsilon \text{ and } \begin{cases} |\boldsymbol{\sigma}_\tau^\epsilon| < \epsilon g^\epsilon \Rightarrow \boldsymbol{u}_\tau^\epsilon = \boldsymbol{0} \\ |\boldsymbol{\sigma}_\tau^\epsilon| = \epsilon g^\epsilon \Rightarrow \exists \lambda \ge 0, \ \boldsymbol{u}_\tau^\epsilon = -\lambda \boldsymbol{\sigma}_\tau^\epsilon \end{cases} \quad \text{on } \ \partial T_\epsilon, \end{cases} \tag{2.4}$$

where $g^\epsilon(x) = g\left(\dfrac{x}{\epsilon}\right)$, with g a non negative Y-periodic function.

Our goal is to derive the macroscopic models corresponding to the above problems. To this end, we shall use the periodic unfolding method which, in fact, requiring no extension operators, allows us to transform any function defined on Ω_ϵ or ∂T_ϵ into a function defined on a fixed domain, $\Omega \times Y^*$ and, respectively, $\Omega \times \partial T$. We recall the definitions of the periodic unfolding operators $\mathcal{T}_\epsilon$ and $\mathcal{T}_\epsilon^b$ for perforated domains and we briefly summarize some of their properties we shall use in our approach. For more details, including complete proofs, we refer the reader to [3], [4], [6], [7], and [8].

In the sequel, we shall use the notation $\widetilde{\varphi}$ for the zero extension to the whole of Ω of a Lebesgue measurable function φ defined on Ω_ϵ. As usual, for a measurable set ω in $\mathbb{R}^n$, $|\omega|$ denotes its Leabesgue measure and $\mathcal{M}_\omega(\varphi)$ denotes the mean value on ω of a Lebesgue measurable function φ, i.e.

$$\mathcal{M}_\omega(\varphi) = \frac{1}{|\omega|} \int_\omega \varphi(x)\,\mathrm{d}x.$$

For any Lebesgue measurable function φ on Ω_ϵ, the periodic unfolding operator $\mathcal{T}_\epsilon$ is the linear operator defined by

$$\mathcal{T}_\epsilon(\varphi)(x,y) = \widetilde{\varphi}\left(\epsilon\left[\frac{x}{\epsilon}\right]_Y + \epsilon y\right) \quad \text{a.e. } (x,y) \in \Omega \times Y^*.$$

Proposition 2.1. *Let $p \in [1,\infty)$. The periodic unfolding operator $\mathcal{T}_\epsilon$ has the following properties:*

1) $\mathcal{T}_\epsilon(\varphi\phi) = \mathcal{T}_\epsilon(\varphi)\mathcal{T}_\epsilon(\phi)$, $\forall\varphi, \phi$ Lebesgue measurable functions on Ω_ϵ.

2) $\mathcal{T}_\epsilon$ is continuous from $L^p(\Omega_\epsilon)$ to $L^p(\Omega \times Y^*)$.

3) $\mathcal{T}_\epsilon(\varphi^\epsilon)(x,y) = \varphi(y)$ a.e. $(x,y) \in \Omega \times Y^*$, $\forall\varphi \in L^p(Y^*)$ a Y-periodic function with $\varphi^\epsilon(x) = \varphi\left(\frac{x}{\epsilon}\right)$.

4) $\displaystyle\int_{\Omega\times Y^*} \mathcal{T}_\epsilon(\varphi)(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\Omega_\epsilon} \varphi(x)\,\mathrm{d}x$, $\forall\varphi \in L^1(\Omega_\epsilon)$.

5) If $\varphi \in H^1(\Omega_\epsilon)$, then $\mathcal{T}_\epsilon(\varphi) \in L^2(\Omega \times H^1(Y^*))$ and

$$\nabla_y \mathcal{T}_\epsilon(\varphi) = \epsilon \mathcal{T}_\epsilon(\nabla_x \varphi).$$

6) If $\{\varphi^\epsilon\}_\epsilon \subset L^2(\Omega)$ is such that $\varphi^\epsilon \to \varphi$ strongly in $L^2(\Omega)$, then

$$\mathcal{T}_\epsilon(\varphi^\epsilon) \to \varphi \text{ strongly in } L^2(\Omega \times Y^*).$$

7) Let $\varphi^\epsilon \in H^1(\Omega_\epsilon)$, $\forall\epsilon > 0$, be such that the sequence $\{\varphi^\epsilon\}_\epsilon$ is bounded in $H^1(\Omega_\epsilon)$. Then, there exist $\varphi^0 \in H^1(\Omega)$ and $\varphi^1 \in L^2(\Omega; H^1_{per}(Y^*))$ with $\mathcal{M}_{Y^*}(\varphi^1)(x) = 0$ a.e. $x \in \Omega$, such that, up to a subsequence,

$$\begin{cases} \mathcal{T}_\epsilon(\varphi^\epsilon) \rightharpoonup \varphi^0 \quad \text{weakly in } L^2(\Omega; H^1(Y^*)), \\ \mathcal{T}_\epsilon(\nabla\varphi^\epsilon) \rightharpoonup \nabla\varphi^0 + \nabla_y\varphi^1 \quad \text{weakly in } L^2(\Omega \times Y^*). \end{cases}$$

In a similar way, we define the periodic unfolding operator on the boundary of the inclusions ∂T_ϵ. This operator allows us to treat the non-homogeneous conditions on the boundary of the inclusions ∂T_ϵ by transforming the surface integrals into volume integrals.

For any Lebesgue measurable function φ defined on ∂T_ϵ, the periodic boundary unfolding operator $\mathcal{T}^b_\epsilon$ is defined by

$$\mathcal{T}^b_\epsilon(\varphi)(x,s) = \varphi\left(\epsilon\left[\frac{x}{\epsilon}\right]_Y + \epsilon s\right) \quad \text{a.e. } (x,s) \in \Omega \times \partial T.$$

We remark that if $\varphi \in H^1(\Omega_\epsilon)$, then $\mathcal{T}^b_\epsilon(\varphi)$ is the trace of $\mathcal{T}_\epsilon(\varphi)$ on ∂T.

The next proposition summarizes the main properties of the boundary unfolding operator defined on the boundary of the inclusions ∂T_ϵ.

Proposition 2.2. *The boundary unfolding operator $\mathcal{T}_\epsilon^b$ is linear. Moreover,*

1) $\mathcal{T}_\epsilon^b(\varphi\phi) = \mathcal{T}_\epsilon^b(\varphi)\mathcal{T}_\epsilon^b(\phi)$, $\forall \varphi, \phi$ Lebesgue measurable functions on ∂T_ϵ.

2) $\mathcal{T}_\epsilon^b$ is continuous from $L^p(\partial T_\epsilon)$ to $L^p(\Omega \times \partial T)$.

3) $\mathcal{T}_\epsilon^b(\varphi^\epsilon)(x,s) = \varphi(s)$ a.e. $(x,s) \in \Omega \times \partial T$, $\forall \varphi \in L^p(\partial T)$ a Y-periodic function with $\varphi^\epsilon(x) = \varphi\Big(\dfrac{x}{\epsilon}\Big)$.

4) $\displaystyle\int_{\Omega\times\partial T} \mathcal{T}_\epsilon^b(\varphi)(x,s)\,\mathrm{d}x\,\mathrm{d}s = \epsilon \int_{\partial T_\epsilon} \varphi(s)\,\mathrm{d}s\,,\ \forall \varphi \in L^1(\partial T_\epsilon)$.

5) $\displaystyle\lim_{\epsilon\to 0} \int_{\Omega\times\partial T} \mathcal{T}_\epsilon^b(\varphi)(x,s)\ \mathrm{d}x\ \mathrm{d}s = |\partial T| \int_{\Omega} \varphi(x)\ \mathrm{d}x\,,\ \forall \varphi \in L^1(\Omega)$.

6) $\mathcal{T}_\epsilon^b(\varphi) \to \widetilde{\varphi}$ strongly in $L^2(\Omega \times \partial T)$, $\forall \varphi \in L^2(\partial T_\epsilon)$.

7) Let $\varphi^\epsilon \in W^{1,p}(\Omega_\epsilon)$, $\forall \epsilon > 0$, and $\varphi \in W^{1,p}(\Omega)$ be such that

$$\mathcal{T}_\epsilon(\varphi^\epsilon) \rightharpoonup \varphi \text{ weakly in } L^p_{loc}(\Omega; W^{1,p}(Y^*))\,.$$

Then

$$\mathcal{T}_\epsilon^b(\varphi^\epsilon) \rightharpoonup \varphi \text{ weakly in } L^p_{loc}(\Omega; W^{1-\frac{1}{p},p}(\partial T))\,.$$

We end this section by recalling a result obtained in [15] for the limit of some two-scale convergent sequences. Taking into account that, for bounded sequences $\{\varphi^\epsilon\}_\epsilon$ in $L^p(\Omega)$, the two-scale convergence of φ^ϵ to a function φ is equivalent with the weakly convergence of $\mathcal{T}_\epsilon(\varphi^\epsilon)$ to φ in $L^p(\Omega \times Y^*)$, the result of [15] can be written in the following suitable form for our paper, which, in fact, shows that the Signorini boundary conditions are preserved under the action of the unfolding operator.

Lemma 2.1. *Let $\{\mathbf{v}^\epsilon\}_\epsilon \subset H^1(\Omega)^n$ be a bounded sequence in $H^1(\Omega)^n$ such that $\mathbf{v}^\epsilon(x) \cdot \boldsymbol{\nu}\Big(\dfrac{x}{\epsilon}\Big) \le 0$ on ∂T_ϵ. Then, there exists $\mathbf{v}^1 \in L^2(\Omega; H^1_{per}(Y^*))^n$ such that $\mathbf{v}^1(x,y) \cdot \boldsymbol{\nu}(y) \le 0$ on ∂T and*

$$\mathcal{T}_\epsilon(\boldsymbol{e}_x(\mathbf{v}^\epsilon)) \rightharpoonup \boldsymbol{e}_y(\mathbf{v}^1) \text{ weakly in } L^2(\Omega \times Y^*)^{n\times n}.$$

3. The Robin problem

This section is devoted to the homogenization of the problem $(\boldsymbol{\mathcal{P}}_{1\epsilon})$. We start by introducing its variational formulation. Then, by using suitable a priori estimates and the periodic unfolding method, we obtain the macroscopic problem.

We define the following Hilbert space:

$$\boldsymbol{V}^\epsilon = \{\mathbf{v}^\epsilon \in H^1(\Omega_\epsilon)^n\,;\, \mathbf{v}^\epsilon = \mathbf{0} \text{ a.e. on } \Gamma_1\}, \tag{3.1}$$

endowed with the scalar product

$$(\boldsymbol{u}^\epsilon, \mathbf{v}^\epsilon)_{\boldsymbol{V}^\epsilon} = \int_{\Omega^\epsilon} e_{ij}(\boldsymbol{u}^\epsilon) e_{ij}(\mathbf{v}^\epsilon)\ \mathrm{d}x \quad \forall \boldsymbol{u}^\epsilon,\, \mathbf{v}^\epsilon \in \boldsymbol{V}^\epsilon.$$

We make the following regularity assumptions on the data:

$$\begin{cases} \boldsymbol{f} \in L^2(\Omega)^n, \\ \boldsymbol{t} \in L^2(\Gamma_2)^n, \\ a_{ijkh} \in L^\infty(Y),\ i,j,k,h = 1,\dots,n, \end{cases} \tag{3.2}$$

$$k_i \in L^2(\partial T), \qquad \mathcal{M}_{\partial T}(k_i) \neq 0\,, \quad \forall i \in \{1,\dots,n\}. \tag{3.3}$$

By using the hypotheses (2.3) and Krasnosel'skii lemma (see, e.g. [16]), it follows that, for any $i \in \{1,\dots,n\}$, the function h_i has the properties:

$$\begin{cases} |h_i(s)| \leq C(1+|s|^q) \quad \forall s \in \mathbb{R}, \\ s h_i(s) \geq 0 \quad \forall s \in \mathbb{R}, \\ h_i(u)v \in L^1(\partial S) \quad \forall S \subset \Omega \text{ compact set}, \forall u,v \in H^1(\Omega \setminus S)\,. \end{cases} \tag{3.4}$$

We are now in the position to write the weak formulation of problem $(\mathcal{P}_{1\epsilon})$.
Problem $(\mathbf{P}_{1\epsilon})$: Find $\boldsymbol{u}^\epsilon \in \boldsymbol{V}^\epsilon$ such that

$$\begin{cases} \displaystyle\int_{\Omega^\epsilon} A^\epsilon \boldsymbol{e}(\boldsymbol{u}^\epsilon)\, \boldsymbol{e}(\boldsymbol{v}^\epsilon)\, \mathrm{d}x + \epsilon \int_{\partial T^\epsilon} \boldsymbol{h}(\boldsymbol{u}^\epsilon)\cdot \boldsymbol{v}^\epsilon\, \mathrm{d}s = \int_{\Omega^\epsilon} \boldsymbol{f}\cdot\boldsymbol{v}^\epsilon\, \mathrm{d}x + \int_{\Gamma_2} \boldsymbol{t}\cdot\boldsymbol{v}^\epsilon\, \mathrm{d}x \\ \displaystyle -\epsilon \int_{\partial T^\epsilon} \boldsymbol{k}^\epsilon \cdot \boldsymbol{v}^\epsilon\, \mathrm{d}s \qquad\qquad \forall\, \boldsymbol{v}^\epsilon \in \boldsymbol{V}^\epsilon\,. \end{cases} \tag{3.5}$$

Proposition 3.1. *Under the above assumptions, there exists a unique solution $\boldsymbol{u}^\epsilon$ of problem $(\mathbf{P}_{1\epsilon})$.*

Proof. Since the operator $\boldsymbol{h}$ is monotone, from the ellipticity of the coefficients a^ϵ and Korn inequality, one gets immediately the uniqueness of the solution of the problem $(\mathbf{P}_{1\epsilon})$. Using the continuity of the Nemytskii operator associated to the function h_i and Minty-Browder theorem (see, for instance, [1] and [8]), we obtain the existence of a solution of $(\mathbf{P}_{1\epsilon})$. ∎

Now, taking $\boldsymbol{v}^\epsilon = \boldsymbol{u}^\epsilon$ in $(\mathbf{P}_{1\epsilon})$ and using Korn inequality, we obtain the following a priori estimate:

$$\|\boldsymbol{u}^\epsilon\|_{\boldsymbol{V}^\epsilon} \leq C, \tag{3.6}$$

with C a positive constant independent on ϵ.

Applying Propositions 2.1, 2.2 and taking into account the definition of the space $\boldsymbol{V}^\epsilon$, it follows that the next result is true.

Proposition 3.2. *There exist $\boldsymbol{u}^0 \in \boldsymbol{V} = \{\mathbf{v} \in H^1(\Omega)^n;\ \mathbf{v} = 0 \text{ a.e. on } \Gamma_1\}$ and $\boldsymbol{u}^1 \in L^2(\Omega; H^1_{per}(Y^*))^n$ with $\mathcal{M}_{Y^*}(\boldsymbol{u}^1)(x) = 0$ a.e. $x \in \Omega$, such that, passing to a subsequence, the following convergences hold:*

$$\begin{aligned} &\mathcal{T}_\epsilon(\boldsymbol{u}^\epsilon) \rightharpoonup \boldsymbol{u}^0 \quad \text{weakly in } L^2(\Omega; H^1(Y^*))^n, \\ &\mathcal{T}_\epsilon(\boldsymbol{e}(\boldsymbol{u}^\epsilon)) \rightharpoonup \boldsymbol{e}(\boldsymbol{u}^0) + \boldsymbol{e}_y(\boldsymbol{u}^1) \quad \text{weakly in } L^2(\Omega \times Y^*)^{n\times n}, \end{aligned} \tag{3.7}$$

$$\mathcal{T}^b_\epsilon(\boldsymbol{h}(\boldsymbol{u}^\epsilon)) \rightharpoonup \boldsymbol{h}(\boldsymbol{u}^0) \quad \text{weakly in } L^{q_1}_{loc}(\Omega; W^{1-\frac{1}{q},q_1}(\partial T))^n, \tag{3.8}$$

where $q_1 = \dfrac{2n}{q(n-2)+n}$ (see, also, [8] and [9]).

The main result of this section is stated in the following theorem.

Theorem 3.1. *Let $\boldsymbol{u}^\epsilon \in \boldsymbol{V}^\epsilon$ be the solution of Problem* $(\mathbf{P}_{1\epsilon})$. *Then, there exists $\boldsymbol{u}^0 \in \boldsymbol{V} = \{\mathbf{v} \in H^1(\Omega)^n;\ \mathbf{v} = 0$ a.e. on $\Gamma_1\}$ such that*

$$\mathcal{T}_\epsilon(\boldsymbol{u}^\epsilon) \rightharpoonup \boldsymbol{u}^0 \quad \text{weakly in } L^2(\Omega; H^1(Y^*))^n, \tag{3.9}$$

where $\boldsymbol{u}^0$ is the unique solution of the homogenized problem

$$\begin{cases} -\operatorname{div} \boldsymbol{\sigma}^{hom}(\boldsymbol{u}^0) + |\partial T|\, \boldsymbol{h}(\boldsymbol{u}^0) = |Y^*|\, \boldsymbol{f} - |\partial T| \mathcal{M}_{\partial T}(\boldsymbol{k}) \quad \text{in } \Omega, \\ \boldsymbol{u}^0 = 0 \quad \text{on } \Gamma_1, \\ \boldsymbol{\sigma}^{hom}(\boldsymbol{u}^0) \cdot \boldsymbol{\nu} = \boldsymbol{t} \quad \text{on } \Gamma_2. \end{cases} \tag{3.10}$$

Here, $\boldsymbol{\nu}$ is the outer unit normal to $\partial\Omega$ and

$$\boldsymbol{\sigma}^{hom}(\boldsymbol{u}^0) = A^{hom} \boldsymbol{e}(\boldsymbol{u}^0),$$

$A^{hom} = (a^{hom}_{ijkh})$ being the homogenized fourth order tensor defined by

$$a^{hom}_{ijkh} = \int_{Y^*} \big(a_{ijkh} - a_{ijlm} e_{lm}(\boldsymbol{\chi}^{kh})\big)\, \mathrm{d}y, \tag{3.11}$$

where, for $k, h \in \{1, \ldots, n\}$, the vector-valued function $\boldsymbol{\chi}^{kh} = (\chi^{kh}_1, \ldots, \chi^{kh}_n)$ has the component $\chi^{kh}_l \in \{v \in H^1_{per}(Y^*);\ \mathcal{M}_{Y^*}(v) = 0\}$, for $l \in \{1, \ldots, n\}$, the solution of the cell problem

$$\begin{cases} \dfrac{\partial}{\partial y_j}\left(a_{ijlm} \dfrac{\partial \chi^{kh}_l}{\partial y_m}\right) = \dfrac{\partial a_{ijkh}}{\partial y_j} \quad \text{in } Y^*,\ i = 1, \ldots, n, \\[2ex] \left(a_{ijkh} - a_{ijlm} \dfrac{\partial \chi^{kh}_l}{\partial y_m}\right) \nu_j = 0 \quad \text{on } \partial T,\ i = 1, \ldots, n. \end{cases} \tag{3.12}$$

Proof. By applying the corresponding unfolding operators $\mathcal{T}_\epsilon$ and $\mathcal{T}^b_\epsilon$ in each term of (3.5), we get

$$\begin{aligned} &\int_{\Omega \times Y^*} \mathcal{T}_\epsilon(A^\epsilon) \mathcal{T}_\epsilon(\boldsymbol{e}(\boldsymbol{u}^\epsilon))\, \mathcal{T}_\epsilon(\boldsymbol{e}(\mathbf{v}^\epsilon))\, \mathrm{d}x\, \mathrm{d}y + \int_{\Omega \times \partial T} \mathcal{T}^b_\epsilon(\boldsymbol{h}(\boldsymbol{u}^\epsilon)) \cdot \mathcal{T}^b_\epsilon(\mathbf{v}^\epsilon)\, \mathrm{d}x\, \mathrm{d}s = \\ &\quad \int_{\Omega \times Y^*} \mathcal{T}_\epsilon(\boldsymbol{f}) \cdot \mathcal{T}_\epsilon(\mathbf{v}^\epsilon)\, \mathrm{d}x\, \mathrm{d}y + \int_{\Gamma_2} \boldsymbol{t} \cdot \mathbf{v}^\epsilon\, \mathrm{d}s - \int_{\Omega \times \partial T} \mathcal{T}^b_\epsilon(\boldsymbol{k}^\epsilon) \cdot \mathcal{T}^b_\epsilon(\mathbf{v}^\epsilon)\, \mathrm{d}x\, \mathrm{d}s\, \forall \boldsymbol{v}^\epsilon \in \boldsymbol{V}^\epsilon. \end{aligned} \tag{3.13}$$

Taking $\boldsymbol{v}^\epsilon(x) = \boldsymbol{\varphi}(x)$, with $\boldsymbol{\varphi} \in \{\mathbf{v} \in C^1(\bar{\Omega})^n;\ \mathbf{v} = 0$ near $\Gamma_1\}$, in (3.13) and passing to the limit, we get

$$\begin{aligned} &\int_{\Omega \times Y^*} A(y) \big(\boldsymbol{e}(\boldsymbol{u}^0)(x) + \boldsymbol{e}_y(\boldsymbol{u}^1)(x, y)\big)\, \boldsymbol{e}(\boldsymbol{\varphi})(x)\, \mathrm{d}x\, \mathrm{d}y + \int_{\Omega \times \partial T} \boldsymbol{h}(\boldsymbol{u}^0)(x) \boldsymbol{\varphi}(x)\, \mathrm{d}x\, \mathrm{d}s = \\ &\quad \int_{\Omega \times Y^*} \boldsymbol{f}(x) \boldsymbol{\varphi}(x)\, \mathrm{d}x\, \mathrm{d}y + \int_{\Gamma_2} \boldsymbol{t} \cdot \boldsymbol{\varphi}\, \mathrm{d}s - \int_{\Omega \times \partial T} \boldsymbol{k}(s) \boldsymbol{\varphi}(x)\, \mathrm{d}x\, \mathrm{d}s\,. \end{aligned} \tag{3.14}$$

Hence, by density, it follows that

$$\int_{\Omega\times Y^*} A(y)\left(\boldsymbol{e}(\boldsymbol{u}^0)(x)+\boldsymbol{e}_y(\boldsymbol{u}^1)(x,y)\right)\boldsymbol{e}(\boldsymbol{\varphi})(x)\,\mathrm{d}x\,\mathrm{d}y+|\partial T|\int_\Omega \boldsymbol{h}(\boldsymbol{u}^0)(x)\boldsymbol{\varphi}(x)\,\mathrm{d}x= \\ |Y^*|\int_\Omega \boldsymbol{f}(x)\boldsymbol{\varphi}(x)\,\mathrm{d}x\,\mathrm{d}y+\int_{\Gamma_2}\boldsymbol{t}\cdot\boldsymbol{\varphi}\,\mathrm{d}s-|\partial T|\mathcal{M}_{\partial T}(\boldsymbol{k})\int_\Omega \boldsymbol{\varphi}(x)\,\mathrm{d}x \quad \forall\,\boldsymbol{\varphi}\in\boldsymbol{V}. \tag{3.15}$$

Now, putting $\boldsymbol{v}^\epsilon(x)=\epsilon\boldsymbol{\psi}\Big(x,\dfrac{x}{\epsilon}\Big)$ with $\boldsymbol{\psi}\in\mathcal{D}(\Omega;H^1_{per}(Y^*))^n$ in (3.13), and taking into account the convergences:

$$\begin{array}{l} \mathcal{T}_\epsilon(\mathbf{v}^\epsilon)\to 0 \quad \text{strongly in } L^2(\Omega;H^1(Y^*))^n,\\ \mathcal{T}^b_\epsilon(\mathbf{v}^\epsilon)\to 0 \quad \text{strongly in } L^2(\Omega\times\partial T)^n,\\ \mathcal{T}_\epsilon(\boldsymbol{e}(\mathbf{v}^\epsilon))\to \boldsymbol{e}_y(\boldsymbol{\psi}) \quad \text{strongly in } L^2(\Omega\times Y^*)^{n\times n}, \end{array} \tag{3.16}$$

by passing to the limit and by using density arguments, we get

$$\int_{\Omega\times Y^*} A(y)\left(\boldsymbol{e}(\boldsymbol{u}^0)(x)+\boldsymbol{e}_y(\boldsymbol{u}^1)(x,y)\right)\boldsymbol{e}_y(\boldsymbol{\psi})(x,y)\,\mathrm{d}x\,\mathrm{d}y=0, \tag{3.17}$$

for all $\boldsymbol{\psi}\in L^2(\Omega;H^1_{per}(Y^*))^n$. This relation gives

$$-\frac{\partial}{\partial y}\left(A(y)\boldsymbol{e}_y(\boldsymbol{u}^1)(x,y)\right)=\frac{\partial}{\partial y}\left(A(y)\boldsymbol{e}(\boldsymbol{u}^0(x)\right) \qquad \text{in } \Omega\times Y^*. \tag{3.18}$$

By standard arguments, this yields

$$\boldsymbol{u}^1(x,y)=-\sum_{k,h=1}^n \boldsymbol{\chi}^{kh}(y)e_{kh}(\boldsymbol{u}^0(x))=-\sum_{k,h=1}^n \boldsymbol{\chi}^{kh}(y)\frac{\partial u^0_k(x)}{\partial x_h}, \tag{3.19}$$

where $\boldsymbol{\chi}^{kh}=(\chi^{kh}_1,\ldots,\chi^{kh}_n)$, with χ^{kh}_l the solution of the local problem (3.12).

We replace now $\boldsymbol{u}^1$ given by (3.19) in (3.15) and we get

$$\int_\Omega A^{hom}\boldsymbol{e}(\boldsymbol{u}^0)(x)\boldsymbol{e}(\boldsymbol{\varphi})(x)\,\mathrm{d}x+|\partial T|\int_\Omega \boldsymbol{h}(\boldsymbol{u}^0)(x)\boldsymbol{\varphi}(x)\,\mathrm{d}x= \\ |Y^*|\int_\Omega \boldsymbol{f}(x)\boldsymbol{\varphi}(x)\,\mathrm{d}x\,\mathrm{d}y+\int_{\Gamma_2}\boldsymbol{t}\cdot\boldsymbol{\varphi}\,\mathrm{d}s-|\partial T|\mathcal{M}_{\partial T}(\boldsymbol{k})\int_\Omega \boldsymbol{\varphi}(x)\,\mathrm{d}x\;\forall\boldsymbol{\varphi}\in\boldsymbol{V}, \tag{3.20}$$

where $A^{hom}=(a^{hom}_{ijkh})$ is given by (3.11). Therefore, taking $\boldsymbol{\varphi}\in\mathcal{D}(\Omega)^n$, we obtain $(3.10)_1$. Multiplying $(3.10)_1$ with $\boldsymbol{\varphi}\in\boldsymbol{V}$, integrating over Ω, and using (3.20), we get $(3.10)_3$.

By standard arguments, it follows that the solution of Problem (3.20) is unique, which implies that the convergence (3.9) holds true on the whole sequence. ∎

We notice that the homogenized elasticity tensor is the classical one appearing in the linear problems and the homogenized equation contains two extra terms generated by the given functions arising in the nonlinear condition on the boundary of the inclusions.

Remark 3.1. *It is not difficult to see that the above results are still true for the case in which*

$$\boldsymbol{h}(\boldsymbol{u}^\epsilon) = (h_{1j}(u_j^\epsilon), h_{2j}(u_j^\epsilon), \ldots, h_{nj}(u_j^\epsilon)),$$

with h_{ij} satisfying the conditions (2.3).

Also, we can work with

$$k_i \in L^\infty(\partial T) \text{ and } k_i(y) \geq \lambda_i > 0\,,\ \forall y \in \partial T\,,$$

which implies (3.3).

4. The Signorini-Tresca problem

This section is devoted to the homogenization of the Problem $(\mathbf{P}_{2\epsilon})$. Let $\mathbf{K}^\epsilon$ be the closed convex cone in $\boldsymbol{V}^\epsilon$ defined by

$$\mathbf{K}^\epsilon = \{\mathbf{v}^\epsilon \in \boldsymbol{V}^\epsilon\,;\, v_\nu^\epsilon \leq 0 \text{ a.e. on } \partial T^\epsilon\},$$

where $\boldsymbol{V}^\epsilon$ is given by (3.1). In order to write the variational formulation of Problem $(\mathbf{P}_{2\epsilon})$, we assume that $\boldsymbol{f}$, $\boldsymbol{t}$ and a_{ijkh} satisfy the regularity conditions (3.2). In addition, we suppose that

$$g \in L^\infty(\partial T) \text{ with } g \geq 0. \tag{4.1}$$

The weak formulation of Problem (2.4) is as follows.

Problem $(\mathbf{P}_{2\epsilon})$: Find $\boldsymbol{u}^\epsilon \in \boldsymbol{K}^\epsilon$ such that

$$\left\{\begin{aligned} &\int_{\Omega^\epsilon} A^\epsilon \boldsymbol{e}(\boldsymbol{u}^\epsilon)\,(\boldsymbol{e}(\mathbf{v}^\epsilon) - \boldsymbol{e}(\boldsymbol{u}^\epsilon))\,\mathrm{d}x + \int_{\partial T_\epsilon} \epsilon g^\epsilon(s)|\mathbf{v}_\tau^\epsilon(s)|\,\mathrm{d}s - \int_{\partial T_\epsilon} \epsilon g^\epsilon(s)|\boldsymbol{u}_\tau^\epsilon(s)|\,\mathrm{d}s \\ &\quad \geq \int_{\Omega^\epsilon} \boldsymbol{f}\cdot(\mathbf{v}^\epsilon - \boldsymbol{u}^\epsilon)\,\mathrm{d}x + \int_{\Gamma_2} \boldsymbol{t}\cdot(\mathbf{v}^\epsilon - \boldsymbol{u}^\epsilon)\,\mathrm{d}s \quad \forall\, \mathbf{v}^\epsilon \in \boldsymbol{K}^\epsilon. \end{aligned}\right. \tag{4.2}$$

By classical results for variational inequalities (see, e.g., [2] or [23]), it follows that Problem $(\mathbf{P}_{2\epsilon})$ has a unique solution. Moreover, taking $\mathbf{v}^\epsilon = 2\boldsymbol{u}^\epsilon$ and $\mathbf{v}^\epsilon = \mathbf{0}$ in (4.2), we deduce that

$$\int_{\Omega^\epsilon} A^\epsilon \boldsymbol{e}(\boldsymbol{u}^\epsilon)\boldsymbol{e}(\boldsymbol{u}^\epsilon)\,\mathrm{d}x + \int_{\partial T_\epsilon} \epsilon g^\epsilon(s)|\boldsymbol{u}_\tau(s)|\,\mathrm{d}s = \int_{\Omega^\epsilon} \boldsymbol{f}\cdot\boldsymbol{u}^\epsilon\,\mathrm{d}x + \int_{\Gamma_2} \boldsymbol{t}\cdot\boldsymbol{u}^\epsilon\,\mathrm{d}s\,.$$

Thus, we obtain the a priori estimate (3.6) and, as a consequence, the convergences (3.7) hold true.

The main result of this section gives the macroscopic behavior of the solution of Problem $(\mathbf{P}_{2\epsilon})$. As we shall see, once again, we obtain at the limit the classical homogenized tensor. Moreover, the Signorini conditions imposed on the boundary of the inclusions become unilateral restrictions defined almost everywhere in the domain Ω.

Theorem 4.1. *Let $\boldsymbol{u}^\epsilon \in \boldsymbol{K}^\epsilon$ be the solution of Problem $(\mathbf{P}_{2\epsilon})$. Then, there exists $\boldsymbol{u}^0 \in \boldsymbol{K} = \{\mathbf{v} \in H^1(\Omega)^n;\ \mathbf{v} = 0$ a.e. on Γ_1, $\mathbf{v}(x)\cdot\boldsymbol{\nu}(y) \le 0$ a.e. $x \in \Omega$, $\forall y \in \partial T\}$ such that*

$$\mathcal{T}_\epsilon(\boldsymbol{u}^\epsilon) \rightharpoonup \boldsymbol{u}^0 \quad \text{weakly in } L^2(\Omega; H^1(Y^*))^n, \tag{4.3}$$

where $\boldsymbol{u}^0$ is the unique solution of the homogenized problem

$$\begin{aligned}
&\int\limits_{\Omega} A^{hom}(y)\boldsymbol{e}(\boldsymbol{u}^0)(\boldsymbol{e}(\mathbf{v})-\boldsymbol{e}(\boldsymbol{u}^0))\,\mathrm{d}x + \int\limits_{\Omega\times\partial T} g(s)|\mathbf{v}_\tau|\,\mathrm{d}x\,\mathrm{d}s - \int\limits_{\Omega\times\partial T} g(s)|\boldsymbol{u}^0_\tau|\,\mathrm{d}x\,\mathrm{d}s\\
&\ge |Y^*|\int\limits_{\Omega}\boldsymbol{f}\cdot(\mathbf{v}-\boldsymbol{u}^0)\,\mathrm{d}x + \int\limits_{\Gamma_2}\boldsymbol{t}\cdot(\mathbf{v}-\boldsymbol{u}^0)\,\mathrm{d}s \quad \forall \mathbf{v}\in\boldsymbol{K},
\end{aligned} \tag{4.4}$$

with A^{hom} defined by (3.11).

Proof. Let $\boldsymbol{u}^0 \in \boldsymbol{V} = \{\mathbf{v} \in H^1(\Omega)^n;\ \mathbf{v} = 0$ a.e. on $\Gamma_1\}$ and $\boldsymbol{u}^1 \in L^2(\Omega; H^1_{per}(Y^*))^n$ be given by Proposition 3.2, i.e. the convergences (3.7) hold. We first prove that $\boldsymbol{u}^0 \in \boldsymbol{K}$. Indeed, as $\boldsymbol{u}^\epsilon \in \boldsymbol{K}^\epsilon$ (i.e. $u^\epsilon_\nu = \boldsymbol{u}^\epsilon\cdot\boldsymbol{\nu}^\epsilon \le 0$ on ∂T_ϵ, where $\boldsymbol{\nu}^\epsilon(x) = \boldsymbol{\nu}\left(\frac{x}{\epsilon}\right)$), then, for any $\delta \in C^\infty(\partial T)$ and $\varphi \in \mathcal{D}(\Omega)$ such that $\delta \ge 0$, $\varphi \ge 0$, we have

$$\begin{aligned}
0 &\ge \int\limits_{\partial T_\epsilon} \epsilon\delta^\epsilon(x)\varphi(x)u^\epsilon_\nu(x)\,\mathrm{d}x\\
&= \int\limits_{\Omega\times\partial T} \mathcal{T}^b_\epsilon(\delta^\epsilon)(x,y)\mathcal{T}^b_\epsilon(\varphi)(x,y)\mathcal{T}^b_\epsilon(\boldsymbol{u}^\epsilon)(x,y)\mathcal{T}^b_\epsilon(\boldsymbol{\nu}^\epsilon)(x,y)\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}$$

with $\delta^\epsilon(x) = \delta\left(\frac{x}{\epsilon}\right)$. Here, we used the properties of the boundary unfolding operator $\mathcal{T}^b_\epsilon$. Passing to the limit, we get

$$\int\limits_{\Omega\times\partial T} \boldsymbol{u}^0(x)\cdot\boldsymbol{\nu}(y)\varphi(x)\delta(y)\,\mathrm{d}x\,\mathrm{d}y \le 0\,\forall(\varphi,\delta)\in\mathcal{D}(\Omega)\times C^\infty(\partial T) \text{ with } \delta\ge 0\,,\ \varphi\ge 0,$$

i.e. $\boldsymbol{u}^0 \in \boldsymbol{K}$.

Now, by using Lemma 2.1, we deduce that there exists $\mathbf{v}^1 \in L^2(\Omega, H^1_{per}(Y^*))^n$ such that $\mathbf{v}^1(x,y)\cdot\boldsymbol{\nu}(y) \le 0$ on ∂T and $\mathcal{T}_\epsilon(\boldsymbol{e}_x(\boldsymbol{u}^\epsilon)) \rightharpoonup \boldsymbol{e}_y(\mathbf{v}^1)$ weakly in $L^2(\Omega\times Y^*)^{n\times n}$. But taking into account that $\mathcal{M}_{Y^*}(\boldsymbol{u}^1) = 0$ a.e. on Ω, from the convergence $(3.7)_2$ and the connectedness of Y^*, it results that there exists a unique element $\mathbf{v}^1 \in L^2(\Omega; H^1((Y^*))^n$ with zero average on Y^* such that

$\boldsymbol{e}_y(\mathbf{v}^1) = \boldsymbol{e}(\boldsymbol{u}^0) + \boldsymbol{e}_y(\boldsymbol{u}^1)$, namely $y_{\mathcal{M}}\boldsymbol{e}(\boldsymbol{u}^0) + \boldsymbol{u}^1$ where $y_{\mathcal{M}} = y - \mathcal{M}_{Y^*}(y)$ (see, also, [4] or [10]). Hence, we have $(y_{\mathcal{M}}\boldsymbol{e}(\boldsymbol{u}^0)(x) + \boldsymbol{u}^1(x,y)) \cdot \boldsymbol{\nu}(y) \le 0$, for a.e. $x \in \Omega$ and for any $y \in \partial T$.

By applying the corresponding unfolding operators in (4.2), we get

$$\begin{aligned}
&\int\limits_{\Omega\times Y^*} \mathcal{T}_\epsilon(A^\epsilon)\mathcal{T}_\epsilon(\boldsymbol{e}(\boldsymbol{u}^\epsilon))\,\mathcal{T}_\epsilon(\boldsymbol{e}(\boldsymbol{u}^\epsilon))\,\mathrm{d}x\,\mathrm{d}y + \int\limits_{\Omega\times\partial T} \mathcal{T}^b_\epsilon(g^\epsilon)\mathcal{T}^b_\epsilon(|\boldsymbol{u}^\epsilon_\tau|)\,\mathrm{d}x\,\mathrm{d}s \\
&\le \int\limits_{\Omega\times Y^*} \mathcal{T}_\epsilon(A^\epsilon)\mathcal{T}_\epsilon(\boldsymbol{e}(\boldsymbol{u}^\epsilon))\,\mathcal{T}_\epsilon(\boldsymbol{e}(\mathbf{v}^\epsilon))\,\mathrm{d}x\,\mathrm{d}y + \int\limits_{\Omega\times\partial T} \mathcal{T}^b_\epsilon(g^\epsilon)\mathcal{T}^b_\epsilon(|\mathbf{v}^\epsilon_\tau|)\,\mathrm{d}x\,\mathrm{d}s \\
&- \int\limits_{\Omega\times Y^*} \mathcal{T}_\epsilon(\boldsymbol{f})\cdot(\mathcal{T}_\epsilon(\mathbf{v}^\epsilon) - \mathcal{T}_\epsilon(\boldsymbol{u}^\epsilon))\,\mathrm{d}x\,\mathrm{d}y - \int\limits_{\Gamma_2} \boldsymbol{t}\cdot(\mathbf{v}^\epsilon - \boldsymbol{u}^\epsilon)\,\mathrm{d}s\,.
\end{aligned} \tag{4.5}$$

Taking $\boldsymbol{v}^\epsilon(x) = \boldsymbol{\varphi}(x) + \epsilon\boldsymbol{\psi}\Big(x, \dfrac{x}{\epsilon}\Big)$, with $\boldsymbol{\varphi} \in C^1(\bar{\Omega})^n$ such that $\boldsymbol{\varphi} = 0$ near Γ_1 and $\boldsymbol{\varphi}(x)\cdot\boldsymbol{\nu}(y) \le 0$, $\forall (x,y) \in \Omega\times\partial T$, and $\boldsymbol{\psi} \in \mathcal{D}(\Omega; H^1_{per}(Y^*))^n$ such that $\boldsymbol{\psi}(x,y)\cdot\boldsymbol{\nu}(y) \le 0$, $\forall (x,y) \in \Omega\times\partial T$ in the last relation and taking into account the convergences (3.7), we obtain

$$\begin{aligned}
&\int\limits_{\Omega\times Y^*} A(y)|\boldsymbol{e}(\boldsymbol{u}^0) + \boldsymbol{e}_y(\boldsymbol{u}^1)|^2\,\mathrm{d}x\,\mathrm{d}y + \int\limits_{\Omega\times\partial T} g(s)|\boldsymbol{u}^0_\tau|\,\mathrm{d}x\,\mathrm{d}s \\
&\le \liminf_{\epsilon\to 0} \int\limits_{\Omega\times Y^*} \mathcal{T}_\epsilon(A^\epsilon)\mathcal{T}_\epsilon(\boldsymbol{e}(\boldsymbol{u}^\epsilon))\,\mathcal{T}_\epsilon(\boldsymbol{e}(\boldsymbol{u}^\epsilon))\,\mathrm{d}x\,\mathrm{d}y + \liminf_{\epsilon\to 0} \int\limits_{\Omega\times\partial T} \mathcal{T}^b_\epsilon(g^\epsilon)\mathcal{T}^b_\epsilon(|\boldsymbol{u}^\epsilon_\tau|)\,\mathrm{d}x\,\mathrm{d}s \\
&\le \limsup_{\epsilon\to 0}\left(\int\limits_{\Omega\times Y^*} \mathcal{T}_\epsilon(A^\epsilon)\mathcal{T}_\epsilon(\boldsymbol{e}(\boldsymbol{u}^\epsilon))\,\mathcal{T}_\epsilon(\boldsymbol{e}(\boldsymbol{u}^\epsilon))\,\mathrm{d}x\,\mathrm{d}y + \int\limits_{\Omega\times\partial T} \mathcal{T}^b_\epsilon(g^\epsilon)\mathcal{T}^b_\epsilon(|\boldsymbol{u}^\epsilon_\tau|)\,\mathrm{d}x\,\mathrm{d}s \right) \\
&\le \int\limits_{\Omega\times Y^*} A(y)(\boldsymbol{e}(\boldsymbol{u}^0) + \boldsymbol{e}_y(\boldsymbol{u}^1))(\boldsymbol{e}(\boldsymbol{\varphi}) + \boldsymbol{e}_y(\boldsymbol{\psi}))\,\mathrm{d}x\,\mathrm{d}y + \int\limits_{\Omega\times\partial T} g(s)|\boldsymbol{\varphi}_\tau|\,\mathrm{d}x\,\mathrm{d}s \\
&- \int\limits_{\Omega\times Y^*} \boldsymbol{f}\cdot(\boldsymbol{\varphi} - \boldsymbol{u}^0)\,\mathrm{d}x\,\mathrm{d}y - \int\limits_{\Gamma_2} \boldsymbol{t}\cdot(\boldsymbol{\varphi} - \boldsymbol{u}^0)\,\mathrm{d}s\,.
\end{aligned} \tag{4.6}$$

Now, by taking $\boldsymbol{\varphi} = \boldsymbol{u}^0$ and $\boldsymbol{\psi} = \boldsymbol{u}^1 + y_{\mathcal{M}}\boldsymbol{e}(\boldsymbol{u}^0) + \lambda\boldsymbol{\theta}$, with $\lambda > 0$, $\boldsymbol{\theta} \in \mathcal{D}(\Omega; H^1_{per}(Y^*))^n$ and $\boldsymbol{\theta}(x,y)\cdot\boldsymbol{\nu}(y) = 0$, $\forall (x,y) \in \Omega\times\partial T$, it follows that

$$\begin{aligned}
&\lambda \int\limits_{\Omega\times Y^*} A(y)\left(\boldsymbol{e}(\boldsymbol{u}^0)(x) + \boldsymbol{e}_y(\boldsymbol{u}^1)(x,y)\right)\boldsymbol{e}_y(\boldsymbol{\theta})(x,y)\,\mathrm{d}x\,\mathrm{d}y + \\
&\int\limits_{\Omega\times Y^*} A(y)\left(\boldsymbol{e}(\boldsymbol{u}^0)(x) + \boldsymbol{e}_y(\boldsymbol{u}^1)(x,y)\right)\boldsymbol{e}(\boldsymbol{u}^0)\,\mathrm{d}x\,\mathrm{d}y \ge 0\,, \forall\lambda > 0
\end{aligned}$$

which implies

$$\int\limits_{\Omega\times Y^*} A(y)\left(\boldsymbol{e}(\boldsymbol{u}^0)(x) + \boldsymbol{e}_y(\boldsymbol{u}^1)(x,y)\right)\boldsymbol{e}_y(\boldsymbol{\theta})(x,y)\,\mathrm{d}x\,\mathrm{d}y \ge 0 \tag{4.7}$$

and

$$\int\limits_{\Omega\times Y^*} A(y)\left(\boldsymbol{e}(\boldsymbol{u}^0)(x)+\boldsymbol{e}_y(\boldsymbol{u}^1)(x,y)\right)\boldsymbol{e}(\boldsymbol{u}^0)\,\mathrm{d}x\,\mathrm{d}y \geq 0. \tag{4.8}$$

Putting $\theta = \pm\theta$ in (4.7), we get

$$\int\limits_{\Omega\times Y^*} A(y)\left(\boldsymbol{e}(\boldsymbol{u}^0)(x)+\boldsymbol{e}_y(\boldsymbol{u}^1)(x,y)\right)\boldsymbol{e}_y(\boldsymbol{\theta})(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0, \tag{4.9}$$

which ensures that (3.17) holds true (see [22]). This implies that $\boldsymbol{u}^1$ is given by (3.19).

Now, using (4.9) in the relation (4.6) and density arguments, we obtain:

$$\begin{aligned}&\int\limits_{\Omega\times Y^*} A(y)(\boldsymbol{e}(\boldsymbol{u}^0)+\boldsymbol{e}_y(\boldsymbol{u}^1))(\boldsymbol{e}(\boldsymbol{\varphi})-\boldsymbol{e}(\boldsymbol{u}^0))\,\mathrm{d}x\,\mathrm{d}y + \int\limits_{\Omega\times\partial T} g(s)|\boldsymbol{\varphi}_\tau|\,\mathrm{d}x\,\mathrm{d}s\\ &-\int\limits_{\Omega\times\partial T} g(s)|\boldsymbol{u}^0_\tau|\,\mathrm{d}x\,\mathrm{d}s \geq \int\limits_{\Omega\times Y^*} \boldsymbol{f}\cdot(\boldsymbol{\varphi}-\boldsymbol{u}^0)\,\mathrm{d}x\,\mathrm{d}y + \int\limits_{\Gamma_2}\boldsymbol{t}\cdot(\boldsymbol{\varphi}-\boldsymbol{u}^0)\,\mathrm{d}s \quad \forall\boldsymbol{\varphi}\in\boldsymbol{K}.\end{aligned}$$

Thus, by (3.19), it follows that $\boldsymbol{u}^0$ satisfies (4.4). Due to the uniqueness of the solution of (4.4), all the above convergences hold true on the whole sequences and this ends the proof. ■

Unfortunately, in this case, we can not decouple the variables in the term

$$\int\limits_{\Omega\times\partial T} g(s)|\boldsymbol{\varphi}_\tau(x,s)|\,\mathrm{d}x\,\mathrm{d}s$$

and, hence, we do not obtain a strong formulation of the homogenized problem as in [5], where the authors perform the homogenization, via the periodic unfolding method, of a contact problem for an elastic body with closed and open cracks. In their problem, since the unilateral contact and given friction conditions involve the jumps of the solutions on the oscillating interface, one can choose suitable test functions such that the nonlinear term, describing the friction, does not appear explicitly in the macroscopic problem. However, a contribution of the given friction is taken into account in the homogenized tensor.

Finally, let us notice that in the case in which $g = 0$, i.e. the frictionless contact problem given by (2.4) with the condition $(2.4)_5$ replaced by $\boldsymbol{\sigma}_\tau = \boldsymbol{0}$, Theorem 4.1 shows that the homogenized solution $\boldsymbol{u}^0 \in \boldsymbol{K}$ satisfies the following inequality:

$$\int\limits_{\Omega} A^{hom}\boldsymbol{e}(\boldsymbol{u}^0)\left(\boldsymbol{e}(\mathbf{v})-\boldsymbol{e}(\boldsymbol{u}^0)\right)\mathrm{d}x \geq \int\limits_{\Omega}\boldsymbol{f}\cdot(\mathbf{v}-\boldsymbol{u}^0)\,\mathrm{d}x + \int\limits_{\Gamma_2}\boldsymbol{t}\cdot(\mathbf{v}-\boldsymbol{u}^0)\,\mathrm{d}s \quad \forall\boldsymbol{v}\in\boldsymbol{K},$$

which is equivalent to

$$\begin{cases} -\operatorname{div}\boldsymbol{\sigma}^{hom}(\boldsymbol{u}^0) = |Y^*|\,\boldsymbol{f} & \text{in } \Omega,\\ \boldsymbol{u}^0 = 0 & \text{on } \Gamma_1,\\ \boldsymbol{\sigma}^{hom}(\boldsymbol{u}^0)\cdot\boldsymbol{\nu} = \boldsymbol{t} & \text{on } \Gamma_2. \end{cases} \tag{4.10}$$

So, for $\boldsymbol{t} = \mathbf{0}$, we regain the result obtained in [14].

This problem was also addressed in [15], by the two-scale convergence method, for more general geometric structures of the inclusions on which the Signorini conditions act.

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