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Boundedness in a three-dimensional chemotaxis–fluid system involving tensor-valued sensitivity with saturation

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ABSTRACT

We consider the chemotaxis–fluid system

$$\begin{cases} n_t + u\cdot\nabla n = \nabla\cdot(D(n)\nabla n) - \nabla\cdot(nS(x,n,c)\cdot\nabla c) + an - bn^2, \\ \quad x\in\Omega, t>0, \\ c_t + u\cdot\nabla c = \Delta c - c + n, \quad x\in\Omega, t>0, \\ u_t + \nabla P = \Delta u + n\nabla\phi + g(x,t), \quad x\in\Omega, t>0, \\ \nabla\cdot u = 0, \quad x\in\Omega, t>0 \end{cases} \tag{CF}$$

under homogeneous Neumann boundary conditions in a smooth bounded domain $\Omega\subseteq\mathbb{R}^3$, where $\phi\in W^{1,\infty}(\Omega)$, $a\geq 0$ and $b>0$. Here $g\in C^1(\bar\Omega\times[0,\infty))\cap L^\infty(\Omega\times(0,\infty))$, $D(n)\geq u^{m-1}$, $|S(x,n,c)|\leq(1+n)^{-\alpha}$, and the parameter $\alpha>0$. If $m>\max\{\frac{6}{5}-\alpha,\frac{1}{3}\}$, then for all reasonably regular initial data, a corresponding initial–boundary value problem for (CF) possesses a globally defined weak solution through the Moser-type iteration.

1. Introduction

Chemotaxis is the directed movement of cells in response to the gradient of a chemical signaling substance [17,27]. A model has been proposed to describe the chemotaxis–fluid interaction in cases where the evolution of the chemoattractant is essentially dominated via its production by cells [2,17]. Based on numerous applications (e.g., see Hillen and Painter [17]), we assume that the cell kinetics follow a logistic-type law determined by the parameters a and b, where allowing for the borderline case when $a=0$, we explicitly include cases where cell proliferation can be neglected. In the present study, we investigate the basic mathematical features of a simple model of chemotaxis–fluid interaction in cases where the evolution of the chemoattractant is essentially governed by it production from cells, as found in the original Keller–Segel system (see [21,22]). In physically relevant cases, the environment of the bacterial cells is more complex

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and additional external influences must be considered. There may be interactions between the bacterial swimming speed and direction, such as the effects of gravity. As noted by Xue and Othmer in [48], this may lead to chemotactic sensitivity, which is actually a tensor type rather than a scalar function. In particular, we consider a chemotaxis–Stokes system with rotational flux and a logistic source

$$\begin{cases} n_t + u\cdot\nabla n = \nabla\cdot(D(n)\nabla n) - \nabla\cdot(nS(x,n,c)\cdot\nabla c) + an - bn^2, \quad x\in\Omega, t>0,\\ c_t + u\cdot\nabla c = \Delta c - c + n, \quad x\in\Omega, t>0,\\ u_t + \nabla P = \Delta u + n\nabla\phi + g(x,t), \quad x\in\Omega, t>0,\\ \nabla\cdot u = 0, \quad x\in\Omega, t>0,\\ (D(n)\nabla n - nS(x,n,c)\cdot\nabla c)\cdot\nu = \nabla c\cdot\nu = 0, u = 0, \quad x\in\partial\Omega, t>0,\\ n(x,0)=n_0(x), c(x,0)=c_0(x), u(x,0)=u_0(x), \quad x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, $\Delta = \sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}$, $\frac{\partial}{\partial\nu}$ denotes the outward unit normal derivative on $\partial\Omega$, $a\geq 0$, $b>0$, $g\in C^1(\bar{\Omega}\times[0,\infty))\cap L^\infty(\Omega\times(0,\infty))$, and $D(n)$ denotes the nonlinear diffusion. In this setting, $n=n(x,t)$ and $c=c(x,t)$ denote the density of the cell population and the oxygen concentration, respectively, while $u=u(x,t)$ and $P=P(x,t)$ represent the fluid velocity and the associated pressure, respectively. Moreover, model (1.1) assumes that in addition to the driving action of cells through buoyant forces within the gravitational field with potential ϕ, the motion of the fluid might be controlled by a given external force g. $S(x,n,c)$ is a chemotactic sensitivity tensor that satisfies

$$S\in C^2(\bar{\Omega}\times[0,\infty)^2;\mathbb{R}^{3\times 3}) \tag{1.2}$$

and

$$|S(x,n,c)|\leq C_S(1+n)^{-\alpha} \quad \text{for all} \quad (x,n,c)\in\Omega\times[0,\infty)^2 \tag{1.3}$$

with some $C_S>0$ and $\alpha>0$.

If $u\equiv 0$ and $S(x,n,c):=S(n)$, then (1.1) transforms into the following quasilinear chemotaxis model, which was proposed in pioneering studies by [35,41]:

$$\begin{cases} n_t = \nabla\cdot(D(n)\nabla n) - \chi\nabla\cdot(S(n)\nabla c) + h(n), \quad x\in\Omega,\ t>0,\\ \tau c_t = \Delta c - c + n, \quad x\in\Omega,\ t>0,\\ \frac{\partial n}{\partial\nu} = \frac{\partial c}{\partial\nu} = 0, \quad x\in\partial\Omega,\ t>0,\\ n(x,0)=n_0(x), \quad c(x,0)=c_0(x), \quad x\in\Omega \end{cases} \tag{1.4}$$

with $\tau=0$ or 1 and $h(n)=an-bn^2$. For problem (1.4) in the absence of a logistic source, i.e.,

$$\begin{cases} n_t = \nabla\cdot(D(n)\nabla n) - \chi\nabla\cdot(S(n)\nabla c), \quad x\in\Omega,\ t>0,\\ c_t = \Delta c - c + n, \quad x\in\Omega,\ t>0,\\ \frac{\partial n}{\partial\nu} = \frac{\partial c}{\partial\nu} = 0, \quad x\in\partial\Omega,\ t>0,\\ n(x,0)=n_0(x), \quad c(x,0)=c_0(x), x\in\Omega, \end{cases} \tag{1.5}$$

many studies have considered the global boundedness or blow-up of the solutions to problem (1.5) (see Burger et al. [3], Cieślak et al. [6–8], Dolbeault and Perthame [9], Hillen [17], Horstmann et al. [18], Ishida et al. [20], Kowalczyk [23], Winkler et al. [31,47,43]).

Moreover, logistic-type growth restrictions have been demonstrated to prevent any chemotactic collapse in some systems that are closely related to (1.4) (e.g., see Winkler [41,35], Wang et al. [38,39], Zheng [50,51]). In particular, if $\tau = 1$, $D(n) \equiv 1$, $S(n) = n$, $h(n)$ satisfies

$$h(n) \leq a - bn^{\lambda} \ \text{ for all } \ n \geq 0 \tag{1.6}$$

with $\lambda = 2$ and sufficiently large b, then Winkler [41] proved that problem (1.4) possesses a unique global in time and bounded classical solution for all sufficiently smooth initial data. Wang et al. [38] also showed that when $D(n) = (n+1)^{-p}$, $S(n) = n(n+1)^{q-1}$ with $0 < p + q < \frac{2}{N}$, and h satisfies (1.6), then problem (1.4) possesses a unique globally bounded classical solution (n, c). When $D(n) \equiv 1$, $S(n) = n$, $h(u)$ satisfies (1.6) with $\lambda = 2$ and some $a \geq 0$, $b > 0$. Thus, Tello and Winkler [35] discussed the existence of global bounded classical solutions under the assumption that either $N \leq 2$, or the logistic damping effect $b > \dfrac{N-2}{N}\chi$. $h(u)$ satisfies (1.6) with some $a \geq 0$, $b > 0$, and in our recent study [51], we proved that if $0 < p + q < \max\{r - 1 + p, \frac{2}{N}\}$ or b is sufficiently large when $q = r - 1$, then the classical solutions to (1.4) are uniformly bounded.

If $S(x, n, c) = \chi S(c)$ with $\chi > 0$, and a given function S, $-c + n$ in the c-equation is replaced by $-nf(c)$ with a given function f, and the u-equation is a (Navier–)Stokes equation, then (1.1) becomes the following chemotaxis–(Navier–)Stokes system in the context of signal consumption by cells

$$\begin{cases} n_t + u \cdot \nabla n = \nabla \cdot (D(n)\nabla n) - \nabla \cdot (nS(c)\nabla c), \quad x \in \Omega, t > 0, \\ c_t + u \cdot \nabla c = \Delta c - nf(c), \quad x \in \Omega, t > 0, \\ u_t + \kappa(u \cdot \nabla)u + \nabla P = \Delta u + n\nabla\phi + g(x,t), \quad x \in \Omega, t > 0, \\ \nabla \cdot u = 0, \quad x \in \Omega, t > 0. \end{cases} \tag{1.7}$$

Systems of this type arise during the modeling of populations of aerobic bacteria suspended in sessile drops of water [10,37]. This model describes the motion of oxygen-driven swimming cells in an incompressible fluid. The motion of the fluid is under the influence of the gravitational force exerted by the aggregation of cells in the fluid [10,37].

In recent years, approaches have been developed based on a natural energy functional, and some local and global solvability aspects of the corresponding initial value problem for (1.7) in either bounded or unbounded domains (e.g., see Chae, Kang, and Lee [4,5], Duan, Lorz, Markowich [11], Liu and Lorz [25,26], Tao and Winkler [34,42,44,45], Zhang and Zheng [49], and the references therein). For instance, if the diffusion of the cells is linear, i.e., $D(n) = 1$, in [26], certain local in time weak solutions were constructed for a boundary value problem for system (1.7). Winkler [42] proved that the system possesses a unique global classical solution for large data in a bounded convex domain $\Omega \subset \mathbb{R}^2$ and the simplified chemotaxis–Stokes system possesses at least one global weak solution for large data in a bounded convex domain $\Omega \subset \mathbb{R}^3$, whereas in [25], global weak solutions for large data were constructed in two space dimensions by Liu and Lorz. In the case of degenerate cell diffusion in a porous medium type, i.e., the diffusion term $\nabla \cdot (D(n)\nabla n)$ in the first equation of (1.7) is replaced by a porous medium-type diffusion term Δn^m, and many results are related to the mathematical analysis of (1.7). In fact, if $\kappa = 0$ in the third equation of (1.7), [13] asserted the global existence of weak solutions for the system with $m \in (\frac{3}{2}, 2]$ and $\Omega \subset \mathbb{R}^2$, whereas Tao and Winkler [32] proved that global bounded weak solutions exist whenever $m > 1$, which fills up the gap $(1, \frac{3}{2}]$. Moreover, [13] established the global existence of weak solutions for the system with $m \in [\frac{7+\sqrt{217}}{12}, 2]$ and $\Omega \subset \mathbb{R}^3$, whereas Tao and Winkler [33] recently proved that the model possesses global weak solutions for large data in a bounded convex domain $\Omega \subset \mathbb{R}^3$ that is locally bounded with $m > \frac{8}{7}$, when $\Omega = \mathbb{R}^3$. Liu and Lorz [25] and Duan and Xiang [12] also established the global existence of weak solutions for $m = \frac{4}{3}$ and $m \geq 1$, respectively.

It has been noted that due to the presence of the tensor-valued sensitivity, the corresponding chemotaxis–Stokes system loses some energy structure, which leads to considerable mathematical difficulties during the analysis. Therefore, very few results appear to be available for chemotaxis–Stokes system with such tensor-valued sensitivities [19,40,46]. In fact, in two space dimensions, assuming that

$$|S(x,n,c)| \le S_0(c) \quad \text{for all} \quad (x,n,c) \in \Omega \times [0,\infty)^2, \tag{1.8}$$

Ishida [19] proved that the following problem

$$\begin{cases} n_t + u \cdot \nabla n = \nabla \cdot (D(n)\nabla n) - \nabla \cdot (nS(x,n,c)\nabla c), & x \in \Omega, t > 0, \\ c_t + u \cdot \nabla c = \Delta c - nf(c), & x \in \Omega, t > 0, \\ u_t + \nabla P = \Delta u + n\nabla \phi, & x \in \Omega, t > 0, \\ \nabla \cdot u = 0, & x \in \Omega, t > 0, \end{cases} \tag{1.9}$$

admits a bounded global weak solution, where S_0 is a nondecreasing function on $[0,\infty)$, $D(n) = mn^{m-1}$ and $f(c) = c$. Winkler [46] developed alternative a priori estimates to prove that in a bounded convex domain, $\Omega \subset \mathbb{R}^3$, the chemotaxis–Stokes system (1.9) possesses at least one bounded weak solution, which stabilizes to the spatially homogeneous equilibrium $(\bar{n}_0, 0, 0)$ with $\bar{n}_0 := \frac{1}{|\Omega|}\int_\Omega n_0$ as $t \to \infty$. In this case, the nonnegative function $f \in C^1([0,\infty))$, S satisfies (1.8),

$$D \in C^{\iota}_{loc}([0,\infty)) \quad \text{for some} \quad \iota > 0, \tag{1.10}$$

as well as

$$D(n) \ge C_D n^{m-1} \quad \text{for all} \quad n > 0 \tag{1.11}$$

with some $m > 1$ and $C_D > 0$. For the case of linear diffusion in (1.1) without a logistic source, the global bounded classical solution was established in a two-dimensional space [40] for arbitrarily large initial data under the assumption that $S(x,n,c)$ satisfies (1.2) and (1.3).

Motivated by these previous studies, our main aim is to consider the effect of the diffusion exponent on the boundedness of solutions to (1.1) in a three-dimensional case. In contrast to the chemical consumption setting considered by [24,46], which leads to a quite easily obtainable L^∞ bound for c, our setting of signal production by cells does not allow the application of the important auxiliary lemma of [24,46], which played a crucial role in the proof of boundedness in previous studies. Therefore we derive a different sequence of a priori estimates to overcome this problem.

The remainder of this paper is organized as follows. In Section 2, we recall some preliminary results, state the main results of this study, and prove the local existence of the classical solution to (1.1). In Section 3, we establish an iteration step to develop the main component of our result. In particular, we give a suitable upper bound of $\int_\Omega n_\varepsilon^p + \int_\Omega |\nabla c_\varepsilon|^{2\beta}$, where $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ is the solution of the regularized problem of (1.1). After this crucial step, only a straightforward rearrangement of the standard arguments is required (e.g., see [1] and Lemma A.1 from [30]), i.e., a Moser-type iteration will allow us to pass to a time-independent a priori estimate for n in $L^\infty(\Omega)$.

2. Preliminaries and main results

Before proving our main results, we provide some preliminary lemmas, which play a crucial role in the following proofs. The proofs of these lemmas are not repeated in this study.

Lemma 2.1. *([15,20]) Let $s \geq 1$ and $q \geq 1$. Assume that $p > 0$ and $a \in (0,1)$ satisfy*

$$\frac{1}{2} - \frac{p}{3} = (1-a)\frac{q}{s} + \frac{a}{6} \quad \text{and} \quad p \leq a.$$

Then, $c_0, c_0' > 0$ exist such that for all $u \in W^{1,2}(\Omega) \cap L^{\frac{s}{q}}(\Omega)$,

$$\|u\|_{W^{p,2}(\Omega)} \leq c_0 \|\nabla u\|^a_{L^2(\Omega)} \|u\|^{1-a}_{L^{\frac{s}{q}}(\Omega)} + c_0' \|u\|_{L^{\frac{s}{q}}(\Omega)}.$$

Lemma 2.2. *([50]) Let $0 < \theta \leq p \leq 6$. A positive constant C_{GN} exists such that for all $u \in W^{1,2}(\Omega) \cap L^\theta(\Omega)$,*

$$\|u\|_{L^p(\Omega)} \leq C_{GN}(\|\nabla u\|^a_{L^2(\Omega)} \|u\|^{1-a}_{L^\theta(\Omega)} + \|u\|_{L^\theta(\Omega)})$$

is valid with $a = \dfrac{\frac{3}{\theta} - \frac{3}{p}}{\frac{3}{\theta} - \frac{1}{2}} \in (0,1)$.

In this study, we assume that

$$\phi \in W^{1,\infty}(\Omega). \tag{2.1}$$

Moreover, let the initial data (n_0, c_0, u_0) satisfy

$$\begin{cases} n_0 \in C^\kappa(\bar{\Omega}) \ \text{ for certain } \ \kappa > 0 \ \text{ with } \ n_0 \geq 0 \ \text{ in } \ \Omega, \\ c_0 \in W^{1,\infty}(\Omega) \ \text{ with } \ c_0 \geq 0 \ \text{ in } \ \bar{\Omega}, \\ u_0 \in D(A_r^\gamma) \ \text{ for some } \ \gamma \in (\frac{1}{2}, 1) \ \text{ and any } \ r \in (1,\infty), \end{cases} \tag{2.2}$$

where A_r denotes the Stokes operator with domain $D(A_r) := W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega) \cap L^r_\sigma(\Omega)$, and $L^r_\sigma(\Omega) := \{\varphi \in L^r(\Omega) | \nabla \cdot \varphi = 0\}$ for $r \in (1,\infty)$ [29].

Theorem 2.1. *Let (1.2), (1.3), and (2.1) hold, and suppose that D satisfies (1.10) and (1.11), with some*

$$m > \max\{\frac{6}{5} - \alpha, \frac{1}{3}\}, \tag{2.3}$$

and $\alpha > 0$. Then for any choice of n_0, c_0, and u_0 that satisfies (2.2), the problem (1.1) possesses at least one global weak solution (n, c, u, P) in the sense of Definition 2.1 below. This solution is bounded in $\Omega \times (0,\infty)$ in the sense that

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|u(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C \quad \text{for all} \quad t > 0. \tag{2.4}$$

Furthermore, c and u are continuous in $\bar{\Omega} \times [0,\infty)$, and

$$n \in C^0_{\omega-*}([0,\infty); L^\infty(\Omega)) \quad \text{in} \quad L^\infty(\Omega \times [0,\infty)), \tag{2.5}$$

i.e., n is continuous on $[0,\infty)$ as an $L^\infty(\Omega)$-valued function with respect to the weak-$*$ topology.

The assumption (2.3) originates from the modeling approaches, including the volume-filling effect, i.e., the movement of cells is inhibited near points where the cells are densely packed (see [28]), and thus the exponent $\alpha > 0$ implies that the role of chemotaxis will weaken when the cell density increases.

Definition 2.1. Let $T > 0$ and (n_0, c_0, u_0) satisfies (2.2). Then, a triple of functions (n, c, u) is called a weak solution of (1.1) if the following conditions are satisfied

$$\begin{cases} n \in L^1_{loc}(\bar{\Omega} \times [0,T)), \\ c \in L^\infty_{loc}(\bar{\Omega} \times [0,T)) \cap L^1_{loc}([0,T); W^{1,1}(\Omega)), \\ u \in L^1_{loc}([0,T); W^{1,1}(\Omega)), \end{cases} \tag{2.6}$$

where $n \geq 0$ and $c \geq 0$ in $\Omega \times (0,T)$ as well as $\nabla \cdot u = 0$ in the distributional sense in $\Omega \times (0,T)$; moreover,

$$H(n), \quad n|\nabla c| \quad \text{and} \quad n|u| \;\text{ belong to }\; L^1_{loc}(\bar{\Omega} \times [0,T)), \tag{2.7}$$

and

$$\begin{aligned} -\int_0^T \int_\Omega n\varphi_t - \int_\Omega n_0 \varphi(\cdot,0) = & \int_0^T \int_\Omega H(n)\Delta\varphi + \int_0^T \int_\Omega n(S(x,n,c)\cdot\nabla c)\cdot\nabla\varphi \\ & + \int_0^T \int_\Omega nu\cdot\nabla\varphi + a\int_0^T \int_\Omega \varphi n - b\int_0^T \int_\Omega \varphi n^2 \end{aligned} \tag{2.8}$$

for any $\varphi \in C_0^\infty(\bar{\Omega} \times [0,T))$ that satisfies $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega \times (0,T)$, as well as

$$-\int_0^T \int_\Omega c\varphi_t - \int_\Omega c_0\varphi(\cdot,0) = -\int_0^T \int_\Omega \nabla c\cdot\nabla\varphi - \int_0^T \int_\Omega c\varphi + \int_0^T \int_\Omega n\varphi + \int_0^T \int_\Omega cu\cdot\nabla\varphi \tag{2.9}$$

for any $\varphi \in C_0^\infty(\bar{\Omega} \times [0,T))$ and

$$-\int_0^T \int_\Omega u\cdot\varphi_t - \int_\Omega u_0\cdot\varphi(\cdot,0) = -\int_0^T \int_\Omega \nabla u\cdot\nabla\varphi - \int_0^T \int_\Omega n\nabla\phi\cdot\varphi + \int_0^T \int_\Omega g\varphi \tag{2.10}$$

for any $\varphi \in C_0^\infty(\bar{\Omega} \times [0,T); \mathbb{R}^3)$ that satisfies $\nabla\varphi \equiv 0$ in $\Omega \times (0,T)$, where we let

$$H(s) = \int_0^s D(\sigma)d\sigma \quad \text{for} \quad s \geq 0.$$

If $\Omega \times (0,\infty) \longrightarrow \mathbb{R}^5$ is a weak solution of (1.1) in $\Omega \times (0,T)$ for all $T > 0$, then we call (n,c,u) a global weak solution of (1.1).

Given Definition 2.1, we can verify that (1.1) is globally solvable by applying the previously established estimates and in the standard manner of passing to the limit.

It is known that the solutions to (1.1) will serve as the limit of the solutions to the corresponding regularized system. Thus, in this section, we first establish the regularized system of (1.1). Thus, we approximate the diffusion coefficient function in (1.1) by a family $(D_\varepsilon)_{\varepsilon\in(0,1)}$ of functions

$$D_\varepsilon \in C^2((0,\infty)) \text{ such that } D_\varepsilon(n) \geq \varepsilon \text{ for all } n > 0$$
$$\text{and } D(n) \leq D_\varepsilon(n) \leq D(n) + 2\varepsilon \text{ for all } n > 0 \text{ and } \varepsilon \in (0,1).$$

Next, we let $(\rho_\varepsilon)_{\varepsilon\in(0,1)} \in C_0^\infty(\Omega)$ be a family of standard cut-off functions that satisfy $0 \le \rho_\varepsilon \le 1$ in Ω and $\rho_\varepsilon \to 1$ in Ω as $\varepsilon \to 0$, and we define

$$S_\varepsilon(x,n,c) = \rho_\varepsilon(x) S(x,n,c), \quad x \in \bar{\Omega}, \quad n \ge 0, \quad c \ge 0 \tag{2.11}$$

for $\varepsilon \in (0,1)$ to approximate the sensitivity tensor S, which implies that $S_\varepsilon(x,n,c) = 0$ on $\partial\Omega$ for each $\varepsilon \in (0,1)$. Therefore, the regularized problem of (1.1) is presented as follows.

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon = \nabla\cdot(D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon) - \nabla\cdot(n_\varepsilon S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon) + a n_\varepsilon - b n_\varepsilon^2, \quad x\in\Omega, t>0, \\ c_{\varepsilon t} + u_\varepsilon\cdot\nabla c_\varepsilon = \Delta c_\varepsilon - c_\varepsilon + n_\varepsilon, \quad x\in\Omega, t>0, \\ u_{\varepsilon t} + \nabla P_\varepsilon = \Delta u_\varepsilon + n_\varepsilon\nabla\phi + g(x,t), \quad x\in\Omega, t>0, \\ \nabla\cdot u_\varepsilon = 0, \quad x\in\Omega, t>0, \\ \nabla n_\varepsilon\cdot\nu = \nabla c_\varepsilon\cdot\nu = 0, u_\varepsilon = 0, \quad x\in\partial\Omega, t>0, \\ n_\varepsilon(x,0) = n_0(x), c_\varepsilon(x,0) = c_0(x), u_\varepsilon(x,0) = u_0(x), \quad x\in\Omega. \end{cases} \tag{2.12}$$

Given the well-established fixed point arguments (see [46], Lemma 2.1 in [28], and Lemma 2.1 in [45]), we can prove that (2.12) is locally solvable in the classical sense, which is stated as the following lemma.

Lemma 2.3. *Assume that $\varepsilon \in (0,1)$. Then, $T_{max} \in (0,\infty]$ and a classical solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ of (2.12) in $\Omega\times(0,T_{max})$ exists such that*

$$\begin{cases} n_\varepsilon \in C^0(\bar{\Omega}\times[0,T_{max})) \cap C^{2,1}(\bar{\Omega}\times(0,T_{max})), \\ c_\varepsilon \in C^0(\bar{\Omega}\times[0,T_{max})) \cap C^{2,1}(\bar{\Omega}\times(0,T_{max})), \\ u_\varepsilon \in C^0(\bar{\Omega}\times[0,T_{max})) \cap C^{2,1}(\bar{\Omega}\times(0,T_{max})), \\ P_\varepsilon \in C^{1,0}(\bar{\Omega}\times(0,T_{max})), \end{cases} \tag{2.13}$$

by classically solving (2.12) in $\Omega\times[0,T_{max})$. Moreover, n_ε and c_ε are nonnegative in $\Omega\times(0,T_{max})$, and

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|A^\gamma u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \to \infty \quad as \quad t\to T_{max}, \tag{2.14}$$

where γ is given by (2.2).

3. A priori estimates

In this section, we establish an iteration step to develop the main component of our result. The iteration step depends on a series of a priori estimates. First, let us derive the following a priori bound for the solutions of model (2.12).

Lemma 3.1. *([46]) Let $l \in [1,+\infty)$ and $r \in [1,+\infty]$ be such that*

$$\begin{cases} l < \frac{3r}{3-r} \quad if \quad r \le 3, \\ l \le \infty \quad if \quad r > 3. \end{cases} \tag{3.1}$$

Then, for all $K > 0$, $C = C(l,r,K)$ exists such that if for some $\varepsilon \in (0,1)$, we have

$$\|n_\varepsilon(\cdot,t)\|_{L^r(\Omega)} \le K \quad for\ all \quad t \in (0,T_{max}), \tag{3.2}$$

then

$$\|Du_\varepsilon(\cdot,t)\|_{L^l(\Omega)} \le C \quad for\ all \quad t\in(0,T_{max}). \tag{3.3}$$

The proof of this lemma is very similar to that of Lemma 2.2–Lemma 2.6 in [34], so we omit its proof.

Lemma 3.2. *$C > 0$ exists such that the solution of (2.12) satisfies*

$$\int_\Omega n_\varepsilon + \int_\Omega |\nabla u_\varepsilon|^2 + \int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega c_\varepsilon \le C \quad \text{for all} \quad t \in (0, T_{max}). \tag{3.4}$$

Lemma 3.3. *Let $p > 1$. Then, the solution of (2.12) from Lemma 2.3 satisfies*

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|n_\varepsilon\|^p_{L^p(\Omega)} + \frac{C_D(p-1)}{2}\int_\Omega n_\varepsilon^{m+p-3}|\nabla n_\varepsilon|^2 \\ &\le \frac{(p-1)C_S^2}{2C_D}\int_\Omega n_\varepsilon^{p+1-m-2\alpha}|\nabla c_\varepsilon|^2 + a\int_\Omega n_\varepsilon^p - b\int_\Omega n_\varepsilon^{p+1}. \end{aligned} \tag{3.5}$$

Proof. By taking n_ε^{p-1} as the test function for the first equation of (2.12), combining with the second equation, and using $\nabla \cdot u_\varepsilon = 0$, we obtain

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|n_\varepsilon\|^p_{L^p(\Omega)} + (p-1)\int_\Omega n_\varepsilon^{p-2}D_\varepsilon(n_\varepsilon)|\nabla n_\varepsilon|^2 \\ &\le (p-1)\int_\Omega n_\varepsilon^{p-1}\nabla n_\varepsilon \cdot (S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon) + a\int_\Omega n_\varepsilon^p - b\int_\Omega n_\varepsilon^{p+1} \\ &\le (p-1)C_S\int_\Omega n_\varepsilon^{p-\alpha-1}|\nabla n_\varepsilon||\nabla c_\varepsilon| + a\int_\Omega n_\varepsilon^p - b\int_\Omega n_\varepsilon^{p+1} \\ &\le \frac{C_D(p-1)}{2}\int_\Omega n_\varepsilon^{m+p-3}|\nabla n_\varepsilon|^2 + \frac{(p-1)C_S^2}{2C_D}\int_\Omega {n_\varepsilon}^{p+1-m-2\alpha}|\nabla c_\varepsilon|^2 \\ &\quad + a\int_\Omega n_\varepsilon^p - b\int_\Omega n_\varepsilon^{p+1}, \end{aligned} \tag{3.6}$$

which together with (1.11) implies that

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|n_\varepsilon\|^p_{L^p(\Omega)} + \frac{C_D(p-1)}{2}\int_\Omega n_\varepsilon^{m+p-3}|\nabla n_\varepsilon|^2 \\ &\le \frac{(p-1)C_S^2}{2C_D}\int_\Omega n_\varepsilon^{p+1-m-2\alpha}|\nabla c_\varepsilon|^2 + a\int_\Omega n_\varepsilon^p - b\int_\Omega n_\varepsilon^{p+1}. \end{aligned} \tag{3.7}$$

This completes the proof of Lemma 3.3. □

Lemma 3.4. *For all $\beta > 1$, the solution of (2.12) from Lemma 2.3 satisfies*

$$\begin{aligned} &\frac{1}{2\beta}\frac{d}{dt}\|\nabla c_\varepsilon\|^{2\beta}_{L^{2\beta}(\Omega)} + \frac{3(\beta-1)}{4\beta^2}\int_\Omega \left|\nabla|\nabla c_\varepsilon|^\beta\right|^2 + \frac{1}{2}\int_\Omega |\nabla c_\varepsilon|^{2\beta-2}|D^2c_\varepsilon|^2 + \int_\Omega |\nabla c_\varepsilon|^{2\beta} \\ &\le (3+2(\beta-1))\int_\Omega n_\varepsilon^2|\nabla c_\varepsilon|^{2\beta-2} + (3+2(\beta-1))\int_\Omega |u_\varepsilon|^2|\nabla c_\varepsilon|^{2\beta} + C \quad \text{for all} \quad t\in(0,T_{max}), \end{aligned} \tag{3.8}$$

where $C > 0$ is a positive constant that is independent of ε.

Proof. Given that $\nabla c_\varepsilon \cdot \nabla \Delta c_\varepsilon = \frac{1}{2}\Delta|\nabla c_\varepsilon|^2 - |D^2 c_\varepsilon|^2$, then by a straightforward computation using the second equation in (2.12) and several integrations by parts, we find that

$$\begin{aligned}
\frac{1}{2\beta}\frac{d}{dt}\|\nabla c_\varepsilon\|^{2\beta}_{L^{2\beta}(\Omega)} &= \int_\Omega |\nabla c_\varepsilon|^{2\beta-2}\nabla c_\varepsilon \cdot \nabla(\Delta c_\varepsilon - c_\varepsilon + n_\varepsilon - u_\varepsilon \cdot \nabla c_\varepsilon) \\
&= \frac{1}{2}\int_\Omega |\nabla c_\varepsilon|^{2\beta-2}\Delta|\nabla c_\varepsilon|^2 - \int_\Omega |\nabla c_\varepsilon|^{2\beta-2}|D^2 c_\varepsilon|^2 - \int_\Omega |\nabla c_\varepsilon|^{2\beta} \\
&\quad - \int_\Omega n_\varepsilon \nabla\cdot(|\nabla c_\varepsilon|^{2\beta-2}\nabla c_\varepsilon) + \int_\Omega (u_\varepsilon\cdot\nabla c_\varepsilon)\nabla\cdot(|\nabla c_\varepsilon|^{2\beta-2}\nabla c_\varepsilon) \\
&= -\frac{\beta-1}{2}\int_\Omega |\nabla c_\varepsilon|^{2\beta-4}\left|\nabla|\nabla c_\varepsilon|^2\right|^2 + \frac{1}{2}\int_{\partial\Omega} |\nabla c_\varepsilon|^{2\beta-2}\frac{\partial|\nabla c_\varepsilon|^2}{\partial\nu} - \int_\Omega |\nabla c_\varepsilon|^{2\beta} \\
&\quad - \int_\Omega |\nabla c_\varepsilon|^{2\beta-2}|D^2 c_\varepsilon|^2 - \int_\Omega n_\varepsilon|\nabla c_\varepsilon|^{2\beta-2}\Delta c_\varepsilon - \int_\Omega n_\varepsilon\nabla c_\varepsilon\cdot\nabla(|\nabla c_\varepsilon|^{2\beta-2}) \\
&\quad + \int_\Omega (u_\varepsilon\cdot\nabla c_\varepsilon)|\nabla c_\varepsilon|^{2\beta-2}\Delta c_\varepsilon + \int_\Omega (u_\varepsilon\cdot\nabla c_\varepsilon)\nabla c_\varepsilon\cdot\nabla(|\nabla c_\varepsilon|^{2\beta-2}) \\
&= -\frac{2(\beta-1)}{\beta^2}\int_\Omega \left|\nabla|\nabla c_\varepsilon|^\beta\right|^2 + \frac{1}{2}\int_{\partial\Omega} |\nabla c_\varepsilon|^{2\beta-2}\frac{\partial|\nabla c_\varepsilon|^2}{\partial\nu} - \int_\Omega |\nabla c_\varepsilon|^{2\beta-2}|D^2 c_\varepsilon|^2 \\
&\quad - \int_\Omega n_\varepsilon|\nabla c_\varepsilon|^{2\beta-2}\Delta c_\varepsilon - \int_\Omega n_\varepsilon\nabla c_\varepsilon\cdot\nabla(|\nabla c_\varepsilon|^{2\beta-2}) - \int_\Omega |\nabla c_\varepsilon|^{2\beta} \\
&\quad + \int_\Omega (u_\varepsilon\cdot\nabla c_\varepsilon)|\nabla c_\varepsilon|^{2\beta-2}\Delta c_\varepsilon + \int_\Omega (u_\varepsilon\cdot\nabla c_\varepsilon)\nabla c_\varepsilon\cdot\nabla(|\nabla c_\varepsilon|^{2\beta-2})
\end{aligned} \tag{3.9}$$

for all $t \in (0, T_{max})$. In this case, since $|\Delta c_\varepsilon| \le \sqrt{3}|D^2 c_\varepsilon|$, by Young's inequality, we can estimate

$$\begin{aligned}
\textstyle\int_\Omega n_\varepsilon|\nabla c_\varepsilon|^{2\beta-2}\Delta c_\varepsilon &\le \sqrt{3}\int_\Omega n_\varepsilon|\nabla c_\varepsilon|^{2\beta-2}|D^2 c_\varepsilon| \\
&\le \frac{1}{4}\int_\Omega |\nabla c_\varepsilon|^{2\beta-2}|D^2 c_\varepsilon|^2 + 3\int_\Omega n_\varepsilon^2|\nabla c_\varepsilon|^{2\beta-2}
\end{aligned} \tag{3.10}$$

and similarly,

$$\begin{aligned}
\textstyle\int_\Omega (u_\varepsilon\cdot\nabla c_\varepsilon)|\nabla c_\varepsilon|^{2\beta-2}\Delta c_\varepsilon &\le \sqrt{3}\int_\Omega |u_\varepsilon\cdot\nabla c_\varepsilon||\nabla c_\varepsilon|^{2\beta-2}|D^2 c_\varepsilon| \\
&\le \frac{1}{4}\int_\Omega |\nabla c_\varepsilon|^{2\beta-2}|D^2 c_\varepsilon|^2 + 3\int_\Omega |u_\varepsilon\cdot\nabla c_\varepsilon|^2|\nabla c_\varepsilon|^{2\beta-2} \\
&\le \frac{1}{4}\int_\Omega |\nabla c_\varepsilon|^{2\beta-2}|D^2 c_\varepsilon|^2 + 3\int_\Omega |u_\varepsilon|^2|\nabla c_\varepsilon|^{2\beta}
\end{aligned} \tag{3.11}$$

for all $t \in (0, T_{max})$. Moreover, by the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
-\int_\Omega n_\varepsilon \nabla c_\varepsilon \cdot \nabla(|\nabla c_\varepsilon|^{2\beta-2}) &= -(\beta-1)\int_\Omega n_\varepsilon |\nabla c_\varepsilon|^{2(\beta-2)}\nabla c_\varepsilon \cdot \nabla |\nabla c_\varepsilon|^2 \\
&\le \frac{\beta-1}{8}\int_\Omega |\nabla c_\varepsilon|^{2\beta-4}\left|\nabla|\nabla c_\varepsilon|^2\right|^2 + 2(\beta-1)\int_\Omega |n_\varepsilon|^2|\nabla c_\varepsilon|^{2\beta-2} \\
&\le \frac{(\beta-1)}{2\beta^2}\int_\Omega \left|\nabla|\nabla c_\varepsilon|^\beta\right|^2 + 2(\beta-1)\int_\Omega |n_\varepsilon|^2|\nabla c_\varepsilon|^{2\beta-2}
\end{aligned}
\tag{3.12}
$$

and

$$
\begin{aligned}
\int_\Omega (u_\varepsilon\cdot\nabla c_\varepsilon)\nabla c_\varepsilon \cdot \nabla(|\nabla c_\varepsilon|^{2\beta-2}) &= (\beta-1)\int_\Omega (u_\varepsilon\cdot\nabla c_\varepsilon)|\nabla c_\varepsilon|^{2(\beta-2)}\nabla c_\varepsilon\cdot\nabla|\nabla c_\varepsilon|^2 \\
&\le \frac{\beta-1}{8}\int_\Omega |\nabla c_\varepsilon|^{2\beta-4}\left|\nabla|\nabla c_\varepsilon|^2\right|^2 \\
&\quad + 2(\beta-1)\int_\Omega |u_\varepsilon\cdot\nabla c_\varepsilon|^2|\nabla c_\varepsilon|^{2\beta-2} \\
&\le \frac{(\beta-1)}{2\beta^2}\int_\Omega \left|\nabla|\nabla c_\varepsilon|^\beta\right|^2 + 2(\beta-1)\int_\Omega |u_\varepsilon|^2|\nabla c_\varepsilon|^{2\beta}.
\end{aligned}
\tag{3.13}
$$

Next, we deal with the integration on $\partial\Omega$. From Lemma 2.1, we can see that

$$
\begin{aligned}
&\int_{\partial\Omega} \frac{\partial|\nabla c_\varepsilon|^2}{\partial\nu}|\nabla c_\varepsilon|^{2\beta-2} \\
&\le C_\Omega \int_{\partial\Omega} |\nabla c_\varepsilon|^{2\beta} \\
&= C_\Omega \||\nabla c_\varepsilon|^\beta\|^2_{L^2(\partial\Omega)}.
\end{aligned}
\tag{3.14}
$$

Let us take $r\in(0,\frac{1}{2})$. The embedding $W^{r+\frac{1}{2},2}(\Omega)\hookrightarrow L^2(\partial\Omega)$ is compact (e.g., see Proposition 4.22 (ii) in [16]), so we have

$$
\||\nabla c_\varepsilon|^\beta\|^2_{L^2(\partial\Omega)} \le C_1\||\nabla c_\varepsilon|^\beta\|^2_{W^{r+\frac{1}{2},2}(\Omega)}.
\tag{3.15}
$$

In order to apply Lemma 2.1 to the right-hand side of (3.15), let us select $a\in(0,1)$ to satisfy

$$
a = \frac{3\beta+2r-2}{3\beta-1}.
$$

We note that $r\in(0,\frac{1}{2})$ and $\beta>1$ implies that $r+\frac{1}{2}\le a<1$, so from the fractional Gagliardo–Nirenberg inequality (Lemma 2.1) and the boundedness of $|\nabla c_\varepsilon|^2$ (see Lemma 3.2), we can see that

$$
\begin{aligned}
&\||\nabla c_\varepsilon|^\beta\|^2_{W^{r+\frac{1}{2},2}(\Omega)} \\
&\le c_0\|\nabla|\nabla c_\varepsilon|^\beta\|^a_{L^2(\Omega)}\||\nabla c_\varepsilon|^\beta\|^{1-a}_{L^{\frac{2}{\beta}}(\Omega)} + c_0'\||\nabla c_\varepsilon|^\beta\|_{L^{\frac{2}{\beta}}(\Omega)} \\
&\le C_2\|\nabla|\nabla c_\varepsilon|^\beta\|^a_{L^2(\Omega)} + C_2.
\end{aligned}
\tag{3.16}
$$

By combining (3.14) and (3.15) with (3.16), we obtain

$$\int\limits_{\partial\Omega} \frac{\partial|\nabla c_\varepsilon|^2}{\partial\nu}|\nabla c_\varepsilon|^{2\beta-2} \leq C_3\|\nabla|\nabla c_\varepsilon|^\beta\|^a_{L^2(\Omega)} + C_3. \tag{3.17}$$

Now, by inserting (3.10)–(3.13) and (3.17) into (3.9), and using Young's inequality, we can obtain the results. This completes the proof of Lemma 3.4. □

By Lemma 3.3–3.4, we can obtain the following Lemma.

Lemma 3.5. *Assume that $m > 1 - \alpha$, and $\beta > 1$. If*

$$\max\{1, \frac{4}{3} - m, m + 2\alpha - \frac{2}{3}\} < p < [2(m+\alpha) - 2](3\beta - 1) + \frac{4}{3} - m,$$

then for all, small $\delta > 0$, we can find a constant $C := C(p, \beta, \delta) > 0$ such that

$$\int\limits_\Omega n_\varepsilon^{p+1-m-2\alpha}|\nabla c_\varepsilon|^2 \leq \delta\int\limits_\Omega |\nabla n_\varepsilon^{\frac{m+p-1}{2}}|^2 + \delta\|\nabla|\nabla c_\varepsilon|^\beta\|^2_{L^2(\Omega)} + C \quad \text{for all } t \in (0, T_{max}). \tag{3.18}$$

Proof. By the Hölder inequality, we have

$$\begin{aligned} J_1 &:= \int\limits_\Omega n_\varepsilon^{p+1-m-2\alpha}|\nabla c_\varepsilon|^2 \\ &\leq \left(\int\limits_\Omega {n_\varepsilon}^{3(p+1-m-2\alpha)}\right)^{\frac{1}{3}} \left(\int\limits_\Omega |\nabla c_\varepsilon|^3\right)^{\frac{2}{3}} \\ &= \|n_\varepsilon^{\frac{p+m-1}{2}}\|^{\frac{2(p+1-m-2\alpha)}{p+m-1}}_{L^{\frac{6(p+1-m-2\alpha)}{p+m-1}}(\Omega)} \|\nabla c_\varepsilon\|^2_{L^3(\Omega)}. \end{aligned} \tag{3.19}$$

Since, $m \geq 1 - \alpha$ and $p \geq m + 2\alpha - \frac{2}{3}$, we have

$$\frac{2}{p+m-1} \leq \frac{6(p+1-m-2\alpha)}{p+m-1} \leq 6,$$

which together with Lemma 2.2 and $m > 1 - \alpha$ implies that

$$\begin{aligned} &\|n_\varepsilon^{\frac{p+m-1}{2}}\|^{\frac{2(p+1-m-2\alpha)}{p+m-1}}_{L^{\frac{6(p+1-m-2\alpha)}{p+m-1}}(\Omega)} \\ &\leq C_4(\|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\|^{\mu_1}_{L^2(\Omega)}\|n_\varepsilon^{\frac{p+m-1}{2}}\|^{1-\mu_1}_{L^{\frac{2}{p+m-1}}(\Omega)} + \|n_\varepsilon^{\frac{p+m-1}{2}}\|_{L^{\frac{2}{p+m-1}}(\Omega)})^{\frac{2(p+1-m-2\alpha)}{p+m-1}} \\ &\leq C_5(\|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\|^{\frac{2(p+1-m-2\alpha)\mu_1}{p+m-1}}_{L^2(\Omega)} + 1) \\ &= C_5(\|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\|^{2(1-\frac{6(m+\alpha-1)}{3p+3m-4})}_{L^2(\Omega)} + 1) \end{aligned} \tag{3.20}$$

with some positive constants C_4, C_5 and

$$\mu_1 = \frac{\frac{3[p+m-1]}{2} - \frac{3(p+m-1)}{6(p+1-m-2\alpha)}}{-\frac{1}{2} + \frac{3[p+m-1]}{2}} = [p+m-1]\frac{\frac{3}{2} - \frac{3}{6(p+1-m-2\alpha)}}{-\frac{1}{2} + \frac{3[p+m-1]}{2}} \in (0,1).$$

In this case, we have use the fact that

$$2(1-\frac{6(m+\alpha-1)}{3p+3m-4})<2.$$

In addition, due to Lemma 2.2 and the fact that $\beta > 1 \geq \frac{1}{2}$, we have

$$\begin{aligned}
\|\nabla c_\varepsilon\|^2_{L^3(\Omega)} &= \||\nabla c_\varepsilon|^\beta\|^{\frac{2}{\beta}}_{L^{\frac{3}{\beta}}(\Omega)} \\
&\leq C_6(\|\nabla|\nabla c_\varepsilon|^\beta\|^{\frac{2\mu_2}{\beta}}_{L^2(\Omega)}\||\nabla c_\varepsilon|^\beta\|^{\frac{2(1-\mu_2)}{\beta}}_{L^{\frac{2}{\beta}}(\Omega)} + \||\nabla c_\varepsilon|^\beta\|^{\frac{2}{\beta}}_{L^{\frac{2}{\beta}}(\Omega)}) \\
&\leq C_7(\|\nabla|\nabla c_\varepsilon|^\beta\|^{\frac{2\mu_2}{\beta}}_{L^2(\Omega)} + 1),
\end{aligned} \tag{3.21}$$

with some positive constants C_6, C_7 and

$$\mu_2 = \frac{\frac{3\beta}{2} - \frac{3\beta}{3}}{-\frac{1}{2} + \frac{3\beta}{2}} = \beta\frac{\frac{3}{2} - 1}{-\frac{1}{2} + \frac{3\beta}{2}} \in (0, 1).$$

By inserting (3.20)–(3.21) into (3.19) and using Young's inequality, we have

$$\begin{aligned}
J_1 &\leq C_8(\|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\|^{2(1-\frac{6(m+\alpha-1)}{3p+3m-4})}_{L^2(\Omega)} + 1)(\|\nabla|\nabla c_\varepsilon|^\beta\|^{\frac{2\mu_2}{\beta}}_{L^2(\Omega)} + 1) \\
&= C_8(\|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\|^{2(1-\frac{6(m+\alpha-1)}{3p+3m-4})}_{L^2(\Omega)} + 1)(\|\nabla|\nabla c_\varepsilon|^\beta\|^{\frac{3(1-\frac{2}{3})}{-\frac{1}{2}+\frac{3\beta}{2}}}_{L^2(\Omega)} + 1) \\
&\leq \delta\int_\Omega |\nabla n_\varepsilon^{\frac{m+p-1}{2}}|^2 + \delta\|\nabla|\nabla c_\varepsilon|^\beta\|^2_{L^2(\Omega)} + C_9 \quad \text{for all} \quad t \in (0, T_{max}),
\end{aligned} \tag{3.22}$$

where we have used the fact that $p < [2(m+\alpha) - 2](3\beta - 1) + \frac{4}{3} - m$ and $p > \frac{4}{3} - m$. □

Lemma 3.6. *Assume that $\beta > 2$ and $\theta \geq \frac{3\beta}{2\beta+1}$. If*

$$p > \max\{1, 1 - m + \frac{2\theta}{3}, \frac{2\theta - 1}{5\theta - 3}(3\beta - 1) + \frac{4}{3} - m\},$$

then for all small $\delta > 0$, we can find a constant $C := C(p, \theta, \beta, \delta) > 0$ such that

$$\int_\Omega n_\varepsilon^2|\nabla c_\varepsilon|^{2\beta-2} \leq \delta\int_\Omega |\nabla n_\varepsilon^{\frac{m+p-1}{2}}|^2 + \delta\|\nabla|\nabla c_\varepsilon|^\beta\|^2_{L^2(\Omega)} + C \quad \textit{for all} \quad t \in (0, T_{max}). \tag{3.23}$$

Proof. First, due to the Hölder inequality, we have

$$\begin{aligned}
J_2 &:= \int_\Omega n_\varepsilon^2|\nabla c_\varepsilon|^{2\beta-2} \\
&\leq \left(\int_\Omega n_\varepsilon^{2\theta}\right)^{\frac{1}{\theta}}\left(\int_\Omega |\nabla c_\varepsilon|^{(2\beta-2)\theta'}\right)^{\frac{1}{\theta'}} \\
&= \|n_\varepsilon^{\frac{p+m-1}{2}}\|^{\frac{4}{p+m-1}}_{L^{\frac{4\theta}{p+m-1}}(\Omega)}\|\nabla c_\varepsilon\|^{(2\beta-2)}_{L^{(2\beta-2)\theta'}(\Omega)},
\end{aligned} \tag{3.24}$$

where θ and θ' satisfy $\frac{1}{\theta} + \frac{1}{\theta'} = 1$.

Moreover, by using $p > 1 - m + \frac{2\theta}{3}$ and Lemma 2.2, we can conclude that

$$\begin{aligned} &\|n_\varepsilon^{\frac{p+m-1}{2}}\|_{L^{\frac{4\theta}{p+m-1}}(\Omega)}^{\frac{4}{p+m-1}} \\ &\le C_{10}(\|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\|_{L^2(\Omega)}^{\mu_3}\|n_\varepsilon^{\frac{p+m-1}{2}}\|_{L^{\frac{2}{p+m-1}}(\Omega)}^{(1-\mu_3)} + \|n_\varepsilon^{\frac{p+m-1}{2}}\|_{L^{\frac{2}{p+m-1}}(\Omega)})^{\frac{4}{p+m-1}} \\ &\le C_{11}(\|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\|_{L^2(\Omega)}^{\frac{4\mu_3}{p+m-1}} + 1) \end{aligned} \tag{3.25}$$

with some positive constants C_{10}, C_{11} and

$$\mu_3 = \frac{\frac{3[p+m-1]}{2} - \frac{3(p+m-1)}{4\theta}}{-\frac{1}{2} + \frac{3[p+m-1]}{2}} = [p+m-1]\frac{\frac{3}{2} - \frac{3}{4\theta}}{-\frac{1}{2} + \frac{3[p+m-1]}{2}} \in (0,1).$$

Then, from $\theta \ge \frac{3\beta}{2\beta+1}$, $\beta > 2$, and Lemma 2.2, it follows that

$$\begin{aligned} \|\nabla c_\varepsilon\|_{L^{(2\beta-2)\theta'}(\Omega)}^{(2\beta-2)} &= \||\nabla c_\varepsilon|^\beta\|_{L^{\frac{(2\beta-2)\theta'}{\beta}}(\Omega)}^{\frac{2\beta-2}{\beta}} \\ &\le C_{12}(\|\nabla|\nabla c_\varepsilon|^\beta\|_{L^2(\Omega)}^{\frac{(2\beta-2)\mu_4}{\beta}}\||\nabla c_\varepsilon|^\beta\|_{L^{\frac{2}{\beta}}(\Omega)}^{\frac{(2\beta-2)(1-\mu_4)}{\beta}} + \||\nabla c_\varepsilon|^\beta\|_{L^{\frac{2}{\beta}}(\Omega)}^{\frac{(2\beta-2)}{\beta}}) \\ &\le C_{13}(\|\nabla|\nabla c_\varepsilon|^\beta\|_{L^2(\Omega)}^{\frac{(2\beta-2)\mu_4}{\beta}} + 1), \end{aligned} \tag{3.26}$$

with some positive constants C_{12}, C_{13} and

$$\mu_4 = \frac{\frac{3\beta}{2} - \frac{3\beta}{(2\beta-2)\theta'}}{-\frac{1}{2} + \frac{3\beta}{2}} = \beta\frac{\frac{3}{2} - \frac{3}{(2\beta-2)\theta'}}{-\frac{1}{2} + \frac{3\beta}{2}} \in (0,1).$$

By inserting (3.30)–(3.31) into (3.29) and using $p > \frac{2\theta-1}{5\theta-3}(3\beta-1) + \frac{4}{3} - m$ and Lemma 2.2, we have

$$\begin{aligned} J_2 &\le C_{14}(\|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\|_{L^2(\Omega)}^{\frac{3(2-\frac{1}{\theta})}{-\frac{1}{2}+\frac{3[p+m-1]}{2}}} + 1)(\|\nabla|\nabla c_\varepsilon|^\beta\|_{L^2(\Omega)}^{\frac{3(\beta-1-\frac{1}{\theta'})}{-\frac{1}{2}+\frac{3\beta}{2}}} + 1) \\ &\le \delta\int_\Omega |\nabla n_\varepsilon^{\frac{m+p-1}{2}}|^2 + \delta\|\nabla|\nabla c_\varepsilon|^\beta\|_{L^2(\Omega)}^2 + C_{15} \quad \text{for all} \quad t \in (0, T_{max}). \quad \square \end{aligned} \tag{3.27}$$

Lemma 3.7. *Assume that $\beta > 1$. If*

$$p > \max\{1, \frac{17}{12} - m, \frac{2}{5}(3\beta - 1) + \frac{4}{3} - m\},$$

then for all small $\delta > 0$, the solution of (2.12) from Lemma 2.3 satisfies

$$\int_\Omega u_\varepsilon^2|\nabla c_\varepsilon|^{2\beta} \le \delta\int_\Omega |\nabla n_\varepsilon^{\frac{m+p-1}{2}}|^2 + \delta\|\nabla|\nabla c_\varepsilon|^\beta\|_{L^2(\Omega)}^2 + C \quad \textit{for all} \quad t \in (0, T_{max}), \tag{3.28}$$

where a positive constant C depends on β and δ.

Proof. First, due to the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ and the Hölder inequality, we have

$$\begin{aligned} J_3 &:= \int_\Omega u_\varepsilon^2 |\nabla c_\varepsilon|^{2\beta} \\ &\le \left(\int_\Omega u_\varepsilon^6\right)^{\frac{1}{3}} \left(\int_\Omega |\nabla c_\varepsilon|^{3\beta}\right)^{\frac{2}{3}} \\ &\le C_\Omega \|Du_\varepsilon\|^2_{L^2(\Omega)} \|\nabla c_\varepsilon\|^{2\beta}_{L^{3\beta}(\Omega)}. \end{aligned} \tag{3.29}$$

$2 < \frac{15}{7}$, and thus Lemma 3.1 and $p \ge \frac{17}{12} - m$ imply that

$$\begin{aligned} &\|Du_\varepsilon\|^2_{L^2(\Omega)} \\ &\le C_{16}(\|n_\varepsilon\|^2_{L^{\frac{5}{4}}(\Omega)} + 1) \\ &= C_{16}(\|n_\varepsilon^{\frac{p+m-1}{2}}\|^{\frac{4}{p+m-1}}_{L^{\frac{\frac{5}{2}}{p+m-1}}(\Omega)} + 1) \\ &\le C_{17}[(\|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\|^{\mu_5}_{L^2(\Omega)} \|n_\varepsilon^{\frac{p+m-1}{2}}\|^{(1-\mu_5)}_{L^{\frac{2}{p+m-1}}(\Omega)} + \|n_\varepsilon^{\frac{p+m-1}{2}}\|_{L^{\frac{2}{p+m-1}}(\Omega)})^{\frac{4}{p+m-1}} + 1] \\ &\le C_{18}(\|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\|^{\frac{4\mu_5}{p+m-1}}_{L^2(\Omega)} + 1) \end{aligned} \tag{3.30}$$

with some positive constants C_{16}, C_{17}, C_{18} and

$$\mu_5 = \frac{\frac{3[p+m-1]}{2} - \frac{3(p+m-1)}{\frac{5}{2}}}{-\frac{1}{2} + \frac{3[p+m-1]}{2}} = [p+m-1]\frac{\frac{3}{2} - \frac{3}{\frac{5}{2}}}{-\frac{1}{2} + \frac{3[p+m-1]}{2}} \in (0,1).$$

In addition, due to Young's inequality, we have

$$\begin{aligned} \|\nabla c_\varepsilon\|^{2\beta}_{L^{3\beta}(\Omega)} &= \||\nabla c_\varepsilon|^\beta\|^2_{L^3(\Omega)} \\ &\le C_{19}(\|\nabla |\nabla c_\varepsilon|^\beta\|^{2\mu_6}_{L^2(\Omega)} \||\nabla c_\varepsilon|^\beta\|^{2(1-\mu_6)}_{L^{\frac{2}{\beta}}(\Omega)} + \||\nabla c_\varepsilon|^\beta\|^2_{L^{\frac{2}{\beta}}(\Omega)}) \\ &\le C_{20}(\|\nabla |\nabla c_\varepsilon|^\beta\|^{2\mu_6}_{L^2(\Omega)} + 1), \end{aligned} \tag{3.31}$$

with some positive constants C_{19}, C_{20} and

$$\mu_6 = \frac{\frac{3\beta}{2} - 1}{-\frac{1}{2} + \frac{3\beta}{2}} \in (0,1).$$

By inserting (3.30)–(3.31) into (3.29), we have

$$\begin{aligned} J_3 &\le C_{21}(\|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\|^{\frac{\frac{6}{5}}{-\frac{1}{2}+\frac{3[p+m-1]}{2}}}_{L^2(\Omega)} + 1)(\|\nabla |\nabla c_\varepsilon|^\beta\|^{\frac{3(\beta-\frac{2}{3})}{-\frac{1}{2}+\frac{3\beta}{2}}}_{L^2(\Omega)} + 1) \\ &\le \delta \int_\Omega |\nabla n_\varepsilon^{\frac{m+p-1}{2}}|^2 + \delta \|\nabla |\nabla c_\varepsilon|^\beta\|^2_{L^2(\Omega)} + C_{22} \quad \text{for all} \quad t \in (0, T_{max}). \quad \square \end{aligned} \tag{3.32}$$

Lemma 3.8. *Assume that $m \ge \frac{1}{3}$ and $p > \max\{1, \frac{3(1-m)}{2}\}$. Then for all small $\delta > 0$, we can find a constant $C := C(\beta, \delta) > 0$ such that*

$$\int_\Omega n_\varepsilon^p \le \delta \int_\Omega |\nabla n_\varepsilon^{\frac{m+p-1}{2}}|^2 + C \quad \textit{for all} \quad t \in (0, T_{max}). \tag{3.33}$$

Proof. $p \geq \max\{1, \frac{3(1-m)}{2}\}$ and $m \geq \frac{1}{3}$, so we have

$$\frac{2}{p+m-1} \leq \frac{2p}{p+m-1} \leq 6,$$

and thus by the Gagliardo–Nirenberg inequality and Young's inequality, we have

$$\begin{aligned} J_4 &:= \int_\Omega n_\varepsilon^p \\ &= \|n_\varepsilon^{\frac{p+m-1}{2}}\|_{L^{\frac{2p}{p+m-1}}(\Omega)}^{\frac{2p}{p+m-1}} \\ &\leq C_{23}(\|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\|_{L^2(\Omega)}^{\mu_7}\|n_\varepsilon^{\frac{p+m-1}{2}}\|_{L^{\frac{2}{p+m-1}}(\Omega)}^{1-\mu_7} + \|n_\varepsilon^{\frac{p+m-1}{2}}\|_{L^{\frac{2}{p+m-1}}(\Omega)})^{\frac{2p}{p+m-1}} \\ &\leq C_{24}(\|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\|_{L^2(\Omega)}^{\frac{3(p-1)}{-\frac{1}{2}+\frac{3[p+m-1]}{2}}} + 1) \\ &\leq \delta \int_\Omega |\nabla n_\varepsilon^{\frac{m+p-1}{2}}|^2 + C_{25}, \end{aligned} \tag{3.34}$$

with some positive constants C_{23}, C_{24}, C_{25} and

$$\mu_7 = \frac{\frac{3[p+m-1]}{2} - \frac{3(p+m-1)}{2p}}{-\frac{1}{2} + \frac{3[p+m-1]}{2}} = [p+m-1]\frac{\frac{3}{2} - \frac{3}{2p}}{-\frac{1}{2} + \frac{3[p+m-1]}{2}} \in (0,1). \quad \square$$

4. Combining previous estimates

Lemma 4.1. *Assume that $m > \{\frac{6}{5} - \alpha, \frac{1}{3}\}$. If*

$$\max\{1, \frac{18\beta-5}{45\beta-15}(3\beta-1) + \frac{4}{3} - m, m + 2\alpha - \frac{2}{3}\} < p < [2(m+\alpha)-2](3\beta-1) + \frac{4}{3} - m,$$

then a number $\bar{\beta} > 2$ exists such that for any $\varepsilon \in (0,1)$,

$$\|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)} + \|\nabla c_\varepsilon(\cdot,t)\|_{L^{2\beta}(\Omega)} \leq C$$

for all $p \geq 1$ and $\beta > \bar{\beta}$.

Proof. First, due to $\beta > 1$, we have

$$\frac{17}{12} - m < \frac{2}{5}(3\beta-1) + \frac{4}{3} - m.$$

Next, since $\frac{18\beta-5}{45\beta-15}$ is a decreasing function with respect to β, then $\lim_{\beta\to+\infty} \frac{18\beta-5}{45\beta-15} = \frac{2}{5} > 0$ and $m+\alpha > \frac{6}{5}$, so $\bar{\beta}_1 > 1$ exists such that

$$\frac{18\beta-5}{45\beta-15} < \frac{2}{5} + \frac{m+\alpha-\frac{6}{5}}{2} < 2(m+\alpha-1) \quad \text{for all} \quad \beta \geq \bar{\beta}_1.$$

Now, by choosing $\theta = \frac{9\beta}{5}$ in Lemma 3.6, we have

$$1 - m + \frac{2 \times \frac{9\beta}{5}}{3} \leq \frac{2 \times \frac{9\beta}{5} - 1}{5 \times \frac{9\beta}{5} - 3}(3\beta-1) + \frac{4}{3} - m = \frac{18\beta-5}{45\beta-15}(3\beta-1) + \frac{4}{3} - m.$$

$\frac{18\beta-5}{45\beta-15}$ is a decreasing function with respect to β and $\lim_{\beta\to+\infty}\frac{18\beta-5}{45\beta-15}=\frac{2}{5}>0$, so we can also derive

$$\frac{2}{5}(3\beta-1)+\frac{4}{3}-m\le\frac{18\beta-5}{45\beta-15}(3\beta-1)+\frac{4}{3}-m.$$

Choosing $\bar{\beta}_2=\max\{\frac{5}{6}m+\frac{1}{18},\frac{5(m+\alpha)}{3}-\frac{4}{3}\}$, and thus for $\beta\ge\bar{\beta}_2$, we have

$$\max\{1,\frac{18\beta-5}{45\beta-15}(3\beta-1)+\frac{4}{3}-m,m+2\alpha-\frac{2}{3}\}=\frac{18\beta-5}{45\beta-15}(3\beta-1)+\frac{4}{3}-m.$$

Now, by choosing $\bar{\beta}=\max\{2,\bar{\beta}_1,\bar{\beta}_2\}$, for any $\beta\ge\bar{\beta}$, the fact that $m>\max\{\frac{6}{5}-\alpha,\frac{1}{3}\}$ implies that

$$\max\{1,\frac{18\beta-5}{45\beta-15}(3\beta-1)+\frac{4}{3}-m,m+2\alpha-\frac{2}{3}\}<p<[2(m+\alpha)-2](3\beta-1)+\frac{4}{3}-m, \tag{4.1}$$

holds. Now, by choosing a sufficiently small δ in Lemma 3.5–Lemma 3.8, the solution of (2.12) satisfies

$$\begin{aligned}&\frac{d}{dt}\left(\int_\Omega n_\varepsilon^p+\int_\Omega|\nabla c_\varepsilon|^{2\beta}\right)+\int_\Omega n_\varepsilon^p+\int_\Omega|\nabla n_\varepsilon^{\frac{m+p-1}{2}}|^2+\int_\Omega|\nabla c_\varepsilon|^{2\beta}+\frac{1}{2}\int_\Omega|\nabla c_\varepsilon|^{2\beta-2}|D^2c_\varepsilon|^2\\&\le C_{26},\end{aligned} \tag{4.2}$$

which implies that

$$\frac{d}{dt}y(t)+C_{27}y(t)\le C_{28},$$

where $y:=\int_\Omega n_\varepsilon^p+\int_\Omega|\nabla c_\varepsilon|^{2\beta}$. Thus, a standard ordinary differential equation comparison argument implies the boundedness of $y(t)$ for all $t\in(0,T_{max})$. Clearly, $\|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)}$ and $\|\nabla c_\varepsilon(\cdot,t)\|_{L^{2\beta}(\Omega)}$ are bounded for all $t\in(0,T_{max})$. Finally, we can obtain the results by using the Hölder inequality. Thus, the proof of Lemma 4.1 is complete. $\square$

Based on the estimates established above, we can derive the following boundedness results by invoking a Moser-type iteration (see Lemma A.1 in [31]) and standard parabolic regularity arguments.

Lemma 4.2. *Let $m>\max\{\frac{6}{5}-\alpha,\frac{1}{3}\}$ and γ be as given in (2.2). Then, we can find $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ such that $\varepsilon_j\to0$ as $j\to\infty$ and such that*

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}\le C\ \ \text{for all}\ \ t\in(0,T_{max}), \tag{4.3}$$

and

$$\|c_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)}\le C\ \ \text{for all}\ \ t\in(0,T_{max}), \tag{4.4}$$

as well as

$$\|u_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)}\le C\ \ \text{for all}\ \ t\in(0,T_{max}). \tag{4.5}$$

Moreover, we have

$$\|A^\gamma u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}\le C\ \ \text{for all}\ \ t\in(0,T_{max}). \tag{4.6}$$

Proof. By the Moser-type iteration (see Lemma A.1 in [31]), we can use Lemma 4.1 to prove (4.3). In addition, (4.4) and (4.5) can be proved by using the standard parabolic regularity arguments.

Next, we prove (4.6). We apply the fractional power A^γ to the variation-of-constant formula

$$u_\varepsilon(\cdot,t) = e^{-tA}u_0 + \int_0^t e^{-(t-\tau)A}\mathcal{P}(n_\varepsilon(\cdot,t)\nabla\phi + g(\cdot,t))d\tau \quad \text{for all} \quad t \in (0, T_{max})$$

and then we have

$$\|A^\gamma u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le \|A^\gamma e^{-tA}u_0\|_{L^2(\Omega)} + \int_0^t \|A^\gamma e^{-(t-\tau)A}\mathcal{P}(n_\varepsilon(\cdot,t)\nabla\phi + g(\cdot,t))\|_{L^2(\Omega)}d\tau$$
$$\text{for all} \quad t \in (0, T_{max}),$$

where $\mathcal{P}$ denotes the Helmholtz project from $L^2(\Omega)$ into $L^2_\sigma(\Omega)$. Due to $u_0 \in D(A^\gamma)$, we can find $C_1 > 0$ such that

$$\|A^\gamma e^{-tA}u_0\|_{L^2(\Omega)} = \|e^{-tA}A^\gamma u_0\|_{L^2(\Omega)} \le C_1 \quad \text{for all} \quad t \in (0, T_{max}). \tag{4.7}$$

In addition, the estimates of the Stokes operator (see [14]) show that for each $\varphi \in L^2_\sigma(\Omega)$ and some $\lambda > 0$,

$$\|A^\gamma e^{-tA}\varphi\|_{L^2(\Omega)} \le C_2 t^{-\gamma}e^{-\lambda t}\|\varphi\|_{L^2(\Omega)} \quad \text{for all} \quad t \in (0, T_{max})$$

is valid. By (2.1) and the boundedness property of $\mathcal{P}$ from $L^2(\Omega)$ to $L^2_\sigma(\Omega)$, we have

$$\begin{aligned}
&\int_0^t \|A^\gamma e^{-(t-\tau)A}\mathcal{P}(n_\varepsilon(\cdot,\tau)\nabla\phi + g(\cdot,\tau))\|_{L^2(\Omega)}d\tau \\
&\le C_3(\|n_\varepsilon\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\Omega)}) \int_0^t (t-s)^{-\gamma}e^{-\lambda(t-s)}ds \\
&\le C_4 \quad \text{for all} \quad t \in (0, T_{max})
\end{aligned}$$

with positive constants C_3 and C_4, which combined with (4.7) gives (4.6). □

By virtue of (2.14) and Lemma 4.2, the local-in-time solution can be extended to the global-in-time solution.

Proposition 4.1. *Let $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)_{\varepsilon\in(0,1)}$ be the classical solutions of (2.12) constructed in Lemma 2.3 on $[0, T_{max})$. Then, the solution is global on $[0, \infty)$.*

Proposition 4.1 allows for an extension of the outcome in Lemma 4.2 from $[0, T_{max})$ to $[0, \infty)$, which is the following lemma.

Lemma 4.3. *Let $m > \max\{\frac{6}{5} - \alpha, \frac{1}{3}\}$ and γ be as given in (2.2). Then, we can find $C > 0$ independent of $\varepsilon \in (0, 1)$ such that*

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C \quad \textit{for all} \quad t \in (0, \infty) \tag{4.8}$$

and

$$\|c_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C \quad \text{for all} \quad t \in (0,\infty), \tag{4.9}$$

as well as

$$\|u_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C \quad \text{for all} \quad t \in (0,\infty). \tag{4.10}$$

Moreover, we have

$$\|A^\gamma u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C \quad \text{for all} \quad t \in (0,\infty). \tag{4.11}$$

As a straightforward result from Lemma 4.3, the following lemma gives the uniform Hölder regularity properties of c_ε, ∇c_ε and u_ε. We only state the lemma and for the detailed proof, readers can refer to the arguments for Lemma 3.18 and Lemma 3.19 in [46].

Lemma 4.4. *Let $m > \max\{\frac{6}{5} - \alpha, \frac{1}{3}\}$. Then, we can find $\mu \in (0,1)$ such that for some $C > 0$*

$$\|c_\varepsilon(\cdot,t)\|_{C^{\mu,\frac{\mu}{2}}(\Omega\times[t,t+1])} \le C \quad \text{for all} \quad t \in (0,\infty) \tag{4.12}$$

as well as

$$\|u_\varepsilon(\cdot,t)\|_{C^{\mu,\frac{\mu}{2}}(\Omega\times[t,t+1])} \le C \quad \text{for all} \quad t \in (0,\infty), \tag{4.13}$$

and such that for any $\tau > 0$ a $C(\tau) > 0$ exists that satisfies

$$\|\nabla c_\varepsilon(\cdot,t)\|_{C^{\mu,\frac{\mu}{2}}(\Omega\times[t,t+1])} \le C \quad \text{for all} \quad t \in (\tau,\infty). \tag{4.14}$$

5. Regularity properties of the time derivatives

Using the previously established estimates, we can obtain the boundedness property of the time derivatives of certain powers of n_ε on a fixed finite time interval, which will contribute to passing to the limit for the first equation in (2.12). Thus, we first derive the uniform bounds and the technical lemmas for n_ε by using an idea derived from [46].

Lemma 5.1. *Let $m > \max\{\frac{6}{5} - \alpha, \frac{1}{3}\}$. Then, we can find $\varepsilon \in (0,1)$ such that for some $C > 0$,*

$$\|\partial_t n_\varepsilon(\cdot,t)\|_{(W_0^{2,2}(\Omega))^*} \le C \quad \text{for all} \quad t \in (0,\infty). \tag{5.1}$$

Moreover, let $\varsigma > m$ and $\varsigma \ge 2(m-1+\alpha)$. Then, for all $T > 0$ and $\varepsilon \in (0,1)$, $C(T) > 0$ exists such that

$$\int_0^T \|\partial_t n_\varepsilon^\varsigma(\cdot,t)\|_{(W_0^{3,2}(\Omega))^*} dt \le C(T). \tag{5.2}$$

Proof. First, by using Lemma 4.4, for all $\varepsilon \in (0,1)$, we can fix a positive constant C_1 such that

$$n_\varepsilon \le C_1, |\nabla c_\varepsilon| \le C_1 \quad \text{and} \quad |u_\varepsilon| \le C_1 \quad \text{in} \quad \Omega \times (0,\infty), \tag{5.3}$$

and due to the fact that $D_\varepsilon \le D + 2\varepsilon$ for all $\varepsilon \in (0,1)$, we also derive

$$D_\varepsilon(n_\varepsilon) \le \quad C_2 \quad \text{in} \quad \Omega \times (0,\infty) \quad \text{for all} \quad \varepsilon \in (0,1), \tag{5.4}$$

where $C_2 := \|D\|_{L^\infty((0,C_1))} + 2$. By recalling (1.3) and $n_\varepsilon \geq 0$ in $\Omega \times (0,\infty)$ for all $\varepsilon \in (0,1)$, we also derive

$$|S_\varepsilon(x, n_\varepsilon, c_\varepsilon)| \leq \frac{C_S}{(1+n_\varepsilon)^2} \leq C_S \quad \text{in} \quad \Omega \times (0,\infty) \quad \text{for all} \quad \varepsilon \in (0,1). \tag{5.5}$$

Now, by testing the first equation with certain $\varphi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}
\int_\Omega n_{\varepsilon,t}(\cdot,t)\cdot\varphi &= \int_\Omega \left[\nabla\cdot(D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon) - \nabla\cdot(n_\varepsilon S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon) - u_\varepsilon\cdot\nabla n_\varepsilon\right]\cdot\varphi \\
&\quad + \int_\Omega \left[a n_\varepsilon - b n_\varepsilon^2\right]\varphi \\
&= \int_\Omega H_\varepsilon(n_\varepsilon)\Delta\varphi + \int_\Omega n_\varepsilon S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon\cdot\nabla\varphi \\
&\quad + \int_\Omega n_\varepsilon u_\varepsilon\cdot\nabla\varphi + a\int_\Omega n_\varepsilon\varphi - b\int_\Omega n_\varepsilon^2\varphi \quad \text{for all} \quad t\in(0,\infty),
\end{aligned} \tag{5.6}$$

where $H_\varepsilon(s) := \int_0^s D_\varepsilon(\tau)d\tau$ for $s \geq 0$. Moreover, by $D_\varepsilon \leq D + 2\varepsilon$ and $n_\varepsilon \geq 0$, we also obtain

$$H_\varepsilon(n_\varepsilon) \leq C_3 := C_1(\|D\|_{L^\infty((0,C_1))} + 2) \quad \text{in} \quad \Omega\times(0,\infty) \quad \text{for all} \quad \varepsilon\in(0,1). \tag{5.7}$$

Therefore, (5.3) and (5.5)–(5.7) imply that

$$|\int_\Omega n_{\varepsilon,t}(\cdot,t)\cdot\varphi| \leq C_3\int_\Omega |\Delta\varphi| + C_1^2(1+C_S)\int_\Omega |\nabla\varphi| + C_1(a+bC_1)\int_\Omega |\varphi| \tag{5.8}$$

for all $t \in (0,\infty)$ and for all $\varepsilon \in (0,1)$. Thus, (5.8) yields (5.1).

Next, we prove (5.2). Thus, for any fixed $\psi \in C_0^\infty(\Omega)$, after multiplying the first equation by $n_\varepsilon^{\varsigma-1}\psi$, we have

$$\begin{aligned}
\frac{1}{\varsigma}\int_\Omega \partial_t n_\varepsilon^\varsigma(\cdot,t)\cdot\psi &= \int_\Omega n_\varepsilon^{\varsigma-1}\left[\nabla\cdot(D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon) - \nabla\cdot(n_\varepsilon S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon) - u_\varepsilon\cdot\nabla n_\varepsilon\right]\cdot\psi \\
&\quad + \int_\Omega n_\varepsilon^{\varsigma-1}\left[a n_\varepsilon - b n_\varepsilon^2\right]\psi \\
&= -(\varsigma-1)\int_\Omega n_\varepsilon^{\varsigma-2}D_\varepsilon(n_\varepsilon)|\nabla n_\varepsilon|^2\psi - \int_\Omega n_\varepsilon^{\varsigma-1}D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon\cdot\nabla\psi \\
&\quad + (\varsigma-1)\int_\Omega n_\varepsilon^{\varsigma-1}\nabla n_\varepsilon\cdot(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon)\psi + \int_\Omega n_\varepsilon^\varsigma S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon\cdot\nabla\psi \\
&\quad + \frac{1}{\varsigma}\int_\Omega n_\varepsilon^\varsigma u_\varepsilon\cdot\nabla\psi + a\int_\Omega n_\varepsilon^\varsigma\psi - b\int_\Omega n_\varepsilon^{\varsigma+1}\psi \quad \text{for all} \quad t\in(0,\infty).
\end{aligned} \tag{5.9}$$

Next, we estimate the right-hand sides of (5.9). Thus, assuming that $p := \varsigma - m + 1$, then $\varsigma > m$ and $\varsigma \geq 2(m-1+\alpha)$ yield $p > 1$ and $p \geq m - 1 + 2\alpha$. Since (5.3), we integrate (3.5) with respect to t over $(0,T)$ for some fixed $T > 0$, and we then have

$$\begin{aligned}
&\frac{1}{p}\int_\Omega n_\varepsilon^p(\cdot,T)+\frac{C_D(p-1)}{2}\int_0^T\int_\Omega n_\varepsilon^{m+p-3}|\nabla n_\varepsilon|^2\\
\le\;& \frac{(p-1)C_S^2}{2C_D}\int_0^T\int_\Omega n_\varepsilon^{p+1-m-2\alpha}|\nabla c_\varepsilon|^2+aC_1^pT+\frac{1}{p}\int_\Omega n_0^p\\
\le\;& \frac{(p-1)C_S^2}{2C_D}C_1^{p+3-m-2\alpha}T+aC_1^pT+\frac{1}{p}\int_\Omega n_0^p.
\end{aligned}\tag{5.10}$$

In addition, by $p=\varsigma-m+1$, we have

$$\int_0^T\int_\Omega n_\varepsilon^{\varsigma-2}|\nabla n_\varepsilon|^2=\int_0^T\int_\Omega n_\varepsilon^{m+p-3}|\nabla n_\varepsilon|^2\le C_3(1+T)\tag{5.11}$$

for some positive constant C_3. Next, by (5.3), we also derive

$$a\int_\Omega n_\varepsilon^\varsigma\psi-b\int_\Omega n_\varepsilon^{\varsigma+1}\psi\le C_1^\varsigma|\Omega|(a+bC_1)\|\psi\|_{L^\infty(\Omega)}\quad\text{for all}\quad \varepsilon\in(0,1).\tag{5.12}$$

Moreover, by (5.3)–(5.5) and Young's inequality, and by employing the same arguments as those given in the proof of Lemma 3.22 in [46], we can conclude that $C_4>0$ exists such that

$$|\int_\Omega \partial_t n_\varepsilon^\varsigma(\cdot,t)\cdot\psi|\le C_4(\int_\Omega n_\varepsilon^{\varsigma-2}|\nabla n_\varepsilon|^2+1)\|\psi\|_{W^{1,\infty}(\Omega)},\tag{5.13}$$

which together with the embedding $W_0^{3,2}(\Omega)\hookrightarrow W^{1,\infty}(\Omega)$ in three-dimensional space implies that $C_5>0$ exists such that

$$\|\partial_t n_\varepsilon^\varsigma(\cdot,t)\|_{(W_0^{3,2}(\Omega))^*}\le C_5(\int_\Omega n_\varepsilon^{\varsigma-2}|\nabla n_\varepsilon|^2+1)\ \text{ for all }\ t\in(0,\infty)\ \text{ and any }\ \varepsilon\in(0,1).\tag{5.14}$$

Now, by combining (5.11) and (5.14), we can obtain (5.2). □

Lemma 5.2. *Assume that $m>\max\{\frac{6}{5}-\alpha,\frac{1}{3}\}$. Then, $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ exists such that $\varepsilon_j\to0$ as $j\to\infty$ and such that*

$$n_\varepsilon\to n\quad a.e.\quad in\quad \Omega\times(0,\infty),\tag{5.15}$$

$$n_\varepsilon\rightharpoonup n\quad weakly\ star\ in\quad L^\infty(\Omega\times(0,\infty)),\tag{5.16}$$

$$n_\varepsilon\to n\quad in\quad C^0_{loc}([0,\infty);(W_0^{2,2}(\Omega))^*),\tag{5.17}$$

$$c_\varepsilon\to c\quad in\quad C^0_{loc}(\bar\Omega\times[0,\infty)),\tag{5.18}$$

$$\nabla c_\varepsilon\to\nabla c\quad in\quad C^0_{loc}(\bar\Omega\times[0,\infty)),\tag{5.19}$$

$$\nabla c_\varepsilon\to\nabla c\quad in\quad L^\infty(\Omega\times(0,\infty)),\tag{5.20}$$

$$u_\varepsilon\to u\quad in\quad C^0_{loc}(\bar\Omega\times(0,\infty)),\tag{5.21}$$

and

$$Du_\varepsilon\rightharpoonup Du\quad weakly\ in\quad L^\infty(\Omega\times[0,\infty))\tag{5.22}$$

with some triple (n, c, u)*, which is a global weak solution of (1.1) in the sense of Definition 2.1. Moreover, n satisfies*

$$n \in C^0_{\omega-*}([0,\infty); L^\infty(\Omega)) \quad \textit{in} \quad L^\infty(\Omega \times [0,\infty)). \tag{5.23}$$

Proof. First, by applying the same arguments as those given in the proof of Lemma 4.1 by [46], we can conclude with (5.17)–(5.23), where the required equicontinuity property used in the proof is implied by (5.1). In addition, by Lemma 4.3, we find that for a certain $n \in L^\infty(\Omega \times (0,\infty))$, (5.16) is true. Furthermore, for each $T > 0$ and $p > 1$, (5.11) implies that

$$\int_0^T \int_\Omega n_\varepsilon^{m+p-3} |\nabla n_\varepsilon|^2 \le C_1 \tag{5.24}$$

for some positive $C_1 := C_1(T)$. Next, we fix $\varsigma > m$ to satisfy $\varsigma \ge 2(m - 1 + \alpha)$ and we set $p := 2\zeta - m + 1$, and thus (5.24) implies that for each $T > 0$, $(n_\varepsilon^\varsigma)_{\varepsilon \in (0,1)}$ is bounded in $L^2((0,T); W^{1,2}(\Omega))$. Using Lemma 5.1, we can also show that

$$(\partial_t n_\varepsilon^\varsigma)_{\varepsilon \in (0,1)} \text{ is bounded in } L^1((0,T); (W_0^{3,2}(\Omega))^*) \text{ for each } T > 0.$$

Hence, applying an Aubin–Lions lemma (e.g., see [36]) to the inequality above yields the strong precompactness of $(n_\varepsilon^\varsigma)_{\varepsilon \in (0,1)}$ in $L^2(\Omega \times (0,T))$. Therefore, we can select a suitable subsequence such that $n_\varepsilon^\varsigma \to z^\varsigma$ for some nonnegative measurable $z : \Omega \times (0, \Omega) \to \mathbb{R}$. Given (5.16) and the Egorov theorem, we necessarily have $z = n$, so (5.15) is valid. The proof of Lemma 5.2 is complete. $\square$

The proof of Theorem 2.1. Using Lemma 4.3 and 5.2, we can obtain Theorem 2.1. $\square$

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