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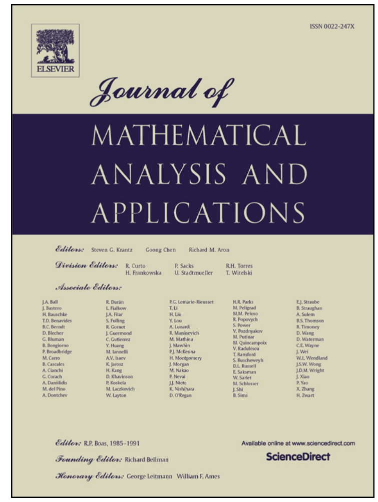


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Oscar Blasco, Georgios Stylogiannis

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# LIPSCHITZ-TYPE CONDITIONS ON HOMOGENEOUS BANACH SPACES OF ANALYTIC FUNCTIONS

OSCAR BLASCO AND GEORGIOS STYLOGIANNIS.

ABSTRACT. In this paper we deal with Banach spaces of analytic functions $X$ defined on the unit disc satisfying that $R_t f \in X$ for any $t > 0$ and $f \in X$, where $R_t f(z) = f(e^{it}z)$. We study the space of functions in $X$ such that $\|P_r(Df)\|_X = O(\frac{\omega(1-r)}{1-r}),\ r \to 1^-$ where $Df(z) = \sum_{n=0}^{\infty}(n+1)a_n z^n$ and $\omega$ is a continuous and non-decreasing weight satisfying certain mild assumptions. The space under consideration is shown to coincide with the subspace of functions in $X$ satisfying any of the following conditions: (a) $||R_t f - f||_X = O(\omega(t))$, (b) $||P_r f - f||_X = O(\omega(1-r))$, (c) $||\Delta_n f||_X = O(\omega(2^{-n}))$, or (d) $||f - s_n f||_X = O(\omega(n^{-1}))$, where $P_r f(z) = f(rz)$, $s_n f(z) = \sum_{k=0}^{n} a_k z^k$ and $\Delta_n f = s_{2^n} f - s_{2^{n-1}} f$. Our results extend those known for Hardy or Bergman spaces and power weights $\omega(t) = t^\alpha$.

## 1. INTRODUCTION

Let $\mathcal{H}(\mathbb{D})$ be the Fréchet space of all analytic functions in the unit disk $\mathbb{D}$ endowed with the topology of uniform convergence on compact subsets of $\mathbb{D}$. For $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $0 \le r < 1$ we write $P_r f$ and $R_t f$ for the dilation and rotation operators, i.e. for $0 \le r < 1$ and $t \in \mathbb{R}$

$$P_r f(z) = f(rz) \text{ and } R_t f(z) = f(e^{it}z).$$

As usual, we use the notation $s_n f = \sum_{k=0}^{n} a_k z^k$, $\Delta_n f = s_{2^n} f - s_{2^{n-1}} f$ and $\sigma_n f = \sum_{k=0}^{n}(1 - \frac{k}{n+1})a_k z^k$ for each $f \in \mathcal{H}(\mathbb{D})$.

A Banach space $X$ is said to be a *Banach space of analytic functions* (called $\mathcal{H}$-admissible in [BlPa11]) if

$$A(\mathbb{D}) \subset X \subset \mathcal{H}(\mathbb{D}),$$

with continuous inclusions, where $A(\mathbb{D})$ stands for the disk algebra.

We shall write $\mathcal{P}$ for the subspace of polynomials and we shall denote by $X_\mathcal{P}$ the closure of $\mathcal{P}$ under the norm in $X$, i.e. $\overline{\mathcal{P}} = X_\mathcal{P}$ or equivalently $\overline{A(\mathbb{D})} = X_\mathcal{P}$. Of course $X_\mathcal{P}$ is also a Banach space of analytic functions and

$$X_\mathcal{P} \subseteq \{f \in X : \lim_{t\to 0^+} ||R_t f - f||_X = 0\}. \tag{1.1}$$

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A Banach space of analytic functions $X$ is said to be *homogeneous* (see [BlPa11, Definition 4.1]) whenever $X$ also satisfies the following properties

$$f \in X \Longrightarrow R_t f \in X \text{ and } ||R_t f||_X = ||f||_X \text{ for every } t \in [0, 2\pi), \tag{1.2}$$

$$f \in X \Longrightarrow P_r f \in X \text{ and } ||P_r f||_X \leq K||f||_X \text{ for every } 0 \leq r < 1, \tag{1.3}$$

for some absolute constant $K \geq 1$.

Most of the classical spaces such as Hardy spaces $H^p$, Bergman spaces $A^p$, $BMOA$, the Bloch space $\mathcal{B}$ and many others are homogeneous spaces of analytic functions.

A basic fact holding for homogeneous spaces to be used in the sequel is that for each $f \in X$ the map $w \to f_w$, where $f_w(z) = f(wz)$ for $w \in \overline{\mathbb{D}}$ defines an $X_{\mathcal{P}}$-valued analytic function i.e. $F(w) = f_w \in \mathcal{H}(\mathbb{D}, X_{\mathcal{P}})$. In particular

$$M_X(r, f) := \sup_{|w|=r} ||f_w||_X$$

is an increasing function of $r$ and $M_X(r, f) = ||P_r f||_X$.

Moreover the function $F$ belongs to $A(\mathbb{D}, X_{\mathcal{P}})$, the space all vector-valued bounded holomorphic functions $F : \mathbb{D} \to X_{\mathcal{P}}$ with continuous extension to the boundary equipped with the norm

$$||F||_{A(\mathbb{D},X_{\mathcal{P}})} = \sup_{|w|\leq 1} ||F(w)||_X = \sup_{|\zeta|=1} ||F(\zeta)||_X = \|f\|_X.$$

Of course if $X$ is a homogeneous Banach space of analytic functions, so it is $X_{\mathcal{P}}$. Actually, for homogeneous Banach spaces of analytic functions, (1.3) together with the fact that $P_r f \in A(\mathbb{D})$ for each $0 \leq r < 1$ and polynomials are dense in $A(\mathbb{D})$ allow us to characterize $X_{\mathcal{P}}$ as

$$X_{\mathcal{P}} = \{f \in X : \lim_{r\to 1^-} ||P_r f - f||_X = 0\}. \tag{1.4}$$

The study of the subspace of $X_{\mathcal{P}}$ with a fixed rate of convergence to zero in (1.1) goes back to the work of Hardy and Littlewood in the twenties for the case $X = H^p$. Their fundamental contribution, proved in a series of papers ([HaLi28a], [HaLi28b] and [HaLi32]), can be condensed in the following result.

**Theorem (H-L)** Let $1 \leq p < \infty$, $0 < \alpha \leq 1$ and $f \in H^p$. Then the following statements are equivalent:

$(a)$ $||R_t f - f||_{H^p} = O(t^\alpha),\ t \to 0^+$,
$(b)$ $M_{H^p}(r, f') = O((1 - r)^{\alpha-1}),\ r \to 1^-$.

Concerning the approximation given in (1.4) and $X = H^p$, much later Komatsu [Ko66] and independently Storozhenko [St82] showed that conditions $(a)$ and $(b)$ are also equivalent to:

$(c)$ $||P_r f - f||_{H^p} = O((1-r)^\alpha),\ r \to 1^-$.

Using also approximation by Fourier series, in 1989 Bourdon, Shapiro, and Sledd [BoShSl89, Theorem 3.1] showed that, also for $X = H^p$ but with the restriction $1 < p < \infty$, the condition $(a)$ in the Hardy Littlewood Theorem is equivalent to each of the following statements:

$(d)$ $||\Delta_n f||_{H^p} = O(2^{-n\alpha}), n \to \infty$,
$(e)$ $||f - s_n f||_{H^p} = O(n^{-\alpha}), n \to \infty$.

It was only quite recently that the same problems were considered for other Banach spaces of analytic functions. In [GaSiSt15, Theorem 4.1]) the analogue of the equivalence between (a), (b) and (c) for Bergman spaces $X = A^p$ was proved and finally in Blasco's paper [Bl16] the above results were generalized for a wide class of homogeneous Banach spaces of analytic functions in the following result which recovers all the cases previously known.

**Theorem A** Let $0 < \alpha \leq 1$ and let $X$ be a homogeneous Banach space of analytic functions such that the dilation and translation semigroups are strongly continuous on $X$, i.e $\lim_{r\to 1^-} ||P_r f - f||_X = 0$ and $\lim_{t\to 0^+} ||R_t f - f||_X = 0$ for every $f \in X$. The following statements are equivalent:

$(i)$ $||R_t f - f||_X = O(t^\alpha),\ t \to 0^+$,
$(ii)$ $||P_r f - f||_X = O((1-r)^\alpha),\ r \to 1^-$,
$(iii)$ $M_X(r, Df) = O((1-r)^{\alpha-1}),\ r \to 1^-$.

where $D(f)(z) = (zf(z))' = \sum_{n=0}^\infty (n+1)a_n z^n$.

In this paper we shall extend the results previously known for homogeneous spaces with strongly continuous dilation and translation semigroups (in particular for Hardy and Bergman spaces) to a wider class of homogeneous spaces and for approximation rates given by a weight more general than $\omega(t) = t^\alpha$.

A *weight* $\omega$ will stand for a continuous and non-decreasing function $\omega : [0,\pi] \to \mathbb{R}^+$ with $\omega(0) = 0$. In the sequel we shall need extra assumptions to be assumed on the weights, namely the existence of constants $m, C_1, C_2 > 0$ so that

$$\inf_{0<t\leq\pi} \frac{\omega(t)}{t} = m > 0. \tag{1.5}$$

$$\int_0^\delta \frac{\omega(t)}{t} dt \leq C_1 \omega(\delta) \text{ for } 0 < \delta < \pi. \tag{1.6}$$

$$\int_\delta^\pi \frac{\omega(t)}{t^2} dt \leq C_2 \frac{\omega(\delta)}{\delta} \text{ for } 0 < \delta < \pi. \tag{1.7}$$

The paper is divided into three sections. In the first one we shall observe that condition (1.3) follows from (1.2) if the map $t \to R_t f$ is assumed to be measurable for any $f \in X$. Using that we shall manage to show the following result.

**Theorem 1.1.** *Let $X$ be a Banach space of analytic functions satisfying*

$$R_t f \in X, \quad \forall t \in [0, 2\pi), \quad \forall f \in X.$$

*Then*

$$X_{\mathcal{P}} = \{f \in X : \lim_{n\to\infty} \|f - \sigma_n f\|_X = 0\} = \{f \in X : \lim_{t\to 0^+} \|f - R_t f\|_X = 0\}.$$

In Section 3 we handle the equivalences $(a)$, $(b)$ and $(c)$ in our setting and in last section we analyze the equivalences $(d)$ and $(e)$ where the condition $1 < p < \infty$ is replaced by the fact that the Riesz projection is bounded.

Our main result, collecting the information from the whole paper, can be stated as follows:

**Theorem 1.2.** *Let $X$ be a homogeneous Banach space of analytic functions such that the polynomials are dense and the Riesz projection is bounded on $X$. Let $\omega : [0, \pi] \to \mathbb{R}^+$ be a weight satisfying* (1.5), (1.6) *and* (1.7). *Then the following statements are equivalent:*

$(i)$ $||R_t f - f||_X = O(\omega(t)),\ t \to 0^+$,
$(ii)$ $||P_r f - f||_X = O(\omega(1-r)),\ r \to 1^-$,
$(iii)$ $M_X(r, Df) = O(\frac{\omega(1-r)}{1-r}),\ r \to 1^-$,
$(iv)$ $||\Delta_n f||_X = O(\omega(2^{-n})),\ n \to \infty$,
$(v)$ $||f - s_n f||_X = O(\omega(n^{-1})),\ n \to \infty$.

In what follows $X$ is always assumed to be a Banach space of analytic functions, $\omega$ be a weight defined on $[0, \pi]$ and the letter $C$ will denote a constant, whose value may change from line to line.

## 2. Measurability and polynomial density

We begin this section by exploiting some properties on the family of operators $\{R_t\}_{t\geq 0}$. Of course it can be extended in the whole real line $\mathbb{R}$ and it admits the following group structure:
(i) $R_0 = I$, the identity operator on $X$,
(ii) $R_t^{-1} = R_{-t}$, for every $t \geq 0$,
(iii) $R_t \circ R_s = R_{t+s}$, for every $t, s \in \mathbb{R}$.

If we assume (1.2) then $R_t : X \to X$ is an isometry on $X$ for every $t \in \mathbb{R}$.

We first notice that due to the group structure of $\{R_t\}$ the measurability of $t \to R_t f$ from $\mathbb{R} \to X$ for any $f \in X$ is actually equivalent to the strong continuity of the semigroup.

**Lemma 2.1.** *Let $X$ be a homogeneous Banach space. The following statements are equivalent:*
*$(i)$ For any $f \in X$ there exists $t_0 > 0$ such that $t \to R_t f$ is right continuous at $t_0$.*
*$(ii)$ The map $t \to R_t f$ is continuous for every $t \in \mathbb{R}$ and any $f \in X$.*
*$(iii)$ The map $t \to R_t f$ is measurable on $(0, \infty)$ for each $f \in X$.*

*Proof.* $(i) \Longrightarrow (ii)$ Since $||R_t f - R_{t_1} f||_X = ||R_{t-t_1} f - f||_X$ for all $t_1 \in \mathbb{R}$, it suffices to show that $t \to R_t f$ is continuous at $0$ for any $f \in X$. Now for each $t > 0$

$$\begin{aligned} ||R_t f - f||_X &= ||R_{-t_0}(R_{t+t_0} f - R_{t_0} f)||_X \\ &= ||R_{t+t_0} f - R_{t_0} f||_X. \end{aligned}$$

Thus $||R_t f - f||_X \to 0$ as $t \to 0^+$ and since $||R_t f - f||_X = ||R_{-t} f - f||_X$ for every $f \in X$ we have that $||R_t f - f||_X \to 0 \quad \text{as} \quad t \to 0$.
$(ii) \Longrightarrow (iii)$ Obvious.
$(iii) \Longrightarrow (i)$ Since $t \to R_t f$ is measurable for every $f \in X$, by [DuSc88, Lemma 3 pp. 616] it follows that

$$\lim_{\varepsilon \to 0^+} ||R_{t_0+\varepsilon} f - R_{t_0} f||_X = 0$$

for every $t_0 \in (0, \infty)$. □

**Proposition 2.2.** *Let $X$ be a Banach space of analytic functions satisfying*

$$R_t f \in X, \quad \forall t \in [0, 2\pi), \quad \forall f \in X. \tag{2.1}$$

*If the map $t \to R_t f$ is measurable on $[0, 2\pi)$ for each $f \in X$ then $X$ can be renormed to become homogeneous.*

*Proof.* Periodicity implies that $R_t f \in X$ for any $f \in X$ and $t \in \mathbb{R}$ and that $t \to R_t f$ is measurable on $(-\infty, \infty)$ for each $f \in X$. Moreover the closed graph theorem implies that $R_t : X \to X$ is a bounded linear operator acting on $X$ for every $t \in \mathbb{R}$. It follows by the proof of [DuSc88, Lemma 3 pp. 616] that $||R_t||$ is bounded on each interval $[\delta, 1/\delta]$, $\delta > 0$. Thus we have $\sup\{||R_t|| : t \in \mathbb{R}\} = \sup\{||R_t|| : t \in [0, 2\pi)\} = \sup\{||R_t|| : t \in [2\pi, 4\pi)\} = K_0 < \infty$.

To show (1.3) we use the formula

$$f(rz) = \frac{1}{2\pi} \int_0^{2\pi} R_t(f)(z) P_r(e^{-it}) dt \tag{2.2}$$

where $P_r(e^{it}) = \frac{1-r^2}{(1-r)^2 + 4r\sin^2(t/2)}$ is the Poisson kernel.

Since $t \to R_t f$ is measurable and bounded, we obtain that it is Bochner integrable with respect to the probability measure $d\mu_r(t) = P_r(e^{-it}) dt$ and, for each $0 < r < 1$, we have that

$$P_r f = \frac{1}{2\pi} \int_0^{2\pi} R_t f \, d\mu_r(t) \in X.$$

Thus

$$||P_r f||_X \leq \int_0^{2\pi} ||R_t f||_X d\mu_r(t) \leq K_0 ||f||_X.$$

Now considering $|||f||| = \sup_{0 \leq t \leq 2\pi} \|R_t f\|_X$ we have that $\|f\|_X \leq |||f||| \leq K_0 \|f\|_X$. For such a norm we have $|||R_t f||| = |||f|||$ and $|||P_r f||| \leq |||f|||$ for all $f \in X$, $t \in \mathbb{R}$ and $0 < r < 1$. Hence $(X, ||| \cdot |||)$ is homogeneous. □

As mentioned in the introduction for $f(z) = \sum_{n=0}^{\infty} a_n z^n \in X$, we denote by $\sigma_n f \in X_{\mathcal{P}}$ the polynomial defined by

$$\sigma_n f(z) = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) a_k z^k,$$

or equivalently

$$\sigma_n f(z) = \frac{1}{2\pi} \int_0^{2\pi} R_t(f)(z) K_n(e^{-it}) dt$$

where $K_n$ stands for Fèjer Kernel

$$K_n(e^{it}) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ikt} = \frac{1}{n+1} \left(\frac{\sin\left(\frac{(n+1)t}{2}\right)}{\sin(\frac{t}{2})}\right)^2.$$

We are now ready to give a proof of Theorem 1.1 mentioned in the introduction. We include also some other equivalent formulations in the next result.

**Theorem 2.3.** *Let $X$ be a Banach space of analytic functions satisfying* (2.1) *and let $f \in X$. The following are equivalent:*
*$(i)$ $t \to R_t f$ is measurable from $[0, \infty) \to X$,*
*$(ii)$ $||R_t f - f||_X \to 0$ as $t \to 0^+$ ,*
*$(iii)$ $||\sigma_n f - f||_X \to 0$ as $n \to \infty$ ,*
*$(iv)$ $||P_r f - f||_X \to 0$ as $r \to 1^-$.*
*$(v)$ $f \in X_{\mathcal{P}}$.*

*Proof.* $(i) \Leftrightarrow (ii)$ It follows from Proposition 2.2 that $X$ can be renormed to be a homogeneous Banach space of analytic functions. Now Lemma 2.1 gives this equivalence.
$(iv) \Leftrightarrow (v)$ It is formula (1.4).
$(i) \Rightarrow (iii)$ Since $t \to R_t f$ is Bochner integrable with respect to any probability measure $\mu$ (because it is bounded and measurable) then

$$f - \sigma_n f = \frac{1}{2\pi} \int_0^{2\pi} (f - R_t f) K_n(e^{-it}) dt. \tag{2.3}$$

makes sense as a Bochner integral with values in $X$.

Given $\varepsilon > 0$ take $0 < \delta < \pi$ such that $\|f - R_t f\|_X < \varepsilon/4$ for $0 < t < \delta$. Therefore

$$\begin{aligned}\|f - \sigma_n f\|_X &\le 2\int_0^\delta \|f - R_t f\|_X K_n(e^{-it})dt + 2\int_\delta^\pi \|f - R_t f\|_X K_n(e^{-it})dt \\ &\le \frac{\varepsilon}{2} + 2\|f\|_X \int_\delta^\pi K_n(e^{-it})dt \\ &\le \frac{\varepsilon}{2} + \frac{2\pi^3\|f\|_X}{(n+1)\delta^2}\end{aligned}$$

where we have used that $K_n(e^{-it}) \le \frac{\pi^2}{(n+1)t^2}$. Now selecting $n_0 \in \mathbb{N}$ big enough one has $\|f - \sigma_n f\|_X < \varepsilon$ for $n \ge n_0$.

$(iii) \Rightarrow (v)$ Obvious.

$(v) \Rightarrow (ii)$ Since $R_t(p) \to p$ as $t \to 0^+$ for every polynomial $p$ in the disk algebra $A(\mathbb{D})$ and $A(\mathbb{D}) \subset X$ continuously, it follows that $\lim_{t\to 0^+} ||R_t(p) - p||_X = 0$ for every polynomial $p$. Let $\varepsilon > 0$ and $f \in X_{\mathcal{P}}$ be given and choose a $p$ polynomial such that $||f - p||_X < \varepsilon/4$. By the triangular inequality

$$||R_t f - f||_X \le 2||f - p||_X + ||R_t p - p||_X \le \varepsilon/2 + ||R_t p - p||_X$$

and selecting $t_0$ so that $||R_t p - p||_X < \varepsilon/2$ for $0 < t < t_0$ we obtain (ii). □

We can actually quantify the rate of convergence of $\|f - \sigma_n f\|_X$ and $\|f - P_r f\|_X$ in terms of the rate of convergence of $\|f - R_t f\|_X$.

**Lemma 2.4.** *Let $X$ be a homogeneous Banach space of analytic functions satisfying (2.1) and $f \in X_{\mathcal{P}}$. Then there exist constants $C_1, C_2 > 0$ such that*

(2.4) $$\|f - P_r f\|_X \le C_1\Big(\frac{1}{1-r}\int_0^\delta \|f - R_t f\|_X dt + \frac{1-r}{r}\int_\delta^\pi \frac{\|f - R_t f\|_X}{t^2}dt\Big)$$

(2.5)
$$\|f - \sigma_n f\|_X \le C_2\Big((n+1)\int_0^\delta \|f - R_t f\|_X dt + \frac{1}{n+1}\int_\delta^\pi \frac{\|f - R_t f\|_X}{t^2}dt\Big)$$

*for each $0 < r < 1, 0 < \delta < \pi$ and $n \in \mathbb{N}$.*

*Proof.* From Theorem 2.3, denoting $d\mu_r(t) = P_r(e^{-it})dt$, we can write

(2.6) $$f - P_r f = \frac{1}{2\pi}\int_0^{2\pi} (f - R_t f) d\mu_r(t).$$

From (2.6) and (2.3), denoting $d\mu_n(t) = K_n(e^{-it})dt$, we have

$$\|f - P_r f\|_X \le 2\int_0^\pi \|f - R_t f\|_X d\mu_r(t)$$

and

$$\|f - \sigma_n f\|_X \le 2\int_0^\pi \|f - R_t f\|_X d\mu_n(t).$$

Since $\sin(t) \geq \frac{2}{\pi}t$ for $0 < t < \pi/2$ one has the well known upper estimates

$$P_r(e^{-it}) \leq 2\frac{1-r}{(1-r)^2 + \frac{4rt^2}{\pi^2}}$$

and

$$K_n(e^{-it}) \leq \min\{n+1, \frac{\pi^2}{(n+1)t^2}\}.$$

Therefore estimating $P_r(e^{-it}) \leq \frac{2}{1-r}$ for $0 < t < \delta$ and $P_r(e^{-it}) \leq \frac{\pi^2(1-r)}{2rt^2}$ for $\delta < t < \pi$ we conclude that

$$\begin{aligned} ||f - P_r f||_X &\leq 2\int_0^\delta ||R_t f - f||_X d\mu_r(t) + 2\int_\delta^\pi ||R_t f - f||_X d\mu_r(t) \\ &\leq \frac{C}{1-r}\int_0^\delta ||R_t f - f||_X dt + C(\frac{1}{r} - 1)\int_\delta^\pi \frac{||R_t f - f||_X}{t^2} dt. \end{aligned}$$

Similarly

$$\begin{aligned} ||f - \sigma_n f||_X &\leq 2\int_0^\delta ||R_t f - f||_X d\mu_n(t) + 2\int_\delta^\pi ||R_t f - f||_X d\mu_n(t) \\ &\leq 2(n+1)\int_0^\delta ||R_t f - f||_X dt + 2\frac{C}{n+1}\int_\delta^\pi \frac{||R_t f - f||_X}{t^2} dt. \end{aligned}$$

The proof is now complete. □

## 3. Lipschitz-type conditions

Let $X$ be a Banach space of analytic functions satisfying (2.1). For each $f \in X$ we write

(3.1) $$w_X(f,t) = \sup_{|s|\leq t} \|f - R_s f\|_X, \quad 0 < t \leq \pi.$$

Notice that in the case $X = X_{\mathcal{P}}$, from Theorem 2.3 one has that $w_X(f,t)$ is continuous and non-decreasing in $[0,\pi]$ with $w_X(f,0) = 0$ for each $f \in X$. We compare the speed of convergence of $w_X(f,t)$ as $t$ goes to $0$ with the way that $f$ is approached by $P_r f$ as $r$ goes to $1$ or by $\sigma_n f$ as $n$ goes to $\infty$.

The first result is an easy consequence of Lemma 2.4.

**Theorem 3.1.** *Let $X$ be a Banach space of analytic functions satisfying* (2.1) *and $\omega$ be a weight satisfying* (1.7). *If $f \in X$ and $w_X(f,t) = O(\omega(t))$ as $t \to 0^+$ then*

(3.2) $$\|f - P_r f\|_X = O(\omega(1-r)) \text{ as } r \to 1^-.$$

(3.3) $$\|f - \sigma_n(f)\|_X = O(\omega(\frac{1}{n})) \text{ as } n \to \infty.$$

*Proof.* Since $f \in X_{\mathcal{P}}$, to show (3.2) we invoke (2.4) in Lemma 2.4 for $\delta = 1-r$ to conclude due to (1.7) that

$$\begin{aligned}\|f - P_r f\|_X &\leq C_1 \frac{1}{1-r} \int_0^{1-r} \|f - R_t f\|_X dt \\ &+ C_1 \frac{1-r}{r} \int_{1-r}^{\pi} \frac{\|f - R_t f\|_X}{t^2} dt \\ &\leq C w_X(f, 1-r) + C(1-r) \int_{1-r}^{\pi} \frac{\omega(t)}{t^2} dt \\ &\leq C\omega(1-r).\end{aligned}$$

Similarly to show (3.3) we use now (2.5) in Lemma 2.4 for $\delta = \frac{1}{n+1}$ to conclude, using (1.7) again, that

$$\begin{aligned}\|f - \sigma_n f\|_X &\leq C_2(n+1) \int_0^{1/(n+1)} \|f - R_t f\|_X dt \\ &+ C_2 \frac{1}{n+1} \int_{1/(n+1)}^{\pi} \frac{\|f - R_t f\|_X}{t^2} dt \\ &\leq C w_X(f, 1/(n+1)) + \frac{C}{n+1} \int_{1/(n+1)}^{\pi} \frac{\omega(t)}{t^2} dt \\ &\leq C\omega(1/(n+1)) \leq C\omega(1/n).\end{aligned}$$

□

Following the ideas of Hardy and Littlewood, our aim is to describe functions such that $w_X(f,t) = O(\omega(t))$ for a given continuous non-increasing function with $\omega(0) = 0$ in terms of growth of $M_X(r, Df)$ as $r$ goes to $1$. We shall need some lemmas.

**Lemma 3.2.** *Let $X$ be a space of analytic functions satisfying* (2.1). *Then there exists $C > 0$ such that*

$$(1-r) M_X(r, Df) \leq C\Big(w_X(f, 1-r) + (1-r) \int_{1-r}^{\pi} \frac{w_X(f,t)}{t^2} dt + (1-r) \log(\frac{1}{1-r}) \|f\|_X\Big)$$

*for all $f \in X_{\mathcal{P}}$ and $0 < r < 1$.*

*Proof.* Let $f \in X_{\mathcal{P}}$ and $0 < r < 1$. We use Cauchy formula to obtain

$$P_r Df = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} R_t f - f}{(e^{it} - r)^2} e^{it} dt. \tag{3.4}$$

Hence

$$\begin{aligned} M_X(r,Df) &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\|e^{it}R_t f - f\|_X}{|e^{it}-r|^2}dt \\ &\leq \frac{1}{\pi(1-r^2)}\int_0^{\pi} \|e^{it}R_t f - f\|_X P_r(e^{it})dt \\ &\leq \frac{1}{\pi(1-r^2)}\int_0^{\pi} (w_X(f,t)+|1-e^{it}|\|f\|_X)P_r(e^{it})dt. \end{aligned}$$

Now use the same argument as the one used in the proof of (2.4) with the choice of $\delta = 1-r$ to obtain

$$\begin{aligned} \int_0^{\pi} w_X(f,t)P_r(e^{it})dt &\leq \frac{C}{1-r}\int_0^{1-r} w_X(f,t)dt + C(1-r)\int_{1-r}^{\pi}\frac{w_X(f,t)}{t^2}dt \\ &\leq Cw_X(f,1-r) + C(1-r)\int_{1-r}^{\pi}\frac{w_X(f,t)}{t^2}dt. \end{aligned}$$

On the other hand

$$\begin{aligned} \int_0^{\pi} |1-e^{it}|P_r(e^{it})dt &\leq \frac{C}{1-r}\int_0^{\pi}\frac{t}{(1+C(\frac{t}{1-r})^2)}dt \\ &\leq C(1-r)\int_0^{\frac{\pi}{1-r}}\frac{s}{1+Cs^2}ds \\ &\leq C(1-r)\log(1+\frac{C}{(1-r)^2}). \end{aligned}$$

Putting together the above estimates we obtain the result. □

**Lemma 3.3.** *Let $X$ be a Banach space of analytic functions satisfying* (2.1). *Then there exists $C>0$ such that*

$$w_X(f,t) \leq C\Big(\int_{1-t}^{1} M_X(r,Df)dr + tM_X(1-t,Df) + t\|f\|_X\Big) \tag{3.5}$$

*for all $f \in X_{\mathcal{P}}$ and $0<t<1$.*

*Proof.* Let $f \in X_{\mathcal{P}}$ and $0<t<1$. Using the fundamental theorem of calculus one has, for each $z\in\mathbb{D}$, $0<\delta<1$ and $t\in[0,\pi]$,

$$zf(z) - e^{it}zf(e^{it}z) = \int_\delta^1 Df(rz)zdr - \int_0^t Df(\delta ze^{is})\delta zie^{is}ds - \int_\delta^1 Df(rze^{it})ze^{it}dr.$$

Hence

$$f - e^{it}R_t f = \int_\delta^1 P_r(Df)dr - \int_0^t R_sP_\delta(Df)i\delta e^{is}ds - \int_\delta^1 R_tP_r(Df)e^{it}dr.$$

Therefore

$$\|f - e^{it}R_t f\|_X \leq 2\int_\delta^1 M_X(r,Df)dr + \int_0^t M_X(\delta,Df)ds.$$

Now choosing $\delta = 1 - t$ we have

$$\begin{aligned}\sup_{|s|\le t} \|f - R_s f\|_X &\le \sup_{|s|\le t} \|f - e^{is} R_s f\|_X + |1 - e^{is}| \|R_s f\|_X \\ &\le C\Big(\int_{1-t}^{1} M_X(r, Df)dr + tM_X(1-t, Df) + t\|f\|_X\Big).\end{aligned}$$

This finishes the proof. □

**Theorem 3.4.** *Let $X$ be a Banach space of analytic functions satisfying* (2.1) *and $f \in X_{\mathcal{P}}$. Let $\omega$ be a weight satisfying* (1.5), (1.6) *and* (1.7). *The following conditions are equivalent:*
*(i)* $w_X(f, t) = O(\omega(t)),\ t \to 0^+$,
*(ii)* $M_X(r, Df) = O(\frac{\omega(1-r)}{1-r}),\ r \to 1^-$.

*Proof.* $(i) \Rightarrow (ii)$ Note that conditions (1.5) and (1.7) give

$$m \log(\frac{1}{\delta}) \le \int_\delta^1 \frac{\omega(t)}{t^2} dt \le C_2 \frac{\omega(\delta)}{\delta}.$$

Applying Lemma 3.2 we obtain

$$\begin{aligned}(1-r)M_X(r, Df) &\le C\Big(\omega(1-r) + (1-r)\int_{1-r}^{\pi} \frac{\omega(t)}{t^2} dt + \omega(1-r)\|f\|_X\Big) \\ &\le C\omega(1-r).\end{aligned}$$

$(ii) \Rightarrow (i)$ Using now Lemma 3.3, (1.5) and (1.6) we have

$$\begin{aligned}w_X(f, t) &\le C\Big(\int_{1-t}^{1} \frac{\omega(1-r)}{1-r} dr + \omega(t) + \frac{\|f\|_X}{m}\omega(t)\Big) \\ &\le C\Big(\int_0^t \frac{\omega(s)}{s} ds + \omega(t)\Big) \\ &\le C\omega(t).\end{aligned}$$

□

We now compare the growth of $M_X(r, Df)$ as $r \to 1^-$ with the behaviour of $\|f - P_r f\|_X$ for functions in $X_{\mathcal{P}}$.

**Lemma 3.5.** *Let $X$ be a space of analytic functions satisfying* (2.1), *$f \in X_{\mathcal{P}}$ and $0 < r < 1$. Then*

$$\|f - P_r f\|_X \le K \int_r^1 \Big(M_X(s, Df) + \|f\|_X\Big) ds \tag{3.6}$$

*where $K$ is the constant in* (1.3).

*Proof.* Since $zf(z) = \int_0^1 Df(sz)z ds$ we conclude that $f = \int_0^1 P_s(Df)ds$. Taking into account that $P_r(Df) = D(P_r f)$ we obtain

$$f - rP_r f = \int_0^1 P_s Df ds - r\int_0^1 P_{sr} Df ds = \int_r^1 P_s(Df) ds. \tag{3.7}$$

Therefore

$$\begin{aligned}\|f - P_r f\|_X &\leq \|f - rP_r f\|_X + (1-r)\|P_r f\|_X \\ &\leq \int_r^1 M_X(s, Df)ds + (1-r)K\|f\|_X\end{aligned}$$

and (3.6) is shown. □

**Lemma 3.6.** *Let $X$ be a Banach space of analytic functions satisfying* (2.1)*, $f \in X_{\mathcal{P}}$ and $0 < r < 1$. Then*

(3.8)

$$(1-r)M_X(r, Df) \leq K\Big(\|f - P_r f\|_X + \frac{1}{1-r}\int_r^1 \|P_s f - f\|_X ds + K(1-r)\|f\|_X\Big)$$

*where $K$ is the constant in* (1.3)*.*

*Proof.* We use (3.7) for $P_u f$, $0 < u < 1$ and that $P_u Df = DP_u f$, to obtain

$$(1-r)P_u Df = P_u f - rP_{ru} f - \int_r^1 \Big(P_u DP_s f - P_u Df\Big)ds.$$

Now using the fact

$$M_X(rs, Df) \leq \frac{M_X(s, f)}{1-r^2}, 0 < r, s < 1, \text{ (see [BlPa11, Lemma 3.2])}$$

we can write

$$\begin{aligned}(1-r)M_X(u, Df) &\leq M_X(u, f - rP_r f) + \int_r^1 M_X(u, D(P_s f - f))ds \\ &\leq K\|f - rP_r f\|_X + \int_r^1 \frac{M_X(\sqrt{u}, P_s f - f)}{1-u}ds \\ &\leq K(\|f - P_r f\|_X + (1-r)K\|f\|_X) + \frac{K}{1-u}\int_r^1 \|P_s f - f\|_X ds.\end{aligned}$$

Choosing $u = r$ we obtain (3.8). □

**Theorem 3.7.** *Let $X$ be a Banach space of analytic functions satisfying* (2.1) *and $f \in X_{\mathcal{P}}$. Let $\omega$ be a weight satisfying* (1.5) *and* (1.6)*. The following conditions are equivalent*

*(i)* $\|f - P_r f\|_X = O(\omega(1-r)),\ r \to 1^-$,

*(ii)* $M_X(r, Df) = O(\frac{\omega(1-r)}{1-r}),\ r \to 1^-$.

*Proof.* $(i) \Rightarrow (ii)$ Using first that $\omega$ is non-decreasing and satisties (1.5), we use Lemma 3.6 to obtain

$$\begin{aligned}(1-r)M_X(r, Df) &\leq K\Big(\omega(1-r) + \frac{1}{1-r}\int_0^{1-r} \omega(s)ds + K(1-r)\|f\|_X\Big) \\ &\leq C\Big(\omega(1-r) + \omega(1-r) + \omega(1-r)\|f\|_X\Big).\end{aligned}$$

$(ii) \Rightarrow (i)$ Using now that $\omega$ satisfies (1.5) and (1.6) together with Lemma 3.5 we obtain

$$\begin{aligned}\|f - P_r f\|_X &\leq K\Big(\int_r^1 \frac{\omega(1-s)}{1-s}ds + (1-r)\|f\|_X\Big)\\ &\leq C\Big(\int_0^{1-r} \frac{\omega(s)}{s}ds + \omega(1-r)\|f\|_X\Big)\\ &\leq C\omega(1-r).\end{aligned}$$

□

## 4. Approximation for partial sums and weights

In this section we shall try to extend the conditions (d) and (e) in the introduction to the setting of homogeneous spaces of analytic functions and for sequences more general than $\omega_n = \frac{1}{n^\alpha}$.

**Definition 4.1.** Let $(\omega_n)_{n=0}^\infty$ be a bounded sequence of real numbers. Define

$$\tilde{\omega}(t) = t\sum_{k=0}^\infty \omega_k (1-t)^k, 0 < t \leq 1.$$

Observe that the function $f(z) = \sum_{n=0}^\infty \omega_n z^n$ converges absolutely in $|z| < 1$. Hence $\tilde{\omega}(t)$ it is well defined for any $0 < t \leq 1$ and it is continuous.

**Proposition 4.2.** *Let $(\omega_n)_{n=0}^\infty \subset \mathbb{R}^+$ be a sequence such that $\omega_n \geq \omega_{n+1}$ for all $n \in \mathbb{N}$ and $\lim_n \omega_n = 0$. Then $\tilde{\omega} : (0,1] \to \mathbb{R}^+$ is a weight which satisfies* (1.5) *and*

$$\int_\delta^1 \frac{\tilde{\omega}(t)}{t^2}dt \leq \frac{\tilde{\omega}(\delta)}{\delta}\log\frac{1}{\delta}, \quad 0 < \delta < 1.$$

*Moreover for all $n \in \mathbb{N}$ and $0 < t < 1$*

$$(1-t)^n \sum_{k=0}^n \omega_k \leq \frac{\tilde{\omega}(t)}{t} \leq \sum_{k=0}^n \omega_k + \frac{(1-t)^{n+1}}{t}\omega_{n+1}. \tag{4.1}$$

*In particular*

$$\frac{C_1}{n+1}\sum_{k=0}^n \omega_k \leq \tilde{\omega}(\frac{1}{n+1}) \leq C_2\Big(\frac{1}{n+1}\sum_{k=0}^n \omega_k + \omega_{n+1}\Big). \tag{4.2}$$

*Proof.* Let us rewrite $\tilde{\omega}$. We have

$$\begin{aligned}\tilde{\omega}(t) &= \omega_0 t + \sum_{n=1}^\infty \omega_n\Big((1-t)^n - (1-t)^{n+1}\Big)\\ &= \omega_1 + (\omega_0 - \omega_1)t - \sum_{n=2}^\infty \Big(\omega_{n-1} - \omega_n\Big)(1-t)^n.\end{aligned}$$

This shows that since $\omega_n$ is a non-increasing sequence that $\tilde{\omega}$ is a non-decreasing function, $\lim_{t\to 0}\tilde{\omega}(t) = 0$ and $\frac{\tilde{\omega}(t)}{t} \geq \omega_0$. Using now that $\frac{\tilde{\omega}(t)}{t}$ is non-increasing we have

$$\int_\delta^1 \frac{\tilde{\omega}(t)}{t^2}dt \leq \frac{\tilde{\omega}(\delta)}{\delta}\int_\delta^1 \frac{dt}{t} \leq \frac{\tilde{\omega}(\delta)}{\delta}\log\frac{1}{\delta}.$$

Finally (4.1) follows from the monotonicity of $\omega_k$ and (4.2) uses that $C \leq (1-\frac{1}{n+1})^n \leq 1$. $\square$

Next result is contained in [Bl92, Lemma 5.1] but we include its proof here for the sake of completeness.

**Proposition 4.3.** *Let $\omega$ be a weight satisfying (1.6) and (1.7). Then there exists $C>0$ such that*

$$\frac{\tilde{\omega}(t)}{t} \leq C\frac{\omega(t)}{t}, \quad 0<t<1$$

*where $\tilde{\omega}(t) = t\sum_{k=0}^\infty \omega(\frac{1}{k+1})(1-t)^k$.*

*Proof.* We notice first that condition (1.7) gives for $s_1 \leq s_2$

$$\frac{\omega(s_2)}{s_2} \leq C\int_{s_2}^1 \frac{\omega(t)}{t^2}dt \leq C\int_{s_1}^1 \frac{\omega(t)}{t^2}dt \leq C\frac{\omega(s_1)}{s_1}.$$

Hence condition (1.6) implies $\omega(\delta) = \delta\frac{\omega(\delta)}{\delta} \leq C\int_0^\delta \frac{\omega(t)}{t}dt \leq C\omega(\delta)$ for any $0<\delta<1$. Therefore

$$\omega(s) \approx \int_0^s \frac{\omega(t)}{t}dt. \tag{4.3}$$

In particular

$$\omega(\frac{1}{n+1}) \leq C\int_{1-\frac{1}{n+1}}^1 \frac{\omega(1-t)}{1-t}t^n dt \leq C\int_0^1 \frac{\omega(1-t)}{1-t}t^n dt.$$

Therefore

$$\begin{aligned}
\frac{\tilde{\omega}(1-r)}{1-r} &\leq C\sum_{n=0}^\infty \int_0^1 \frac{\omega(1-t)}{1-t}(rt)^n dt \\
&= C\int_0^1 \frac{\omega(1-t)}{1-t}(\sum_{n=0}^\infty (rt)^n)dt \\
&= C\int_0^1 \frac{\omega(1-t)}{(1-t)(1-rt)}dt \\
&= C\int_0^1 \frac{\omega(u)}{(u(1-r)+ru^2)}du \\
&\leq \frac{C}{1-r}\int_0^{1-r} \frac{\omega(u)}{u}du + \frac{C}{r}\int_{1-r}^1 \frac{\omega(u)}{u^2}du \\
&\leq C\frac{\omega(1-r)}{1-r}.
\end{aligned}$$

This completes the proof. □

**Lemma 4.4.** *Let $X$ be a Banach space of analytic functions satisfying* (2.1) *and let $f(z) = \sum_{k=m}^{n} a_k z^k \in \mathcal{P}$. Then*

$$r^m ||f||_X \leq M_X(r, f) \leq r^n ||f||_X, \tag{4.4}$$

$$(m+1)||f||_X \leq ||Df||_X \leq (n+1)||f||_X. \tag{4.5}$$

*Proof.* Due to Proposition 2.2 there is no lost of generality in assuming that $X$ is homogeneous. We now recall that (see [BoShSl89, Lemma 3.4] or [MaPa82, Lemma 3.1])

$$r^m ||\phi||_\infty \leq M_\infty(r, \phi) \leq r^n ||\phi||_\infty, \tag{4.6}$$

and

$$m||\phi||_\infty \leq ||\phi'||_\infty \leq n||\phi||_\infty$$

for any polynomial $\phi(z) = \sum_{k=m}^{n} b_k z^k$. In particular

$$(m+1)||\phi||_\infty \leq ||D\phi||_\infty \leq (n+1)||\phi||_\infty. \tag{4.7}$$

Let $\Phi(z) = \sum_{k=m}^{n} x_k z^k$ be the vector-valued polynomial with coefficients $x_k = a_k u_k \in X$ where $a_k \in \mathbb{C}$ and $u_k(z) = z^k \in X$. Hence $\Phi(z) = \sum_{k=m}^{n} a_k u_k z^k = f_z \in X$ and $D\Phi(z) = (Df)_z$. Note that $||\Phi||_{A(\mathbb{D},X)} = ||f||_X$ and $||D\Phi||_{A(\mathbb{D},X)} = ||Df||_X$. On the other hand

$$||\Phi||_{A(\mathbb{D},X)} = \sup_{||x^*||\leq 1} \sup_{|z|=1} |\sum_{k=m}^{n} \langle x_k, x^* \rangle z^k|$$

and thus the proof of (4.4) and (4.5) follow directly from (4.6) and (4.7) respectively. □

We now point out that for homogeneous spaces of analytic functions the behaviour of $s_n f$ is strongly connected with the growth of the dyadic blocks $\Delta_n f$ in any space $X$. Recall that $\Delta_0 f = a_0 + a_1 z$ and

$$\Delta_n f = \sum_{k=2^{n-1}+1}^{2^n} a_k z^k, \quad n \geq 1.$$

**Proposition 4.5.** *Let $r_n = 1 - 2^{-n-1}$ for $n \in \mathbb{N}$ and let $X$ be a Banach space of analytic functions satisfying* (2.1) *and for any $g \in X_{\mathcal{P}}$*

$$||\Delta_n g||_X = O(M_X(r_n, g)), \quad n \to \infty. \tag{4.8}$$

*Let $\omega$ be a weight satisfying* (1.6), (1.7) *and $f \in X_{\mathcal{P}}$. Then the following are equivalent:*

(*i*) $||\Delta_n f||_X = O(\omega(2^{-n})),\ n \to \infty,$

(*ii*) $M_X(r, Df) = O(\frac{\omega(1-r)}{1-r}),\ r \to 1^-.$

*Proof.* $(i) \Rightarrow (ii)$. Consider $\tilde{\omega}(t) = t\sum_{n=0}^{\infty}\omega(\frac{1}{n+1})(1-t)^n$. Since $\Delta_n(Df) = D(\Delta_n f)$ then making use of (4.4) and (4.5) we obtain

$$\begin{aligned}
M_X(r, Df) &\leq \sum_n M_X(r, \Delta_n(Df)) \\
&\leq C\sum_{n=0}^{\infty} r^{2^n}||D(\Delta_n f)||_X \leq C\sum_{n=0}^{\infty} r^{2^n}2^n||\Delta_n f||_X \\
&\leq C\sum_{n=0}^{\infty} r^{2^n}2^n\omega(2^{-n}) \leq C\sum_{k=0}^{\infty} r^k\omega(\frac{1}{k+1}) \\
&\leq C\frac{\tilde{\omega}(1-r)}{1-r}.
\end{aligned}$$

Now use Proposition 4.3 to complete this implication.

$(ii) \Rightarrow (i)$ Using (4.8) and Lemma 4.4 for $r_n = 1 - 2^{-n-1}$ one concludes that

$$\begin{aligned}
||\Delta_n f||_X &\leq C2^{-n}||D(\Delta_n f)||_X = C2^{-n}||\Delta_n(Df)||_X \\
&\leq C2^{-n}M_X(r_n, Df) \leq C2^{-n}\frac{\omega(1-r_n)}{1-r_n} \leq C\omega(2^{-n}).
\end{aligned}$$

This finishes the result. $\square$

Let us finally give another description using approximation by Fourier series $s_n f$. Note that the fact that $\|f - s_n f\|_X \to 0$ as $n \to \infty$ for each $f \in X$, in the case $X = X_{\mathcal{P}}$ is equivalent to assume that $\|s_n f\| \leq C\|f\|$ for all $n \in \mathbb{N}$, i.e. the Riesz projection being bounded on $X$.

**Proposition 4.6.** *Let $X$ be a Banach space of analytic functions satisfying* (2.1) *such that the Riesz projection is bounded on $X$. Let $(\omega_n)_n$ be a non-increasing sequence of non-negative numbers such that $\lim_n \omega_n = 0$ and there exists $C > 0$ such that*

$$\omega_m \leq C\omega_{2m}, \quad m \in \mathbb{N}, \tag{4.9}$$

$$\sum_{k=m+1}^{\infty}\frac{\omega_k}{k} \leq C\omega_m, \quad m \in \mathbb{N}. \tag{4.10}$$

*For $f \in X_{\mathcal{P}}$, the following statements are equivalent:*
*$(i)$ $\|\Delta_n f\|_X = O(\omega_{2^n}),\ n \to \infty$,*
*$(ii)$ $||f - s_n f||_X = O(\omega_n),\ n \to \infty$.*

*Proof.* $(i) \Rightarrow (ii)$ Given $n \in \mathbb{N}$ consider $k(n)$ such that $2^{k(n)-1} + 1 \leq n < 2^{k(n)}$. Now

$$\begin{aligned}||f-s_nf||_X &= ||\sum_{k=2^{k(n)-1}+1}^{\infty} a_kz^k - \sum_{k=2^{k(n)-1}+1}^{n} a_kz^k||_X \\ &\le ||s_n(\Delta_{k(n)}f)||_X + \sum_{m\ge k(n)} ||\Delta_m f||_X \\ &\le C||\Delta_{k(n)}f||_X + \sum_{m\ge k(n)} ||\Delta_m f||_X \\ &\le C\sum_{m\ge k(n)} \omega_{2^m} \\ &\le C\sum_{k\ge 2^{k(n)}} \frac{\omega_k}{k} \\ &\le C\omega_{2^{k(n)}} \le C\omega_n.\end{aligned}$$

$(ii) \Rightarrow (i)$ It follows trivially, since (4.9) implies

$$||\Delta_k f||_X \le ||f-s_{2^{k-1}}f||_X + ||f-s_{2^k}f||_X \le C\omega_{2^k}.$$

The proof is then complete. □

*Remark* 4.7. If there exist $\gamma, \beta > 0$ such that $k^\gamma\omega_k$ and $k^\beta\omega_k$ are non-decreasing and non-increasing respectively then (4.9) and (4.10) holds. Indeed, on the one hand $\omega_m \le \frac{(2m)^\gamma}{m^\gamma}\omega_{2m} \le C\omega_{2m}$. On the other hand

$$\begin{aligned}\sum_{k\ge m+1}\frac{\omega_k}{k} &= \sum_{k\ge m+1}\frac{k^\beta\omega_k}{k^{1+\beta}} \\ &\le m^\beta\omega_m\sum_{k\ge m+1}\frac{1}{k^{1+\beta}} \\ &\le C\omega_m.\end{aligned}$$

**Proposition 4.8.** *Let $X$ be a Banach space of analytic functions satisfying* (2.1) *such that the Riesz projection is bounded on $X$. Let $(\omega_n)_n$ be a non-increasing sequence of non-negative numbers such that $\lim_m \omega_n = 0$ and there exists $C > 0$ such that*

$$\frac{1}{n+1}\sum_{k=0}^{n}\omega_k \le C\omega_n, \quad n\in\mathbb{N}. \tag{4.11}$$

*For $f \in X$, the following statements are equivalent:*
*(i) $||f-s_nf||_X = O(\omega_n),\ n\to\infty$,*
*(ii) $||f-\sigma_nf||_X = O(\omega_n),\ n\to\infty$.*

*Proof.* $(i) \Rightarrow (ii)$ Since $\sigma_n f = \frac{1}{n+1}\sum_{k=0}^n s_k f$ this implication follows from $(4.11)$ and the inequality

$$||f - \sigma_n f||_X \leq \frac{1}{n+1}\sum_{k=0}^{n} ||f - s_k f||_X.$$

$(ii) \Rightarrow (i)$ It follows from the estimate

$$||f - s_n f||_X \leq ||f - \sigma_n f||_X + ||s_n(f - \sigma_n f)||_X \leq (1 + ||s_n||)||f - \sigma_n f||_X.$$

□

**Theorem 4.9.** *Let $X$ be a Banach space of analytic functions satisfying* $(2.1)$ *such that the polynomials are dense and the Riesz projection is bounded on $X$. Let $\omega : [0,\pi] \to \mathbb{R}^+$ be a weight satisfying* $(1.5), (1.6)$ *and* $(1.7)$. *Then the following statements are equivalent:*
$(i)$ $M_X(r, Df) = O(\frac{\omega(1-r)}{1-r}),\ r \to 1^-$,
$(ii)$ $||\Delta_n f||_X = O(\omega(2^{-n})),\ n \to \infty$,
$(iii)$ $||f - s_n f||_X = O(\omega(n^{-1})),\ n \to \infty$,
$(iv)$ $||f - \sigma_n f||_X = O(\omega(n^{-1})),\ n \to \infty$.

*Proof.* We first notice that (4.8) is satisfied. Indeed by Lemma 4.4

$$||\Delta_n f||_X \approx M_X(r_n, \Delta_n f) = ||\Delta_n(P_{r_n} f)||_X \leq C||P_{r_n} f||_X = CM_X(r_n, f).$$

Hence the equivalence between $(i)$ and $(ii)$ follows from Proposition 4.5.

Let us now show that $\omega_n = \omega(\frac{1}{n+1})$ satisfies (4.9), (4.10) and (4.11).

Using (4.3) we have

$$\begin{aligned}
\omega_m &\leq C\int_0^{\frac{1}{m+1}} \frac{\omega(t)}{t}dt \\
&\leq C\left(\int_0^{\frac{1}{2m+1}} \frac{\omega(t)}{t}dt + \frac{1}{m+1}\int_{\frac{1}{2m+1}}^{\frac{1}{m+1}} \frac{\omega(t)}{t^2}dt\right) \\
&\leq C\omega_{2m}.
\end{aligned}$$

Using now $\int_{\frac{1}{k+1}}^{\frac{1}{k}} \frac{dt}{t} \approx \frac{C}{k}$ and $\int_{\frac{1}{k+1}}^{\frac{1}{k}} \frac{dt}{t^2} \approx C$ we obtain

$$\sum_{k=n+1}^{\infty} \frac{1}{k}\omega(\frac{1}{k+1}) \leq C\sum_{k=n+1}^{\infty}\int_{\frac{1}{k+1}}^{\frac{1}{k}} \frac{\omega(t)}{t}dt \leq C\int_0^{\frac{1}{n+1}} \frac{\omega(t)}{t}dt \leq C\omega(\frac{1}{n+1})$$

and

$$\sum_{k=1}^{n} \omega(\frac{1}{k+1}) \leq C\sum_{k=1}^{n}\int_{\frac{1}{k+1}}^{\frac{1}{k}} \frac{\omega(t)}{t^2}dt \leq C\int_{\frac{1}{n+1}}^{1} \frac{\omega(t)}{t^2}dt \leq C(n+1)\omega(\frac{1}{n+1}).$$

Hence the equivalences $(ii) \iff (iii)$ and $(iii) \iff (iv)$ follow from Proposition 4.6 and Proposition 4.8 respectively. □

The reader needs to combine Theorems 3.4, 3.7 and 4.9 to complete all the characterizations given in Theorem 1.2 in the introduction.

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DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, UNIVERSIDAD DE VALENCIA, 46100 BURJASSOT, VALENCIA, SPAIN

*E-mail address*: oblasco@uv.es

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF THESSALONIKI, THESSALONIKI 54124,, GREECE

*E-mail address*: stylog@math.auth.gr & g.stylog@gmail.com