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The frequency-localization technique and minimal decay-regularity for Euler–Maxwell equations

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ABSTRACT

Dissipative hyperbolic systems of *regularity-loss* have been recently received increasing attention. Extra higher regularity is usually assumed to obtain the optimal decay estimates, in comparison with the global-in-time existence of solutions. In this paper, we develop a new frequency-localization time-decay property, which enables us to overcome the technical difficulty and improve the minimal decay-regularity for dissipative systems. As an application, it is shown that the optimal decay rate of $L^1(\mathbb{R}^3)$–$L^2(\mathbb{R}^3)$ is available for Euler–Maxwell equations with the critical regularity $s_c = 5/2$, that is, the extra higher regularity is not necessary.

1. Introduction

In this paper, we are interested in compressible isentropic Euler–Maxwell equations in plasmas physics (see, for example, [4,17]), which are given by the form

$$\begin{cases} \partial_t n + \nabla \cdot (nu) = 0, \\ \partial_t (nu) + \nabla \cdot (nu \otimes u) + \nabla p(n) = -n(E + u \times B) - nu, \\ \partial_t E - \nabla \times B = nu, \\ \partial_t B + \nabla \times E = 0, \end{cases} \tag{1.1}$$

with constraints

$$\nabla \cdot E = n_\infty - n, \quad \nabla \cdot B = 0 \tag{1.2}$$

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for $(t,x) \in [0,+\infty) \times \mathbb{R}^3$. Here the unknowns $n > 0, u \in \mathbb{R}^3$ are the density and the velocity of electrons, and $E \in \mathbb{R}^3, B \in \mathbb{R}^3$ denote the electric field and magnetic field, respectively. The pressure $p(n)$ is a given smooth function of n satisfying $p'(n) > 0$ for $n > 0$. For the sake of simplicity, n_∞ is assumed to be a positive constant, which stands for the density of positively charged background ions. Observe that system (1.1) admits a constant equilibrium state $(n_\infty, 0, 0, B_\infty)$, which is regarded as vector in $\mathbb{R}^{10}$. $B_\infty \in \mathbb{R}^3$ is an arbitrary fixed constant vector. The main objective of the present paper is to investigate the large-time behavior for the corresponding Cauchy problem. For this purpose, system (1.1) is supplemented with the initial data

$$(n,u,E,B)|_{t=0} = (n_0,u_0,E_0,B_0)(x), \quad x \in \mathbb{R}^3. \tag{1.3}$$

It is not difficult to see that (1.2) can hold for any $t > 0$ if the initial data satisfy the following compatible conditions

$$\nabla \cdot E_0 = n_\infty - n_0, \quad \nabla \cdot B_0 = 0, \quad x \in \mathbb{R}^3. \tag{1.4}$$

Set $w = (n,u,E,B)^\top$ ($\top$ transpose) and $w_0 = (n_0,u_0,E_0,B_0)^\top$. Then (1.1) can be written in the vector form

$$A^0(w)w_t + \sum_{j=1}^{3} A^j(w)w_{x_j} + L(w)w = 0, \tag{1.5}$$

where the coefficient matrices are given explicitly as

$$A^0(w) = \begin{pmatrix} a(n) & 0 & 0 & 0 \\ 0 & nI & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \; L(w) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & n(I-\Omega_B) & nI & 0 \\ 0 & -nI & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\sum_{j=1}^{3} A^j(w)\xi_j = \begin{pmatrix} a(n)(u\cdot\xi) & p'(n)\xi & 0 & 0 \\ p'(n)\xi^\top & n(u\cdot\xi)I & 0 & 0 \\ 0 & 0 & 0 & -\Omega_\xi \\ 0 & 0 & \Omega_\xi & 0 \end{pmatrix}.$$

Here, $a(n) := p'(n)/n$ is the enthalpy function, I is the identity matrix of third order. For any $\xi = (\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3$, Ω_ξ is the skew-symmetric matrix defined by

$$\Omega_\xi = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}$$

such that $\Omega_\xi E^\top = (\xi \times E)^\top$ (as a column vector in $\mathbb{R}^3$).

Clearly, (1.5) is a symmetric hyperbolic system, since $A^0(w)$ is real symmetric and positive definite and $A^j(w)(j = 1,2,3)$ are real symmetric. Generally, the main feature of (1.5) is the finite time blowup of classical solutions even when the initial data are smooth and small. In one dimensional space, Chen, Jerome and Wang [4] first constructed global weak solutions by using the Godunov scheme of the fractional step. By using the dissipative effect of damping terms, Peng Wang and Gu [28] established the global existence of smooth solutions in the periodic domain. Duan [7] analyzed the regularity-loss mechanism in the dissipation part and constructed the global existence and time-decay estimates of smooth solutions. The first author [37] made the best use of the coupling structure of each equation in (1.5) and constructed global classical solutions in spatially critical Besov spaces. So far there are a number of efforts on the Euler–Maxwell system (1.1) with or without dissipation, see [9,11,12,23,24,32–34,41] and therein references.

For the Cauchy problem (1.1)–(1.3), in this paper, we focus on the quantitative decay estimates of solutions toward the equilibrium state $w_\infty = (n_\infty, 0, 0, B_\infty)$. Ueda, Wang and the second author [34] studied the dissipative structure of (1.5), which is weaker than the standard one characterized in [2,13,19–21,29, 35,38,39,43]. More precisely, the dissipative matrix $L(w)$ is nonnegative definite, however, $L(w)$ is not real symmetric, which leads to the regularity-loss not only in the dissipation part of the energy estimate but also in the decay estimate for the linearized system. To clarify it, let us reformulate (1.1) as the linearized perturbation form around the equilibrium state w_∞:

$$\begin{cases} \partial_t n + n_\infty \mathrm{div} v = 0, \\ \partial_t v + a_\infty \nabla n + E + v \times B + v = (\mathrm{div} q_2 + r_2)/n_\infty, \\ \partial_t E - \nabla \times B - n_\infty v = 0, \\ \partial_t B + \nabla \times E = 0, \end{cases} \tag{1.6}$$

where $v = nu/n_\infty,\ a_\infty = p'(n_\infty)/n_\infty$,

$$q_2 = -n_\infty^2 v \otimes v/n - [p(n) - p(n_\infty) - p'(n_\infty)(n - n_\infty)]I$$

and

$$r_2 = -(n - n_\infty)E - n_\infty v \times (B - B_\infty).$$

We put $z := (\rho, v, E, h)^\top$, where $\rho = n - n_\infty$ and $h = B - B_\infty$. The corresponding initial data are given by

$$z|_{t=0} = (\rho_0, v_0, E_0, h_0)^\top(x) \tag{1.7}$$

with $\rho_0 = n_0 - n_\infty,\ v_0 = n_0 u_0/n_\infty$ and $h_0 = B_0 - B_\infty$. System (1.6) can be also rewrite in the vector form as

$$A^0 z_t + \sum_{j=1}^{3} A^j z_{x_j} + Lz = \sum_{j=1}^{3} Q_{x_j} + R, \tag{1.8}$$

where A^0, A^j and L are the constant matrices in (1.5) with $w = w_\infty$, $Q(z) = (0, q_2^j/n_\infty, 0, 0)^\top$ and $R(z) = (0, r_2/n_\infty, 0, 0)^\top$. Observe that $Q(z) = O(|(\rho, v)|^2)$ and $R(z) = O(\rho|E| + |v||h|)$. The corresponding linear form reads as

$$A^0 \partial_t z_{\mathcal{L}} + \sum_{j=1}^{3} A^j \partial_{x_j} z_{\mathcal{L}} + L z_{\mathcal{L}} = 0, \tag{1.9}$$

and the initial data $z_0 := (\rho_0, v_0, E_0, h_0)^\top$ satisfy

$$\mathrm{div} E_0 = -\rho_0, \quad \mathrm{div} h_0 = 0. \tag{1.10}$$

In [33], Ueda and the second author showed that the Fourier image of $z_{\mathcal{L}}$ satisfies the following pointwise estimate

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$$|\widehat{z_{\mathcal{L}}}(t,\xi)| \lesssim e^{-c_0\eta_0(\xi)t}|\hat{z}_0| \tag{1.11}$$

for any $t \geq 0$, $c_0 > 0$ and $\xi \in \mathbb{R}^3$, where the dissipative rate $\eta_0(\xi) := |\xi|^2/(1+|\xi|^2)^2$. Furthermore, the decay estimate of $z_{\mathcal{L}}$ was followed:

$$\|\partial_x^k z_{\mathcal{L}}\|_{L^2} \lesssim (1+t)^{-3/4-k/2}\|z_0\|_{L^1} + (1+t)^{-\ell/2}\|\partial_x^{k+\ell} z_0\|_{L^2}, \tag{1.12}$$

where k and ℓ are non-negative integers.

Remark 1.1. The decay (1.12) is of the regularity-loss type, since $(1+t)^{-\ell/2}$ is created by assuming the additional ℓ-th order regularity on the initial data. As a matter of fact, similar dissipative mechanisms also appear in the study of other dissipative systems which were investigated by the second author and his collaborators in recent several years, for instance, Timoshenko systems in [15,16,22], hyperbolic-elliptic systems of radiating gas in [14], a plate equation with rotational inertia effect in [31], hyperbolic systems of viscoelasticity in [5,6], as well as the Vlasov–Maxwell–Boltzmann system studied by Duan and Strain et al. (see for example, [8,10]).

Furthermore, based on the decay estimate (1.12) of linearized solutions, the decay estimates of Euler–Maxwell equations (1.6)–(1.7) can be obtained by a combination of the time weighted energy method and the semigroup approach. To overcome the major difficulty arising from the weaker mechanism of regularity-loss, the decay regularity index is usually needed to be sufficiently large, for instance, $s \geq 13$ in [7], $s \geq 6$ in [33] and therein references. Very recently, Tan, Wang and Wang [32] obtained various decay rates of the solution and its derivatives for (1.1)–(1.3) by a regularity interpolation trick. However, their decay results may be available in the price of higher regularity of initial data. In the present paper, we investigate a different but interesting problem in comparison with previous efforts. That is, which regularity index does characterize the minimal decay regularity for (1.1)–(1.3)? For this motivation, we formulate a definition on the "minimal decay regularity".

Definition 1.1. If the optimal decay rate of $L^1(\mathbb{R}^n)$–$L^2(\mathbb{R}^n)$ type is achieved under the lowest regularity assumption, then the lowest index is called the minimal decay regularity for dissipative systems of regularity-loss, which is labelled as s_D.

Obviously, following from Definition 1.1, we found $s_D \leq 13$ in [7] and $s_D \leq 6$ in [33]. Based on the recent works [37,41], in Besov space with the regularity 5/2, is it possible to get optimal decay rates in functional spaces with relatively lower regularity? The present paper attempts to seek the minimal decay regularity for (1.1)–(1.3) such that $s_D \leq 5/2$. To achieve it, actually, we need to develop a general frequency-localization time-decay inequality not only for the dissipative rate $\eta_0(\xi) = |\xi|^2/(1+|\xi|^2)^2$, but also for more integer couple (a,b)-type: $\eta(\xi) = |\xi|^{2a}/(1+|\xi|^2)^b$ which has appeared in other dissipative systems of regularity-loss, see [5,6,8,10,14–16,22,31]. Precisely, we have

Theorem 1.1. *Let $\eta(\xi)$ be a positive, continuous and real-valued function in $\mathbb{R}^n$ satisfying*

$$\eta(\xi) \sim \begin{cases} |\xi|^{\sigma_1}, & |\xi| \to 0; \\ |\xi|^{-\sigma_2}, & |\xi| \to \infty; \end{cases} \tag{1.13}$$

for $\sigma_1, \sigma_2 > 0$.

If $f \in \dot{B}^{s+\ell}_{r,\alpha}(\mathbb{R}^n) \cap \dot{B}^{-\varrho}_{2,\infty}(\mathbb{R}^n)$ for $s \in \mathbb{R}, \varrho \in \mathbb{R}$ and $1 \leq \alpha \leq \infty$ such that $s + \varrho > 0$, then it holds that

$$\begin{aligned}&\Big\|2^{qs}\|\widehat{\dot{\Delta}_q f}e^{-\eta(\xi)t}\|_{L^2}\Big\|_{l^\alpha_q}\\ &\lesssim \underbrace{(1+t)^{-\frac{s+\varrho}{\sigma_1}}\|f\|_{\dot{B}^{-\varrho}_{2,\infty}}}_{Low\text{-}frequency\ Estimate} + \underbrace{(1+t)^{-\frac{\ell}{\sigma_2}+\gamma_{\sigma_2}(r,2)}\|f\|_{\dot{B}^{s+\ell}_{r,\alpha}}}_{High\text{-}frequency\ Estimate},\end{aligned} \tag{1.14}$$

for $\ell > n(\frac{1}{r} - \frac{1}{2})$[1] *with $1 \leq r \leq 2$, where $\gamma_{\sigma_2}(r,2) := \frac{n}{\sigma_2}(\frac{1}{r} - \frac{1}{2})$ and l^α_q stands for the α-type summation over $q \in \mathbb{Z}$.*

Remark 1.2. Compared to (1.12), more general time-decay estimate (1.14) is endowed with some novelty. To the best of our knowledge, not only does the high-frequency part decay in time with algebraic rates of any order as long as the function is spatially regular enough, but also decay information related to the localized integrability r is available. Indeed, different values of r (for example, $r = 1$ or $r = 2$) enable us to obtain the desired minimal decay regularity for dissipative systems. Secondly, note that the embedding $L^p(\mathbb{R}^n) \hookrightarrow \dot{B}^{-\varrho}_{2,\infty}(\mathbb{R}^n)(\varrho = n(1/p - 1/2),\ 1 \leq p < 2)$ in Lemma 2.3, it is shown that the low-frequency regularity is less restrictive than the usual L^p space. Additionally, the regularity index $s \in \mathbb{R}$ relaxes as the negative real constant rather than non-negative integers in (1.12) only.

For the convenience of reader, we give the direct consequence for the dissipative rate of $(1, 2)$ type, *i.e.*, $\sigma_1 = \sigma_2 = 2$ in the sense of (1.13).

Corollary 1.1. *Let $\eta(\xi) = |\xi|^2/(1 + |\xi|^2)^2$. If $f \in \dot{B}^{s+\ell}_{r,\alpha}(\mathbb{R}^n) \cap \dot{B}^{-\varrho}_{2,\infty}(\mathbb{R}^n)$ for $s \in \mathbb{R}, \varrho \in \mathbb{R}$ and $1 \leq \alpha \leq \infty$ such that $s + \varrho > 0$, then it holds that*

$$\begin{aligned}&\Big\|2^{qs}\|\widehat{\dot{\Delta}_q f}e^{-\eta(\xi)t}\|_{L^2}\Big\|_{l^\alpha_q}\\ &\lesssim \underbrace{(1+t)^{-\frac{s+\varrho}{2}}\|f\|_{\dot{B}^{-\varrho}_{2,\infty}}}_{Low\text{-}frequency\ Estimate} + \underbrace{(1+t)^{-\frac{\ell}{2}+\frac{n}{2}(\frac{1}{r}-\frac{1}{2})}\|f\|_{\dot{B}^{s+\ell}_{r,\alpha}}}_{High\text{-}frequency\ Estimate},\end{aligned} \tag{1.15}$$

for $\ell > n(\frac{1}{r} - \frac{1}{2})$ with $1 \leq r < 2$, or $\ell \geq 0$ with $r = 2$, where l^α_q stands for the α-type summation over $q \in \mathbb{Z}$.

Recalling those results in [37,41], we have the global-in-time existence of classical solutions to (1.1)–(1.3) in spatially critical Besov spaces.

Theorem 1.2. *([37,41]) Suppose that the initial data satisfy $w_0 - w_\infty \in B^{5/2}_{2,1}(\mathbb{R}^3)$ and compatible conditions (1.4). There exists a positive constant δ_0 such that if*

$$I_0 := \|w_0 - w_\infty\|_{B^{5/2}_{2,1}} \leq \delta_0, \tag{1.16}$$

then the system (1.1)–(1.3) admits a unique global solution $w(t, x)$ satisfying

$$w(t, x) \in \mathcal{C}^1([0, \infty) \times \mathbb{R}^3)$$

and

[1] Let us remark that $\ell \geq 0$ in the case of $r = 2$.

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$$w - w_\infty \in \widetilde{\mathcal{C}}(B^{5/2}_{2,1}(\mathbb{R}^3)) \cap \widetilde{\mathcal{C}}^1(B^{3/2}_{2,1}(\mathbb{R}^3)).$$

Moreover, the following energy inequality holds

$$N_0(t) + D_0(t) \lesssim I_0, \tag{1.17}$$

where $N_0(t) := \|w - w_\infty\|_{\widetilde{L}^\infty_t(B^{5/2}_{2,1})}$ *and* $D_0(t) := \|(n - n_\infty, u)\|_{\widetilde{L}^2_t(B^{5/2}_{2,1})} + \|E\|_{\widetilde{L}^2_t(B^{3/2}_{2,1})} + \|\nabla B\|_{\widetilde{L}^2_t(B^{1/2}_{2,1})}$ *for any* $t > 0$.

The reader is referred to Sect. 2 for the Besov space notations. With such low regularity, some decay techniques used in [33] fail. For instance, Ueda and the second author performed the time-weighted energy estimate related to the norm

$$W^\perp(t) := \sup_{0\le\tau\le t} (1 + \tau)\|(\rho, v, E)\|_{W^{1,\infty}}$$

to eliminate the difficulty of regularity-loss in the semigroup approach. Consequently, higher regularity was needed to bound the norm according to Sobolev embedding theorems. In the present paper, we introduce a different strategy to get around the main obstruction. Based on the Littlewood–Paley pointwise energy estimate (4.6), we could skip the usual semigroup approach as in [33,39]. Here, localized energy estimates for (1.6)–(1.7) are mainly performed in terms of high-frequency and low-frequency decompositions. It is worth noting that the time-decay inequality in Corollary 1.1 does play the key role to overcome the weak dissipative mechanism of regularity-loss at the high-frequency. More precisely, we make the best of advantages of (1.15) rather than (1.12). The high-frequency part is divided into two parts, and on each part, different values of r are chosen to obtain desired decay estimates with the assumption regularity $s_c = 5/2$, see, *e.g.*, (4.11) and (4.13) for more details. One can now state the main result in this position.

Theorem 1.3. *Assume that the initial data satisfy* $w_0 - w_\infty \in B^{5/2}_{2,1} \cap \dot{B}^{-3/2}_{2,\infty}$ *and (1.4). Set* $I_1 := \|w_0 - w_\infty\|_{B^{5/2}_{2,1}\cap\dot{B}^{-3/2}_{2,\infty}}$. *Then there exists a constant* $\delta_1 > 0$ *such that if* $I_1 \le \delta_1$, *then the classical solution to (1.1)–(1.3) admits the optimal decay estimate*

$$\|w - w_\infty\|_{L^2} \lesssim I_1(1 + t)^{-3/4}. \tag{1.18}$$

In Theorem 1.3, the low frequency regularity assumption is posted in $\dot{B}^{-3/2}_{2,\infty}$, which is less restrictive than the usual L^1 space, so a direct consequence is available immediately by Lemma 2.3.

Corollary 1.2. *Assume that the initial data satisfy* $w_0 - w_\infty \in B^{5/2}_{2,1} \cap L^1$. *Set* $\widetilde{I}_1 := \|w_0 - w_\infty\|_{B^{5/2}_{2,1}\cap L^1}$. *Then there exists a constant* $\delta_1 > 0$ *such that if* $\widetilde{I}_1 \le \delta_1$, *then the classical solution to (1.1)–(1.3) satisfies the optimal decay estimate (1.18), where* I_1 *is replaced by* $\widetilde{I}_1$.

Remark 1.3. In the present paper, we achieve that $s_D \le 5/2$, where the regularity $5/2$ is the same as that for global solutions in [37,41]. Compared to [40], it's the first time to show the optimal decay rate in critical Besov spaces, which gives a *sharp* minimal decay regularity for compressible Euler–Maxwell equations. Also, the assumptions for the regularity of the initial data are reduced heavily in comparison with previous works as [7,32,33] and therein references.

Remark 1.4. It is worth noting that Theorems 1.1–1.2 encourages us to investigate the minimal decay regularity for other dissipative systems with regularity-loss as in [5,6,8,10,14–16,22,31] in near future.

Finally, would like to note that all physical parameters are normalized to be one in (1.1). If considered, there are rigorous justifications on the singular parameter limits for (1.1), such as the nonrelativistic limit, quasi-neutral limit as well as combined nonrelativistic and quasi-neutral limits in [25–27], diffusive relaxation limits in [42] and so on.

The rest of this paper unfolds as follows. In Sect. 2, we review Littlewood–Paley decomposition, Besov spaces and Chemin–Lerner spaces to make the context possibly self-contained. Sect. 3 is devoted to prove the frequency-localization time-decay inequality. Furthermore, in Sect. 4, we deduce the optimal decay estimate for (1.6)–(1.7) by employing energy approaches in terms of high-frequency and low-frequency decompositions.

Notations. Throughout the paper, $f \lesssim g$ denotes $f \leq Cg$, where $C > 0$ is a generic constant. $f \approx g$ means $f \lesssim g$ and $g \lesssim f$. Denote by $\mathcal{C}([0,T], X)$ (resp., $\mathcal{C}^1([0,T], X)$) the space of continuous (resp., continuously differentiable) functions on $[0,T]$ with values in a Banach space X. Also, $\|(f,g,h)\|_X$ means $\|f\|_X + \|g\|_X + \|h\|_X$, where $f, g, h \in X$.

2. Preliminary

In this section, we briefly review the Littlewood–Paley decomposition, Besov spaces and Chemin–Lerner spaces in $\mathbb{R}^n (n \geq 1)$, see [1] for more details.

Firstly, we give an improved Bernstein inequality (see [36]), which allows the case of fractional derivatives.

Lemma 2.1. *Let $0 < R_1 < R_2$ and $1 \leq a \leq b \leq \infty$.*

(i) If $\mathrm{Supp}\mathcal{F}f \subset \{\xi \in \mathbb{R}^n : |\xi| \leq R_1\lambda\}$, then

$$\|\Lambda^\alpha f\|_{L^b} \lesssim \lambda^{\alpha + n(\frac{1}{a} - \frac{1}{b})} \|f\|_{L^a}, \quad \textit{for any} \quad \alpha \geq 0;$$

(ii) If $\mathrm{Supp}\mathcal{F}f \subset \{\xi \in \mathbb{R}^n : R_1\lambda \leq |\xi| \leq R_2\lambda\}$, then

$$\|\Lambda^\alpha f\|_{L^a} \approx \lambda^\alpha \|f\|_{L^a}, \quad \textit{for any} \quad \alpha \in \mathbb{R}.$$

Let (φ, χ) is a couple of smooth functions valued in $[0,1]$ such that φ is supported in the shell $\mathcal{C}(0, \frac{3}{4}, \frac{8}{3}) = \{\xi \in \mathbb{R}^n | \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$, χ is supported in the ball $\mathcal{B}(0, \frac{4}{3}) = \{\xi \in \mathbb{R}^n | |\xi| \leq \frac{4}{3}\}$ satisfying

$$\chi(\xi) + \sum_{q \in \mathbb{N}} \varphi(2^{-q}\xi) = 1, \qquad q \in \mathbb{N}, \quad \xi \in \mathbb{R}^n$$

and

$$\sum_{k \in \mathbb{Z}} \varphi(2^{-k}\xi) = 1, \qquad k \in \mathbb{Z}, \quad \xi \in \mathbb{R}^n \setminus \{0\}.$$

For $f \in \mathcal{S}'$ (the set of temperate distributions which is the dual of the Schwarz class $\mathcal{S}$), define

$$\begin{aligned} \Delta_{-1} f &:= \chi(D) f = \mathcal{F}^{-1}(\chi(\xi)\mathcal{F}f), \ \Delta_q f := 0 \ \text{ for } \ q \leq -2; \\ \Delta_q f &:= \varphi(2^{-q}D) f = \mathcal{F}^{-1}(\varphi(2^{-q}\xi)\mathcal{F}f) \ \text{ for } \ q \geq 0; \\ \dot{\Delta}_q f &:= \varphi(2^{-q}D) f = \mathcal{F}^{-1}(\varphi(2^{-q}\xi)\mathcal{F}f) \ \text{ for } \ q \in \mathbb{Z}, \end{aligned}$$

where $\mathcal{F}f$, $\mathcal{F}^{-1}f$ represent the Fourier transform and the inverse Fourier transform on f, respectively. Observe that the operator $\dot{\Delta}_q$ coincides with Δ_q for $q \geq 0$.

Denote by $\mathcal{S}_0' := \mathcal{S}'/\mathcal{P}$ the tempered distributions modulo polynomials $\mathcal{P}$. Furthermore, Besov spaces can be characterized by Littlewood–Paley decompositions.

Definition 2.1. For $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, the homogeneous Besov spaces $\dot{B}^s_{p,r}$ are defined by

$$\dot{B}^s_{p,r} = \{f \in S_0' : \|f\|_{\dot{B}^s_{p,r}} < \infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,r}} = \begin{cases} \left(\sum_{q\in\mathbb{Z}}(2^{qs}\|\dot{\Delta}_q f\|_{L^p})^r\right)^{1/r}, & r < \infty, \\ \sup_{q\in\mathbb{Z}} 2^{qs}\|\dot{\Delta}_q f\|_{L^p}, & r = \infty. \end{cases}$$

Similarly, one also has the definition of inhomogeneous Besov spaces.

Definition 2.2. For $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, the inhomogeneous Besov spaces $B^s_{p,r}$ are defined by

$$B^s_{p,r} = \{f \in S' : \|f\|_{B^s_{p,r}} < \infty\},$$

where

$$\|f\|_{B^s_{p,r}} = \begin{cases} \left(\sum_{q=-1}^{\infty}(2^{qs}\|\Delta_q f\|_{L^p})^r\right)^{1/r}, & r < \infty, \\ \sup_{q\geq -1} 2^{qs}\|\Delta_q f\|_{L^p}, & r = \infty. \end{cases}$$

Besov spaces obey various inclusion relations. Precisely,

Lemma 2.2. *Let $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, then*

(i) If $s > 0$, then $B^s_{p,r} = L^p \cap \dot{B}^s_{p,r}$;
(ii) If $\tilde{s} \leq s$, then $B^s_{p,r} \hookrightarrow B^{\tilde{s}}_{p,r}$. This inclusion relation is false for the homogeneous Besov spaces;
(iii) If $1 \leq r \leq \tilde{r} \leq \infty$, then $\dot{B}^s_{p,r} \hookrightarrow \dot{B}^s_{p,\tilde{r}}$ and $B^s_{p,r} \hookrightarrow B^s_{p,\tilde{r}}$;
(iv) If $1 \leq p \leq \tilde{p} \leq \infty$, then $\dot{B}^s_{p,r} \hookrightarrow \dot{B}^{s-n(\frac{1}{p}-\frac{1}{\tilde{p}})}_{\tilde{p},r}$ and $B^s_{p,r} \hookrightarrow B^{s-n(\frac{1}{p}-\frac{1}{\tilde{p}})}_{\tilde{p},r}$;
(v) $\dot{B}^{n/p}_{p,1} \hookrightarrow \mathcal{C}_0$, $B^{n/p}_{p,1} \hookrightarrow \mathcal{C}_0 (1 \leq p < \infty)$;

where $\mathcal{C}_0$ is the space of continuous bounded functions which decay at infinity.

In the recent work [30], Sohinger and Strain first introduced the Besov space of negative order to investigate the optimal time-decay rate of Boltzmann equation. Here, we give the simplification as follows.

Lemma 2.3. *Suppose that $\varrho > 0$ and $1 \leq p < 2$. It holds that*

$$\|f\|_{\dot{B}^{-\varrho}_{r,\infty}} \lesssim \|f\|_{L^p}$$

with $1/p - 1/r = \varrho/n$. In particular, this holds with $\varrho = n/2, r = 2$ and $p = 1$.

Usually, Moser-type product estimates play an important role in the estimate of bilinear terms.

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Proposition 2.1. *Let $s>0$ and $1\le p,r\le\infty$. Then $\dot{B}^s_{p,r}\cap L^\infty$ is an algebra and*

$$\|fg\|_{\dot{B}^s_{p,r}}\lesssim\|f\|_{L^\infty}\|g\|_{\dot{B}^s_{p,r}}+\|g\|_{L^\infty}\|f\|_{\dot{B}^s_{p,r}}.$$

Let $s_1,s_2\le n/p$ such that $s_1+s_2>n\max\{0,\frac{2}{p}-1\}$. Then one has

$$\|fg\|_{\dot{B}^{s_1+s_2-n/p}_{p,1}}\lesssim\|f\|_{\dot{B}^{s_1}_{p,1}}\|g\|_{\dot{B}^{s_2}_{p,1}}.$$

In the analysis of decay estimates, we use the general form of Moser-type product estimates, which was shown by [44].

Proposition 2.2. *Let $s>0$ and $1\le p,r,p_1,p_2,p_3,p_4\le\infty$. Assume that $f\in L^{p_1}\cap\dot{B}^s_{p_4,r}$ and $g\in L^{p_3}\cap\dot{B}^s_{p_2,r}$ with*

$$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}.$$

Then it holds that

$$\|fg\|_{\dot{B}^s_{p,r}}\lesssim\|f\|_{L^{p_1}}\|g\|_{\dot{B}^s_{p_2,r}}+\|g\|_{L^{p_3}}\|f\|_{\dot{B}^s_{p_4,r}}.$$

On the other hand, we need space–time mixed Besov spaces initiated by J.-Y. Chemin and N. Lerner in [3], which can be regarded as the refinement of the usual space–time mixed spaces $L^\theta_T(\dot{B}^s_{p,r})$ or $L^\theta_T(B^s_{p,r})$.

Definition 2.3. For $T>0, s\in\mathbb{R}, 1\le r,\theta\le\infty$, the homogeneous Chemin–Lerner spaces $\widetilde{L}^\theta_T(\dot{B}^s_{p,r})$ are defined by

$$\widetilde{L}^\theta_T(\dot{B}^s_{p,r}):=\{f\in L^\theta(0,T;\mathcal{S}'_0):\|f\|_{\widetilde{L}^\theta_T(\dot{B}^s_{p,r})}<+\infty\},$$

where

$$\|f\|_{\widetilde{L}^\theta_T(\dot{B}^s_{p,r})}:=\Big(\sum_{q\in\mathbb{Z}}(2^{qs}\|\dot{\Delta}_qf\|_{L^\theta_T(L^p)})^r\Big)^{\frac{1}{r}}$$

with the usual convention if $r=\infty$.

Definition 2.4. For $T>0, s\in\mathbb{R}, 1\le r,\theta\le\infty$, the inhomogeneous Chemin–Lerner spaces $\widetilde{L}^\theta_T(B^s_{p,r})$ are defined by

$$\widetilde{L}^\theta_T(B^s_{p,r}):=\{f\in L^\theta(0,T;\mathcal{S}'):\|f\|_{\widetilde{L}^\theta_T(B^s_{p,r})}<+\infty\},$$

where

$$\|f\|_{\widetilde{L}^\theta_T(B^s_{p,r})}:=\Big(\sum_{q\ge-1}(2^{qs}\|\Delta_qf\|_{L^\theta_T(L^p)})^r\Big)^{\frac{1}{r}}$$

with the usual convention if $r=\infty$.

We further define

$$\widetilde{\mathcal{C}}_T(B^s_{p,r}):=\widetilde{L}^\infty_T(B^s_{p,r})\cap\mathcal{C}([0,T],B^s_{p,r})$$

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and

$$\widetilde{\mathcal{C}}^1_T(B^s_{p,r}) := \{f \in \mathcal{C}^1([0,T], B^s_{p,r}) | \partial_t f \in \widetilde{L}^\infty_T(B^s_{p,r})\},$$

where the index T will be omitted when $T = +\infty$.

By Minkowski's inequality, Chemin–Lerner spaces can be linked with $L^\theta_T(X)$ with $X = B^s_{p,r}$ or $\dot{B}^s_{p,r}$.

Remark 2.1. It holds that

$$\|f\|_{\widetilde{L}^\theta_T(X)} \leq \|f\|_{L^\theta_T(X)} \text{ if } r \geq \theta; \quad \|f\|_{\widetilde{L}^\theta_T(X)} \geq \|f\|_{L^\theta_T(X)} \text{ if } r \leq \theta.$$

3. The proof of Theorem 1.1

In the recent decade, harmonic analysis tools, especially for techniques based on Littlewood–Paley decomposition and paradifferential calculus have proved to be very efficient in the study of partial differential equations, see for example [1]. It is well-known that the frequency-localization operator $\dot{\Delta}_q f$ (or $\Delta_q f$) has a smoothing effect on the function f, even though f is quite rough. Moreover, the L^p-norm of $\dot{\Delta}_q f$ can be preserved if $f \in L^p(\mathbb{R}^n)$. However, so far there are few efforts on the time-decay property related to the block operator, so Theorem 1.1 seems to be a suitable candidate for the motivation, which enables us to overcome the outstanding difficulty of *regularity-loss* in Besov spaces with relatively lower regularity.

Proof. Indeed, we proceed the proof for the inequality (1.14) with the aid of Littlewood–Paley frequency-localization techniques. It follows from the assumption (1.13) that there exist constants $c_0 > 0$ and $R_0 > 0$ such that

$$\begin{aligned} &\|\widehat{\dot{\Delta}_q f} e^{-\eta(\xi)t}\|_{L^2} \\ &\leq \|\widehat{\dot{\Delta}_q f} e^{-c_0|\xi|^{\sigma_1}t}\|_{L^2(|\xi|\leq R_0)} + \|\widehat{\dot{\Delta}_q f} e^{-c_0|\xi|^{-\sigma_2}t}\|_{L^2(|\xi|\geq R_0)}. \end{aligned} \tag{3.1}$$

We set $R_0 = 2^{q_0}(q_0 \in \mathbb{Z})$ without the loss of generality.

(i) If $q \geq q_0$, then $|\xi| \sim 2^q \geq R_0$, which leads to

$$\begin{aligned} &\|\widehat{\dot{\Delta}_q f} e^{-c_0|\xi|^{-\sigma_2}t}\|_{L^2(|\xi|\geq R_0)} \\ &= \Big\| |\xi|^\ell |\widehat{\dot{\Delta}_q f}| \frac{e^{-c_0 t|\xi|^{-\sigma_2}}}{|\xi|^\ell} \Big\|_{L^2(|\xi|\geq R_0)} \\ &\leq \||\xi|^\ell \widehat{\dot{\Delta}_q f}\|_{L^{r'}} \Big\| \frac{e^{-c_0 t|\xi|^{-\sigma_2}}}{|\xi|^\ell} \Big\|_{L^m(|\xi|\geq R_0)} \quad \Big(\frac{1}{r'} + \frac{1}{m} = \frac{1}{2},\ r' \geq 2\Big) \\ &\lesssim 2^{q\ell} \|\dot{\Delta}_q f\|_{L^r} \Big\| \frac{e^{-c_0 t|\xi|^{-\sigma_2}}}{|\xi|^\ell} \Big\|_{L^m(|\xi|\geq R_0)} \quad \Big(\frac{1}{r} + \frac{1}{r'} = 1\Big), \end{aligned} \tag{3.2}$$

where the Hausdorff–Young's inequality was used in the last line. By performing the change of variable, we can arrive at

$$\Big\| \frac{e^{-c_0 t|\xi|^{-\sigma_2}}}{|\xi|^\ell} \Big\|_{L^m(|\xi|\geq R_0)} \lesssim (1+t)^{-\frac{\ell}{\sigma_2} + \frac{n}{\sigma_2}(\frac{1}{r} - \frac{1}{2})} \tag{3.3}$$

for $\ell > n(\frac{1}{r}-\frac{1}{2})$. Besides, it can be also bounded by $(1+t)^{-\frac{\ell}{\sigma_2}}$ for $\ell \geq 0$ if $r=2$. Then it follows from (3.2)–(3.3) that

$$2^{qs}\|\widehat{\dot{\Delta}_q f}e^{-c_0|\xi|^{-\sigma_2}t}\|_{L^2} \lesssim 2^{q(s+\ell)}(1+t)^{-\frac{\ell}{\sigma_2}+\frac{n}{\sigma_2}(\frac{1}{r}-\frac{1}{2})}\|\dot{\Delta}_q f\|_{L^r}. \tag{3.4}$$

(ii) If $q < q_0$, then $|\xi| \sim 2^q \leq R_0$, which implies that

$$\|\widehat{\dot{\Delta}_q f}e^{-c_0|\xi|^{\sigma_1}t}\|_{L^2(|\xi|\leq R_0)} \lesssim \|\widehat{\dot{\Delta}_q f}\|_{L^2}e^{-c_0(2^q\ \sqrt[\sigma_1]{t})^{\sigma_1}}. \tag{3.5}$$

Furthermore, we can obtain

$$2^{qs}\|\widehat{\dot{\Delta}_q f}e^{-c_0|\xi|^{\sigma_1}t}\|_{L^2} \lesssim \|f\|_{\dot{B}^{-\varrho}_{2,\infty}}(1+t)^{-\frac{s+\varrho}{\sigma_1}}[(2^q\ \sqrt[\sigma_1]{t})^{s+\varrho}e^{-c_0(2^q\ \sqrt[\sigma_1]{t})^{\sigma_1}}] \tag{3.6}$$

for $s\in\mathbb{R}, \varrho\in\mathbb{R}$ such that $s+\varrho>0$. Note that

$$\left\|(2^q\ \sqrt[\sigma_1]{t})^{s+\varrho}e^{-c_0(2^q\ \sqrt[\sigma_1]{t})^{\sigma_1}}\right\|_{l^\alpha_q} \lesssim 1, \tag{3.7}$$

for any $\alpha\in[1,+\infty]$. Combining (3.4), (3.6)–(3.7), we conclude that

$$\begin{aligned}&\left\|2^{qs}\|\widehat{\dot{\Delta}_q f}e^{-\eta(\xi)t}\|_{L^2}\right\|_{l^\alpha_q}\\ &\lesssim \|f\|_{\dot{B}^{-\varrho}_{2,\infty}}(1+t)^{-\frac{s+\varrho}{\sigma_1}} + \|f\|_{\dot{B}^{s+\ell}_{r,\alpha}}(1+t)^{-\frac{\ell}{\sigma_2}+\frac{n}{\sigma_2}(\frac{1}{r}-\frac{1}{2})},\end{aligned} \tag{3.8}$$

which is just the inequality (1.14). □

4. The proof of Theorem 1.2

Due to the dissipative mechanism of regularity-loss, extra higher regularity is usually needed to obtain the optimal decay rate for (1.1)–(1.3). To achieve the minimal decay regularity $s_D \leq 5/2$, we skip the usual semigroup approach as in [33,39]. Consequently, the nonlinear energy estimate in Fourier spaces for (1.6)–(1.7) is performed. We would like to mention that similar estimates were first given by the second author in [18] for the Boltzmann equation, then well developed in [21] for hyperbolic systems of balance laws. In the following, our decay analysis focuses on (1.6)–(1.7). As a matter of fact, by Theorem 1.2, we can obtain a similar global existence for the solution z to (1.6)–(1.7). For simplification, allow us to abuse the notations $N_0(t)$ and $D_0(t)$ a little, which means that the corresponding functional norms with respect to z is still labelled as $N_0(t)$ and $D_0(t)$. Then it follows that these norms can be bounded by $\|z_0\|_{B^{5/2}_{2,1}}$.

Define

$$N(t) = \sup_{0\leq\tau\leq t}(1+\tau)^{\frac{3}{4}}\|z(\tau)\|_{L^2}, \tag{4.1}$$

$$D(t) = \|(\rho,v)\|_{L^2_t(B^{5/2}_{2,1})} + \|E\|_{L^2_t(B^{3/2}_{2,1})} + \|\nabla h\|_{L^2_t(B^{1/2}_{2,1})}. \tag{4.2}$$

The optimal decay estimate lies in a nonlinear time-weighted energy inequality, which is included in the following

Lemma 4.1. *Let $z=(\rho,v,E,h)^\top$ be the global classical solutions, which is similar to that in Theorem 1.2. Additionally, if $z_0\in\dot{B}^{-3/2}_{2,\infty}$, then it holds that*

$$N(t) \lesssim \|z_0\|_{B^{5/2}_{2,1}\cap\dot{B}^{-3/2}_{2,\infty}} + N(t)D(t) + N(t)^2. \tag{4.3}$$

Proof. It follows from the nonlinear energy method in [40] that

$$\frac{d}{dt}\mathcal{E}[\hat{z}] + c_1\eta_0(\xi)\mathcal{E}[\hat{z}] \lesssim (|\xi|^2|\hat{Q}|^2 + |\hat{R}|^2), \tag{4.4}$$

for $c_1 > 0$, where $\eta_0(\xi) = |\xi|^2/(1+|\xi|^2)^2$ and $\mathcal{E}[\hat{z}] \approx |\hat{z}|^2$. As a matter of fact, the corresponding Littlewood–Paley pointwise energy estimate is also available according to the derivation of (4.4):

$$\frac{d}{dt}\mathcal{E}[\widehat{\dot{\Delta}_q z}] + c_1\eta_0(\xi)\mathcal{E}[\widehat{\dot{\Delta}_q z}] \lesssim (|\xi|^2|\widehat{\dot{\Delta}_q Q}|^2 + |\widehat{\dot{\Delta}_q R}|^2), \tag{4.5}$$

where $\mathcal{E}[\widehat{\dot{\Delta}_q z}] \approx |\widehat{\dot{\Delta}_q z}|^2$. The standard Gronwall's inequality implies that

$$|\widehat{\dot{\Delta}_q z}|^2 \lesssim e^{-c_1\eta_0(\xi)t}|\widehat{\dot{\Delta}_q z_0}|^2 + \int_0^t e^{-c_1\eta_0(\xi)(t-\tau)}\Big(|\xi|^2|\widehat{\dot{\Delta}_q Q}|^2 + |\widehat{\dot{\Delta}_q R}|^2\Big)d\tau. \tag{4.6}$$

It follows from Fubini and Plancherel theorems that

$$\begin{aligned}
\|z\|_{L^2}^2 &= \sum_{q\in\mathbb{Z}} \|\dot{\Delta}_q z\|_{L^2}^2 \\
&\lesssim \sum_{q\in\mathbb{Z}} \|\widehat{\dot{\Delta}_q z_0} e^{-\frac{1}{2}c_1\eta_0(\xi)t}\|_{L^2}^2 \\
&\quad + \int_0^t \sum_{q\in\mathbb{Z}} \Big(\||\xi|\widehat{\dot{\Delta}_q Q} e^{-\frac{1}{2}c_1\eta_0(\xi)(t-\tau)}\|_{L^2}^2 + \|\widehat{\dot{\Delta}_q R} e^{-\frac{1}{2}c_1\eta_0(\xi)(t-\tau)}\|_{L^2}^2\Big)d\tau \\
&\triangleq J_1 + J_2 + J_3.
\end{aligned} \tag{4.7}$$

For J_1, by taking $r = \alpha = 2, s = 0, \varrho = 3/2$ and $\ell = 2$ in Corollary 1.1, we arrive at

$$\begin{aligned}
J_1 &= \Big(\sum_{q<q_0} + \sum_{q\geq q_0}\Big)\Big(\cdots\Big) \\
&\lesssim \|z_0\|_{\dot{B}^{-3/2}_{2,\infty}}^2 (1+t)^{-\frac{3}{2}} + \sum_{q\geq q_0} 2^{2q}\|\dot{\Delta}_q z_0\|_{L^2}^2 (1+t)^{-2} \\
&\lesssim \|z_0\|_{\dot{B}^{-3/2}_{2,\infty}}^2 (1+t)^{-\frac{3}{2}} + \|z_0\|_{\dot{B}^{2}_{2,2}}^2 (1+t)^{-2} \\
&\lesssim \|z_0\|_{\dot{B}^{-3/2}_{2,\infty}\cap B^{5/2}_{2,1}}^2 (1+t)^{-\frac{3}{2}},
\end{aligned} \tag{4.8}$$

where we have used the embedding relation in Lemma 2.2. Next, we begin to bound nonlinear terms on the right-hand side of (4.7). For J_2, it can be written as the sum of low-frequency and high-frequency:

$$J_2 = \int_0^t \Big(\sum_{q<q_0} + \sum_{q\geq q_0}\Big)\Big(\cdots\Big) \triangleq J_{2L} + J_{2H}. \tag{4.9}$$

For J_{2L}, by taking $\alpha = 2, s = 1, \varrho = 3/2$ in Corollary 1.1, we have

$$\begin{aligned}
J_{2L} &\leq \int_0^t (1+t-\tau)^{-\frac{5}{2}} \|Q(\tau)\|^2_{\dot{B}^{-3/2}_{2,\infty}} d\tau \\
&\lesssim \int_0^t (1+t-\tau)^{-\frac{5}{2}} \|Q(\tau)\|^2_{L^1} d\tau \\
&\lesssim \int_0^t (1+t-\tau)^{-\frac{5}{2}} \|z^{\perp}(\tau)\|^4_{L^2} d\tau \\
&\lesssim N^4(t) \int_0^t (1+t-\tau)^{-\frac{5}{2}} (1+\tau)^{-3} d\tau \\
&\lesssim N^4(t)(1+t)^{-\frac{5}{2}}, \qquad (4.10)
\end{aligned}$$

where we have used the embedding $L^1(\mathbb{R}) \hookrightarrow \dot{B}^{-3/2}_{2,\infty}(\mathbb{R}^3)$ in Lemma 2.3 and the fact $Q(z) = O(|z^{\perp}|^2)$ with $z^{\perp} := (\rho, \upsilon)$.

For the high-frequency part J_{2H}, we need more elaborate decay analysis to achieve the aim of $s_D \leq 5/2$. To do this, we write

$$J_{2H} = \Big(\int_0^{t/2} + \int_{t/2}^t\Big)\Big(\cdots\Big) \triangleq J_{2H1} + J_{2H2}.$$

For J_{2H1}, taking $r = \alpha = 2, s = 1$ and $\ell = 3/2$ in Corollary 1.1 gives

$$\begin{aligned}
J_{2H1} &= \int_0^{t/2} \sum_{q \geq q_0} \||\xi|\widehat{\dot{\Delta}_q Q} e^{-\frac{1}{2}c_1\eta_0(\xi)(t-\tau)}\|^2_{L^2} d\tau \\
&\leq \int_0^{t/2} (1+t-\tau)^{-\frac{3}{2}} \|Q(\tau)\|^2_{\dot{B}^{5/2}_{2,2}} d\tau \\
&\leq \int_0^{t/2} (1+t-\tau)^{-\frac{3}{2}} \|z^{\perp}\|^2_{L^\infty} \|z^{\perp}\|^2_{B^{5/2}_{2,1}} d\tau \\
&\leq \sup_{0 \leq \tau \leq t/2} \{(1+t-\tau)^{-\frac{3}{2}} \|z\|^2_{L^\infty}\} \int_0^{t/2} \|(\rho, \upsilon)\|^2_{B^{5/2}_{2,1}} d\tau \\
&\lesssim (1+t)^{-\frac{3}{2}} \|z_0\|^2_{B^{5/2}_{2,1}} D^2(t) \\
&\lesssim (1+t)^{-\frac{3}{2}} \|z_0\|^4_{B^{5/2}_{2,1}}, \qquad (4.11)
\end{aligned}$$

where we have used Proposition 2.1 and Lemma 2.2. Additionally, we would like to explain a little for the last step of (4.11). It follows from Remark 2.1 that

$$D(t) \lesssim D_0(t) \lesssim \|z_0\|_{B^{5/2}_{2,1}}, \qquad (4.12)$$

where the energy inequality (1.17) in Theorem 1.2 was used.

By choosing $\alpha = 2, r = s = 1$ and $\ell = 3/2$ in Corollary 1.1, I_{2H2} is proceeded as

$$
\begin{aligned}
J_{2H2} &= \int_{t/2}^{t} \sum_{q \geq q_0} \| |\xi| \widehat{\dot{\Delta}_q Q} e^{-\frac{1}{2}c_1 \eta_0(\xi)(t-\tau)} \|_{L^2}^2 d\tau \\
&\leq \int_{t/2}^{t} \sum_{q \geq q_0} 2^{2q(\frac{3}{2}+1)} \|\dot{\Delta}_q Q\|_{L^1}^2 (1+t-\tau)^{-\frac{3}{2}+3(1-\frac{1}{2})} d\tau \\
&= \int_{t/2}^{t} \|Q(\tau)\|^2_{\dot{B}^{5/2}_{1,2}} d\tau.
\end{aligned}
\tag{4.13}
$$

Recalling $Q(z) = O(|z^{\perp}|^2)$, it follows from Proposition 2.2 that

$$
\|Q(z)\|_{\dot{B}^{5/2}_{1,2}} \leq \|Q(z)\|_{\dot{B}^{5/2}_{1,1}} \lesssim \|z^{\perp}\|_{L^2} \|z^{\perp}\|_{\dot{B}^{5/2}_{2,1}}.
\tag{4.14}
$$

Together with (4.13)–(4.14), we are led to

$$
\begin{aligned}
J_{2H2} &\leq \int_{t/2}^{t} \|z^{\perp}(\tau)\|^2_{L^2} \|z^{\perp}(\tau)\|^2_{\dot{B}^{5/2}_{2,1}} d\tau \\
&\lesssim N(t)^2 \int_{t/2}^{t} (1+\tau)^{-\frac{3}{2}} \|z^{\perp}(\tau)\|^2_{\dot{B}^{5/2}_{2,1}} d\tau \\
&\lesssim N(t)^2 \sup_{t/2 \leq \tau \leq t} \left\{ (1+\tau)^{-\frac{3}{2}} \right\} \int_{t/2}^{t} \|(\rho, \upsilon)\|^2_{B^{5/2}_{2,1}} d\tau \\
&\lesssim (1+t)^{-\frac{3}{2}} N(t)^2 D(t)^2.
\end{aligned}
\tag{4.15}
$$

Hence, combine (4.11) and (4.15) to get

$$
J_{2H} \lesssim (1+t)^{-\frac{3}{2}} \|z_0\|^4_{B^{5/2}_{2,1}} + (1+t)^{-\frac{3}{2}} N(t)^2 D(t)^2.
\tag{4.16}
$$

Furthermore, it follows from (4.10) and (4.16) that

$$
J_2 \lesssim (1+t)^{-\frac{3}{2}} \|z_0\|^4_{B^{5/2}_{2,1}} + (1+t)^{-\frac{3}{2}} N(t)^2 D(t)^2 + N(t)^4 (1+t)^{-\frac{5}{2}}.
\tag{4.17}
$$

For J_3, we can perform the similar decay estimates as J_2. Firstly, we write

$$
J_3 = \int_0^t \Big(\sum_{q<q_0} + \sum_{q \geq q_0} \Big) \Big(\cdots \Big) \triangleq J_{3L} + J_{3H}.
\tag{4.18}
$$

Note that $R(z) = O(\rho|E| + |\upsilon||h|)$, by taking $\alpha = 2, s = 0$ and $\varrho = 3/2$ in Corollary 1.1, we obtain

$$
\begin{aligned}
J_{3L} &= \int_0^t (1+t-\tau)^{-\frac{3}{2}} \|R(\tau)\|^2_{\dot{B}^{-3/2}_{2,\infty}} d\tau \\
&\le \int_0^t (1+t-\tau)^{-\frac{3}{2}} \|R(\tau)\|^2_{L^1} d\tau \\
&\le \int_0^t (1+t-\tau)^{-\frac{3}{2}} \|z(\tau)\|^4_{L^2} d\tau \\
&\lesssim N(t)^4 \int_0^t (1+t-\tau)^{-\frac{3}{2}} (1+\tau)^{-3} d\tau \\
&= N(t)^4 (1+t)^{-\frac{3}{2}},
\end{aligned} \tag{4.19}
$$

where Lemma 2.3 was well used. For the high-frequency part, we separate it into two parts

$$
J_{3H} = \Big(\int_0^{t/2} + \int_{t/2}^t\Big)\Big(\cdots\Big) \triangleq J_{3H1} + J_{3H2}.
$$

For J_{3H1}, taking $r=\alpha=2, s=0$ and $\ell=3/2$ in Corollary 1.1 gives

$$
J_{3H1} \lesssim \int_0^{t/2} (1+t-\tau)^{-\frac{3}{2}} \|R(\tau)\|^2_{\dot{B}^{3/2}_{2,2}} d\tau. \tag{4.20}
$$

It follows from Lemma 2.1 and Proposition 2.1 that

$$
\begin{aligned}
\|R\|_{\dot{B}^{3/2}_{2,2}} &\le \|R\|_{\dot{B}^{3/2}_{2,1}} \\
&\lesssim \|(\varrho,\upsilon)\|_{L^\infty}\Big(\|E\|_{\dot{B}^{3/2}_{2,1}} + \|\nabla h\|_{\dot{B}^{1/2}_{2,1}}\Big) \\
&\quad + \|(h,E)\|_{L^\infty}\|(\varrho,\upsilon)\|_{\dot{B}^{3/2}_{2,1}} \\
&\lesssim \|z\|_{L^\infty}\Big(\|(\varrho,\upsilon,E)\|_{\dot{B}^{3/2}_{2,1}} + \|\nabla h\|_{\dot{B}^{1/2}_{2,1}}\Big).
\end{aligned} \tag{4.21}
$$

Therefore, substituting (4.21) into (4.20) gives

$$
\begin{aligned}
J_{3H1} &\lesssim \int_0^{t/2} (1+t-\tau)^{-\frac{3}{2}} \|z(\tau)\|^2_{L^\infty}\Big(\|(\varrho,\upsilon,E)\|_{\dot{B}^{3/2}_{2,1}} + \|\nabla h\|_{\dot{B}^{1/2}_{2,1}}\Big)^2 d\tau \\
&\lesssim \sup_{0\le\tau\le t/2}\Big\{(1+t-\tau)^{-\frac{3}{2}}\|z(\tau)\|^2_{L^\infty}\Big\} \\
&\quad \times \int_0^{t/2}\Big(\|(\varrho,\upsilon)\|^2_{B^{5/2}_{2,1}} + \|E\|^2_{B^{3/2}_{2,1}} + \|\nabla h\|^2_{B^{1/2}_{2,1}}\Big) d\tau \\
&\lesssim (1+t)^{-\frac{3}{2}}\|z_0\|^2_{B^{5/2}_{2,1}} D(t)^2 \\
&\lesssim (1+t)^{-\frac{3}{2}}\|z_0\|^2_{B^{5/2}_{2,1}},
\end{aligned} \tag{4.22}
$$

where we have used Lemma 2.2 and (4.12).

On the other hand, by taking $\alpha = 2, r = 1, s = 0$ and $\ell = 3/2$ in Corollary 1.1, we arrive at

$$J_{3H2} = \int_{t/2}^{t} \sum_{q \geq q_0} 2^{3q} \|\dot{\Delta}_q R(\tau)\|_{L^1}^2 d\tau$$
$$\lesssim \int_{t/2}^{t} \|R(\tau)\|^2_{\dot{B}^{3/2}_{1,2}} d\tau, \tag{4.23}$$

where Lemma 2.1, Lemma 2.2 and Proposition 2.2 enable us to obtain

$$\|R\|_{\dot{B}^{3/2}_{1,2}} \leq \|R\|_{\dot{B}^{3/2}_{1,1}}$$
$$\lesssim \|(\varrho, \upsilon)\|_{L^2} \Big(\|E\|_{\dot{B}^{3/2}_{2,1}} + \|\nabla h\|_{\dot{B}^{1/2}_{2,1}}\Big)$$
$$+ \|(h, E)\|_{L^2} \|(\varrho, \upsilon)\|_{\dot{B}^{3/2}_{2,1}}$$
$$\lesssim \|z\|_{L^2} \Big(\|(\varrho, \upsilon, E)\|_{\dot{B}^{3/2}_{2,1}} + \|\nabla h\|_{\dot{B}^{1/2}_{2,1}}\Big). \tag{4.24}$$

Together with (4.23)–(4.24), we are led to

$$J_{3H2} \lesssim \int_{t/2}^{t} \|z(\tau)\|^2_{L^2} \Big(\|(\varrho, \upsilon, E)\|_{\dot{B}^{3/2}_{2,1}} + \|\nabla h\|_{\dot{B}^{1/2}_{2,1}}\Big)^2 d\tau$$
$$\lesssim N(t)^2 \int_{t/2}^{t} (1+\tau)^{-\frac{3}{2}} \Big(\|(\varrho, \upsilon)\|^2_{B^{5/2}_{2,1}} + \|E\|^2_{B^{3/2}_{2,1}} + \|\nabla h\|^2_{B^{1/2}_{2,1}}\Big) d\tau$$
$$\lesssim N(t)^2 \sup_{t/2 \leq \tau \leq t} \Big\{(1+\tau)^{-\frac{3}{2}}\Big\} D(t)^2$$
$$\lesssim (1+t)^{-\frac{3}{2}} N(t)^2 D(t)^2. \tag{4.25}$$

Then, it follows from inequalities (4.19), (4.22) and (4.25) that

$$J_3 \lesssim (1+t)^{-\frac{3}{2}} \|z_0\|^2_{B^{5/2}_{2,1}} + (1+t)^{-\frac{3}{2}} N(t)^2 D(t)^2 + (1+t)^{-3} N(t)^4. \tag{4.26}$$

Finally, combine (4.8), (4.17) and (4.26) to obtain

$$\|z\|^2_{L^2} \lesssim (1+t)^{-\frac{3}{2}} \|z_0\|^2_{B^{5/2}_{2,1} \cap \dot{B}^{-\frac{3}{2}}_{2,\infty}}$$
$$+ (1+t)^{-\frac{3}{2}} N(t)^2 D(t)^2 + (1+t)^{-3} N(t)^4, \tag{4.27}$$

where we have used the smallness of $\|z_0\|_{B^{5/2}_{2,1}}$. This leads to the desired inequality (4.3). □

Proof of Theorem 1.3. Note that (4.12), we arrive at

$$D(t) \lesssim \|z_0\|_{B^{5/2}_{2,1}} \lesssim I_1. \tag{4.28}$$

Then the inequality (4.3) can be solved as $N(t) \lesssim I_1$ by the standard argument, provided that I_1 is sufficiently small. Consequently, the desired decay estimate

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$$\|w - w_\infty\|_{L^2} \lesssim I_1(1+t)^{-\frac{3}{4}} \tag{4.29}$$

follows immediately. Hence, the proof of Theorem 1.3 is complete. □

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