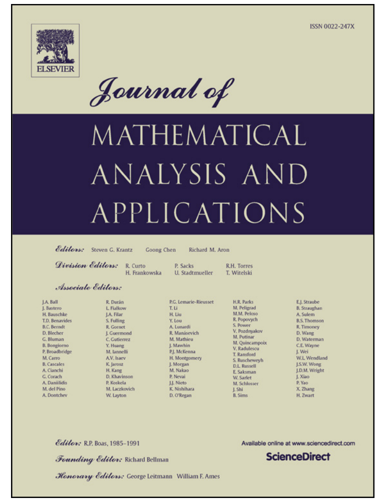
# Accepted Manuscript

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PII: S0022-247X(16)30799-5
DOI: http://dx.doi.org/10.1016/j.jmaa.2016.12.013
Reference: YJMAA 20952

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 1 July 2015

# Existence and regularity of linear nonlocal Fokker-Planck equation with growing drift

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**Abstract**

The nonlocal Fokker-Planck equations for a class of stochastic differential equations with non-Gaussian $\alpha-$stable Lévy motion in Euclidean space are studied. The existence and uniqueness of weak solution are obtained with rough drift. The solution is shown to be smooth on spatial variable if all derivatives of the drift are bounded. Moreover, the solution is jointly smooth on spatial and time variable if we assume further that the drift grows like a power of logarithm function at infinity.

*Keywords:*
Fractional Laplacian operator, Non-Gaussian Lévy noise, Nonlocal Fokker-Planck equation

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## 1. Introduction

In this work, we consider the following nonlocal Fokker-Planck (NFP) equation defined on $\mathbf{R}^n$

$$\begin{cases} u_t + \Lambda^\alpha u + \nabla \cdot (\boldsymbol{a}(x)u) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

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where $\boldsymbol{a} : \mathbf{R}^n \mapsto \mathbf{R}^n$ is a time independent function. The fractional Laplacian $\Lambda^\alpha, \alpha \in (0,2)$, is defined by

$$\Lambda^\alpha f(x) = c_{\alpha,n} P.V. \int_{\mathbf{R}^n} \frac{f(x) - f(y)}{|x-y|^{n+\alpha}} dy, \tag{1.2}$$

where $c_{\alpha,n}$ is a constant depending on $n, \alpha$.

The NFP equation has attracted many people's attention due to following reasons. First of all, the probability density function of anomalous diffusion in a given position can be affected by distant points in space. In this case, Eq.(1.1) is more appropriate than the usual Fokker-Planck equation, see e.g. [2, 3, 11]. Moreover, since Eq.(1.1) can be regraded as a linearized Quasi-geostrophic (QG) equation, the study of (1.1) will help us understand the evolution of QG [10, 12]. Finally, Eq. (1.1) is connected with a stochastic differential equation with a random source denoted by $\widehat{X_t}$ and a drift term given by a deterministic function $\boldsymbol{a}(x)$:

$$dX_t = \boldsymbol{a}(X_t)dt + d\widehat{X_t}, \tag{1.3}$$

where $\widehat{X_t}$ is the $\alpha-$stable Lévy process (non-Gaussian process), and the solution of (1.1) is the probability density of $X_t$. For more background of non-Gaussian process, we refer the readers to [6, 15].

The existence and regularity of solutions for (1.1) with bounded $\boldsymbol{a}(x)$ were studied in [5, 9]. The heat kernel estimate of the semigroup generated by the operator $\Lambda^\alpha + \nabla \cdot (\boldsymbol{a}\cdot)$ was obtained in [1, 16] if $\boldsymbol{a}(x)$ belongs to some Kato class. In particular, the drift $\boldsymbol{a}(x)$ is necessary to satisfy that

$$\sup_{x\in\mathbf{R}^n} \int_{B(x,1)} |\boldsymbol{a}(x)| dx < \infty, \tag{1.4}$$

where $B(x,r)$ denotes the ball centered at $x$ with radius $r$. In other words, the average on a unit ball does not grow at infinity.

Recently, the solution of Eq.(1.1) with Ornstein-Uhlenbeck (OU) drift $\boldsymbol{a}(x) = \boldsymbol{x}$ was shown to be smooth in [17]. Note that the OU drift does not satisfy (1.4). Motivated by this work, we are interested in the well posedness and smoothing effect of (1.1) with general growing drift on the whole space $\mathbf{R}^n$. In section 2, we establish the existence and uniqueness of weak solution of (1.1) in $L^2$ under some regularity assumptions on $\boldsymbol{a}(x)$. In section 3, we first show that the unique solution is smooth on $x$ if the derivatives of $\boldsymbol{a}(x)$

of arbitrarily order are bounded. It should be noted that $\boldsymbol{a}(x)$ allows to be growing like $|x|$ at infinity. Second, the solution is jointly smooth on $x$ and $t$ if we assume further that $\boldsymbol{a}(x)$ grows at most like a power of logarithm function at infinity.

Before leaving this section, we say a few words about the notations. We use $H^s, L^p$ to denote the usual Sobolev spaces on $\mathbf{R}^n$. The domain $\mathbf{R}^n$ is also omitted for other function spaces in some places. We denote by $\langle x\rangle = (1+|x|^2)^{1/2}$. $A \lesssim B$ means $A \leq CB$ for some absolute constant $C$, $A \sim B$ means $A \lesssim B$ and $B \lesssim A$, and $A \gg B$ means $A/B$ is very big, say $A/B \geq 1000$.

## 2. Well posedness

This section is devoted to the existence of weak solutions of (1.1) with general potential function $\boldsymbol{a}(x)$. We assume that $\boldsymbol{a}(x) = (a_1, a_2, \cdots, a_n)$ satisfy the following conditions:

$$(A) \quad a_j \in L^1_{loc}, \partial_{x_i} a_j \in L^p_u, p > \max\{1, n/\alpha\}, 1 \leq i, j \leq n.$$

Here $L^p_u$, $1 \leq p < \infty$, denotes the uniformly local $L^p-$integrable space consisting of functions such that

$$\|f\|_{L^p_u} := \sup_{x\in\mathbf{R}^n} \left( \int_{|x-y|<1} |f(y)|^p dy \right)^{1/p} < \infty.$$

Let $0 \leq \varphi \leq 1$ be a smooth cutoff function such that $\varphi = 1$ if $|x| \leq 1$ and $\varphi = 0$ if $|x| \geq 2$. Set $\varphi_j = \varphi(x-j), j \in \mathbb{Z}^n$. Then it's easy to see that $L^p_u$ has the following equivalent norm

$$\|f\|_{L^p_u} = \sup_{j\in\mathbb{Z}^n} \|\varphi_j f\|_{L^p}.$$

For $\varepsilon > 0$, let $\eta_\varepsilon = \varepsilon^{-n}\varphi(\frac{x}{\varepsilon})/\|\varphi\|_{L^1}$. Then $\int \eta_\varepsilon dx = 1$, $\varepsilon > 0$. The proof of the following lemma is standard, we give it for completeness.

**Lemma 2.1.** *If $f \in L^p_u$, then*

$$\lim_{\varepsilon\to 0} \|\eta_\varepsilon * f - f\|_{L^p_u} = 0.$$

*Proof.* The lemma follows if one can show

$$\lim_{\varepsilon\to 0}\sup_{j\in\mathbb{Z}^n}\|\varphi_j(x)(\eta_\varepsilon * f - f)\|_{L^p} = 0. \tag{2.5}$$

It's well known that $\eta_\varepsilon$ is an approximate unity in $L^p$, note that $\varphi_j f$ are uniformly bounded in $L^p$ we have

$$\lim_{\varepsilon\to 0}\sup_{j\in\mathbb{Z}^n}\|\eta_\varepsilon * (\varphi_j f) - \varphi_j f\|_{L^p} = 0.$$

Now (2.5) is reduced to proving

$$\lim_{\varepsilon\to 0}\sup_{j\in\mathbb{Z}^n}\|\varphi_j(x)(\eta_\varepsilon * f) - \eta_\varepsilon * (\varphi_j(x)f)\|_{L^p} = 0. \tag{2.6}$$

In fact, since the support of $\eta_\varepsilon$ is contained in $\{x : |x| \leq 2\varepsilon\}$, we have

$$\begin{aligned}
&\varphi_j(x)(\eta_\varepsilon * f) - \eta_\varepsilon * (\varphi_j(x)f) \\
&\qquad = \int_{\mathbf{R}^n}(\varphi_j(x) - \varphi_j(y))\eta_\varepsilon(x-y)\chi_{|y-j|\leq 2+2\varepsilon}(y)f(y)dy,
\end{aligned} \tag{2.7}$$

where $\chi_A$ is the characteristic function of the set $A$. Since $\varphi_j$ is smooth and $\varphi_j'$ is bounded, we find

$$\begin{aligned}
|(2.7)| &\lesssim \int_{\mathbf{R}^n}|\nabla\varphi_j|_{L^\infty}|x-y|\eta_\varepsilon(x-y)\chi_{|y-j|\leq 2+2\varepsilon}(y)|f(y)|dy \\
&\lesssim \varepsilon\eta_\varepsilon * (\chi_{|\cdot-j|\leq 2+2\varepsilon}|f|).
\end{aligned}$$

Using Young's inequality we obtain

$$\|\varphi_j(x)(\eta_\varepsilon * f) - \eta_\varepsilon * (\varphi_j(x)f)\|_{L^p} \lesssim \varepsilon\|\eta_\varepsilon\|_{L^1}\|\chi_{|\cdot-j|\leq 2+2\varepsilon}|f|\|_{L^p} \lesssim \varepsilon\|f\|_{L^p_u} \to 0$$

as $\varepsilon$ goes to 0. Hence (2.6) follows, and the proof is complete. □

From Lemma 2.1, we find smooth functions are dense in $L^p_u$. As an application, we prove that, under the assumption $(A)$, $a_j$ is a tempered distribution.

**Lemma 2.2.** *Assume that $(A)$ holds. Then for $j = 1, 2, \cdots$, and $R > 0$*

$$\|a_j\|_{L^p(|x|\leq R)} \lesssim \langle R\rangle^{1+\frac{n}{p}}.$$

*Proof.* Set $a_j^\varepsilon = \eta_\varepsilon * a_j$, then $a_j^\varepsilon$ is smooth. Let $|x_0| \le 1, |x| \le R > 0$, by Taylor's formulae, we have

$$a_j^\varepsilon(x) = a_j^\varepsilon(x_0) + (x - x_0) \cdot \nabla a_j^\varepsilon(x_0 + \theta(x - x_0))$$

for some $\theta \in (0,1)$. Taking $L^p(|x| \le R)$ norm with respect to $x$ on both side implies that

$$\|a_j^\varepsilon(x)\|_{L^p(|x|\le R)} \lesssim |a_j^\varepsilon(x_0)| R^{\frac{n}{p}} + \langle R\rangle \|\nabla a_j^\varepsilon\|_{L^p(|x|\le R+1)}.$$

Integrating on $\{x_0 : |x_0| \le 1\}$ with respect to $x_0$, using Lemma 2.1 yield that

$$\begin{aligned}\|a_j^\varepsilon(x)\|_{L^p(|x|\le R)} &\lesssim |a_j^\varepsilon|_{L^1(|x|\le 1)} R^{\frac{n}{p}} + \langle R\rangle \|\eta_\varepsilon * \nabla a_j\|_{L^p(|x|\le R+1)} \\ &\lesssim R^{\frac{n}{p}} + \langle R\rangle \|\nabla a_j\|_{L^p(|x|\le R+1)} \lesssim \langle R\rangle^{1+\frac{n}{p}}\end{aligned}$$

for $\varepsilon > 0$ small enough. Since $a_j^\varepsilon \to a_j$ in $L^p(|x| \le R)$, it follows from Fatou's lemma that

$$\|a_j\|_{L^p(|x|\le R)} \le \varliminf_{\varepsilon\to 0} \|a_j^\varepsilon\|_{L^p(|x|\le R)} \lesssim \langle R\rangle^{1+\frac{n}{p}}$$

as desired. □

**Lemma 2.3.** *Assume that (A) holds. Then for $j = 1, 2, \cdots, n$*

$$\|\langle x\rangle^{-(n+1)} a_j\|_{L^2} \lesssim 1.$$

*Proof.* Since $p > 1$, it follows from Lemma 2.2 that for $j = 1, 2, \cdots,$

$$\|a_j\|_{L^p} \lesssim \langle R\rangle^{n+1}.$$

Note the implicit constant is independent of $R$, we find

$$\|\langle x\rangle^{-(n+1)} a_j\|_{L^p} \lesssim 1.$$

Also, thanks to the fact $\partial_i a_j \in L_u^p$, it is easy to see

$$\|\langle x\rangle^{-(n+1)} \partial_i a_j\|_{L^p} \lesssim \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{-(n+1)} \|\partial_i a_j\|_{L_u^p} \lesssim 1.$$

Hence

$$\|\langle x\rangle^{-(n+1)} a_j\|_{H^{1,p}} \lesssim 1.$$

The lemma follows from Sobolev embedding theorems if we use the fact that $p > 1$ when $n \leq 2$ and $p > \frac{n}{\alpha}, 0 < \alpha < 2$ when $n \geq 3$. □

It follows from Lemma 2.2 that $a_j \in L^2_{loc}$. Under the assumption $(A)$, if $u \in L^2_{loc}((0,T) \times \mathbf{R}^n)$ then $a_j u \in L^1_{loc}((0,T) \times \mathbf{R}^n)$, $j = 1, 2, \cdots$. Thus we can understand (1.1) in distribution sense. This leads to the following definition.

**Definition 2.1.** *Assume that $(A)$ holds. A function $u \in L^2_{loc}((0,T) \times \mathbf{R}^n)$ is said to be a weak solution of* (1.1) *on $((0,T) \times \mathbf{R}^n)$ if*

$$u_t + \Lambda^\alpha u + \nabla \cdot (\boldsymbol{a}(x) u) = 0$$

*holds in $\mathscr{D}'((0,T) \times \mathbf{R}^n)$.*

*Furthermore, if $T$ can be arbitrary large, we say the weak solution is global.*

We shall construct a weak solution by the Galerkin method. Let $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$ be a mult-index of nonnegative integers and

$$e_\beta(x_1, x_2, \cdots, x_n) := \prod_{j=1}^{n} H_{\beta_j}(x_j) e^{-\frac{1}{2}(x_1^2 + x_2^2 + \cdots + x_n^2)},$$

where $H_{\beta_j}$ are Hermite polynomials of degree $\beta_j$. Then $e_\beta$ form an orthonormal basis of $L^2$. Moreover, it is easy to check that the $e_\beta$ are eigenvectors of the operator $-\triangle + |x|^2$ with eigenvalues $2|\beta| + n$, $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n$. Denote the eigenvalues by $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots \to \infty$, and the corresponding eigenvectors by $e_j$ satisfying

$$(-\triangle + |x|^2) e_j = \lambda_j e_j, j = 1, 2, \cdots .$$

Then $\{e_j(x)\}_{j=1}^{\infty}$ form an orthonormal basis of $L^2$ and $e_j$ are Schwartz functions. Let $P^N$ be the orthogonal projection in $L^2$ onto the span of the basis $e_j(x)$ with $1 \leq j \leq N$.

**Lemma 2.4.** *If $\varphi \in \mathscr{S}$, then for any $k > 0$ and mult-index $\beta$ of nonnegative integers*

$$\lim_{N \to \infty} \|\langle x \rangle^k D^\beta (P^N \varphi - \varphi)\|_{L^2} = 0.$$

*Proof.* Let $m = [k] + |\beta| + 1$. It is easy to check that $\langle x\rangle^k \xi^\beta/(|\xi|^2 + |x|^2)^m$ is a pseudo-differential operator of order 0, thus $\langle x\rangle^k D^\beta(-\triangle + |x|^2)^{-m}$ is bounded from $L^2$ to $L^2$. Then the lemma follows if one can show

$$\lim_{N\to\infty} \|(-\triangle + |x|^2)^m (P^N\varphi - \varphi)\|_{L^2} = 0.$$

In fact, since $(-\triangle + |x|^2)^m$ commutates with $P^N$ and $(-\triangle + |x|^2)^{m+1}\varphi$ is bounded in $L^2$, we find

$$\|(-\triangle + |x|^2)^m (P^N\varphi - \varphi)\|_{L^2} \le \frac{1}{\lambda_{N+1}} \|(-\triangle + |x|^2)^{m+1}\varphi\|_{L^2} \lesssim \frac{1}{\lambda_{N+1}} \to 0$$

as $N$ goes to infinity. □

The assumption $(A)$ implies that $div\boldsymbol{a} \in L^p_u, p > \max\{1, \frac{n}{\alpha}\}$. This enables us to control $div\boldsymbol{a}$ by $\Lambda^\alpha$ in the sense of quadratic form. In fact, by Theorem 4.2 in Zheng [18], for any $\varepsilon > 0$, there exists $c_\varepsilon$ such that

$$\left|\int_{\mathbf{R}^n} (div\boldsymbol{a})|f|^2 dx\right| \le \varepsilon \int_{\mathbf{R}^n} |\Lambda^{\frac{\alpha}{2}} f|^2 dx + c_\varepsilon \int_{\mathbf{R}^n} |f|^2 dx \tag{2.8}$$

for all $f \in H^{\frac{\alpha}{2}}$. Now we state the main result in this section.

**Theorem 2.2 (Existence and uniqueness).** *Assume that $(A)$ holds, $u_0 \in L^2$. Then problem (1.1) has a unique global weak solution $u \in L^\infty(0,\infty; L^2) \bigcap L^2(0,\infty; H^{\frac{\alpha}{2}})$.*

*Proof.* For $R \gg 1$, we choose a smooth function $\chi_R$ such that

$$\chi_R(x) = \begin{cases} 1, & |x| \le R, \\ 0, & |x| \ge R+1, \end{cases}$$

and $|\chi_R'| \le C$.

Now we approximate problem (1.1) by the following ODE system

$$\partial_t u_{N,R} + P^N(\Lambda^\alpha u_{N,R}) + P^N(\nabla\cdot(a\chi_R u_{N,R})) = 0, \quad u_{N,R}(0,x) = P^N u_0. \tag{2.9}$$

Denote by $Ff = P^N(\Lambda^\alpha f) + P^N(\nabla\cdot(a\chi_R f)), f \in L^2$. Then we claim that

$$\|Ff\|_{L^2} \lesssim C(N,R)\|f\|_{L^2}.$$

In fact, on one hand

$$\|P^N(\Lambda^\alpha f)\|_{L^2} \lesssim \sum_{j=1}^N |(\Lambda^\alpha f, e_j)| \lesssim \|f\|_{L^2} \sum_{j=1}^N \|\Lambda^\alpha e_j\|_{L^2} \lesssim_N \|f\|_{L^2}.$$

On the other hand, by Lemma 2.3 we have $a\chi_R \in L^2$, thus

$$\|P^N(\nabla \cdot (a\chi_R f))\|_{L^2} \lesssim \sum_{j=1}^N \|a\chi_R \cdot \nabla e_j\|_{L^2}\|f\|_{L^2} \lesssim_{N,R} \|f\|_{L^2}.$$

Thus $F$ maps $L^2$ into $L^2$. Since $F$ is linear, $F$ is Lipschitz continuous on $L^2$. Thanks to the Cauchy-Lipschitz theorem, for any $u_0 \in L^2$, there exists a unique solution $u_{N,R} \in C([0,\infty), L^2)$ to problem (2.9).

Now we give some uniform bounds of $u_{N,R}$. From the equation (2.9), $u_{N,R} = P^N u_{N,R}$, thus $u_{N,R} \in \mathscr{S}$. Multiplying (2.9) with $u_{N,R}$ and integrating yield that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^n} |u_{N,R}|^2 dx + \int_{\mathbf{R}^n} |\Lambda^{\frac{\alpha}{2}} u_{N,R}|^2 dx + \int_{\mathbf{R}^n} (\nabla \cdot (a\chi_R u_{N,R}))u_{N,R} dx = 0. \tag{2.10}$$

Using integration by parts and (2.8) we get

$$\begin{aligned}\left|\int_{\mathbf{R}^n} (\nabla \cdot (a\chi_R u_{N,R}))u_{N,R} dx\right| &\le \left|\int_{\mathbf{R}^n} div\, \boldsymbol{a} |u_{N,R}|^2 dx\right| \\ &\le \frac{1}{2}\int_{\mathbf{R}^n} |\Lambda^{\frac{\alpha}{2}} u_{N,R}|^2 dx + c\int_{\mathbf{R}^n} |u_{N,R}|^2 dx.\end{aligned}$$

Then (2.10) becomes

$$\frac{d}{dt}\int_{\mathbf{R}^n} |u_{N,R}|^2 dx + \int_{\mathbf{R}^n} |\Lambda^{\frac{\alpha}{2}} u_{N,R}|^2 dx \lesssim \int_{\mathbf{R}^n} |u_{N,R}|^2 dx.$$

By Gronwall's inequality, we obtain

$$\int_{\mathbf{R}^n} |u_{N,R}(t)|^2 dx \le e^{Ct}\int_{\mathbf{R}^n} |u_0|^2 dx,$$

and then

$$\int_0^t \int_{\mathbf{R}^n} |\Lambda^{\frac{\alpha}{2}} u_{N,R}|^2 dx dt \le 2e^{Ct}\int_{\mathbf{R}^n} |u_0|^2 dx.$$

Thus we conclude that for $T > 0$ arbitrary, $u_{N,R}$ is bounded independent of $N$ and $R$ in $L^\infty(0,T;L^2)$ and $L^2(0,T;H^{\frac{\alpha}{2}})$. By weak compactness we find a subsequence denoted as $u_m =: u_{m,m}$ and $u \in L^\infty(0,T;L^2)\bigcap L^2(0,T;H^{\frac{\alpha}{2}})$ such that

$$u_m \to u \text{ in } L^\infty(0,T;L^2) \text{ weak-star},$$

$$u_m \to u \text{ in } L^2(0,T;H^{\frac{\alpha}{2}}) \text{ weakly}.$$

It follows that for any $\varphi(t,x) \in C_0^\infty((0,T)\times \mathbf{R}^n)$

$$\langle \partial_t u_m, \varphi\rangle \to \langle \partial_t u, \varphi\rangle$$

and

$$\langle P^m \Lambda^\alpha u_m, \varphi\rangle \to \langle \Lambda^\alpha u, \varphi\rangle$$

as $m \to \infty$. Combining this and Lemma 2.5, we can pass the limit to obtain

$$\langle u_t, \varphi\rangle + \langle \Lambda^\alpha u, \varphi\rangle + \langle \nabla\cdot(au), \varphi\rangle = 0$$

for all $\varphi \in C_0^\infty((0,T)\times\mathbf{R}^n)$. Thus the limit $u$ is a weak solution of (1.1). Since for any $T > 0$, $u$ is bounded in $L^\infty(0,T;L^2)$, the solution is global. Note that if $u_0 = 0$ then $u_{N,R} = 0$, and of course the limit $u$ must be zero. This gives the uniqueness of the weak solution as problem (1.1) is linear. □

**Lemma 2.5.** *Let $a_m, u_m$ and $\varphi$ be the same as that in the proof of Theorem 2.2. Then*

$$\lim_{m\to\infty} \langle P^m(\nabla\cdot(a\chi_m u_m)), \varphi\rangle = \langle \nabla\cdot(au)), \varphi\rangle.$$

*Proof.* Using integration by parts, it suffices to show

$$\lim_{m\to\infty} \langle a_j\chi_m u_m, \partial_j P^m\varphi\rangle = \langle au, \partial_j\varphi\rangle$$

holds for $j = 1, 2, \cdots$. Now write

$$\begin{aligned}\langle a_j\chi_m u_m, \partial_j P^m\varphi\rangle - \langle au, \partial_j\varphi\rangle &= \langle a_j\chi_m u_m, \partial_j(P^m\varphi - \varphi)\rangle + \langle a_j(\chi_m - 1)u_m, \partial_j\varphi\rangle \\ &\quad + \langle a_j(u_m - u), \partial_j\varphi\rangle \\ &= I_1 + I_2 + I_3.\end{aligned}$$

First, note that $u_m$ is bounded in $L^\infty(0,T;L^2)$, utilizing Lemma 2.3 and Sobolev embedding $H^n \hookrightarrow L^\infty$, we obtain

$$\begin{aligned}|I_1| &\le \|\langle x\rangle^{-(n+1)} a_j\chi_m u_m\|_{L^2(0,T;L^1)}\|\langle x\rangle^{n+1}\partial_j(P^m\varphi-\varphi)\|_{L^2(0,T;L^\infty)}\\ &\le \|\langle x\rangle^{-(n+1)} a_j\chi_m\|_{L^\infty(0,T;L^2)}\|u_m\|_{L^2(0,T;L^2)}\|\langle x\rangle^{n+1}\partial_j(P^m\varphi-\varphi)\|_{L^2(0,T;H^n)}\\ &\lesssim \sum_{0\le k,|\beta|\le n+1}\|\langle x\rangle^{k} D^\beta(P^m\varphi-\varphi)\|_{L^2(0,T;L^2)}\\ &\to 0\end{aligned}$$

as $m\to\infty$, we used Lemma 2.4 in the last step. Similarly,

$$\begin{aligned}|I_2| &\lesssim \|\langle x\rangle^{-(n+1)} a_j(\chi_m-1)\|_{L^2}\|u_m\|_{L^2(0,T;L^2)}\|\langle x\rangle^{n+1}\partial_j\varphi\|_{L^2(0,T;H^n)}\\ &\lesssim \|\langle x\rangle^{-(n+1)} a_j(\chi_m-1)\|_{L^2}\to 0\end{aligned}$$

as $m\to\infty$. Finally, since $a_j\varphi$ is bounded in $L^2(0,T;L^2)$ and $u_m\to u$ weakly in $L^2(0,T;L^2)$, thus

$$\lim_{m\to\infty} I_3 = 0.$$

This completes the proof. □

## 3. Regularity of solutions

In this section, we shall show the smoothing effect of the evolution (1.1). We split the discussion into two subsections for clarity.

### *3.1. Smoothness on spatial variable $x$*

The main result in this subsection is the following theorem. It says that if the drift vector $\boldsymbol{a}$ is smooth with bounded arbitrary order derivatives, then the solution is smooth on $x$. It should be noted that the vector $\boldsymbol{a}$ allows to be unbounded. In particular, if $\boldsymbol{a}=\boldsymbol{x}$ we recover the result in [17].

**Theorem 3.1 (Smoothing effect on $x$).** *Assume that $u_0\in L^2$, and for all mult-index $\beta$ of nonnegative integers with $|\beta|\neq 0$,*

$$\|D^\beta a_j\|_{L^\infty}\le C_\beta,\quad j=1,2,\cdots.$$

*Then the unique solution of (1.1) $u(t)\in C^\infty(\mathbf{R}^n)$ for all $t>0$.*

To prove this result, we first recall the definitions of Littlewood-Paley decomposition operator.

Let $\psi : \mathbf{R}^n \mapsto [0,1]$ be a smooth radial cut-off function, say

$$\psi(\xi) = \begin{cases} 1, & |\xi| \le 1, \\ \text{smooth}, & 1 < |\xi| < 2, \\ 0, & |\xi| \ge 2. \end{cases} \tag{3.11}$$

Denote by $\varphi(\xi) = \psi(\xi) - \psi(2\xi)$ and

$$\varphi_j(\xi) = \varphi(2^{-j}\xi), \quad j = 1, 2, \cdots,$$

$$\varphi_0(\xi) = 1 - \sum_{j=1}^{\infty} \varphi_j(\xi) = \psi(\xi).$$

One can easily check that supp$\varphi_j \subset \{\xi \in \mathbf{R}^n : 2^{j-1} \le |\xi| \le 2^{j+1}\}$ and supp $\varphi_0 \subset \{\xi \in \mathbf{R}^n : |\xi| \le 2\}$. The frequency localization operators $\{\triangle_j\}_{j=0}^{\infty}$ and $S_j$ are defined by

$$\triangle_j f = \mathscr{F}^{-1}\varphi_j \mathscr{F} f, \quad j = 0, 1, \cdots,$$
$$S_j f = \sum_{0 \le k \le j} \triangle_k f.$$

Acting $\triangle_j$ on both sides of (1.1) gives

$$\partial_t \triangle_j u + \Lambda^\alpha \triangle_j u + \nabla \cdot \triangle_j(\boldsymbol{a}(x)u) = 0. \tag{3.12}$$

Similar to [4], we use Bony's decomposition to write

$$\begin{aligned} \triangle_j(\boldsymbol{a}u) = & \sum_{|k-j|\le 2} [\triangle_j, S_{k-1}\boldsymbol{a}]\triangle_k u + \sum_{|k-j|\le 2} S_j \boldsymbol{a} \triangle_k \triangle_j u \\ & + \sum_{|k-j|\le 2} ((S_{k-1}\boldsymbol{a} - S_j\boldsymbol{a})\triangle_k \triangle_j u + \sum_{|k-j|\le 2} \triangle_j(\triangle_k \boldsymbol{a} S_{k-1} u) \\ & + \sum_{k \ge j-1} \sum_{|k-l|\le 1} \triangle_j(\triangle_k u \triangle_l \boldsymbol{a}). \end{aligned}$$

Multiplying (3.12) with $2\triangle_j u$, using integration by parts, Plancherel theorem and the fact $\sum_{|k-j|\le 2} \triangle_k \triangle_j = \triangle_j$ implies that

$$\frac{d}{dt}\|\triangle_j u\|_{L^2}^2 + c2^{j\alpha}\|\triangle_j u\|_{L^2}^2 \le I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$
\begin{aligned}
I_1 &= 2\sum_{|k-j|\le 2}\int [\triangle_j, S_{k-1}\boldsymbol{a}]\triangle_k u\cdot\nabla\triangle_j u,\\
I_2 &= 2\int S_j\boldsymbol{a}\triangle_j u\cdot\nabla\triangle_j u,\\
I_3 &= 2\int\sum_{|k-j|\le 2}(S_{k-1}\boldsymbol{a}-S_j\boldsymbol{a})\triangle_k\triangle_j u\cdot\nabla\triangle_j u,\\
I_4 &= 2\int\sum_{|k-j|\le 2}\triangle_j(\triangle_k\boldsymbol{a}S_{k-1}u)\cdot\nabla\triangle_j u,\\
I_5 &= 2\int\sum_{k\ge j-1}\sum_{|k-l|\le 1}\triangle_j(\triangle_k u\triangle_l\boldsymbol{a})\cdot\nabla\triangle_j u.
\end{aligned}
$$

In what follows, we bound $I_1,\cdots,I_5$ individually. For $I_1$, by Hölder inequality,

$$|I_1|\lesssim 2^j\|\triangle_j u\|_{L^2}\sum_{|k-j|\le 2}\|[\triangle_j, S_{k-1}\boldsymbol{a}]\triangle_k u\|_{L^2}.$$

It's easy to check that

$$[\triangle_j, S_{k-1}\boldsymbol{a}]\triangle_k u=\int(\mathscr{F}^{-1}\varphi_j)(x-y)[S_{k-1}\boldsymbol{a}(y)-S_{k-1}\boldsymbol{a}(x)]\triangle_k u(y)dy.$$

Note that

$$\|S_{k-1}\boldsymbol{a}(y)-S_{k-1}\boldsymbol{a}(x)\|_{L^\infty}\lesssim\|\nabla\boldsymbol{a}\|_{L^\infty}|x-y|.$$

Since the converse Fourier transform of $\varphi_j$ is $2^{jn}\mathscr{F}^{-1}\varphi(2^j\cdot)$, we use Young's inequality to get that

$$
\begin{aligned}
\|[\triangle_j, S_{k-1}\boldsymbol{a}]\triangle_k u\|_{L^2} &\lesssim \|\boldsymbol{a}\|_{L^\infty}\|2^{jn}\mathscr{F}^{-1}\varphi(2^j x)|x|*\triangle_k u\|_{L^2}\\
&\lesssim 2^{-j}\|\triangle_k u\|_{L^2}.
\end{aligned}
$$

Hence

$$|I_1|\lesssim\|\triangle_j u\|_{L^2}\sum_{|k-j|\le 2}\|\triangle_k u\|_{L^2}.$$

For $I_2$, using integration by parts we have

$$|I_2|\lesssim\|\triangle_j u\|_{L^2}^2.$$

For $I_3$, by Hölder inequality,

$$\begin{aligned}|I_3| &\lesssim \sum_{|k-j|\le 2} \|S_{k-1}\boldsymbol{a} - S_j\boldsymbol{a}\|_{L^\infty} \|\triangle_k\triangle_j u\|_{L^2} \|\nabla\triangle_j u\|_{L^2} \\ &\lesssim 2^j \sum_{j-2\le k\le j+1} \|\triangle_k \boldsymbol{a}\|_{L^\infty} \|\triangle_j u\|_{L^2}^2.\end{aligned}$$

Since arbitrary order derivatives of $\boldsymbol{a}$ is bounded, by [7, Theorem 6.3.6], for any given $K > 0$,

$$\|\triangle_j \boldsymbol{a}\|_{L^\infty} \lesssim 2^{-Kj}. \tag{3.13}$$

Thus

$$|I_3| \lesssim 2^{-(K-1)j} \|\triangle_j u\|_{L^2}^2.$$

Similarly, using (3.13), we obtain

$$\begin{aligned}|I_4| &\lesssim 2^j \sum_{|k-j|\le 2} \|\triangle_k \boldsymbol{a}\|_{L^2} \|S_{k-1}u\|_{L^2} \|\triangle_j u\|_{L^2} \\ &\lesssim 2^{-(K-1)j} \|u\|_{L^2} \|\triangle_j u\|_{L^2},\end{aligned}$$

and

$$\begin{aligned}|I_5| &\lesssim \sum_{k\ge j-1} \sum_{|k-l|\le 1} \|\triangle_l \boldsymbol{a}\|_{L^\infty} \|\triangle_k u\|_{L^2} 2^j \|\triangle_j u\|_{L^2} \\ &\lesssim 2^{-(K-1)j} \|u\|_{L^2} \|\triangle_j u\|_{L^2}.\end{aligned}$$

Hence

$$\frac{d}{dt}\|\triangle_j u\|_{L^2}^2 + c2^{j\alpha}\|\triangle_j u\|_{L^2}^2 \lesssim \|\triangle_j u\|_{L^2} \sum_{|k-j|\le 2} \|\triangle_k u\|_{L^2} + 2^{-(K-1)j}\|u\|_{L^2}\|\triangle_j u\|_{L^2}.$$

Applying Hölder inequality to the terms on the right hand side, we have

$$\|\triangle_j u\|_{L^2} \sum_{|k-j|\le 2} \|\triangle_k u\|_{L^2} \le \frac{c}{4} 2^{j\alpha} \|\triangle_j u\|_{L^2}^2 + C' 2^{-j\alpha} \sum_{|k-j|\le 2} \|\triangle_k u\|_{L^2}^2$$

and

$$2^{-(K-1)j}\|u\|_{L^2}\|\triangle_j u\|_{L^2} \le \frac{c}{4} 2^{j\alpha}\|\triangle_j u\|_{L^2}^2 + C'' 2^{-(K-1+\alpha)j}\|u\|_{L^2}^2.$$

Combining these inequalities and the fact $\|u(t)\|_{L^2} \lesssim e^{ct}\|u_0\|_{L^2}$ implies that

$$\frac{d}{dt}\|\triangle_j u\|_{L^2}^2 + \frac{c}{2}2^{j\alpha}\|\triangle_j u\|_{L^2}^2 \lesssim 2^{-j\alpha}\sum_{|k-j|\le 2}\|\triangle_k u\|_{L^2}^2 + 2^{-(K-1+\alpha)j}e^{ct}\|u_0\|_{L^2}^2. \tag{3.14}$$

Based on (3.14), we have the following lemma.

**Lemma 3.1.** *Assume that* $u$ *is the solution of* (1.1) *with initial data* $u_0 \in L^2$. *Then for any* $m \ge 0, \varepsilon > 0$

$$\|u(t)\|_{H^{m-\varepsilon}} \lesssim t^{-m}e^{ct}, \quad t > 0.$$

*Proof.* We first note that the lemma follows from the claim:

$$\|\triangle_j u(t)\|_{L^2} \lesssim t^{-m}e^{ct}2^{-mj\alpha}$$

for all $t > 0$ and nonnegative integer $m$, the implicit constant is independent of $j$. In fact, for any $\varepsilon > 0$

$$\|\triangle_j u(t)\|_{H^{m-\varepsilon}} \lesssim \sum_{j\ge 0} 2^{(m-\varepsilon)j}\|\triangle_j u\|_{L^2} \lesssim \sum_{j\ge 0} 2^{-\varepsilon j}t^{-m}e^{ct} \lesssim t^{-m}e^{ct}.$$

Now we shall prove the claim by induction argument. By Theorem 2.2, we find the claim holds in case $m = 0$. Now let $0 < t_0 < t$. Using Gronwall lemma to (3.14) on the interval $[t_0, t]$ gives that

$$\|\triangle_j u(t)\|_{L^2}^2 \lesssim e^{-\frac{c2^{j\alpha}(t-t_0)}{2}}\|\triangle_j u(t_0)\|_{L^2}^2 + \int_{t_0}^t e^{-\frac{c2^{j\alpha}(t-\tau)}{2}}\Big[2^{-j\alpha}\sum_{|k-j|\le 2}\|\triangle_k u\|_{L^2}^2 + 2^{-(K-1+\alpha)j}e^{ct}\|u_0\|_{L^2}^2\Big]d\tau.$$

For $t > t_0$, by Taylor formula, it's easy to see

$$e^{-\frac{c2^{j\alpha}(t-t_0)}{2}} \lesssim (t-t_0)^{-2}2^{-2j\alpha}.$$

Thus, by the induction hypothesis of $\|\triangle_j u(t)\|_{L^2}$,

$$\begin{aligned}\|\triangle_j u(t)\|_{L^2}^2 &\lesssim (t-t_0)^{-2}t_0^{-2m}e^{2ct}2^{-2(m+1)j\alpha}\\ &+ \int_{t_0}^t e^{-\frac{c2^{j\alpha}(t-\tau)}{2}}\Big[\sum_{|k-j|\le 2}\tau^{-2m}e^{2c\tau}2^{-(2m+1)k\alpha} + 2^{-(K-1+\alpha)j}e^{2ct}\|u_0\|_{L^2}^2\Big]d\tau.\end{aligned} \tag{3.15}$$

The integral on the right hand side of (3.15) is bounded by

$$\begin{aligned}
&\lesssim \int_{t_0}^{t} e^{-\frac{c2^{j\alpha}(t-\tau)}{2}} \left[ t_0^{-2m} e^{2ct} 2^{-(2m+1)j\alpha} + 2^{-(K-1+\alpha)j} e^{2ct} \|u_0\|_{L^2}^2 \right] d\tau \\
&\lesssim t_0^{-2m} e^{2ct} 2^{-(2m+2)j\alpha} + 2^{-(K-1+2\alpha)j} e^{2ct} \|u_0\|_{L^2}^2.
\end{aligned}$$

Inserting this into (3.15) and setting $t_0 = t/2$ yields

$$\begin{aligned}
\|\triangle_j u(t)\|_{L^2} &\lesssim t^{-(m+1)} e^{ct} 2^{-(m+1)j\alpha} + t^{-m} e^{ct} 2^{-(m+1)j\alpha} + 2^{-(K-1+2\alpha)j/2} e^{ct} \|u_0\|_{L^2} \\
&\lesssim t^{-(m+1)} e^{ct} 2^{-(m+1)j\alpha} [1 + t^2 + t^{m+1} 2^{-j(\frac{K-1}{2}-m)} \|u_0\|_{L^2}].
\end{aligned}$$

Let $K = 2m+2$, choose $N_0 = 2\ln^{-1} 2 \ln \|u_0\|_{L^2}$ such that

$$2^{-N_0(\frac{K-1}{2}-m)} \|u_0\|_{L^2} = 1.$$

Then for $j \geq N_0$, we obtain

$$\|\triangle_j u(t)\|_{L^2} \lesssim t^{-(m+1)} e^{c't} 2^{-(m+1)j\alpha},$$

with some different constant $c' > c$. For $j < N_0$,

$$\|\triangle_j u(t)\|_{L^2} \lesssim \|u(t)\|_{L^2} \lesssim e^{ct} \lesssim t^{-(m+1)} e^{c't} 2^{-(m+1)j\alpha},$$

where the implicit constant depends on $\|u_0\|_{L^2}$, and is independent of $j$. This completes the proof. ◻

**Proof of Theorem 3.1.** It follows from Lemma 3.1 and Sobolev embedding theorems.

*3.2. Smoothness on time variable $t$*

In this subsection we shall show the solution is smooth with respect to time $t$. To prove this result, a extra growth restriction on $\boldsymbol{a}$ is needed compared to the assumptions of Theorem 3.1. The main reason is that we benefit less from the cancelation property of $\nabla \cdot (\boldsymbol{a}u)$ now while which is very important in the proof of Theorem 3.1. The main result in this subsection is the following

**Theorem 3.2 (Smoothing effect jointly on $x, t$).** *Under the assumptions of Theorem 3.1, we assume further that*

$$|a| \lesssim \ln^{\gamma}(2 + |x|^2)$$

*for some $\gamma > 0$, then the solution of (1.1) $u \in C^{\infty}((0,\infty) \times \boldsymbol{R}^n)$.*

**Remrak 3.1.** Theorem 3.2 holds for smaller $\gamma$. In particular, if $\gamma \leq 0$, Theorem 3.2 is valid.

To prove the theorem, we first recall the definition of Muckenhoupt weights.

**Definition 3.3.** *Let $1 < p < \infty$ and a weight $w(x)$ be a locally integrable function. We say $w$ is an $A_p$ weight, namely $w \in A_p$, if*

$$\sup_{B \textit{ Balls in } \boldsymbol{R}^n} \frac{1}{|B|}\int_B w(x)dx \cdot \left[\frac{1}{|B|}\int_B w(x)^{-p'/p}dx\right]^{p/p'} < \infty.$$

The importance of $A_p$ can be seen from the following proposition, see e.g. the Corollary in [13, p.205] and Proposition 2 in [13, p.245].

**Proposition 3.1.** *Let $w \in A_p, 1 < p < \infty$, $m(\xi) \in C^\infty(\mathbf{R}^n \backslash \{0\})$ be a bounded function such that for $\beta$*

$$|D^\beta m| \leq C_\beta |\xi|^{-|\beta|},$$

*then the Fourier multiplier $m(D)$ is bounded from $L^p(wdx)$ to $L^p(wdx)$.*

In what follows, we denote by

$$\phi_\gamma = \ln^\gamma(2 + |x|^2).$$

**Lemma 3.2.** *For all $\gamma \in \mathbf{R}$, $\phi_\gamma \in A_2$.*

*Proof.* It's easy to see that $w \in A_2$ if and only $w^{-1} \in A_2$. So we assume $\gamma > 0$ now. Let $B(x_0, r) = \{x : |x - x_0| \leq r\}$, it suffices to show

$$\sup_{x_0 \in \mathbf{R}^n, r>0} r^{-2n}\int_{B(x_0,r)} \phi_\gamma(x)dx \cdot \int_{B(x_0,r)} \phi_\gamma^{-1}(x)dx < \infty. \tag{3.16}$$

We divide the proof into two cases. If $|x_0| > 3r$, then $|x| \sim |x_0|$ for all $x \in B(x_0, r)$. Then

$$LHS(3.16) \lesssim \sup_{x_0 \in \mathbf{R}^n, r>0} r^{-2n}\int_{B(x_0,r)} \phi_\gamma(x_0)dx \cdot \int_{B(x_0,r)} \phi_\gamma^{-1}(x_0)dx \lesssim 1$$

as desired. If $|x_0| \leq 3r$, then $B(x_0, r)$ is contained in $B(0, 4r)$. Since $\phi_\gamma$ is radial and increasing on $|x|$, thus

$$LHS(3.16) \lesssim \sup_{r>0} r^{-2n}\int_{B(0,4r)} \phi_\gamma(x)dx \cdot \int_{B(0,4r)} \phi_\gamma^{-1}(x)dx. \tag{3.17}$$

Since (3.17) is bounded for $r \le r_0$ with some fixed $r_0$, it suffices to show for $r \gg 1$

$$r^{-2n} \int_{B(0,4r)} \phi_\gamma(x)dx \cdot \int_{B(0,4r)} \phi_\gamma^{-1}(x)dx \lesssim 1. \tag{3.18}$$

On one hand, it's obvious that

$$\int_{B(0,4r)} \phi_\gamma(x)dx \lesssim r^n \ln^\gamma r.$$

On the other hand, let $\eta > 0$, we make the splitting

$$\int_{B(0,4r)} \phi_\gamma^{-1}(x)dx = \int_{B(0,\eta)} \phi_\gamma^{-1}(x)dx + \int_{B(0,4r)\setminus B(0,\eta)} \phi_\gamma^{-1}(x)dx.$$

It's easy to see that

$$\int_{B(0,4r)} \phi_\gamma^{-1}(x)dx \lesssim \eta^n + r^n \ln^{-\gamma} \eta. \tag{3.19}$$

Minimizing the right hand side of (3.19) with respect to $\eta$ gives

$$\int_{B(0,4r)} \phi_\gamma^{-1}(x)dx \lesssim r^n \ln^{-\gamma} \eta(1 + \frac{\gamma}{n} \ln^{-1} \eta), \tag{3.20}$$

where

$$n\eta^n \ln^{\gamma+1} = \gamma r^n.$$

Note that $r \gg 1$, we find $\eta \sim r$. Then (3.20) becomes

$$\int_{B(0,4r)} \phi_\gamma^{-1}(x)dx \lesssim r^n \ln^{-\gamma} r.$$

Thus (3.18) holds. This completes the proof. $\square$

**Lemma 3.3.** *For all $\gamma \in \mathbf{R}$, $k \in \mathbb{N}$, it holds that*

$$\|\phi_\gamma \Lambda^\alpha u\|_{H^k} \lesssim \|\phi_\gamma u\|_{H^{k+\alpha}}.$$

*Proof.* By Leibniz rule and the fact $|D^\mu \phi_\gamma| \lesssim \phi_\gamma$, the lemma is reduced to proving

$$\|\phi_\gamma \Lambda^\alpha D^\beta u\|_{L^2} \lesssim \|\phi_\gamma u\|_{H^{m+\alpha}}$$

for all mult-index $|\beta| \le k$. It is equivalent to

$$\|\Lambda^\alpha D^\beta \langle D\rangle^{-(k+\alpha)} u\|_{L^2(\phi_{2\gamma}dx)} \lesssim \|u\|_{L^2(\phi_{2\gamma}dx)}. \tag{3.21}$$

In fact, the symbol of $\Lambda^\alpha D^\beta \langle D\rangle^{-(k+\alpha)}$ is $m(\xi) = \xi^\beta |\xi|^\alpha (1+|\xi|^2)^{-(k+|\alpha|)}$. It's easy to check that for all $\mu$

$$|D^\mu m(\xi)| \lesssim |\xi|^{-|\mu|}.$$

Then (3.21) follows from Lemma 3.2 and Proposition 3.1. □

**Lemma 3.4.** *Assume that* $u$ *is the solution of* (1.1) *with initial data* $u_0 \in L^2$. *Then for every* $0 < t_0 < T < \infty$, *we have*

$$\|\phi_{-k\gamma}\partial_t^k u\|_{L^\infty([t_0,T];H^m)} < \infty$$

*for any integers* $m, k > 0$.

*Proof.* Let $v_k = \phi_{-k\gamma}\partial_t^k u$. Acting $\partial_t^k$ on both sides of (1.1) and multiplying $\phi_{-(k+1)\gamma}$ give

$$v_{k+1} + \phi_{-(k+1)\gamma}\Lambda^\alpha \partial_t^k u + \phi_{-(k+1)\gamma}\nabla(\boldsymbol{a}\partial_t^k u) = 0.$$

Thus, for all integers $m > 0$,

$$\|v_{k+1}\|_{H^m} \lesssim \|\phi_{-(k+1)\gamma}\Lambda^\alpha \partial_t^k u\|_{H^m} + \|\phi_{-(k+1)\gamma}\nabla(\boldsymbol{a}\partial_t^k u)\|_{H^m}.$$

On one hand, it follows from Lemma 3.3 that

$$\|\phi_{-(k+1)\gamma}\Lambda^\alpha \partial_t^k u\|_{H^m} \lesssim \|\phi_{-k\gamma}\Lambda^\alpha \partial_t^k u\|_{H^m} \lesssim \|v_k\|_{H^{m+\alpha}} \lesssim \|v_k\|_{H^{m+2}}.$$

On the other hand, we write $\phi_{-(k+1)\gamma}\nabla(\boldsymbol{a}\partial_t^k u) = \nabla(\phi_{-(k+1)\gamma}a\partial_t^k u) - \nabla\phi_{-(k+1)\gamma}\cdot \boldsymbol{a}\partial_t^k u$ to obtain

$$\begin{aligned}\|\phi_{-(k+1)\gamma}\nabla(\boldsymbol{a}\partial_t^k u)\|_{H^m} &\lesssim \|\phi_{-\gamma}\boldsymbol{a}v_k)\|_{H^{m+1}} + \|\nabla\phi_{-(k+1)\gamma}\cdot \boldsymbol{a}\phi_{k\gamma}v_k\|_{H^m} \\ &\lesssim \|v_k\|_{H^{m+1}},\end{aligned}$$

where in the last step we have used the fact that for all $\mu$,

$$|D^{\mu}\phi_{-\gamma}\boldsymbol{a}| \lesssim 1, |D^{\mu}\nabla\phi_{-(k+1)\gamma}\cdot\boldsymbol{a}\phi_{k\gamma}| \lesssim 1.$$

Thus, we conclude that

$$\|v_{k+1}\|_{H^m} \lesssim \|v_k\|_{H^{m+2}}.$$

It follows from Theorem 3.1 that $v_0 \in H^m$ for all $m \geq 0$, $t > 0$. Then an induction argument implies the desired conclusion. □

**Proof of Theorem 3.2.** It follows from Lemma 3.1 and Lemma 3.4 that for all $0 < t_0 < T < \infty$, $0 \leq m \in \mathbb{N}$, $u \in H^m((t_0, T) \times \mathbf{R}^n)$. Then by Sobolev embedding we find that $u \in C^{\infty}((t_0, T) \times \mathbf{R}^n)$. Since $t_0$ can be arbitrarily small and $T$ can be arbitrarily large, Theorem 3.2 follows.

**Remrak 3.2.** The well posedness of nonlocal Fokker-Planck equation with bounded drift on bounded domain is established in [8]. The restriction on the drift in Theorem 3.2 is relaxed in a more recent work [14].

**Acknowledgments** The Project was Supported by the NSFC (No. 11426209), and the Fundamental Research Funds for the Central Universities, China University of Geosciences(Wuhan), Grant G1323521638.

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