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Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa



On the entropy minimization problem in Statistical Mechanics

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ARTICLE INFO

Article history:

Received 9 February 2016

Available online xxxx

Submitted by H. Frankowska

Keywords:

Entropy minimization

Conjugate function

Series of convex functions

Value function

Statistical mechanics

ABSTRACT

In many works on Statistical Mechanics and Statistical Physics, when deriving the distribution of particles of ideal gases, one uses the method of Lagrange multipliers in a formal way. In this paper we treat rigorously this problem for Bose–Einstein, Fermi–Dirac and Maxwell–Boltzmann entropies and present a complete study in the case of the Maxwell–Boltzmann entropy. Our approach is based on recent results on series of convex functions.

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1. Introduction

In Statistical Mechanics and Statistical Physics, when studying the distribution of the particles of an ideal gas, one considers the problem of maximizing

$$\sum_i \left[n_i \ln \left(\frac{g_i}{n_i} - a \right) - \frac{g_i}{a} \ln \left(1 - a \frac{n_i}{g_i} \right) \right] \quad (1.1)$$

with the constraints $\sum_i n_i = N$ and $\sum_i n_i \varepsilon_i = E$, where, as mentioned in [5, pp. 141–144], ε_i denotes the average energy of a level, g_i the (arbitrary) number of levels in the i th cell, and, in a particular situation, n_i is the number of particles in the i th cell. Moreover, $a = -1$ for the Bose–Einstein case, $+1$ for the Fermi–Dirac case, and 0 for the (classical) Maxwell–Boltzmann case. Even if nothing is said explicitly about the set I of the indices i , from several examples in the literature, I is (or may be) an infinite countable set; the examples

$$\varepsilon_l = l(l+1)h^2/2I, \quad g_l = (2l+1); \quad l = 0, 1, 2, \dots \quad (1.2)$$

$$\varepsilon_{vK} = \varepsilon_0 + h\omega(v + \frac{1}{3}) + h^2 K(K+1)/2I; \quad v, K = 0, 1, 2, \dots \quad (1.3)$$

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$$\varepsilon(n_x, n_y, n_z) = \frac{h^2}{8mL^2}(n_x^2 + n_y^2 + n_z^2); \quad n_x, n_y, n_z = 1, 2, 3, \dots \quad (1.4)$$

are considered in [3, p. 76], [4, p. 138] and [5, p. 10], respectively.

Relation (1.1) suggests the consideration of the following functions defined on \mathbb{R} with values in $\overline{\mathbb{R}}$, called, respectively, Bose–Einstein, Fermi–Dirac and Maxwell–Boltzmann entropies:

$$E_{BE}(u) := \begin{cases} u \ln u - (1+u) \ln(1+u) & \text{if } u \in \mathbb{R}_+, \\ \infty & \text{if } u \in \mathbb{R}_-^*, \end{cases} \quad (1.5)$$

$$E_{FD}(u) := \begin{cases} u \ln u + (1-u) \ln(1-u) & \text{if } u \in [0, 1], \\ \infty & \text{if } u \in \mathbb{R} \setminus [0, 1], \end{cases} \quad (1.6)$$

$$E_{MB}(u) := \begin{cases} u(\ln u - 1) & \text{if } u \in \mathbb{R}_+, \\ \infty & \text{if } u \in \mathbb{R}_-^*, \end{cases} \quad (1.7)$$

where $0 \ln 0 := 0$ and $\mathbb{R}_+ := [0, \infty[$, $\mathbb{R}_+^* :=]0, \infty[$, $\mathbb{R}_- := -\mathbb{R}_+$, $\mathbb{R}_-^* := -\mathbb{R}_+^*$. We have that

$$E'_{BE}(u) = \ln \frac{u}{1+u} \quad \forall u \in \mathbb{R}_+^*, \quad E'_{FD}(u) = \ln \frac{u}{1-u} \quad \forall u \in]0, 1[, \quad E'_{MB}(u) = \ln u \quad \forall u \in \mathbb{R}_+^*.$$

Observe that E_{BE} , E_{MB} , E_{FD} are convex (even strictly convex on their domains), derivable on the interiors of their domains with increasing derivatives, and $E_{BE} \leq E_{MB} \leq E_{FD}$ on \mathbb{R} . The (convex) conjugates of these functions are

$$E_{MB}^*(t) = e^t \quad \forall t \in \mathbb{R}, \quad E_{FD}^*(t) = \ln(1+e^t) \quad \forall t \in \mathbb{R}, \quad E_{BE}^*(t) = \begin{cases} -\ln(1-e^t) & \text{if } t \in \mathbb{R}_-^*, \\ \infty & \text{if } t \in \mathbb{R}_+. \end{cases}$$

Moreover, for $W \in \{E_{BE}, E_{MB}, E_{FD}\}$ we have that $\partial W(u) = \{W'(u)\}$ for $u \in \text{int}(\text{dom } W)$ and $\partial W(u) = \emptyset$ elsewhere; furthermore,

$$(W^*)'(t) = \frac{e^t}{1+a_W e^t} \quad \forall t \in \text{dom } W^*, \quad (1.8)$$

where (as above)

$$a_W := \begin{cases} -1 & \text{if } W = E_{BE}, \\ 0 & \text{if } W = E_{MB}, \\ 1 & \text{if } W = E_{FD}. \end{cases} \quad (1.9)$$

The maximization of (1.1) subject to the constraints $\sum_i n_i = N$ and $\sum_i n_i \varepsilon_i = E$ is equivalent to the minimization problem

$$\text{minimize } \sum_i g_i W\left(\frac{n_i}{g_i}\right) \quad \text{s.t.} \quad \sum_i n_i = N, \quad \sum_i n_i \varepsilon_i = E,$$

where W is one of the functions E_{BE} , E_{FD} , E_{MB} defined in (1.5), (1.6), (1.7), and $g_i \geq 1$.

In many books treating this subject (see [4, pp. 119, 120], [3, pp. 15, 16], [5, p. 144], [1, p. 39]) the above problem is solved using the Lagrange multipliers method in a formal way.

Our aim is to treat rigorously the minimization of Maxwell–Boltzmann, Bose–Einstein and Fermi–Dirac entropies with the constraints $\sum_{i \in I} u_i = u$, $\sum_{i \in I} \sigma_i u_i = v$ in the case in which I is an infinite countable set. Unfortunately, we succeed to do a complete study only for the Maxwell–Boltzmann entropy. For a short description of the results see Conclusions.

Our approach is based on the results of X.Y. Zheng [9] on the subdifferential of an infinite countable sum of convex functions and on our recent results in [7]¹ for the conjugate of such a function.

We shall use standard notations and results from convex analysis (see e.g. [6,8]).

2. Properties of the marginal functions associated to the entropy minimization problems of Statistical Mechanics

Throughout the paper we consider the sequences $(p_n)_{n \geq 1} \subset [1, \infty[$ and $(\sigma_n)_{n \geq 1} \subset \mathbb{R}$, and set

$$S(u, v) := S_{(\sigma_n)}(u, v) := \left\{ (u_n)_{n \geq 1} \subset \mathbb{R}_+ \mid u = \sum_{n \geq 1} u_n, v = \sum_{n \geq 1} \sigma_n u_n \right\} \quad (2.1)$$

for each $(u, v) \in \mathbb{R}^2$. It is clear that $S(tu, tv) = tS(u, v)$ for all $(u, v) \in \mathbb{R}^2$ and $t \in \mathbb{R}_+^*$, $S(u, v) = \emptyset$ if either $u < 0$ or $u = 0 \neq v$, and $S(0, 0) = \{(0)_{n \geq 1}\}$. We also set

$$\rho_n := \sum_{k=1}^n p_k \quad (2.2)$$

$$\eta_n^1 := \min \{ \sigma_k \mid k \in \overline{1, n} \}, \quad \eta_n^2 := \max \{ \sigma_k \mid k \in \overline{1, n} \}, \quad (2.3)$$

$$\eta_1 := \inf \{ \sigma_k \mid n \geq 1 \} \in [-\infty, \infty[, \quad \eta_2 := \sup \{ \sigma_k \mid n \geq 1 \} \in]-\infty, \infty]; \quad (2.4)$$

of course, $\lim_{n \rightarrow \infty} \rho_n = \infty$ (because $p_k \geq 1$ for $n \geq 1$).

The entropy minimization problem (EMP for short) of Statistical Mechanics and Statistical Physics associated to $W \in \{E_{BE}, E_{MB}, E_{FD}\}$ and $(u, v) \in \mathbb{R}^2$ is

$$(EMP)_{u,v} \text{ minimize } \sum_{n \geq 1} p_n W\left(\frac{u_n}{p_n}\right) \text{ s.t. } (u_n)_{n \geq 1} \in S(u, v),$$

where $\sum_{n \geq 1} \beta_n := \lim_{n \rightarrow \infty} \sum_{k=1}^n \beta_k$ when this limit exists in $\overline{\mathbb{R}}$ and $\sum_{n \geq 1} \beta_n := \infty$ otherwise. With the preceding convention, it is easy to see that $\alpha_n \leq \beta_n$ for $n \geq 1$ imply that $\sum_{n \geq 1} \alpha_n \leq \sum_{n \geq 1} \beta_n$.

Remark 2.1. Note that for $(u_n)_{n \geq 1} \in S(u, v)$ one has that $\lim_{n \rightarrow \infty} \sum_{k=1}^n p_k W\left(\frac{u_k}{p_k}\right)$ exists in: $[-\infty, 0]$ when $W = E_{BE}$, in $[-\infty, 0] \cup \{\infty\}$ when $W = E_{FD}$, and in $[-\infty, \infty[$ when $W = E_{MB}$.

The value (marginal) function associated to problems $(EMP)_{u,v}$ is

$$H_W : \mathbb{R}^2 \rightarrow \overline{\mathbb{R}}, \quad H_W(u, v) := \inf \left\{ \sum_{n \geq 1} p_n W\left(\frac{u_n}{p_n}\right) \mid (u_n)_{n \geq 1} \in S(u, v) \right\}, \quad (2.5)$$

with the usual convention $\inf \emptyset := \infty$. We shall write $H_{(\sigma_n), W}^{(p_n)}$ instead of H_W when we want to emphasize the sequences $(p_n)_{n \geq 1}$ and $(\sigma_n)_{n \geq 1}$; moreover, we shall write simply H_{BE} , H_{MB} , H_{FD} when W is E_{BE} , E_{MB} , or E_{FD} , respectively. Therefore,

$$\text{dom } H_W = \text{dom } H_{(\sigma_n), W}^{(p_n)} \subset \text{dom } S = \text{dom } S_{(\sigma_n)} := \{(u, v) \in \mathbb{R}^2 \mid S_{(\sigma_n)}(u, v) \neq \emptyset\};$$

hence $H_W(u, v) = \infty$ if either $u < 0$ or $u = 0 \neq v$, and $H_W(0, 0) = 0$. Taking into account that $E_{BE} \leq E_{MB} \leq E_{FD}$, and using Remark 2.1, we get

¹ See the preprint arXiv:1506.01216v1.

$$H_{BE} \leq H_{MB} \leq H_{FD}, \quad (2.6)$$

$$\text{dom } H_{FD} \subset \text{dom } H_{MB} = \text{dom } H_{BE} = \text{dom } S. \quad (2.7)$$

The results in the next two lemmas are surely known. For their proofs one uses the Lagrange multipliers method.

Lemma 2.2. *Let $n \geq 2$ be fixed. Then for $W \in \{E_{BE}, E_{MB}, E_{FD}\}$ we have*

$$\inf \left\{ \sum_{k=1}^n p_k W \left(\frac{u_k}{p_k} \right) \mid (u_k)_{k \in \overline{1, n}} \subset \mathbb{R}_+, \sum_{k=1}^n u_k = u \right\} = \rho_n \cdot W(u/\rho_n) \quad \forall u \in \mathbb{R}_+, \quad (2.8)$$

the infimum being attained for $u_k := up_k/\rho_n$ ($k \in \overline{1, n}$), where ρ_n is defined in (2.2).

Proof. Consider

$$\widetilde{W}_n : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}, \quad \widetilde{W}_n(u_1, \dots, u_n) := \sum_{k=1}^n p_k W \left(\frac{u_k}{p_k} \right). \quad (2.9)$$

Then $\text{dom } \widetilde{W}_n = \mathbb{R}_+^n$ for $W \in \{E_{MB}, E_{BE}\}$ and $\text{dom } \widetilde{W}_n = \prod_{k=1}^n [0, p_k]$ for $W = E_{FD}$. Of course, \widetilde{W}_n is convex, lower semicontinuous (lsc for short), continuous on $\text{int}(\text{dom } \widetilde{W}_n)$, and strictly convex on $\text{dom } \widetilde{W}_n$. Let $S'_n(u) := \{(u_1, \dots, u_n) \in \mathbb{R}_+^n \mid \sum_{k=1}^n u_k = u\}$. Since $S'_n(0) = \{(0)_{k \in \overline{1, n}}\}$, the conclusion is obvious for $u = 0$.

Consider first $W \in \{E_{MB}, E_{BE}\}$, and take $u \in \mathbb{R}_+^*$. Then $u\rho_n^{-1}(p_1, \dots, p_n) \in S'_n(u) \cap \text{int}(\text{dom } \widetilde{W}_n)$. Since $S'_n(u)$ is a compact set and \widetilde{W}_n is lsc, there exists a unique $(\bar{u}_1, \dots, \bar{u}_n) \in S'_n(u)$ minimizing \widetilde{W}_n on $S'_n(u)$. Using (for example) [8, Th. 2.9.6], there exists $\alpha \in \mathbb{R}$ such that $\alpha(1, \dots, 1) \in \partial \widetilde{W}_n(\bar{u}_1, \dots, \bar{u}_n) = \partial W(\frac{\bar{u}_1}{p_1}) \times \dots \times \partial W(\frac{\bar{u}_n}{p_n})$. Since $\partial W(0) = \emptyset$, it follows that $\bar{u}_k/p_k > 0$ for $k \in \overline{1, n}$. Hence $\partial W(\frac{\bar{u}_k}{p_k}) = \{W'(\frac{\bar{u}_k}{p_k})\}$ for $k \in \overline{1, n}$, whence $\frac{\bar{u}_k}{p_k} =: \eta$. Thus, $u = \sum_{k=1}^n \bar{u}_k = \eta\rho_n$, that is $\eta = u/\rho_n$, and so $\bar{u}_k = up_k/\rho_n$ for $k \in \overline{1, n}$. It follows that the infimum in (2.8) is $\rho_n \cdot W(u/\rho_n)$.

Consider now $W = E_{FD}$. For $u = \rho_n$ we have that $S'_n(\rho_n) = \{p\}$, where $p := (p_1, \dots, p_n)$, and $S'_n(u) = \emptyset$ for $u > \rho_n$; hence, the conclusion is trivial for $u \geq \rho_n$.

Let $u \in]0, \rho_n[$. Then $u\rho_n^{-1}p \in S'_n(u) \cap \text{int}(\text{dom } \widetilde{W}_n)$. The rest of the proof is the same as that of the preceding case. The proof is complete. \square

For $(u, v) \in \mathbb{R}^2$ and $n \geq 1$ let us set

$$S''_n(u, v) := \left\{ (u_1, \dots, u_n) \in \mathbb{R}_+^n \mid \sum_{k=1}^n u_k = u, \sum_{k=1}^n u_k \sigma_k = v \right\},$$

and

$$\text{dom } S''_n := \{(u, v) \in \mathbb{R}^2 \mid S''_n(u, v) \neq \emptyset\} = T_n(\mathbb{R}_+^n),$$

where

$$T_n : \mathbb{R}^n \rightarrow \mathbb{R}^2, \quad T_n(u_1, \dots, u_n) := \sum_{k=1}^n u_k(1, \sigma_k). \quad (2.10)$$

It follows that

$$\begin{aligned} \text{dom } S''_n &= \sum_{k=1}^n \mathbb{R}_+ \cdot (1, \sigma_k) = \mathbb{R}_+(1, \eta_n^1) + \mathbb{R}_+(1, \eta_n^2) \\ &= \{(u, v) \in \mathbb{R}_+ \times \mathbb{R} \mid \eta_n^1 u \leq v \leq \eta_n^2 u\}, \end{aligned} \quad (2.11)$$

where η_n^1, η_n^2 are defined in (2.3). Hence $\text{ri}(\text{dom } S_n'') = \mathbb{R}_+^*(1, \eta_n^1)$ when $\eta_n^1 = \eta_n^2$; if $\eta_n^1 < \eta_n^2$ then T_n is surjective, and so

$$\begin{aligned} \text{int}(\text{dom } S_n'') &= \text{int}(T_n(\mathbb{R}_+^n)) = T_n(\text{int } \mathbb{R}_+^n) = \mathbb{R}_+^*(1, \eta_n^1) + \mathbb{R}_+^*(1, \eta_n^2) \\ &= \{(u, v) \in \mathbb{R}_+^* \times \mathbb{R} \mid \eta_n^1 u < v < \eta_n^2 u\}. \end{aligned} \quad (2.12)$$

Observe that $S_n''(u, v) = \mathbb{R}_+^n \cap T_n^{-1}(\{(u, v)\})$ and $S_n''(u, v)$ is convex and compact for each $(u, v) \in \text{dom } S_n''$.

In the next result we characterize the solutions of the minimization problem

$$(EMP)_{u,v}^n \text{ minimize } \widetilde{W}_n(u_1, \dots, u_n) \text{ s.t. } (u_k)_{k \in \overline{1, n}} \in S_n''(u, v),$$

where \widetilde{W}_n is defined in (2.9).

Lemma 2.3. *Let $n \geq 2$ be fixed, and $W \in \{E_{BE}, E_{MB}\}$. Assume that $\eta_n^1 < \eta_n^2$ [see (2.3)] and take $(u, v) \in \mathbb{R}_+ \times \mathbb{R}$ such that $\eta_n^1 u \leq v \leq \eta_n^2 v$. Then $(EMP)_{u,v}^n$ has a unique solution $(\bar{u}_1, \dots, \bar{u}_n)$. Moreover, the following assertions are true:*

- (i) *If $u = 0$ then $\bar{u}_k = 0$ for every $k \in \overline{1, n}$.*
- (ii) *If $\eta_n^1 u < v < \eta_n^2 u$ then $\bar{u}_k > 0$ for every $k \in \overline{1, n}$. Moreover, there exist (and they are unique) $\alpha, \beta \in \mathbb{R}$ such that $W'(\bar{u}_k/p_k) = \alpha + \beta \sigma_k$ for every $k \in \overline{1, n}$.*
- (iii) *If $u \in \mathbb{R}_+^*$ and $v = \eta_n^i u$ for some $i \in \{1, 2\}$, then $\Sigma_i := \{k \in \overline{1, n} \mid \sigma_k = \eta_n^i\} \neq \emptyset$ and $\bar{u}_k = up_k / \sum_{l \in \Sigma_i} p_l$ for $k \in \Sigma_i$, $\bar{u}_k = 0$ for $k \in \overline{1, n} \setminus \Sigma_i$.*

Proof. Since $S_n''(u, v) (\subset \text{dom } \widetilde{W}_n = \mathbb{R}_+^n)$ is a nonempty [see (2.11)] compact set and \widetilde{W}_n is lsc and strictly convex on $\text{dom } \widetilde{W}_n$, $(EMP)_{u,v}^n$ has a unique solution $(\bar{u}_1, \dots, \bar{u}_n)$.

(i) The assertion is obvious.

(ii) By (2.12) we have that $(u, v) \in \text{int}(\text{dom } S_n'') = T_n(\text{int}(\text{dom } \widetilde{W}_n))$. Using again [8, Th. 2.9.6], there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha(1, \dots, 1) + \beta(\sigma_1, \dots, \sigma_n) \in \partial \widetilde{W}_n(\bar{u}_1, \dots, \bar{u}_n) = \partial W(\bar{u}_1/p_1) \times \dots \times \partial W(\bar{u}_n/p_n).$$

It follows that $\bar{u}_k/p_k > 0$, and so $\alpha + \beta \sigma_k = W'(\bar{u}_k/p_k)$ for $k \in \overline{1, n}$. Since $\eta_n^1 \neq \eta_n^2$, α and β are unique.

(iii) Consider the case $v = \eta_n^1 u$ with $u \in \mathbb{R}_+^*$ (the case $i = 2$ being similar). Take $(u_1, \dots, u_n) \in S_n''(u, v)$. Then $\sum_{k=1}^n u_k(\sigma_k - \eta_n^1) = 0$, whence $u_k(\sigma_k - \eta_n^1) = 0$ for every $k \in \overline{1, n}$. It follows that $u_k = 0$ for $k \in \overline{1, n} \setminus \Sigma_1$. Therefore, problem $(EMP)_{u,v}^n$ is equivalent to minimizing $\sum_{k \in \Sigma_1} p_k W(\frac{u_k}{p_k})$ with the constraint $\sum_{k \in \Sigma_1} u_k = u$. Using Lemma 2.2, the unique solution of this problem is $(\bar{u}_k)_{k \in \Sigma_1}$ with $\bar{u}_k = up_k / \sum_{l \in \Sigma_1} p_l$. \square

The argument for the proof of the next result is very similar to that in the proof of the preceding one, so we omit it.

Lemma 2.4. *Let $n \geq 2$ be fixed and $W = E_{FD}$. Assume that $(u, v) \in \mathbb{R}_+ \times \mathbb{R}$ is such that $S_n''(u, v) \cap \text{dom } \widetilde{W}_n \neq \emptyset$. Then $(EMP)_{u,v}^n$ has a unique solution $(\bar{u}_1, \dots, \bar{u}_n)$. Moreover, if $S_n''(u, v) \cap \text{int}(\text{dom } \widetilde{W}_n) \neq \emptyset$, then there exist (and they are unique) $\alpha, \beta \in \mathbb{R}$ such that $W'(\bar{u}_k/p_k) = \alpha + \beta \sigma_k$ for every $k \in \overline{1, n}$; in particular, $(\bar{u}_1, \dots, \bar{u}_n) \in \text{int}(\text{dom } \widetilde{W}_n)$.*

Note that α and β from Lemma 2.3 in case $W = E_{MB}$ can be obtained (quite easily) using Lemma 2.5 (below); indeed, $\beta = (\varphi_n)^{-1}(v/u)$ and $\alpha = \ln(u / \sum_{k=1}^n p_k e^{\sigma_k \beta})$.

Lemma 2.5. Let $n \geq 2$ and suppose that $\eta_n^1 < \eta_n^2$ (see (2.3)). Consider the function

$$\varphi_n : \mathbb{R} \rightarrow \mathbb{R}, \quad \varphi_n(t) := \frac{\sum_{k=1}^n p_k \sigma_k e^{\sigma_k t}}{\sum_{k=1}^n p_k e^{\sigma_k t}}. \quad (2.13)$$

Then φ_n is increasing and $\lim_{t \rightarrow -\infty} \varphi_n(t) = \eta_n^1$, $\lim_{t \rightarrow \infty} \varphi_n(t) = \eta_n^2$. Therefore, $\varphi_n(\mathbb{R}) =]\eta_n^1, \eta_n^2[$.

Proof. We have that

$$\varphi'_n(t) := \frac{\sum_{k=1}^n p_k \sigma_k^2 e^{\sigma_k t} \cdot \sum_{k=1}^n p_k e^{\sigma_k t} - (\sum_{k=1}^n p_k \sigma_k e^{\sigma_k t})^2}{(\sum_{k=1}^n p_k e^{\sigma_k t})^2} \quad \forall t \in \mathbb{R}.$$

By Cauchy–Bunyakovsky inequality we have that

$$\left(\sum_{k=1}^n \left[\sigma_k (p_k e^{\sigma_k t})^{\frac{1}{2}} \right] \cdot (p_k e^{\sigma_k t})^{\frac{1}{2}} \right)^2 < \sum_{k=1}^n p_k \sigma_k^2 e^{\sigma_k t} \cdot \sum_{k=1}^n p_k e^{\sigma_k t} \quad \forall t \in \mathbb{R}$$

(the inequality being strict because $\eta_n^1 < \eta_n^2$), and so $\varphi'_n(t) > 0$ for every $t \in \mathbb{R}$.

Set $\Sigma_i := \{k \in \overline{1, n} \mid \sigma_k = \eta_n^i\}$ for $i \in \{1, 2\}$. Since $\lim_{t \rightarrow -\infty} e^{\sigma t} = 0$ for $\sigma \in \mathbb{R}_+^*$, we obtain that

$$\lim_{t \rightarrow -\infty} \varphi_n(t) = \lim_{t \rightarrow -\infty} \frac{\sum_{k=1}^n p_k \sigma_k e^{(\sigma_k - \eta_n^1)t}}{\sum_{k=1}^n p_k e^{(\sigma_k - \eta_n^1)t}} = \frac{\sum_{k \in \Sigma_1} p_k \sigma_k}{\sum_{k \in \Sigma_1} p_k} = \eta_n^1.$$

Similarly, $\lim_{t \rightarrow \infty} \varphi_n(t) = \eta_n^2$. Because φ_n is increasing and continuous we obtain that $\varphi_n(\mathbb{R}) =]\eta_n^1, \eta_n^2[$. \square

In the next result we establish the convexity of H_W and estimate its domain for $W \in \{E_{MB}, E_{BE}, E_{FD}\}$.

Proposition 2.6. Let η_1 and η_2 be defined in (2.4). Set $S := S_{(\sigma_n)}$ and $H_W := H_{(\sigma_n), W}^{(p_n)}$ for $W \in \{E_{BE}, E_{MB}, E_{FD}\}$. The following assertions hold:

- (i) The marginal function H_W is convex; moreover, (2.6) and (2.7) hold.
- (ii) Assume that $\eta_1 = \eta_2$. Then $\text{dom } H_W = \text{dom } S = \mathbb{R}_+ \cdot (1, \sigma_1)$; in particular, $\text{ri}(\text{dom } H_W) = \mathbb{R}_+^* (1, \sigma_1) \neq \emptyset = \text{int}(\text{dom } H_W)$.
- (iii) Assume that $\eta_1 < \eta_2$ and take $\bar{n} \geq 2$ such that $\{\sigma_k \mid k \in \overline{1, \bar{n}}\}$ is not a singleton. Then for $W \in \{E_{BE}, E_{MB}\}$ one has

$$C := \bigcup_{n \geq 1} \sum_{k=1}^n \mathbb{R}_+ \cdot (1, \sigma_k) \subset \text{dom } H_W = \text{dom } S \subset \text{cl } C, \quad (2.14)$$

$$\text{int}(\text{dom } H_W) = \text{int}(\text{dom } S) = \text{int } C = \bigcup_{n \geq \bar{n}} \sum_{k=1}^n \mathbb{R}_+^* \cdot (1, \sigma_k) = \mathbb{R}_+^* \cdot (\{1\} \times]\eta_1, \eta_2[). \quad (2.15)$$

Moreover,

$$A := \bigcup_{n \geq 1} \sum_{k=1}^n [0, p_k] \cdot (1, \sigma_k) \subset \text{dom } H_{FD} \subset \text{cl } A, \quad (2.16)$$

$$\text{int}(\text{dom } H_{FD}) = \text{int } A = \bigcup_{n \geq \bar{n}} \sum_{k=1}^n]0, p_k[\cdot (1, \sigma_k). \quad (2.17)$$

Proof. (i) Let $(u, v), (u', v') \in \text{dom } H_W$ and $\lambda \in]0, 1[$. Take $\mu, \mu' \in \mathbb{R}$ with $H_W(u, v) < \mu$, $H_W(u', v') < \mu'$; there exist $(u_n)_{n \geq 1} \in S(u, v)$, $(u'_n)_{n \geq 1} \in S(u', v')$ such that $\sum_{n \geq 1} p_n W(\frac{u_n}{p_n}) < \mu$, $\sum_{n \geq 1} p_n W(\frac{u'_n}{p_n}) < \mu'$. Clearly, $(\lambda u_n + (1 - \lambda)u'_n)_{n \geq 1} \in S(\lambda(u, v) + (1 - \lambda)(u', v'))$. Since $p_n W(\frac{\lambda u_n + (1 - \lambda)u'_n}{p_n}) \leq \lambda p_n W(\frac{u_n}{p_n}) + (1 - \lambda)p_n W(\frac{u'_n}{p_n})$, summing up term by term for $n \geq 1$, we get

$$H_W(\lambda(u, v) + (1 - \lambda)(u', v')) \leq \lambda \sum_{n \geq 1} p_n W(\frac{u_n}{p_n}) + (1 - \lambda) \sum_{n \geq 1} p_n W(\frac{u'_n}{p_n}) < \lambda\mu + (1 - \lambda)\mu'.$$

Letting $\mu \rightarrow H_W(u, v)$ and $\mu' \rightarrow H_W(u', v')$ we get $H_W(\lambda(u, v) + (1 - \lambda)(u', v')) \leq \lambda H_W(u, v) + (1 - \lambda)H_W(u', v')$. Hence H_W is convex.

(ii) Assume that $\eta_1 = \eta_2$; hence $\sigma_n = \sigma_1$ for $n \geq 1$. Then $\text{dom } S = \mathbb{R}_+ \cdot (1, \sigma_1)$.

Taking into account (2.7), it is sufficient to show that $\text{dom } S \subset \text{dom } H_{FD}$. Take $u \in \mathbb{R}_+$. If $u < p_1$, take $u_1 := u$ and $u_k := 0$ for $n \geq 2$. If $u \geq p_1$, there exists $n \geq 1$ such that $\rho_n := \sum_{k=1}^n p_k \leq u < \rho_{n+1}$. Take $u_k = p_k$ for $k \in \overline{1, n}$, $u_{n+1} := u - \rho_n < p_{n+1}$, $u_k := 0$ for $k \geq n + 1$. In both cases we have that $u = \sum_{k \geq 1} u_k$ and $\sum_{k \geq 1} p_k W(u_k/p_k) \leq 0$, and so $(u, u\sigma_1) \in \text{dom } H_W$. Hence $\text{dom } S = \mathbb{R}_+ \cdot (1, \sigma_1) \subset \text{dom } H_{FD}$.

From the expression of $\text{dom } H_W$, the last assertion is obvious.

(iii) Assume that $\eta_1 < \eta_2$. The first three inclusions in (2.14) are obvious. For the last one, take $(u, v) \in \text{dom } S$; then there exists $(u_n)_{n \geq 1} \in S(u, v)$. Since $C \ni \sum_{k=1}^n u_k(1, \sigma_k) \rightarrow (u, v)$, we have that $(u, v) \in \text{cl } C$.

Set $C_n := \sum_{k=1}^n \mathbb{R}_+ \cdot (1, \sigma_k)$. Clearly $C_n \subset C_{n+1}$ and $C = \cup_{n \geq 1} C_n$; hence C is convex. The first two equalities in (2.15) follow from (2.14) because C is convex. Take $n \geq \bar{n}$. Since the linear operator $T_n : \mathbb{R}^n \rightarrow \mathbb{R}^2$ defined in (2.10) is surjective and $C_n = T_n(\mathbb{R}_+^n)$, we have that

$$\text{int } C_n = T_n(\text{int } \mathbb{R}_+^n) = T_n(\prod_{k=1}^n \mathbb{R}_+^*) = \sum_{k=1}^n \mathbb{R}_+^* \cdot (1, \sigma_k).$$

Since $(C_n)_{n \geq \bar{n}}$ is an increasing sequence of convex sets with nonempty interior and $C = \cup_{n \geq \bar{n}} C_n$, we obtain that $\text{int } C = \cup_{n \geq \bar{n}} \text{int } C_n$. Hence the third equality in (2.15) holds.

Take now $(u, v) \in \sum_{k=1}^n \mathbb{R}_+^* \cdot (1, \sigma_k)$ for some $n \geq \bar{n}$. Then $(u, v) = \sum_{k=1}^n u_k(1, \sigma_k)$ with $(u_k)_{k \in \overline{1, n}} \subset \mathbb{R}_+^*$. It follows that $(u, v) = \alpha \cdot (1, w)$, where $\alpha := \sum_{i=1}^n u_i \in \mathbb{R}_+^*$ and $w := \sum_{k=1}^n \frac{u_k}{\alpha} \sigma_k$. Because $\{\sigma_k \mid k \in \overline{1, n}\}$ is not a singleton, $w \in]\eta_1, \eta_2[$, and so $(u, v) \in B := \mathbb{R}_+^* \cdot (\{1\} \times]\eta_1, \eta_2[)$; clearly, B is open. Conversely, take $(u, v) \in B$, that is $(u, v) = \alpha \cdot (1, w)$ with $\alpha \in \mathbb{R}_+^*$ and $w \in]\eta_1, \eta_2[$. Then there exists $n_1, n_2 \geq 2$ such that $\sigma_{n_1} < w < \sigma_{n_2}$; hence $w = \lambda \sigma_{n_1} + (1 - \lambda) \sigma_{n_2}$ for some $\lambda \in]0, 1[$. Consider $n = \max\{\bar{n}, n_1, n_2\}$. It follows that $(u, v) \in C_n \subset C$. Therefore, $B \subset C$, whence $B = \text{int } B \subset \text{int } C$. Hence the last equality in (2.15) holds, too.

The first inclusion in (2.16) is obvious. Take $(u, v) \in \text{dom } H_{FD}$; then there exists $(u_n)_{n \geq 1} \in S(u, v)$ such that $\sum_{n \geq 1} p_n H_{FD}(\frac{u_n}{p_n}) < \infty$. It follows that $u_n \in [0, p_n]$ for $n \geq 1$. Since $A \ni \sum_{k=1}^n u_k(1, \sigma_k) \rightarrow (u, v)$, we have that $(u, v) \in \text{cl } A$. Hence (2.16) holds.

Set $A_n := \sum_{k=1}^n [0, p_k] \cdot (1, \sigma_k)$. Clearly $A_n \subset A_{n+1}$ and $A = \cup_{n \geq 1} A_n$, and so A is convex. The first equality in (2.17) follows from (2.16) because A is convex. Take $n \geq \bar{n}$. Since the linear operator T_n is surjective and $A_n = T_n(\prod_{k=1}^n [0, p_k])$, we have that

$$\text{int } A_n = T_n(\text{int } \prod_{k=1}^n [0, p_k]) = T_n(\prod_{k=1}^n]0, p_k]) = \sum_{k=1}^n]0, p_k[\cdot (1, \sigma_k).$$

Since $(A_n)_{n \geq \bar{n}}$ is an increasing sequence of convex sets with nonempty interior and $A = \cup_{n \geq \bar{n}} A_n$, we obtain that $\text{int } A = \cup_{n \geq \bar{n}} \text{int } A_n$. Therefore, the last equality in (2.17) holds, too. The proof is complete. \square

Proposition 2.7. Consider $W \in \{E_{BE}, E_{MB}, E_{FD}\}$.

- (i) Assume that $\eta_1 = \eta_2$. Then $H_W(u, v) = -\infty$ for all $(u, v) \in \text{ri}(\text{dom } H_W) = \mathbb{R}_+^* \cdot (1, \sigma_1)$.
(ii) Assume that the series $\sum_{n \geq 1} p_n e^{\sigma_n x}$ is divergent for every $x \in \mathbb{R}$ and $\eta_1 < \eta_2$. Then

$$H_W(u, v) = -\infty \quad \forall (u, v) \in \text{int}(\text{dom } H_W). \quad (2.18)$$

Proof. (i) Clearly, $\sigma_n = \sigma_1$ for $n \geq 1$. Take $u > 0$. Then there exists $\bar{n} \geq 1$ such that $u < \rho_n = \sum_{k=1}^n p_k$ for $n \geq \bar{n}$. Having in view Lemma 2.2, consider $u_k := up_k/\rho_n$ for $k \in \overline{1, n}$ and $u_k := 0$ for $k \geq n+1$. Then $u = \sum_{k \geq 1} u_k$, and so

$$H_{FD}(u, u\sigma_1) \leq \sum_{k=1}^n p_k E_{FD}(u/\rho_n) = u \ln u - u \ln \rho_n - u \left(1 - \frac{u}{\rho_n}\right) \frac{\ln \left(1 - \frac{u}{\rho_n}\right)}{-\frac{u}{\rho_n}}$$

for all $n \geq \bar{n}$. Since $\rho_n \rightarrow \infty$, we obtain that $H_{FD}(u, u\sigma_1) = -\infty$. Using (2.6), we have that $H_W(u, v) = -\infty$ for $(u, v) \in \text{ri}(\text{dom } H_W) = \mathbb{R}_+^* \cdot (1, \sigma_1)$ for $W \in \{E_{BE}, E_{MB}, E_{FD}\}$.

(ii) Take first $W = E_{MB}$. Because $S_{(\sigma_{\phi(n)})}(u, v) = \{(u_{\phi(n)}) \mid (u_n) \in S_{(\sigma_n)}(u, v)\}$ and $H_{(\sigma_n), W}^{(p_n)}(u, v) = H_{(\sigma_{\phi(n)}), W}^{(p_{\phi(n)})}(u, v)$ for every bijection $\phi: \mathbb{N}^* \rightarrow \mathbb{N}^*$ with $\phi(n) = n$ for large n , we may (and do) assume that $\sigma_1 < \sigma_2$. Even more, because for $a \in \mathbb{R}$ and $\sigma'_n := \sigma_n - a$ ($n \geq 1$) we have that $S_{(\sigma_n)}(u, v) = S_{(\sigma'_n)}(u, v - au)$ and, consequently, $H_{(\sigma_n), W}^{(p_n)}(u, v) = H_{(\sigma'_n), W}^{(p_n)}(u, v - au)$, we may (and do) assume that $0 \in]\sigma_1, \sigma_2[$. For $n \geq 2$ consider the function φ_n defined in (2.13). By Lemma 2.5 there exists (a unique) $y_n \in \mathbb{R}$ such that $\varphi_n(y_n) = 0$. We claim that $\lim_{n \rightarrow \infty} \sum_{k=1}^n p_k e^{\sigma_k y_n} = \infty$. In the contrary case there exist an increasing sequence $(n_m)_{m \geq 1} \subset \mathbb{N}^* \setminus \{1, 2\}$ and $M \in \mathbb{R}_+^*$ such that $\sum_{k=1}^{n_m} p_k e^{\sigma_k y_{n_m}} \leq M$ for every $m \geq 1$. In particular, $p_1 e^{\sigma_1 y_{n_m}} \leq M$ (whence $y_{n_m} \geq (\ln M - \ln p_1)/\sigma_1$) and $e^{\sigma_2 y_{n_m}} \leq M$ (whence $y_{n_m} \leq (\ln M - \ln p_2)/\sigma_2$) for $m \geq 1$; hence $(y_{n_m})_{m \geq 1}$ is bounded. Passing if necessary to a subsequence, we may (and do) assume that $y_{n_m} \rightarrow y \in \mathbb{R}$. For $q \geq 2$ there exists $m_q \geq 1$ such that $n_m \geq q$ for every $m \geq m_q$. Hence $\sum_{k=1}^q p_k e^{\sigma_k y_{n_m}} \leq M$ for every $m \geq m_q$. Letting $(m_q \leq) m \rightarrow \infty$ we obtain that $\sum_{k=1}^q p_k e^{\sigma_k y} \leq M$ for every $q \geq 2$, and so we get the contradiction that the series $\sum_{k \geq 1} p_k e^{\sigma_k y}$ is convergent. Therefore, our claim is true.

Set $x_n := -\ln(\sum_{k=1}^n p_k e^{\sigma_k y_n}) \rightarrow -\infty$ for $n \rightarrow \infty$. Set $u_k := p_k e^{x_n + \sigma_k y_n} > 0$ for $k \in \overline{1, n}$ and $u_k := 0$ for $k \geq n+1$. Then

$$\sum_{k \geq 1} u_k = \sum_{k=1}^n p_k e^{x_n + \sigma_k y_n} = 1, \quad \sum_{k \geq 1} u_k \sigma_k = \sum_{k=1}^n p_k \sigma_k e^{x_n + \sigma_k y_n} = 0,$$

and so $(u_k)_{k \geq 1} \in S(1, 0)$. Hence $u_k = p_k e^{x_n + \sigma_k y_n} \leq 1 \leq p_k$, and so $e^{x_n + \sigma_k y_n} \leq 1$, for every $k \in \overline{1, n}$. It follows that

$$\begin{aligned} H_{FD}(1, 0) &\leq \sum_{k \geq 1} p_k E_{FD}\left(\frac{u_k}{p_k}\right) = \sum_{k=1}^n u_k \ln e^{x_n + \sigma_k y_n} + \sum_{k=1}^n (p_k - u_k) \ln(1 - e^{x_n + \sigma_k y_n}) \\ &\leq \sum_{k=1}^n u_k (x_n + \sigma_k y_n) = x_n \quad \forall n \geq 2, \end{aligned}$$

and so $H_{FD}(1, 0) = -\infty$. Using (2.6) we obtain that (2.18) holds. The proof is complete. \square

The previous result shows the lack of interest of the EMP when the sequence $(\sigma_n)_{n \geq 1}$ is constant. Also, it gives a hint on the importance of knowing the properties of the function

$$f: \mathbb{R} \rightarrow \overline{\mathbb{R}}, \quad f(x) = \sum_{n \geq 1} p_n e^{\sigma_n x}. \quad (2.19)$$

The next result, with $p_n = 1$ for $n \geq 1$, is practically [7, Prop. 12]; the adaptation of its proof for the present case is easy.

Proposition 2.8. *Let $f_n(x) := p_n e^{\sigma_n x}$ for $n \geq 1$, $x \in \mathbb{R}$, and set $f = \sum_{n \geq 1} f_n$.*

- (i) *If $\bar{x} \in \text{dom } f$ then $\sigma_n \bar{x} \rightarrow -\infty$, and so either $\bar{x} > 0$ and $\sigma_n \rightarrow -\infty$, or $\bar{x} < 0$ and $\sigma_n \rightarrow \infty$. Furthermore, assume that $(A_{\sigma f})$ holds, where*

$$(A_{\sigma f}) \quad (\sigma_n)_{n \geq 1} \subset \mathbb{R}_+^*, \sigma_n \rightarrow \infty, \text{ and } \text{dom } f \neq \emptyset.$$

- (ii) *Then there exists $\alpha \in \mathbb{R}_+$ such that $I :=]-\infty, -\alpha[\subset \text{dom } f \subset \mathbb{R}_+^* \cap \text{cl } I$, f is strictly convex and increasing on $\text{dom } f$, and $\lim_{x \rightarrow -\infty} f(x) = 0 = \inf f$. Moreover,*

$$f'(x) = \sum_{n \geq 1} f'_n(x) = \sum_{n \geq 1} p_n \sigma_n e^{\sigma_n x} \quad \forall x \in \text{int}(\text{dom } f) = I,$$

f' is increasing and continuous on I , $\lim_{x \rightarrow -\infty} f'(x) = 0$, and

$$\lim_{x \uparrow -\alpha} f'(x) = \sum_{n \geq 1} p_n \sigma_n e^{-\sigma_n \alpha} =: \gamma \in]0, \infty].$$

In particular, $\partial f(\text{int}(\text{dom } f)) = f'(I) =]0, \gamma[$.

- (iii) *Let α, I, γ be as in (ii). Assume that $\alpha \in \mathbb{R}_+^*$. Then either (a) $\text{dom } f = I$ and $\gamma = \infty$, or (b) $\text{dom } f = \text{cl } I$ and $\gamma = \infty$, in which case $f'_-(-\alpha) = \gamma$, $\partial f(-\alpha) = \emptyset$ and the series $\sum_{n \geq 1} f'_n(-\alpha)$ is divergent, or (c) $\text{dom } f = \text{cl } I$ and $\gamma < \infty$, in which case $f'_-(-\alpha) = \gamma$ and*

$$\sum_{n \geq 1} f'_n(-\alpha) = \gamma \in [\gamma, \infty[= \partial f(-\alpha).$$

Let us consider the following functions for $W \in \{E_{MB}, E_{FD}, E_{BE}\}$:

$$h_n^W : \mathbb{R}^2 \rightarrow \overline{\mathbb{R}}, \quad h_n^W(x, y) := p_n W^*(x + \sigma_n y) > 0 \quad (n \geq 1, x, y \in \mathbb{R}),$$

$$h_W := h_{W, (\sigma_n)}^{(p_n)} : \mathbb{R}^2 \rightarrow \overline{\mathbb{R}}, \quad h_W := \sum_{n \geq 1} h_n^W;$$

we write simply h_{MB}, h_{FD}, h_{BE} instead of $h_{E_{MB}}, h_{E_{FD}}, h_{E_{BE}}$, respectively. Because $h_n^W = (p_n W^*) \circ A_n$, where $A_n : \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined by $A_n(x, y) := x + \sigma_n y$ [and so $A_n^* w = w(1, \sigma_n)$], we have that

$$\begin{aligned} (h_n^W)^*(u, v) &= \min \{ (p_n W^*)^*(w) \mid A_n^* w = (u, v) \} = \min \left\{ p_n W\left(\frac{w}{p_n}\right) \mid A_n^* w = (u, v) \right\} \\ &= \begin{cases} p_n W\left(\frac{u}{p_n}\right) & \text{if } u \geq 0 \text{ and } v = \sigma_n u, \\ \infty & \text{otherwise,} \end{cases} \end{aligned} \quad (2.20)$$

and so $(h_n^W)^*$ is strictly convex on its domain.

The expression of $(h_n^W)^*$ (above) in connection with [7, Prop. 15(i)] shows the interest of studying the properties of the functions h_W .

3. Properties of the functions h_W

Because $p_n \geq 1$ for $n \geq 1$, we have that

$$\begin{aligned} (x, y) \in \text{dom } h_W &\Rightarrow p_n W^*(x + \sigma_n y) \rightarrow 0 \Rightarrow W^*(x + \sigma_n y) \rightarrow 0 \Leftrightarrow \sigma_n y \rightarrow -\infty \\ &\Leftrightarrow [y > 0 \text{ and } \sigma_n \rightarrow -\infty] \text{ or } [y < 0 \text{ and } \sigma_n \rightarrow \infty]. \end{aligned} \quad (3.1)$$

Of course, $h_n^W(x, y) > 0$ for all $(x, y) \in \mathbb{R}^2$, $n \geq 1$ and $W \in \{E_{MB}, E_{FD}, E_{BE}\}$; because for $\sigma_n y \rightarrow -\infty$ we have

$$\lim_{n \rightarrow \infty} \frac{h_n^{E_{FD}}(x, y)}{h_n^{E_{MB}}(x, y)} = \lim_{n \rightarrow \infty} \frac{h_n^{E_{BE}}(x, y)}{h_n^{E_{MB}}(x, y)} = 1,$$

we obtain that

$$\text{dom } h_{FD} = \text{dom } h_{MB}, \quad \text{dom } h_{BE} = \text{dom } h_{MB} \cap \{(x, y) \in \mathbb{R}^2 \mid x + \sigma_n y < 0 \ \forall n \geq 1\}.$$

Since $h_{MB}(x, y) = \sum_{n \geq 1} p_n e^{x + \sigma_n y} = e^x f(y)$, where f is defined by (2.19), clearly $\text{dom } h_{MB} = \mathbb{R} \times \text{dom } f$. It follows that

$$\text{dom } h_{FD} \neq \emptyset \Leftrightarrow \text{dom } h_{MB} \neq \emptyset \Leftrightarrow \text{dom } h_{BE} \neq \emptyset \Leftrightarrow \text{dom } f \neq \emptyset.$$

Because it is natural to consider the case in which $\text{dom } h_W$ is nonempty, in the sequel, we assume that $(A_{\sigma f})$ holds. (The general case can be reduced to this one replacing $(\sigma_n)_{n \geq 1}$ by $(-\sigma_n)_{n \geq 1}$ if $\sigma_n \rightarrow -\infty$, then replacing $(\sigma_n)_{n \geq 1}$ by $(\sigma_n - a)_{n \geq 1}$ with $a < \min_{n \geq 1} \sigma_n$.)

Proposition 3.1. Assume that $(A_{\sigma f})$ holds, and take $\alpha \in \mathbb{R}_+$ such that $I :=]-\infty, -\alpha[\subset \text{dom } f \subset \text{cl } I$, where f is defined in (2.19). Let $W \in \{E_{MB}, E_{FD}, E_{BE}\}$.

(i) Then h_W is convex, lower semicontinuous, positive, and

$$\text{dom } h_{FD} = \text{dom } h_{MB} = \mathbb{R} \times \text{dom } f, \quad \text{dom } h_{BE} = \{(x, y) \in \mathbb{R} \times \text{dom } f \mid x + \theta_1 y < 0\},$$

where $\theta_1 := \min\{\sigma_n \mid n \geq 1\}$.

(ii) h_W is differentiable at any $(x, y) \in \text{int}(\text{dom } h_W)$ and

$$\nabla h_W(x, y) = \sum_{n \geq 1} \nabla h_n^W(x, y) = \sum_{n \geq 1} p_n \frac{e^{x + \sigma_n y}}{1 + a_W e^{x + \sigma_n y}} \cdot (1, \sigma_n), \quad (3.2)$$

a_W being defined in (1.9). Moreover, assume that $(x, -\alpha) \in \text{dom } h_W$; in particular, $-\alpha \in \text{dom } f \subset \mathbb{R}_-^*$. Then

$$\partial h_W(x, -\alpha) \neq \emptyset \iff \sum_{n \geq 1} \nabla h_n^W(x, -\alpha) \text{ converges} \iff \gamma := \sum_{n \geq 1} p_n \sigma_n e^{x - \sigma_n \alpha} \in \mathbb{R}; \quad (3.3)$$

if $(\bar{u}, \bar{v}) := \sum_{n \geq 1} \nabla h_n^W(x, -\alpha)$ exists in \mathbb{R}^2 , then

$$\partial h_W(x, -\alpha) = \{\bar{u}\} \times [\bar{v}, \infty[= \{(\bar{u}, \bar{v})\} + \{0\} \times \mathbb{R}_+. \quad (3.4)$$

Proof. The existence of $\alpha \in \mathbb{R}_+$ such that $I :=]-\infty, -\alpha[\subset \text{dom } f \subset \text{cl } I$ is ensured by Proposition 2.8.

(i) Since $h_{MB}(x, y) = e^x f(y)$ for $(x, y) \in \mathbb{R}^2$, we have that $\text{dom } h_{MB} = \mathbb{R} \times \text{dom } f$. Taking into account (3.1) and the fact that $\lim_{t \rightarrow -\infty} e^{-t} W(t) = 1$ for $W \in \{E_{MB}, E_{FD}, E_{BE}\}$, we obtain that

$$\text{dom } h_W = (\mathbb{R} \times \text{dom } f) \cap \bigcap_{n \geq 1} \text{dom } h_n^W \subset (\mathbb{R} \times \mathbb{R}_-^*) \cap \bigcap_{n \geq 1} \text{dom } h_n^W. \quad (3.5)$$

Since $\text{dom } h_n^W = \mathbb{R}^2$ for $n \geq 1$ and $W \in \{E_{MB}, E_{FD}\}$, we get $\text{dom } h_W = \mathbb{R} \times \text{dom } f$ for $W \in \{E_{MB}, E_{FD}\}$. Let $W = E_{BE}$; then $\text{dom } h_n^W = \{(x, y) \in \mathbb{R}^2 \mid x + \sigma_n y < 0\}$. For $x \in \mathbb{R}$ and $y \in \text{dom } f$ (hence $y < 0$), $(x, y) \in \bigcap_{n \geq 1} \text{dom } h_n^W$ if and only if $x + \theta_1 y < 0$. From (3.5) we get the given expression of $\text{dom } h_{BE}$.

Since h_n^W is convex, continuous and positive, we obtain that h_W is convex, lower semicontinuous and positive.

(ii) Using [7, Cor. 11] (and (1.8)), we obtain that h_W is differentiable on $\text{int}(\text{dom } h_W)$ and (3.2) holds.

Assume that $(x, -\alpha) \in \text{dom } h_W$. Hence $-\alpha \in \text{dom } f \subset \mathbb{R}_-^*$; moreover, $x \in \mathbb{R}$ for $W \in \{E_{MB}, E_{FD}\}$, and $x < \theta_1 \alpha$ ($\leq \sigma_n \alpha$ for $n \geq 1$) for $W = E_{BE}$.

Because $\lim_{n \rightarrow \infty} (1 + a_W e^{x+\sigma_n y}) = 1$, the last equivalence in (3.3) holds.

In order to prove the first equivalence in (3.3), suppose first that $(u, v) \in \partial h_W(x, -\alpha)$. Then, for $(x', y) \in \text{dom } h_W$,

$$u \cdot (x' - x) + v \cdot (y + \alpha) \leq h_W(x', y) - h_W(x, -\alpha). \quad (3.6)$$

Taking $y := -\alpha$ we obtain that $u \in \partial h_W(\cdot, -\alpha)(x)$. Since $h_W(\cdot, -\alpha) = \sum_{n \geq 1} h_n^W(\cdot, -\alpha)$ and $x \in \text{int}(\text{dom } h_W(\cdot, -\alpha))$, using again [7, Cor. 11] we obtain that $h_W(\cdot, -\alpha)$ is derivable at x and

$$u = (h_W(\cdot, -\alpha))'(x) = \sum_{n \geq 1} (h_n^W(\cdot, -\alpha))'(x).$$

Take now $x' = x$ and $y < -\alpha$ in (3.6). Dividing by $y + \alpha$ (< 0), and taking into account that W^* is increasing on its domain, we get

$$v \geq \sum_{n \geq 1} p_n \frac{W^*(x + \sigma_n y) - W^*(x - \sigma_n \alpha)}{y + \alpha} \geq \sum_{k=1}^n p_k \frac{W^*(x + \sigma_k y) - W^*(x - \sigma_k \alpha)}{y + \alpha} \quad \forall n \geq 1.$$

Taking the limit for $y \uparrow -\alpha$ in the second inequality, we get $v \geq \sum_{k=1}^n p_k \sigma_k (W^*)'(x - \sigma_k \alpha)$. Since $(W^*)'(x - \sigma_k \alpha) > 0$ for every $k \geq 1$, we obtain that the series $\sum_{n \geq 1} p_n \sigma_n (W^*)'(x - \sigma_n \alpha) = \sum_{n \geq 1} (h_n^W(x, \cdot))'(-\alpha)$ is convergent and $v \geq \sum_{n \geq 1} (h_n^W(x, \cdot))'(-\alpha)$. Hence the series $\sum_{n \geq 1} \nabla h_n^W(x, -\alpha)$ is convergent and for its sum (\bar{u}, \bar{v}) we have that $u = \bar{u}$ and $v \geq \bar{v}$.

Conversely, assume that the series $\sum_{n \geq 1} \nabla h_n^W(x, -\alpha)$ is convergent with sum (\bar{u}, \bar{v}) ; take $v \geq \bar{v}$ and $(x', y) \in \text{dom } h_W$. Using [7, Prop. 15(iii)] we obtain that $(\bar{u}, \bar{v}) \in \partial h_W(x, -\alpha)$. Hence (3.6) holds for (u, v) replaced by (\bar{u}, \bar{v}) . Since $v \geq \bar{v}$ and $y \leq -\alpha$, we have that $v \cdot (y + \alpha) \leq \bar{v} \cdot (y + \alpha)$, and so (3.6) also holds for u replaced by \bar{u} . It follows that $(\bar{u}, v) \in \partial h_W(x, -\alpha)$. Therefore, (3.4) holds. \square

Theorem 3.2. Let $W \in \{E_{MB}, E_{FD}, E_{BE}\}$ and a_W be defined in (1.9). Then for every $(x, y) \in \cap_{n \geq 1} \text{dom } h_n^W$ such that the series $\sum_{n \geq 1} p_n \frac{e^{x+\sigma_n y}}{1+a_W e^{x+\sigma_n y}} \cdot (1, \sigma_n)$ is convergent [this is the case, for example, when $(x, y) \in \text{int}(\text{dom } h_W)]$ with sum $(u, v) \in \mathbb{R}^2$, the problem $(EMP)_{u,v}$ has the unique optimal solution $\left(p_n \frac{e^{x+\sigma_n y}}{1+a_W e^{x+\sigma_n y}}\right)_{n \geq 1}$. Moreover, the value of the problem $(EMP)_{u,v}$ is $h_W^*(u, v)$, that is $H_W(u, v) = h_W^*(u, v)$.

Proof. Taking into account that h_n^W is a proper convex function for $n \geq 1$ with

$$\partial h_n^W(x, y) = \{\nabla h_n^W(x, y)\} = \{p_n (W^*)'(e^{x+\sigma_n y}) \cdot (1, \sigma_n)\} = \left\{p_n \frac{e^{x+\sigma_n y}}{1+a_W e^{x+\sigma_n y}} \cdot (1, \sigma_n)\right\}$$

for $(x, y) \in \text{dom } h_n^W$, and $h_W = \sum_{n \geq 1} h_n^W$, as well as the expression $(h_n^W)^*$ given in (2.20), we get the conclusion using [7, Prop. 15(iii)]. The fact that the series $\sum_{n \geq 1} p_n \frac{e^{x+\sigma_n y}}{1+a_W e^{x+\sigma_n y}} \cdot (1, \sigma_n)$ is convergent for $(x, y) \in \text{int}(\text{dom } h_W)$ is ensured by [7, Cor. 11(i)]. \square

The result in Theorem 3.2 is obtained generally using the Lagrange multipliers method in a formal way.

A complete solution to EMP for the Maxwell–Boltzmann entropy is provided in the next section.

4. Complete solution to EMP in the case of the Maxwell–Boltzmann entropy

In this section $W = E_{MB}$; to simplify the writing, we set $h_n := h_n^{E_{MB}}$, $h := h_{MB}$, $H := H_{MB}$; we mention also the sequences (σ_n) and (p_n) if necessary. We assume that $(A_{\sigma f})$ holds if not stated explicitly otherwise. Moreover, we use I , α , γ as in Proposition 2.8.

From (2.20) we have that

$$h_n^*(u, v) = \begin{cases} u(\ln \frac{u}{p_n} - 1) & \text{if } u \geq 0 \text{ and } v = \sigma_n u, \\ \infty & \text{otherwise,} \end{cases}$$

and so h_n^* is strictly convex on its domain.

Let us compute h^* . From the definition of the conjugate, for $(u, v) \in \mathbb{R}^2$ we have that

$$\begin{aligned} h^*(u, v) &= \sup_{(x, y) \in \mathbb{R}^2} [ux + vy - e^x f(y)] = \sup_{y \in \text{dom } f} \left[vy + \sup_{x \in \mathbb{R}} [ux - e^x f(y)] \right] \\ &= \sup_{y \in \text{dom } f} \left[vy + f(y) \sup_{x \in \mathbb{R}} \left(x \frac{u}{f(y)} - e^x \right) \right] = \sup_{y \in \text{dom } f} \left[vy + f(y) \exp^* \left(\frac{u}{f(y)} \right) \right]. \end{aligned}$$

It follows that $h^*(u, v) = \infty$ for $u \in \mathbb{R}_-^*$ and $h^*(0, v) = \iota_{\text{dom } f}^*(v)$; hence $h^*(0, v) = \infty$ for $v \in \mathbb{R}_-^*$ and $h^*(0, v) = -\alpha v$ for $v \in \mathbb{R}_+$.

Fix $(u, v) \in \mathbb{R}_+^* \times \mathbb{R}$. From the above expression of $h^*(u, v)$ we get

$$\begin{aligned} h^*(u, v) &= \sup_{y \in I} [vy + u \ln u - u \ln [f(y)] - u] = u(\ln u - 1) + u \sup_{y \in I} \left[\frac{v}{u} y - \ln [f(y)] \right] \\ &= u(\ln u - 1) + u \cdot (\ln f)^*(v/u). \end{aligned}$$

Below we show that $\ln f := \ln \circ f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ is convex; we also calculate its conjugate. For these consider

$$\varphi : I \rightarrow \mathbb{R}_+^*, \quad \varphi(y) := f'(y)/f(y). \quad (4.1)$$

Observe, using Schwarz' inequality in ℓ_2 , that

$$[f'(y)]^2 = \left[\sum_{n \geq 1} \sigma_n (p_n e^{\sigma_n y})^{\frac{1}{2}} \cdot (p_n e^{\sigma_n y})^{\frac{1}{2}} \right]^2 < \sum_{n \geq 1} \sigma_n^2 p_n e^{\sigma_n y} \sum_{n \geq 1} p_n e^{\sigma_n y} = f(y) \cdot f''(y) \quad \forall y \in I$$

(the inequality being strict because $\sigma_n \rightarrow \infty$). Hence $\varphi'(y) = (f(y) \cdot f''(y) - [f'(y)]^2)/[f(y)]^2 > 0$ for $y \in I$. Therefore, φ is increasing, and so

$$0 < \theta_1 := \min_{n \geq 1} \sigma_n = \lim_{y \rightarrow -\infty} \varphi(y) < \lim_{y \uparrow -\alpha} \varphi(y) =: \theta_2 \leq \infty.$$

Restricting the co-domain of φ to $]\theta_1, \theta_2[$ we get an increasing bijective function denoted also by φ . Since $(\ln f)' = f'/f = \varphi$ on I , $\ln f$ is (strictly) convex on its domain. Observe that $\theta_2 < \infty$ is equivalent to $-\alpha \in \text{dom } f$ and $\gamma := f'_-(-\alpha) < \infty$, in which case $\theta_2 = f'_-(-\alpha)/f(-\alpha)$. Indeed, assume that $\theta_2 < \infty$ and fix $y_0 \in I$. Then for $y_0 < y < -\alpha$ we have that $\ln \frac{f(y)}{f(y_0)} = \int_{y_0}^y \varphi(t) dt \leq \theta_2(y - y_0)$, whence $f(y) \leq f(y_0)e^{\theta_2(y - y_0)}$, and so $f(-\alpha) \leq f(y_0)e^{-\theta_2(\alpha + y_0)} < \infty$; then, because $f'(y) = f(y) \cdot \varphi(y)$ for $y \in I$, we get $\gamma = \theta_2 \cdot f(-\alpha)$. The converse implication is obvious. Because $\text{int}(\text{dom}(\ln f)) = I$, it follows that

$$(\ln f)^*(w) = \sup_{y \in I} (wy - \ln[f(y)]) = \begin{cases} \infty & \text{if } w < \theta_1, \\ w\varphi^{-1}(w) - \ln[f(\varphi^{-1}(w))] & \text{if } \theta_1 < w < \theta_2, \\ -\alpha w - \ln[f(-\alpha)] & \text{if } \theta_2 \leq w, \end{cases}$$

where the last line has to be taken into consideration only if $\theta_2 < \infty$. Let us set

$$\Sigma := \{k \in \mathbb{N}^* \mid \sigma_k = \theta_1\};$$

of course, Σ is finite and nonempty, and so we may (and do) suppose that $\Sigma = \overline{1, q}$ for some $q \in \mathbb{N}^*$ and $\sigma_{q+1} \leq \sigma_n$ for $n \geq q+1$. Because $(\ln f)^*$ is convex and lsc, we have that

$$(\ln f)^*(\theta_1) = \lim_{w \downarrow \theta_1} (\ln f)^*(w) = \lim_{w \downarrow \theta_1} [w\varphi^{-1}(w) - \ln[f(\varphi^{-1}(w))]] = \lim_{y \rightarrow -\infty} \psi(y),$$

where $\psi(y) = y\varphi(y) - \ln[f(y)]$ for $y \in I$. But

$$\begin{aligned} \psi(y) &= y \frac{\sigma_1 \sum_{n=1}^q p_n + \sum_{n \geq q+1} p_n \sigma_n e^{(\sigma_n - \sigma_1)y}}{\sum_{n=1}^q p_n + \sum_{n \geq q+1} p_n e^{(\sigma_n - \sigma_1)y}} - \sigma_1 y - \ln \left(\sum_{n=1}^q p_n + \sum_{n \geq q+1} p_n e^{(\sigma_n - \sigma_1)y} \right) \\ &= y e^{(\sigma_{q+1} - \sigma_1)y} \cdot \frac{\sum_{n \geq q+1} p_n (\sigma_n - \sigma_1) e^{(\sigma_n - \sigma_{q+1})y}}{\sum_{n=1}^q p_n + \sum_{n \geq q+1} p_n e^{(\sigma_n - \sigma_1)y}} - \ln \left(\sum_{n=1}^q p_n + \sum_{n \geq q+1} p_n e^{(\sigma_n - \sigma_1)y} \right). \end{aligned}$$

It follows that $(\ln f)^*(\theta_1) = -\ln(\sum_{n=1}^q p_n) = -\ln(\sum_{n \in \Sigma} p_n)$. Therefore, $\text{dom}(\ln f)^* = [\theta_1, \infty[$.

Summing up the preceding computations we get

$$h^*(u, v) = \begin{cases} u \ln u - u + u \cdot (\ln f)^*(v/u) & \text{if } v \geq \theta_1 u > 0 \\ -\alpha v & \text{if } u = 0 \leq v, \\ \infty & \text{if } u \in \mathbb{R}_+^*, \text{ or } v \in \mathbb{R}_+^*, \text{ or } 0 \leq v < \theta_1 u. \end{cases}$$

It follows that

$$\{(u, v) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \mid v > \theta_1 u\} = \text{int}(\text{dom } h^*) \subset \text{dom } h^* = \{(u, v) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid v \geq \theta_1 u\}.$$

Moreover,

$$\partial h(\text{int}(\text{dom } h)) = \nabla h(\mathbb{R} \times I) = \{(u, v) \mid u \in \mathbb{R}_+^*, \theta_1 u < v < \theta_2 u\} =: E, \quad (4.2)$$

as a simple argument shows.

For each $(u, v) \in \mathbb{R}^2$, $S(u, v) := S_{(\sigma_n)}(u, v)$ and $H(u, v) := H_{(\sigma_n)}^{(p_n)}(u, v)$ are defined in (2.1) and (2.5), respectively. By [7, Prop. 15 (i)] we have that $h^*(u, v) \leq H(u, v)$ for all $(u, v) \in \text{dom } h^*$.

In the sequel we determine the set A of those $(u, v) \in \text{dom } h^*$ such that

$$h^*(u, v) = \min \left\{ \sum_{n \geq 1} u_n \left(\ln \frac{u_n}{p_n} - 1 \right) \mid (u_n)_{n \geq 1} \in S(u, v) \right\}, \quad (4.3)$$

and the set B of those $(u, v) \in \text{dom } h^*$ such that $h^*(u, v) = H(u, v)$; of course, $A \subset B$.

Under our working hypothesis $(A_{\sigma f})$, we have that $\eta_1 = \theta_1 > 0$, $\eta_2 = \infty$ and

$$\text{dom } H = \text{dom } S = \{(0, 0)\} \cup \{(u, v) \in \mathbb{R}^2 \mid v \geq \theta_1 u > 0\}.$$

Moreover, $S(0, 0) = \{(0)_{n \geq 1}\}$, while for $u > 0$ and $\Sigma := \{k \in \mathbb{N}^* \mid \sigma_k = \theta_1\} = \overline{1, q}$ with $q \in \mathbb{N}^*$,

$$S(u, \theta_1 u) = \{(u_n)_{n \geq 1} \subset \mathbb{R}_+ \mid u_n = 0 \ \forall n \geq q + 1, \ u = u_1 + \dots + u_q\}.$$

It follows that $h^*(0, v) = -\alpha v < \infty = H(0, v)$, whence $(0, v) \notin B$, for $v > 0$, and (4.3) holds for $(u, v) = (0, 0)$, and so $(0, 0) \in A$. Using [7, Prop. 15 (v)], we obtain that (4.3) holds for all $(u, v) \in E$ (with E defined in (4.2)), with attainment for $u_n := p_n e^{x + \sigma_n y}$ ($n \geq 1$), where $y := \varphi^{-1}(v/u)$ and $x := \ln[u/f(y)]$; hence $E \subset A$.

It remains to analyze the case of those $(u, v) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ with $v/u = \theta_1$ or $v/u \geq \theta_2$.

Let first $v = \theta_1 u > 0$. By Lemma 2.2, $H(u, v)$ is attained for $\bar{u}_n = p_n u / \sum_{k=1}^q p_k = p_n u / \rho_q$ if $n \in \overline{1, q}$, $\bar{u}_n := 0$ if $n \geq q + 1$, and so

$$H(u, v) = \sum_{n=1}^q \frac{p_n u}{\rho_q} \left(\ln \frac{u}{\rho_q} - 1 \right) = u (\ln u - 1 - \ln \rho_q).$$

Since

$$h^*(u, v) = u \ln u - u + u \cdot (\ln f)^*(\theta_1) = u \ln u - u - u \ln \rho_q = H(u, v),$$

we have that (4.3) holds with attainment for $(\bar{u}_n)_{n \geq 1}$ mentioned above; in particular $(u, v) \in A$.

Assume now that $\theta_2 < \infty$, and so $-\alpha \in \text{dom } f$, $\gamma < \infty$, and $\theta_2 = \gamma/f(-\alpha)$. Take now $v \geq \theta_2 u > 0$, and assume that $(u, v) \in A$. Since $v \geq \theta_2 u > 0$, by (3.4), we have that $(u, v) \in \partial h(x, -\alpha)$ with $x := \ln \frac{u}{f(-\alpha)}$. Since $(u, v) \in A$, there exists $(u_n)_{n \geq 1} \subset \mathbb{R}_+$ such that $(u, v) = \sum_{n \geq 1} (u_n, \sigma_n u_n)$ and $h^*(u, v) = \sum_{n \geq 1} u_n (\ln \frac{u_n}{p_n} - 1) = \sum_{n \geq 1} h_n^*(u_n, \sigma_n u_n)$. Using [7, Prop. 15 (iv)], we obtain that $(u_n, \sigma_n u_n) \in \partial h_n(x, -\alpha) = \{(p_n e^{x - \sigma_n \alpha}, p_n \sigma_n e^{x - \sigma_n \alpha})\}$, that is $u_n = p_n e^{x - \sigma_n \alpha} = p_n u e^{-\sigma_n \alpha} / f(-\alpha)$ ($n \geq 1$). It follows that $v = \sum_{n \geq 1} \sigma_n u_n = \frac{u}{f(-\alpha)} \sum_{n \geq 1} p_n \sigma_n e^{-\sigma_n \alpha} = \frac{u}{f(-\alpha)} \gamma = \theta_2 u$. Conversely, if $v = \theta_2 u > 0$ and setting again $x := \ln \frac{u}{f(-\alpha)}$, the calculus above shows that $(p_n e^{x - \sigma_n \alpha})_{n \geq 1} \in A$.

Take now $v > \theta_2 u > 0$; we claim that $(u, v) \in B$, and so $(u, v) \in B \setminus A$.

Let us set $x := \ln \frac{u}{f(-\alpha)}$; then

$$h^*(u, v) = u \ln u - u + u \cdot (\ln f)^*(v/u) = u \ln u - u - \alpha v - u \ln f(-\alpha) = (x - 1)u - \alpha v. \quad (4.4)$$

Take $\bar{n} > \max \Sigma (= q)$. Using Lemma 2.5 we have that for $n \geq \bar{n}$, φ_n [defined in (2.13)] is an increasing bijection from \mathbb{R} to $] \eta_n^1, \eta_n^2[=] \theta_1, \eta_n^2[\subset] \theta_1, \infty[$. Moreover,

$$\lim_{n \rightarrow \infty} \varphi_n(0) = \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n p_k \sigma_k}{\sum_{k=1}^n p_k} = \lim_{n \rightarrow \infty} \frac{p_{n+1} \sigma_{n+1}}{p_{n+1}} = \lim_{n \rightarrow \infty} \sigma_{n+1} = \infty > \frac{v}{u},$$

and

$$\lim_{n \rightarrow \infty} \varphi_n(-\alpha) = \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n p_k \sigma_k e^{-\sigma_k \alpha}}{\sum_{k=1}^n p_k e^{-\sigma_k \alpha}} = \frac{\sum_{k \geq 1} p_k \sigma_k e^{-\sigma_k \alpha}}{\sum_{k \geq 1} p_k e^{-\sigma_k \alpha}} = \frac{\gamma}{f(-\alpha)} = \theta_2 < \frac{v}{u}.$$

Increasing \bar{n} if necessary, we may (and do) assume that $\varphi_n(-\alpha) < v/u < \varphi_n(0)$ for $n \geq \bar{n}$. Hence, for every $n \geq \bar{n}$ there exists a unique $\lambda_n \in]0, \alpha[$ with $\varphi_n(-\lambda_n) = v/u$. Set $v_n := \ln(u / \sum_{k=1}^n p_k e^{-\sigma_k \lambda_n})$. Define $u_k := p_k e^{v_n - \sigma_k \lambda_n}$ for $k \in \overline{1, n}$ and $u_k := 0$ for $k > n$. Then

$$\sum_{k \geq 1} u_k = \sum_{k=1}^n p_k e^{v_n - \sigma_k \lambda_n} = u, \quad \sum_{k \geq 1} \sigma_k u_k = \sum_{k=1}^n p_k \sigma_k e^{v_n - \sigma_k \lambda_n} = v.$$

Because $\lambda_n < \alpha$ for $n \geq \bar{n}$, we get $\sum_{k=1}^n p_k e^{-\sigma_k \lambda_n} \geq \sum_{k=1}^n p_k e^{-\sigma_k \alpha}$, whence

$$\limsup_{n \rightarrow \infty} v_n \leq \limsup_{n \rightarrow \infty} \ln \frac{u}{\sum_{k=1}^n p_k e^{-\sigma_k \alpha}} = \ln \frac{u}{\sum_{k \geq 1} p_k e^{-\sigma_k \alpha}} = \ln \frac{u}{f(-\alpha)} = x.$$

Moreover,

$$\begin{aligned} h^*(u, v) &\leq H(u, v) \leq \sum_{k \geq 1} u_k \left(\ln \frac{u_k}{p_k} - 1 \right) = \sum_{k=1}^n u_k (v_n - \sigma_k \lambda_n - 1) \\ &= (v_n - 1) \sum_{k=1}^n u_k - \lambda_n \sum_{k=1}^n \sigma_k u_k = (v_n - 1)u - \lambda_n v \end{aligned} \quad (4.5)$$

Assume that $\liminf_{n \rightarrow \infty} \lambda_n < \alpha$. Then, for some $\mu \in]0, \alpha[$ and some subsequence $(\lambda_{n_m})_{m \geq 1}$ we have that $\lambda_{n_m} \leq \mu$ for every $m \geq 1$. Then $\sum_{k=1}^{n_m} p_k e^{-\sigma_k \lambda_{n_m}} \geq \sum_{k=1}^{n_m} p_k e^{-\sigma_k \mu}$, whence

$$\liminf_{m \rightarrow \infty} \sum_{k=1}^{n_m} p_k e^{-\sigma_k \lambda_{n_m}} \geq \lim_{m \rightarrow \infty} \sum_{k=1}^{n_m} p_k e^{-\sigma_k \mu} = \sum_{k \geq 1} p_k e^{-\sigma_k \mu} = \infty$$

because $-\mu \notin \text{dom } f$. Since $v_{n_m} = \ln u - \ln \sum_{k=1}^{n_m} p_k e^{-\sigma_k \lambda_{n_m}}$, we get $\lim_{m \rightarrow \infty} v_{n_m} = -\infty$. Replacing n by n_m in (4.5), then passing to the limit for $m \rightarrow \infty$, we get the contradiction $-\infty < h^*(u, v) \leq u \cdot (-\infty) = -\infty$. Hence $\lim \lambda_n = \alpha$ and $\limsup_{n \rightarrow \infty} v_n \leq x$. Passing to \limsup in (4.5) for $n \rightarrow \infty$ we get $H(u, v) \leq (x - 1)u - \alpha v$, and so $h^*(u, v) = H(u, v)$.

Summing up the above discussion we get the next result.

Theorem 4.1. *Let $(p_n)_{n \geq 1} \subset [1, \infty[$, $(\sigma_n)_{n \geq 1} \subset \mathbb{R}_+^*$ with $\sigma_n \rightarrow \infty$, and $h_n : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $h_n(x, y) := p_n e^{x + \sigma_n y}$ for $n \geq 1$ and $x, y \in \mathbb{R}$; set $h = \sum_{n \geq 1} h_n$. Assume that $\text{dom } h \neq \emptyset$. Clearly, h, h_n ($n \geq 1$) are convex and*

$$h(x, y) = e^x \sum_{n \geq 1} p_n e^{\sigma_n y} = e^x f(y) \quad \forall (x, y) \in \mathbb{R}^2,$$

where f is defined in (2.19). Since $\text{dom } h = \mathbb{R} \times \text{dom } f \neq \emptyset$, using Proposition 2.8, we have that $I :=]-\infty, -\alpha[\subset \text{dom } f \subset \text{cl } I$ for some $\alpha \in \mathbb{R}_+$. It follows that $\text{int}(\text{dom } h) = \mathbb{R} \times I \subset \cap_{n \geq 1} \text{dom } h_n = \mathbb{R}^2$.

(i) *We have that h is differentiable on $\text{int}(\text{dom } h)$ and*

$$\partial h(\text{int}(\text{dom } h)) = \nabla h(\mathbb{R} \times I) = \{(u, v) \in \mathbb{R}^2 \mid u \in \mathbb{R}_+^*, \theta_1 u < v < \theta_2 u\},$$

where $\theta_1 := \min\{\sigma_n \mid n \geq 1\}$ and $\theta_2 := \lim_{y \uparrow -\alpha} f'(y)/f(y) \in]\theta_1, \infty[$; $\theta_2 < \infty$ iff $-\alpha \in \text{dom } f$ and $\gamma := f'_-(-\alpha) = \sum_{n \geq 1} p_n \sigma_n e^{-\sigma_n \alpha} < \infty$. Moreover, if $-\alpha \in \text{dom } f$ and $\gamma < \infty$ then

$$e^x(f(-\alpha), \gamma) = \sum_{n \geq 1} \nabla h_n(x, -\alpha) \in \partial h(x, -\alpha) = \{e^x f(-\alpha)\} \times [e^x \gamma, \infty[.$$

(ii) *The function $\varphi : I \rightarrow]\theta_1, \theta_2[$, $\varphi(y) := f'(y)/f(y)$, is bijective (and increasing), $\ln f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ is convex (even strictly convex and increasing on its domain), and*

$$(\ln f)^*(w) = \begin{cases} \infty & \text{if } w < \theta_1, \\ -\ln \sum_{n \in \Sigma} p_n & \text{if } w = \theta_1, \\ w\varphi^{-1}(w) - \ln[f(\varphi^{-1}(w))] & \text{if } \theta_1 < w < \theta_2, \\ -\alpha w - \ln[f(-\alpha)] & \text{if } \theta_2 \leq w, \end{cases}$$

where $\Sigma := \{n \in \mathbb{N}^* \mid \sigma_n = \theta_1\}$. Moreover, $\text{dom } h^* = \{(u, v) \in \mathbb{R}^2 \mid v \geq \theta_1 u \geq 0\}$ and

$$h^*(u, v) = \begin{cases} u \ln u - u + u \cdot (\ln f)^*(v/u) & \text{if } v \geq \theta_1 u > 0, \\ -\alpha v & \text{if } u = 0 \leq v. \end{cases}$$

(iii) Take $(u, v) \in \text{dom } h^*$. Then

$$h^*(u, v) = \min \left\{ \sum_{n \geq 1} u_n \left(\ln \frac{u_n}{p_n} - 1 \right) \mid (u_n)_{n \geq 1} \in S(u, v) \right\} = H(u, v)$$

iff $(u, v) \in A := \{(0, 0)\} \cup \{(u, v) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \mid \theta_1 u \leq v \leq \theta_2 u\}$, where $S(u, v) := S_{(\sigma_n)}(u, v)$ is defined in (2.1) and $H := H_{(\sigma_n), E_{MB}}^{(p_n)}$ is defined in (2.5). More precisely, for $(u, v) \in A$ the minimum is attained at a unique sequence $(\bar{u}_n)_{n \geq 1} \in S(u, v)$, as follows: (a) $(\bar{u}_n)_{n \geq 1} = (0)_{n \geq 1}$ if $(u, v) = (0, 0)$; (b) $\bar{u}_n := p_n u / \sum_{k \in \Sigma} p_k$ if $n \in \Sigma$, $\bar{u}_n := 0$ if $n \in \mathbb{N}^* \setminus \Sigma$ provided $u \in \mathbb{R}_+^*$ and $v = \theta_1 u$; (c) $(\bar{u}_n)_{n \geq 1} = (p_n e^{x + \sigma_n y})_{n \geq 1}$ if $u \in \mathbb{R}_+^*$ and $\theta_1 u < v < \theta_2 u$, where $y := \varphi^{-1}(v/u)$ and $x := \ln[u/f(y)]$; (d) $(\bar{u}_n)_{n \geq 1} = (p_n e^{x - \sigma_n \alpha})_{n \geq 1}$ if $\theta_2 < \infty$, $u \in \mathbb{R}_+^*$ and $v = \theta_2 u$, where $x := \ln[u/f(-\alpha)]$.

Moreover, $S(0, v) = \emptyset$ if $v \in \mathbb{R}_+^*$, and $h^*(u, v) = H(u, v)$ whenever $0 < \theta_2 u < v$ (for $\theta_2 < \infty$).

Corollary 4.2. Consider the sequences $(p_n)_{n \geq 1} \subset [1, \infty[$, $(\sigma_n)_{n \geq 1} \subset \mathbb{R}$ and let $H := H_{(\sigma_n)}^{(p_n)} := H_{(\sigma_n), E_{MB}}^{(p_n)}$. Then $H^*(x, y) = \sum_{n \geq 1} p_n e^{x + \sigma_n y} = e^x \sum_{n \geq 1} p_n e^{\sigma_n y}$ for every $(x, y) \in \mathbb{R}^2$.

Proof. Assume first that the series $\sum_{n \geq 1} p_n e^{\sigma_n y}$ is divergent for each $y \in \mathbb{R}$. Then $h(x, y) := h_{(\sigma_n), E_{MB}}^{(p_n)}(x, y) := \sum_{n \geq 1} p_n e^{x + \sigma_n y} = \infty$ for all $(x, y) \in \mathbb{R}^2$. Using Proposition 2.7 we have that H takes the value $-\infty$, and so $H^*(x, y) = \infty = h(x, y)$ for every $(x, y) \in \mathbb{R}^2$.

Assume now that the series $\sum_{n \geq 1} p_n e^{\sigma_n y}$ is convergent for some $y \in \mathbb{R}$. By Proposition 2.8 (i) we have that $\sigma_n \rightarrow \infty$ or $\sigma_n \rightarrow -\infty$.

In the first case, if $\eta_1 := \theta_1 := \min_{n \geq 1} \sigma_n > 0$, using Theorem 4.1 we have that $\text{dom } H \subset \text{dom } h^* = (\{0\} \times \mathbb{R}_+^*) \cup \text{dom } H$, $h^* \leq H$, and $h^*(u, v) = H(u, v)$ for all $(u, v) \in \text{dom } H \supset \text{int}(\text{dom } h^*) = \text{int}(\text{dom } H)$. Because for a convex function $f : E \rightarrow \overline{\mathbb{R}}$ with $D := \text{int}(\text{dom } f) \neq \emptyset$ one has $f^* = (f + \iota_D)^*$, it follows that $h = (h^*)^* = H^*$. If $\eta_1 \leq 0$, take $\sigma'_n := \sigma_n + a$ ($n \geq 1$) with $a > -\eta_1$. Then $(H_{(\sigma'_n)}^{(p_n)})^* = h_{(\sigma'_n)}^{(p_n)}$. But $h_{(\sigma_n)}^{(p_n)}(x, y) = h_{(\sigma'_n)}^{(p_n)}(x - ay, y)$ for $(x, y) \in \mathbb{R}^2$ and $H_{(\sigma_n)}^{(p_n)}(u, v) = H_{(\sigma'_n)}^{(p_n)}(u, au + v)$ for $(u, v) \in \mathbb{R}^2$, whence

$$\left(H_{(\sigma_n)}^{(p_n)} \right)^* (x, y) = \left(H_{(\sigma'_n)}^{(p_n)} \right)^* (x - ay, y) = h_{(\sigma'_n)}^{(p_n)}(x - ay, y) = h_{(\sigma_n)}^{(p_n)}(x, y) \quad \forall (x, y) \in \mathbb{R}^2.$$

If $\sigma_n \rightarrow -\infty$ take $\sigma'_n := -\sigma_n$ ($n \geq 1$). A similar argument as above shows that $(H_{(\sigma_n)}^{(p_n)})^* = h_{(\sigma_n)}^{(p_n)}$. The proof is complete. \square

Remark 4.3. In [7, Section 4] the Entropy Minimization Problem (EMP) for Maxwell–Boltzmann entropy is solved completely when $I = \mathbb{N}^* \times \mathbb{N}^* \times \mathbb{N}^*$ and $\sigma_{k,l,m} := k^2 + l^2 + m^2$, from where one gets rapidly the solution of (EMP) for the example in (1.4). Observe that in this case $\theta_2 = \infty$.

5. Conclusions

The Entropy Minimization Problem (EMP) is considered in Statistical Mechanics and Statistical Physics for W one of the functions E_{MB} , E_{BE} , E_{FD} . In general one obtains the optimal solutions using the Lagrange multipliers method (LMM), method used by us in the proofs of Lemmas 2.2, 2.3 and 2.4. When the number of variables is infinite this method can not be generally used because the function to be minimized is not differentiable and the linear restrictions are not provided by continuous (linear) operators (in this sense see the recent survey paper [2]). Even more, although the solutions found using LMM are indeed solutions of

the EMP, LMM does not provide always the solutions even in the case of a finite numbers of variables as seen in Lemma 2.3 (iii). Observe that in the works on Statistical Mechanics nothing is said about the value of $(EMP)_{u,v}$ when the problem has not optimal solutions, and, of course, if this value could be $-\infty$ or not; maybe this is not interesting in Physics.

In the present paper, for $W = E_{MB}$, that is the Maxwell–Boltzmann entropy, a complete study of the EMP is realized (when $p_n \geq 1$ for $n \geq 1$). More precisely,

- the set of those $(u, v) \in \mathbb{R}^2$ for which $(EMP)_{u,v}$ has feasible solutions is described (see Proposition 2.6);
- it is shown that H (the value function of the EMP) takes the value $-\infty$ if and only if the series $\sum_{n \geq 1} p_n e^{x + \sigma_n y}$ is divergent for all $(x, y) \in \mathbb{R}^2$ (see Proposition 2.7);
- when $\sum_{n \geq 1} p_n e^{x + \sigma_n y}$ is convergent for some $(x, y) \in \mathbb{R}^2$, it is confirmed that the solution found using LMM in a formal way is indeed a solution of problem $(EMP)_{u,v}$; however, it is shown that either there are situations in which $(EMP)_{u,v}$ has optimal solutions not found using LMM, or there are situations in which $(EMP)_{u,v}$ has finite values but not optimal solutions (see Theorem 4.1).

Acknowledgment

We thank Prof. M. Durea for his remarks on a previous version of the manuscript. We also thank the three anonymous referees for their useful comments. This work was supported partially by CNCS-UEFISCDI (Romania) [grant number PN-II-ID-PCE-2011-3-0084].

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