



## Note

Blow-up at infinity of solutions to a semilinear heat equation with logarithmic nonlinearity<sup>☆</sup>

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## ABSTRACT

In this short note, the author establishes a blow-up result for a semilinear heat equation with logarithmic nonlinearity, by using the logarithmic Sobolev inequality. This improves a recent blow-up result obtained in Chen et al. (2015) [1].

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## 1. Main result and its proof

In this short note, we are concerned with the following initial boundary value problem of a semilinear heat equation with logarithmic nonlinearity

$$\begin{cases} u_t - \Delta u = u \ln |u|, & (x, t) \in \Omega \times (0, T), \\ u = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \quad (1.1)$$

where  $\Omega \subset \mathbb{R}^n (n \geq 1)$  is a bounded domain with smooth boundary  $\partial\Omega$ , and  $u_0 \in H_0^1(\Omega)$ .

Partial differential equations with logarithmic nonlinearities have attracted much attention in recent years, due to their wide applications in physics and other applied sciences. We refer the interested reader to [1–10] for the study of different kinds of partial differential equations with logarithmic nonlinearity.

In particular, in [1] the authors investigated the global well-posedness and blow-up properties of solutions to problem (1.1), by using the logarithmic Sobolev inequality and a family of potential wells. Among some other interesting results, they showed that problem (1.1) admits a global weak solution  $u(x, t)$  when

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$J(u_0) \leq M$  and  $I(u_0) \geq 0$ , while the weak solution  $u(x, t)$  blows up at infinity when  $J(u_0) \leq M$  and  $I(u_0) < 0$ . Here  $M > 0$  is a constant and  $J(u)$  and  $I(u)$  are defined, respectively, by

$$J(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{2} \int_{\Omega} u^2 \ln |u| dx + \frac{1}{4} \int_{\Omega} u^2 dx, \quad (1.2)$$

$$I(u) = \int_{\Omega} |\nabla u|^2 dx - \frac{1}{2} \int_{\Omega} u^2 \ln |u| dx. \quad (1.3)$$

By observing the monotonicity of  $I(u(x, t))$  with respect to  $t$  when  $I(u_0) < 0$ , we will show that the condition  $J(u_0) \leq M$  is unnecessary for the weak solution of problem (1.1) to blow up at infinity. We first present the definition of weak solutions to problem (1.1).

**Definition 1.1** (Weak solution) [1]. Let  $T > 0$ . A function  $u = u(x, t) \in L^\infty(0, T; H_0^1(\Omega))$  with  $u_t \in L^2(0, T; L^2(\Omega))$  is called a weak solution to problem (1.1) in  $\Omega \times [0, T)$ , if  $u(x, 0) = u_0(x) \in H_0^1(\Omega)$  and  $u(x, t)$  satisfies (1.1) in the sense of distribution, i.e.

$$(u_t, \phi) + (\nabla u, \nabla \phi) = (u \ln |u|, \phi), \quad \text{for any } \phi \in H_0^1(\Omega), \quad t \in (0, T), \quad (1.4)$$

where  $(\cdot, \cdot)$  means the inner product in  $L^2(\Omega)$ .

Throughout this paper, we will denote by  $\|v\|_2$  the  $L^2(\Omega)$  norm of  $v \in L^2(\Omega)$  and by  $T \in (0, +\infty]$  the maximum existence time of  $u(x, t)$ . Our main result is the following

**Theorem 1.1.** Assume that  $u_0 \in H_0^1(\Omega)$  and  $I(u_0) < 0$ . Then the weak solution  $u = u(x, t)$  of problem (1.1) blows up at infinity, i.e.

$$\lim_{t \rightarrow +\infty} \|u(\cdot, t)\|_2 = +\infty. \quad (1.5)$$

To prove the main result, we need the following two lemmas. The first one is a well known energy identity (see (3.1) in [1]) and the second one is the logarithmic Sobolev inequality.

**Lemma 1.1.** Assume that  $u(x, t)$  is a weak solution to problem (1.1). Then  $J(u(x, t))$  is non-increasing with respect to  $t$  and satisfies

$$\int_0^t \|u_\tau(\cdot, \tau)\|_2^2 d\tau + J(u(x, t)) = J(u_0), \quad t \in (0, T). \quad (1.6)$$

**Lemma 1.2.** Let  $u$  be any function in  $H^1(\mathbb{R}^n)$  and  $a > 0$  be any number. Then

$$2 \int_{\mathbb{R}^n} |u(x)|^2 \ln \left( \frac{|u(x)|}{\|u\|_{L^2(\mathbb{R}^n)}} \right) dx + n(1 + \ln a) \|u\|_{L^2(\mathbb{R}^n)}^2 \leq \frac{a^2}{\pi} \int_{\mathbb{R}^n} |\nabla u(x)|^2 dx. \quad (1.7)$$

For any  $u \in H_0^1(\Omega)$ , define  $u(x) = 0$  for  $x \in \mathbb{R}^n \setminus \Omega$ . Then  $u \in H^1(\mathbb{R}^n)$ , and therefore the following inequality holds

$$2 \int_{\Omega} |u(x)|^2 \ln \left( \frac{|u(x)|}{\|u\|_2} \right) dx + n(1 + \ln a) \|u\|_2^2 \leq \frac{a^2}{\pi} \|\nabla u\|_2^2. \quad (1.8)$$

**Proof of Theorem 1.1.** Let  $u(x, t)$  be any weak solution to problem (1.1). We first show that  $I(u(x, t))$  is decreasing with respect to  $t$  when  $I(u_0) < 0$ . Taking  $\phi = u$  in (1.4) and recalling (1.3) we have

$$\frac{d}{dt} \|u(\cdot, t)\|_2^2 = -2I(u(x, t)), \quad t \in (0, T). \quad (1.9)$$

Combining (1.2), (1.3), (1.9) with Lemma 1.6 one obtains

$$\begin{aligned} \frac{d}{dt} I(u(x, t)) &= \frac{d}{dt} \left[ 2J(u(x, t)) - \frac{1}{2} \|u(\cdot, t)\|_2^2 \right] \\ &= -2\|u_t\|_2^2 + I(u(x, t)) \leq I(u(x, t)), \end{aligned} \quad (1.10)$$

which, by recalling Gronwall's inequality, yields

$$I(u(x, t)) \leq I(u_0)e^t \leq I(u_0), \quad t \in (0, T). \quad (1.11)$$

Therefore,  $I(u(x, t)) < 0$  for all  $t \geq 0$  and  $I(u(x, t))$  is decreasing with respect to  $t$ .

Set  $G(t) = \int_0^t \|u(\cdot, \tau)\|_2^2 d\tau$ . Then

$$G'(t) = \|u(\cdot, t)\|_2^2, \quad G''(t) = 2(u_t, u) = -2I(u(x, t)). \quad (1.12)$$

By taking  $a = \sqrt{2\pi}$  in (1.8) and recalling (1.12) we have

$$\begin{aligned} G'(t) \ln G'(t) - G''(t) &= 2\|u(\cdot, t)\|_2^2 \ln \|u(\cdot, t)\|_2 + 2I(u(x, t)) \\ &= 2\|u(\cdot, t)\|_2^2 \ln \|u(\cdot, t)\|_2 + 2\|\nabla u(\cdot, t)\|_2^2 - 2 \int_{\Omega} u^2 \ln |u| dx \\ &\geq \frac{n(2 + \ln(2\pi))}{2} \|u(\cdot, t)\|_2^2 \geq 0, \end{aligned} \quad (1.13)$$

which, in turn, implies that

$$(\ln G'(t))' \leq \ln G'(t).$$

After integration, one has

$$\ln G'(t) \leq e^t \ln G'(0) = e^t \ln \|u_0\|_2^2,$$

and

$$\|u(\cdot, t)\|_2 \leq \|u_0\|_2^{e^t}, \quad t \geq 0.$$

This means that  $u(x, t)$  does not blow up in finite time.

On the other hand, from (1.2), (1.3) and the energy identity (1.6) it follows that

$$G''(t) = -4J(u(x, t)) + G'(t) = -4J(u_0) + 4 \int_0^t \|u(\cdot, \tau)\|_2^2 d\tau + G'(t). \quad (1.14)$$

Noticing that

$$(G'(t))^2 = 4 \left( \int_0^t \int_{\Omega} u_{\tau} u dx d\tau \right)^2 + 2 \|u_0\|_2^2 G'(t) - \|u_0\|_2^4,$$

we have

$$\begin{aligned} G''(t)G(t) - G'(t)^2 &= 4 \left[ \int_0^t \|u_{\tau}(\cdot, \tau)\|_2^2 d\tau \int_0^t \|u(\cdot, \tau)\|_2^2 d\tau - \left( \int_0^t \int_{\Omega} u_{\tau} u dx d\tau \right)^2 \right] \\ &\quad - 4J(u_0)G(t) + G'(t)G(t) - 2\|u_0\|_2^2 G'(t) + \|u_0\|_2^4. \end{aligned}$$

With the help of Cauchy–Schwarz inequality

$$\int_0^t \int_{\Omega} u_{\tau} u dx d\tau \leq \int_0^t \|u_{\tau}(\cdot, \tau)\|_2^2 d\tau \int_0^t \|u(\cdot, \tau)\|_2^2 d\tau,$$

we further obtain

$$\begin{aligned} G''(t)G(t) - G'(t)^2 &\geq G'(t)G(t) - 4J(u_0)G(t) - 2\|u_0\|_2^2 G'(t) \\ &= G(t) \left( \frac{G'(t)}{2} - 4J(u_0) \right) + G'(t) \left( \frac{G(t)}{2} - 2\|u_0\|_2^2 \right). \end{aligned} \quad (1.15)$$

From  $G''(t) = -2I(u(x, t))$  and  $I(u(x, t)) \leq I(u_0) < 0$ , it follows that

$$\begin{aligned} G'(t) &\geq G'(0) - 2I(u_0)t \geq -2I(u_0)t, \quad t \geq 0, \\ G(t) &\geq G(0) - I(u_0)t^2, \quad t \geq 0. \end{aligned}$$

Therefore, there exists a  $t_0 > 0$  such that for  $t \geq t_0$  it holds that

$$G''(t)G(t) - G'(t)^2 > 0.$$

By direct computation,

$$(\ln G(t))' = \frac{G'(t)}{G(t)}, \quad (\ln G(t))'' = \left( \frac{G'(t)}{G(t)} \right)' = \frac{G''(t)G(t) - G'(t)^2}{G^2(t)}. \quad (1.16)$$

After integration, one obtains, for any  $t > t_0$ , that

$$(\ln G(t))' = (\ln G(t_0))' + \int_{t_0}^t \frac{G''(\tau)G(\tau) - G'(\tau)^2}{G^2(\tau)} d\tau \geq (\ln G(t_0))',$$

and

$$\ln G(t) - \ln G(t_0) = \int_{t_0}^t (\ln G(\tau))' d\tau \geq \frac{G'(t_0)}{G(t_0)}(t - t_0),$$

or equivalently

$$G(t) \geq G(t_0) e^{\frac{G'(t_0)}{G(t_0)}(t-t_0)}. \quad (1.17)$$

In view of (1.12), (1.16)–(1.17), we have

$$\begin{aligned}\|u(\cdot, t)\|_2^2 &= G'(t) \geq \frac{G'(t_0)}{G(t_0)} G(t) \geq G'(t_0) e^{\frac{G'(t_0)}{G(t_0)}(t-t_0)} \\ &= \|u(\cdot, t_0)\|_2^2 e^{\frac{G'(t_0)}{G(t_0)}(t-t_0)} \geq \|u_0\|_2^2 e^{\frac{G'(t_0)}{G(t_0)}(t-t_0)}, \quad t \geq t_0.\end{aligned}\tag{1.18}$$

This implies that  $\lim_{t \rightarrow +\infty} \|u(\cdot, t)\|_2^2 = +\infty$ , i.e.  $u(x, t)$  blows up at  $+\infty$ . The proof is complete.

**Remark 1.1.** In [1], the authors proved that if  $J(u_0) \leq M$  and  $I(u_0) < 0$ , then  $u(x, t)$  blows up at infinity. To achieve the inequality  $G''(t)G(t) - G'(t)^2 > 0$  for sufficiently large  $t$ , they introduced some sets  $\mathcal{N}_\delta$ ,  $W_\delta$ ,  $V_\delta$  and the depth of the potential wells  $d(\delta)$ , whose properties must be verified through a difficult process. By making full use of the monotonicity of  $I(u(x, t))$ , we give a much simpler proof under weaker assumptions.

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