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Weak solution of a stochastic 3D Cahn-Hilliard-Navier-Stokes model driven by jump noise

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ABSTRACT

We investigate a stochastic 2D and 3D Cahn-Hilliard-Navier-Stokes system with a multiplicative noise of Lévy type. The model consists of the Navier-Stokes equations for the velocity, coupled with a Cahn-Hilliard system for the order (phase) parameter. We prove that the system the existence of weak martingale solution for both 2D and 3D cases. The proof of the existence is based on a classical Galerkin approximation as well as some compactness methods. In the 2D case, we prove the pathwise uniqueness of the weak solution.

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1. Introduction

Stochastic partial differential equations (SPDE) are used to model physical systems subjected to influence of internal, external or environmental noises or to describe systems that are too complex to be described deterministically, e.g. a flow of a chemical substance in a river subjected by wind and rain, an airflow around an airplane wing perturbed by the random state of the atmosphere and weather, a laser beam subjected to turbulent movement of the atmosphere, spread of an epidemic in some regions and the spatial spread of infectious diseases. SPDEs are also used in the physical sciences (e.g. in plasmas turbulence, physics of growth phenomena such as molecular beam epitaxy and fluid flow in porous media with applications to the production of semiconductors and to the oil industry) and biology (e.g. bacteria growth and DNA structure). Models related to the so called passive scalar equations have potential applications to the understanding of waste (e.g. nuclear) convection under the earth's surface, [11,9,35,36].

The presence of noise can lead to new and important phenomena. For example, the 2-dimensional Navier—Stokes equations with sufficiently degenerate noise have a unique invariant measure and hence exhibit ergodic behavior in the sense that the time average of a solution is equal to the average over all possible initial data. Despite continuous efforts in the last thirty years, such a property has so far not been found for the

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deterministic counterpart of these equations. This property could lead to profound understanding of the nature of turbulence. The aforementioned Navier-Stokes Equations (NSE) are now a widely accepted model of fluid motion, see for instance the well known monograph [45,44]. The theory of NSE is reasonable well understood. For instance, in the case of 2-dimensional domains, it is known since the pioneering works of Lions and Prodi in the 1960s (see for instance [34]) that the solutions exist for all times and are unique. In the 3-dimensional case it is known that the weak solutions exist for all times, see celebrated work of Leray [32], and that the strong solutions are unique. However, despite many efforts in the recent years the questions whether the weak solutions are unique or strong solutions exist for all times, remain unresolved, see for instance [46]. To our best knowledge, the first work on the stochastic NSE (SNSE) written from the mathematical point of view is a paper [4]. Later the motivation for the large deviations paper of Faris and Jona-Lasinio [22] was clearly the stochastic fluid dynamics as they wrote, roughly speaking, “the motion of a viscous incompressible fluid is described by the Navier-Stokes equations. However, these equations are only approximate. In particular, they take into account only the macroscopic nature of the fluid motion.

It is well accepted that the incompressible Navier-Stokes equation governs the motions of single-phase fluids such as air or water. On the other hand, we are faced with the difficult problem of understanding the motion of binary fluid mixtures, that is fluids composed by either two phases of the same chemical species or phases of different composition. Diffuse interface models are well-known tools to describe the dynamics of complex (e.g., binary) fluids, [25,24]. For instance, this approach is used in [5] to describe cavitation phenomena in a flowing liquid. The model consists of the NSE equation coupled with the phase-field system, [15,25,24,26]. In the isothermal compressible case, the existence of a global weak solution is proved in [23]. In the incompressible isothermal case, neglecting chemical reactions and other forces, the model reduces to an evolution system which governs the fluid velocity v and the order parameter ϕ . This system can be written as a NSE equation coupled with a convective Allen-Cahn equation, [24]. The associated initial and boundary value problem was studied in [24] in which the authors proved that the system generated a strongly continuous semigroup on a suitable phase space which possesses a global attractor. When the two fluids have the same constant density, the temperature differences are negligible and the diffuse interface between the two phases has a small but non-zero thickness, a well-known model is the so-called “Model H” (cf. [30,27]). This is a system of equations where an incompressible Navier-Stokes equation for the (mean) velocity v is coupled with a convective Cahn-Hilliard equation for the order parameter ϕ , which represents the relative concentration of one of the fluids.

There are few notable works available on the stochastic CH-NSE driven by Gaussian noise. In [19], the authors considered the stochastic 3D globally modified Cahn-Hilliard-Navier-Stokes equations with multiplicative Gaussian noise. They proved the existence and uniqueness of strong solution (in the sense of partial differential equations and stochastic analysis). Moreover, they studied the asymptotic behavior of the unique solution and obtained the existence of a probabilistic weak solution for the stochastic 3D Cahn-Hilliard-Navier-Stokes equations. In [18], they also considered the asymptotic stability of the unique strong solution for the 3D globally modified Cahn-Hilliard-Navier-Stokes equations. The second author of the paper has proved the existence and uniqueness of the probabilistic strong solution for the stochastic 2D CH-NSE with multiplicative noise, [43].

In recent years, introducing a jump-type noises as Lévy-type or Poisson-type perturbations has become extremely popular for modeling natural phenomena, because these noises are very nice choice to reproduce the performance of some natural phenomena in real world models, such as some large moves and unpredictable events. There is a large amount of literature on the existence and uniqueness solutions for stochastic partial differential equations driven by jump-type noises. We refer the reader to [41,42,33,20,21,9,47,48,38,40,39]. However, the existing results in the literature do not cover the situation considered in this paper.

The aim of this article is to study a class of stochastic coupled 2D or 3D CH-NSE driven by jump noise of Lévy type. To the best of our knowledge, this is the first work dealing with the stochastic version of the CH-NSE driven by jump noise. The model includes an abstract and general form of random external forces

depending eventually on the velocity v of the fluid and the order parameter ϕ . We prove that the system the existence of weak martingale solution for both 2D and 3D cases. The proof of the existence is based on a classical Galerkin approximation as well as some compactness methods. In the 2D case, we prove the pathwise uniqueness of the weak solution.

We prove the existence of martingale solutions for both the 2D and 3D case. The proof of the existence of solution is based on a Galerkin scheme similar to that in [38,40,13] in the case of the Navier-Stokes and the Nematic liquid crystal flow. In the 2D case, we proved the uniqueness of solution.

The article is divided as follows. In the next section we present the stochastic Cahn-Hilliard-Navier-Stokes model and its mathematical setting. We also give most of the notations and necessary preliminary used throughout this work. In Section 3, we recall from [38,40,13] some compactness and tightness results. The Galerkin approximation and some a priori estimates are given in Section 4. In Section 5, we prove the tightness of laws of the approximate sequence. The existence of a martingale solution is given in Section 6, while in Section 7, we prove the pathwise uniqueness in the 2D case.

2. The stochastic CH-NSE and its mathematical setting

2.1. Governing equations

We assume that the domain \mathcal{M} of the fluid is a bounded domain in \mathbb{R}^d , $d = 2, 3$. Then, we consider the system

$$\left\{ \begin{array}{l} dv + [-\nu_1 \Delta v + (v \cdot \nabla)v + \nabla p + \mathcal{K} \operatorname{div}(\nabla \phi \otimes \nabla \phi)] dt = g_1(t, v, \phi) dt + \int_Z \sigma(t, v, \phi, z) \tilde{\eta}(dt, dz) \\ \quad \text{in } (0, T) \times \mathcal{M}, \\ \operatorname{div} v = 0 \quad \text{in } (0, T) \times \mathcal{M}, \\ \frac{\partial \phi}{\partial t} + v \cdot \nabla \phi - \nu_3 \Delta \mu = 0 \quad \text{in } (0, T) \times \mathcal{M}, \\ \mu = -\nu_2 \Delta \phi + \alpha f(\phi) \quad \text{in } (0, T) \times \mathcal{M}. \end{array} \right. \quad (2.1)$$

In (2.1), the unknown functions are the velocity v of the fluid, its pressure p and the order (phase) parameter ϕ . The terms $g_1(t, v, \phi)$ and $\int_Z \sigma(t, v, \phi, z) \tilde{\eta}(dt, dz)$ respectively represent the deterministic and the random external forces that eventually depend on (v, ϕ) , and $\tilde{\eta}$ is a compensated time homogeneous Poisson random measure on a measurable space (Z, \mathcal{Z}) . Precise assumption on the data are given below. The model (2.1) describes the motion of a binary fluid exited by random forces.

The quantity μ is the variational derivative of the following free energy functional

$$\mathcal{F}_p(\phi) = \int_{\mathcal{M}} \left(\frac{\nu_2}{2} |\nabla \phi|^2 + \alpha F(\phi) \right) ds, \quad (2.2)$$

where, e.g., $F(r) = \int_0^r f(\zeta) d\zeta$. Here, the constants $\nu_1 > 0$, $\nu_3 > 0$ and $\mathcal{K} > 0$ correspond to the kinematic viscosity of the fluid, the mobility constant and the capillarity (stress) coefficient respectively. Here ν_2 , $\alpha > 0$ are two physical parameters describing the interaction between the two phases. In particular, ν_2 is related with the thickness of the interface separating the two fluids. The stress tensor $\nabla \phi \otimes \nabla \phi$ is considered the

main contribution modeling capillary forces due to surface tension at the interface between the two phases of the fluid.

A typical example of potential F is that of logarithmic type.

However, this potential is often replaced by a polynomial approximation of the type $F(r) = \gamma_1 r^4 - \gamma_2 r^2$, γ_1, γ_2 being positive constants.

Regarding the boundary conditions for these models, we assume that the boundary conditions for ϕ are the natural no-flux condition

$$\partial_\eta \phi = \partial_\eta \Delta \phi = 0, \quad \text{on } \partial\mathcal{M} \times (0, \infty), \quad (2.3)$$

where $\partial\mathcal{M}$ is the boundary of \mathcal{M} and η is the outward normal to $\partial\mathcal{M}$. These conditions ensure the mass conservation. Note that (2.3) implies that

$$\partial_\eta \mu = 0, \quad \text{on } \partial\mathcal{M} \times (0, \infty). \quad (2.4)$$

From (2.4), we deduce the conservation of the following quantity

$$\langle \phi(t) \rangle = \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} \phi(x, t) dx, \quad (2.5)$$

where $|\mathcal{M}|$ stands for the Lebesgue measure of \mathcal{M} . More precisely, we have

$$\langle \phi(t) \rangle = \langle \phi(0) \rangle, \quad \forall t \geq 0. \quad (2.6)$$

Concerning the boundary condition for v , we assume the Dirichlet (no-slip) boundary condition

$$v = 0, \quad \text{on } \partial\mathcal{M} \times (0, \infty). \quad (2.7)$$

Therefore we assume that there is no relative motion at the fluid-solid interface.

The initial condition is given by

$$(v, \phi)(0) = (v_0, \phi_0), \quad \text{in } \mathcal{M}. \quad (2.8)$$

2.2. Mathematical setting

We first recall from [24] a weak formulation of (2.1), (2.3), (2.7)-(2.8). Hereafter, we assume that the domain \mathcal{M} is bounded with a smooth boundary $\partial\mathcal{M}$ (e.g., of class \mathcal{C}^3). We also assume that $f \in \mathcal{C}^2(\mathbb{R})$ satisfies

$$\begin{cases} \lim_{|r| \rightarrow +\infty} f'(r) > 0, \\ |f'(r)| \leq c_f(1 + |r|^k), \quad \forall r \in \mathbb{R}, \end{cases} \quad (2.9)$$

where c_f is some positive constant and $k \in [1, +\infty)$ in the 2D case, $k \in [1, 2]$ in the 3D case is fixed. It follows from (2.9) that

$$|f(r)| \leq c_f(1 + |r|^{k+1}), \quad \forall r \in \mathbb{R}. \quad (2.10)$$

Note that the derivative of the typical double-well potential f satisfies conditions similar to (2.9). Let us now recall from [24] the functional set up of the model (2.1), (2.3), (2.7), (2.8).

If X is a real Hilbert space with inner product $(\cdot, \cdot)_X$, we will denote the induced norm by $|\cdot|_X$, while X^* will indicate its dual. We set

$$\mathcal{V}_1 = \{u \in (C_c^\infty(\mathcal{M}))^d : \operatorname{div} u = 0, \text{ in } \mathcal{M}\}.$$

We denote by H_1 , V_1 and V_{1s} the closure of \mathcal{V}_1 in $(L^2(\mathcal{M}))^d$, $(H_0^1(\mathcal{M}))^d$ and $(H^s(\mathcal{M}))^d$ respectively, where $s > 0$. The scalar product in H_1 is denoted by $(\cdot, \cdot)_{L^2}$ and the associated norm by $|\cdot|_{L^2}$. Moreover, the space V_1 is endowed with the scalar product

$$((u, v)) = \sum_{i=1}^d (\partial_{x_i} u, \partial_{x_i} v)_{L^2}, \quad \|u\| = ((u, u))^{1/2}, \quad d = 2, 3.$$

We now define the operator A_0 by

$$A_0 u = -\mathcal{P}_1 \Delta u, \quad \forall u \in D(A_0) = (H^2(\mathcal{M}))^d \cap V_1,$$

where \mathcal{P}_1 is the Leray-Helmoltz projector in $(L^2(\mathcal{M}))^d$ onto H_1 . Then, A_0 is a self-adjoint positive unbounded operator in H_1 which is associated with the scalar product defined above. Furthermore, A_0^{-1} is a compact linear operator on H_1 and $|A_0 \cdot|_{L^2}$ is a norm on $D(A_0)$ that is equivalent to the H^2 -norm.

We introduce the linear nonnegative unbounded operator on $L^2(\mathcal{M})$

$$A_1 \phi = -\Delta \phi, \quad \forall \phi \in D(A_1) = \{\phi \in H^2(\mathcal{M}), \partial_\eta \phi = 0, \text{ on } \partial \mathcal{M}\}, \quad (2.11)$$

and we endow $D(A_1)$ with the norm $|A_1 \cdot|_{L^2} + |\langle \cdot \rangle|_{L^2}$, which is equivalent to the H^2 -norm. We also define the linear positive unbounded operator on the Hilbert space $L_0^2(\mathcal{M})$ of the L^2 -functions with null mean

$$B_n \phi = -\Delta \phi, \quad \forall \phi \in D(B_n) = D(A_1) \cap L_0^2(\mathcal{M}). \quad (2.12)$$

Note that B_n^{-1} is a compact linear operator on $L_0^2(\mathcal{M})$. More generally, we can define B_n^s , for any $s \in \mathfrak{R}$, noting that $|B_n^{s/2} \cdot|_{L^2}$, $s > 0$, is an equivalent norm to the canonical H^s -norm on $D(B_n^{s/2}) \subset H^s(\mathcal{M}) \cap L_0^2(\mathcal{M})$. Also note that $A_1 = B_n$ on $D(B_n)$. If ϕ is such that $\phi - \langle \phi \rangle \in D(B_n^{s/2})$, we have that $|B_n^{s/2}(\phi - \langle \phi \rangle)|_{L^2} + |\langle \phi \rangle|_{L^2}$ is equivalent to the H^s -norm. Moreover, we set $H^{-s}(\mathcal{M}) = (H^s(\mathcal{M}))^*$, whenever $s < 0$.

Finally we set

$$H_2 = D(B_n^{1/2}), \quad V_2 = D(B_n). \quad (2.13)$$

The norms in H_2 and V_2 are denoted respectively by $\|\cdot\|$ and $\|\cdot\|_{H^2}$, where $\|\psi\| = |B_n^{1/2} \psi|_{L^2}$ and $\|\psi\|_{H^2} = |B_n \psi|_{L^2}$.

We introduce the bilinear operators B_0 , B_1 (and their associated trilinear forms b_0, b_1) as well as the coupling mapping R_0 , which are defined from $D(A_0) \times D(A_0)$ into H_1 , $D(A_0) \times D(A_1)$ into $L^2(\mathcal{M})$, and $D(A_1) \times D(A_1)$ into V_1^* , respectively. More precisely, we set

$$\begin{aligned} \langle B_0(u, v), w \rangle &= \int_{\mathcal{M}} [(u \cdot \nabla) v] \cdot w dx = b_0(u, v, w), \quad \forall u, v, w \in D(A_0), \\ \langle B_1(u, \phi), \rho \rangle &= \int_{\mathcal{M}} [(u \cdot \nabla) \phi] \rho dx = b_1(u, \phi, \rho), \quad \forall u \in D(A_0), \phi, \rho \in D(A_1). \end{aligned}$$

Now we introduce the trilinear form r_1 defined by (see also [13,12]):

$$r_0(\phi, \psi, v) = - \sum_{i,j}^2 \int_{\mathcal{M}} \frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx, \quad \forall \phi, \psi \in W^{1,4}, \quad v \in V_1. \quad (2.14)$$

Proposition 2.1. *There exists a constant $c > 0$ such that*

$$|r_0(\phi, \psi, v)| \leq c \|\phi\|^{1/4} |A_1 \phi|_{L^2}^{3/4} \|\psi\|^{1/4} |A_1 \psi|_{L^2}^{3/4} \|v\|, \quad \forall \phi, \psi \in D(A_1), \quad v \in V_1. \quad (2.15)$$

Proof. From (2.14), we easily derive that for any $\phi, \psi \in D(A_1)$, $v \in V_1$, we have

$$\begin{aligned} |r_0(\phi, \psi, v)| &\leq c |\nabla \phi|_{L^4} |\nabla \psi|_{L^4} |\nabla v|_{L^2} \\ &\leq c \|\phi\|^{1/4} |A_1 \phi|_{L^2}^{3/4} \|\psi\|^{1/4} |A_1 \psi|_{L^2}^{3/4} \|v\|, \end{aligned} \quad (2.16)$$

and (2.15) is proved. \square

Proposition 2.2. *There exists a bilinear operator R_0 defined on $D(A_1) \times D(A_1)$ with values in V_1^* such that*

$$\langle R_0(\phi, \psi), v \rangle = r_0(\phi, \psi, v), \quad \forall \phi, \psi \in D(A_1), \quad v \in V_1. \quad (2.17)$$

Proof. The first part of the proposition follows from the fact that for any $v \in V_1$, the mapping $r_0(\cdot, \cdot, v)$ defined on $D(A_1) \times D(A_1)$ with values in \mathfrak{R} is continuous. \square

Proposition 2.3. *For any $v \in V_1$ and $\phi \in D(A_1)$, we have*

$$\langle B_1(v, \phi), A_1 \phi \rangle = - \langle R_0(\phi, \phi), v \rangle. \quad (2.18)$$

Proof. To simplify the notations, we assume the summation over repeated indexes. Taking into account the fact that $\operatorname{div} v = 0$ as well as the boundary conditions, we derive that

$$\begin{aligned} \langle B_1(v, \phi), -A_1 \phi \rangle &= \int_{\mathcal{M}} v_i \frac{\partial \phi}{\partial x_i} \frac{\partial^2 \phi}{\partial x_j \partial x_j} dx \\ &= - \int_{\mathcal{M}} \frac{\partial v_i}{\partial x_j} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} dx - \int_{\mathcal{M}} v_i \frac{\partial^2 \phi}{\partial x_i \partial x_j} \frac{\partial \phi}{\partial x_j} dx \\ &= - \int_{\mathcal{M}} \frac{\partial v_i}{\partial x_j} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} dx - \frac{1}{2} \int_{\mathcal{M}} v_i \frac{\partial |\nabla \phi|^2}{\partial x_i} dx \\ &= - \int_{\mathcal{M}} \frac{\partial v_i}{\partial x_j} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} dx \\ &= r_0(\phi, \phi, v) = \langle R_0(\phi, \phi), v \rangle. \end{aligned} \quad (2.19)$$

Note that in (2.19), we use the fact that $\langle B_1(v, \phi), f(\phi) \rangle = 0$. \square

Remark 2.1. We recall from [24] that

$$\operatorname{div}(\nabla \phi \otimes \nabla \phi) = \nabla \left(\frac{1}{2} |\nabla \phi|^2 + \alpha F(\phi) \right) - \mu \nabla \phi. \quad (2.20)$$

It is clear that $\operatorname{div}(\nabla\phi \otimes \nabla\psi) \in L^2(\mathcal{M})$ for $\phi \in D(A_1), \psi \in D(A_1^{3/2})$. Therefore

$$R_0(\phi, \psi) = \mathcal{P}_1 [\operatorname{div}(\nabla\phi \otimes \nabla\psi)], \quad \forall \phi \in D(A_1), \psi \in D(A_1^{3/2}). \quad (2.21)$$

It follows from (2.20)-(2.21) that

$$\begin{aligned} R_0(\phi, \phi) &= \mathcal{P}_1 [\operatorname{div}(\nabla\phi \otimes \nabla\phi)] \\ &= \mathcal{P}_1 \left[\nabla \left(\frac{1}{2} |\nabla\phi|^2 + \alpha F(\phi) \right) \right] - \mathcal{P}_1 [\mu \nabla\phi] \\ &= -\mathcal{P}_1 [\mu \nabla\phi]. \end{aligned} \quad (2.22)$$

From (2.14), we can easily check that for $s > \frac{d}{2} + 1$, R_0 can be uniquely extended to a bounded bilinear operator (still) denoted $R_0 : H_2 \times H_2 \rightarrow V_{1s}^*$ and we have

$$|R_0(\phi, \psi)|_{V_{1s}^*} \leq c |\phi|_{L^2} |\psi|_{L^2}. \quad (2.23)$$

Note that

$$R_0(\phi, \phi) = \mathcal{P}_1 \mu \nabla\phi.$$

We recall from [24] (see also [25,26]) that B_0 , B_1 and R_0 satisfy the following estimates

$$\langle B_0(u, v), v \rangle = 0, \quad \forall u, v \in V_1, \quad (2.24)$$

$$\langle B_1(v, \phi), \phi \rangle = 0, \quad \forall (v, \phi) \in \mathcal{Y}, \quad (2.25)$$

$$|B_1(v, \phi)|_{D(A_1)^*} \leq c |v|_{L^2} \|\phi\|, \quad \forall (v, \phi) \in \mathcal{Y}.$$

$$\langle R_0(\phi, \phi), u \rangle = -\langle B_1(u, \phi), A_1\phi \rangle, \quad \forall (u, \phi) \in V_1 \times V_2. \quad (2.26)$$

We recall from [38,40] that for $s > \frac{d}{2} + 1$, B_0 can be uniquely extended to a bounded bilinear operator (still) denoted $B_0 : H_1 \times H_1 \rightarrow V_{1s}^*$ and we have

$$|B_0(u, v)|_{V_{1s}^*} \leq c |u|_{L^2} |v|_{L^2}. \quad (2.27)$$

We recall that (due to the mass conservation) we have

$$\langle \phi(t) \rangle = \langle \phi(0) \rangle = M_0, \quad \forall t > 0. \quad (2.28)$$

Thus, up to a shift of the order parameter field, we can always assume that the mean of ϕ is zero at the initial time and, therefore it will remain zero for all positive times. Hereafter, we assume that

$$\langle \phi(t) \rangle = \langle \phi(0) \rangle = 0, \quad \forall t > 0. \quad (2.29)$$

We set

$$\mathcal{Y} = H_1 \times D(B_n^{1/2}). \quad (2.30)$$

The space \mathcal{Y} is a complete metric space with respect to the norm

$$|(v, \phi)|_{\mathcal{Y}}^2 = \mathcal{K}^{-1}|v|_{L^2}^2 + \nu_2|\nabla\phi|_{L^2}^2. \quad (2.31)$$

We define the Hilbert space \mathcal{U} by

$$\mathcal{U} = V_1 \times D(B_n^{3/2}), \quad (2.32)$$

endowed with the scalar product whose associated norm is

$$\|(v, \phi)\|_{\mathcal{U}}^2 = \|v\|^2 + |B_n^{3/2}\phi|_{L^2}^2. \quad (2.33)$$

We will also denote by c a generic positive constant that depends on the domain \mathcal{M} . To simplify the notations, we set (without loss of generality) $\nu_1 = \nu_2 = \nu_3 = \alpha = \mathcal{K} = 1$.

Using the notations above, we rewrite (2.1), (2.3), (2.7)-(2.5) as

$$\begin{cases} dv + [A_0v + B_0(v, v) + R_0(\phi, \phi)] dt = g_1(t, v, \phi)dt + \int_Z \sigma(t, v, \phi, z)\tilde{\eta}(dt, dz), \\ \frac{d\phi}{dt} + A_1\mu + B_1(v, \phi) = 0, \quad \mu = A_1\phi + f(\phi), \\ (v, \phi)(0) = (v_0, \phi_0). \end{cases} \quad (2.34)$$

Remark 2.2. For the sake of convenience, as in [24] we will replace μ in (2.34)₂ by $\bar{\mu} = \mu - \langle \mu \rangle$, that is $\bar{\mu} = A_1\phi + f(\phi) - \langle f(\phi) \rangle$, a.e., in $\mathcal{M} \times (0, T)$. Obviously we have $\langle \bar{\mu}(t) \rangle = 0$, $\forall t > 0$.

Notations. We first recall from [29,11] some notations and stochastic preliminaries.

Hereafter, by \mathbb{N} we denote the set of nonnegative integers, i.e. $\mathbb{N} = \{0, 1, 2, \dots\}$ and by $\bar{\mathbb{N}}$ we denote the set $\mathbb{N} \cup \{+\infty\}$. Whenever we speak about \mathbb{N} (or $\bar{\mathbb{N}}$)-valued measurable functions we implicitly assume that the set is equipped with the trivial Σ -field $2^{\mathbb{N}}$ (or $2^{\bar{\mathbb{N}}}$). By \mathbb{R}_+ we will denote the interval $[0, \infty)$ and by \mathbb{R}_* the set $\mathbb{R} \setminus \{0\}$. If X is a topological space, then by $\mathcal{B}(X)$ we will denote the Borel Σ -field on X . By λ_d we will denote the Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, by λ the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

If (S, \mathfrak{S}) is a measurable space then by $M(S)$ we denote the set of all real valued measures on (S, \mathfrak{S}) , and by $M(S)$ the Σ -field on $M(S)$ generated by the functions $i_B : M(S) \ni \varsigma \mapsto \varsigma(B) \in \mathbb{R}, B \in S$. By $M_+(S)$ we denote the set of all nonnegative measures on S , and by $M(S)$ the Σ -field on $M_+(S)$ generated by the functions $i_B : M_+(S) \ni \varsigma \mapsto \varsigma(B) \in \mathbb{R}_+, B \in S$. Finally, by $M_I(S)$ we denote the family of all $\bar{\mathbb{N}}$ valued measures on (S, \mathfrak{S}) , and by $M_I(S)$ the Σ -field on $M_I(S)$ generated by functions $i_B : M(S) \ni \varsigma \mapsto \varsigma(B) \in \bar{\mathbb{N}}, B \in S$. If (S, \mathfrak{S}) is a measurable space then we will denote by $\mathfrak{S} \otimes \mathcal{B}(\mathbb{R}_+)$ the product Σ -field on $S \times \mathbb{R}_+$ and by $\nu \otimes \lambda$ the product measure of ν and the Lebesgue measure λ .

Preliminaries. As mentioned earlier we will study a stochastic model for a CH-NSE excited by random forces. We first describe the forces acting on the fluids. Let (Z, \mathcal{Z}) be a separable metric space and let ν be a Σ -finite positive measure on it. Suppose that $\mathfrak{P} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space, where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is a filtration satisfying the usual conditions, and $\eta : \Omega \times \mathcal{B}(\mathbb{R}_+) \times Z \rightarrow \bar{\mathbb{N}}$ is a time homogeneous Poisson random measure, with intensity measure ν , defined over the filtered probability space \mathfrak{P} . A time homogeneous Poisson random measure defined over \mathfrak{P} is given in the following definition.

Definition 2.1. Let Z be a metric space and \mathcal{Z} its Borel Σ -algebra, ν a positive Σ -finite measure on (Z, \mathcal{Z}) . A Poisson random measure, with intensity measure ν defined on (Z, \mathcal{Z}) over \mathfrak{P} is a measurable map $\eta : (\Omega, \mathcal{F}) \rightarrow (M_I(Z \times \mathbb{R}_+), M_I(Z \times \mathbb{R}_+))$ satisfying the following conditions:

- (i) for all $O \in \mathcal{B}(Z \otimes \mathbb{R}_+)$, $\eta(O) : \Omega \rightarrow \bar{\mathbb{N}}$ is a Poisson random measure with parameter $\mathbb{E}[\eta(O)]$;
- (ii) η is independently scattered, i.e., if the sets $O_j \in \mathcal{B}(Z \otimes \mathbb{R}_+)$, $j = 1, \dots, n$, are disjoint then the random variables $\eta(O_j)$, $j = 1, \dots, n$, are independent;
- (iii) for all $U \in \mathcal{Z}$ and $I \in \mathcal{B}(\mathbb{R}_+)$

$$\mathbb{E}[\eta(U \times I)] = \lambda(I)\nu(U);$$

- (iv) for all $U \in \mathcal{Z}$ the $\bar{\mathbb{N}}$ -valued process $(N(U, t))_{t \geq 0}$ defined by $N(U, t) := \eta(U \times (0, t])$, $t \geq 0$, is \mathbb{F} -adapted and its increments are independent of the past, i.e., if $t > s \geq 0$, then the random variable $N(U, t) - N(U, s) = \eta(U \times (s, t])$ is independent of \mathcal{F}_s .

We will denote by $\tilde{\eta}$ the compensated Poisson random measure defined by

$$\tilde{\eta} := \eta - \gamma,$$

where the compensator $\gamma : \mathcal{B}(Z \times \mathbb{R}_+) \rightarrow \mathbb{R}_+$ is defined by

$$\gamma(A \times I) = \lambda(I)\nu(A), I \in \mathcal{B}(\mathbb{R}_+), A \in \mathcal{Z}.$$

As noted in [29], while items (i) and (ii) are the classical definition, see for e.g. Definition 6.1 in [41], of a Poisson Random measure η , the remaining items implicitly indicate that η is associated to a certain Lévy process \tilde{L} ; see, for instance [[41], Proposition 4.16].

Let $\mathcal{M}^2(\mathbb{R}_+, L^2(Z, \nu, H_1))$ be the class of all progressively measurable processes $\xi : \mathbb{R}_+ \times Z \times \Omega \rightarrow H_1$ satisfying the condition

$$\mathbb{E} \int_0^T \int_Z |\xi(r, z)|_{L^2}^2 \nu(dz) dr < \infty, \quad \forall T > 0. \quad (2.35)$$

If $T > 0$, the class of all progressively measurable processes $\xi : [0, T] \times Z \times \Omega \rightarrow H_1$ satisfying the condition (2.35) just for this one T , will be denoted by $\mathcal{M}^2(0, T, L^2(Z, \nu, H_1))$.

Also, let $\mathcal{M}_{step}^2(\mathbb{R}_+, L^2(Z, \nu, H_1))$ be the space of all processes $\xi \in \mathcal{M}^2(\mathbb{R}_+, L^2(Z, \nu, H_1))$ such that

$$\xi(r) = \sum_{j=1}^n 1_{(t_{j-1}, t_j]}(r) \xi_j, \quad 0 \leq r, \quad (2.36)$$

where $\{0 = t_0 < t_1 < \dots < t_n < \infty\}$ is a partition of $[0, \infty)$, and for all j , ξ_j is an $\mathcal{F}_{t_{j-1}}$ -measurable random variable.

For any $\xi \in \mathcal{M}_{step}^2(\mathbb{R}_+, L^2(Z, \nu, H_1))$, we set

$$\tilde{I}(\xi) = \sum_{j=1}^n \int_Z \xi_j(z) \tilde{\eta}(dz, (t_{j-1}, t_j]).$$

This is basically the definition of stochastic integral of a random step process ξ with respect to the compound random Poisson measure $\tilde{\eta}$. The extension of this integral on $\mathcal{M}^2(\mathbb{R}_+, L^2(Z, \nu, H_1))$ is possible thanks to the following result which is taken from [41], Theorem C.1.

Theorem 2.4. *There exists a unique bounded linear operator*

$$I : \mathcal{M}^2(\mathbb{R}_+, L^2(Z, \nu; H_1)) \rightarrow L^2(\Omega, \mathcal{F}; H_1)$$

such that for $\xi \in \mathcal{M}_{step}^2(\mathbb{R}_+, L^2(Z, \nu, H_1))$ we have $\tilde{I}(\xi) = I(\xi)$.

In particular, there exists a constant $C = C(H_1)$ such that for any $\xi \in \mathcal{M}^2(\mathbb{R}_+, L^2(Z, \nu, H_1))$,

$$\mathbb{E} \left| \int_0^t \int_Z \xi(r, z) \tilde{\eta}(dz, dr) \right|_{L^2}^2 \leq C \mathbb{E} \int_0^t \int_Z |\xi(r, z)|_{L^2}^2 \nu(dz) dr, \quad t > 0.$$

Moreover, for each $\xi \in \mathcal{M}^2(\mathbb{R}_+, L^2(Z, \nu, H_1))$, the process $I(1_{[0,t]}\xi)$, $t \geq 0$, is an H_1 -valued càdlàg martingale. The process $1_{[0,t]}\xi$ is defined by $[1_{[0,t]}\xi](r, z, \omega) := 1_{[0,t]}(r)\xi(r, z, \omega)$, $t \geq 0, r \in \mathbb{R}_+, z \in Z$ and $\omega \in \Omega$.

As usual we will write

$$\int_0^t \int_Z \xi(r, z) \tilde{\eta}(dz, dr) := I(\xi)(t), \quad t \geq 0.$$

If $T > 0$, we denote by $\mathbb{D}(0, T; H_1)$ the space of all càdlàg paths from $[0, T]$ into H_1 .

Now we introduce the main set of hypotheses used in this article. As in [29,11], we suppose that we are given a function σ satisfying the following set of constraints:

Assumption 1. We assume that

- (1) $\tilde{\eta}$ is a compensated time homogeneous Poisson random measure on a measurable space $(Z, \mathcal{B}(Z))$ over $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with a σ -finite intensity measure ν .
- (2) $\sigma : [0, T] \times \mathcal{Y} \times Z \rightarrow H_1$ is a measurable function and there exist nonnegative constants l_0, l_1, l_2 such that, for any $t \in [0, T]$ and all $(v_1, \phi_1), (v_2, \phi_2) \in \mathcal{Y}$, we have

$$\begin{aligned} |\sigma(t, v_1, \phi_1)|_{L^2(Z, \nu; H_1)}^p &\leq l_0 + l_1 |(v_1, \phi_1)|_{\mathcal{Y}}^p; \quad \text{for any } p \geq 2, \\ |\sigma(t, v_1, \phi_1) - \sigma(t, v_2, \phi_2)|_{L^2(Z, \nu; H_1)}^2 &\leq l_2 |(v_1, \phi_1) - (v_2, \phi_2)|_{\mathcal{Y}}^2. \end{aligned} \quad (2.37)$$

We assume that the external forcing g_1 is a measurable Lipschitz and sublinear mappings from $\Omega \times (0, T) \times H_1$ into V_1^* . More precisely, for all $(v_1, \phi_1), (v_2, \phi_2) \in V_1$, $g_1(\cdot, v_1, \phi_1)$ is \mathcal{F} -progressively measurable, and $d\mathbb{P} \times dt$ -a.e. in $\Omega \times (0, T)$

$$\begin{aligned} \|g_1(t, v_1, \phi_1) - g_1(t, v_2, \phi_2)\|_{V_1^*} &\leq L_1 |(v_1, \phi_1) - (v_2, \phi_2)|_{\mathcal{Y}}, \\ g_1(t, 0, 0) &\in M_{\mathcal{F}}^2(0, T; V_1^*). \end{aligned} \quad (2.38)$$

Finally, we assume that

$$(v_0, \phi_0) \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathcal{Y}). \quad (2.39)$$

Hereafter, for any $(w, \psi) \in \mathcal{Y}$, we set

$$\mathcal{E}(w, \psi) = |w|_{L^2}^2 + \|\psi\|^2 + 2\langle F(\psi), 1 \rangle + c_1, \quad (2.40)$$

where $c_1 > 0$ is a constant large enough and independent on (w, ψ) such that $\mathcal{E}(w, \psi)$ is nonnegative (note that F is bounded from below).

We can check that (see [24]) there exists a monotone non-decreasing function Q_0 (independent on time and the initial condition) such that

$$|(w, \psi)|_{\mathcal{Y}}^2 \leq \mathcal{E}(w, \psi) \leq Q_0(|(w, \psi)|_{\mathcal{Y}}^2), \quad \forall (w, \psi) \in \mathcal{Y}. \quad (2.41)$$

Definition 2.2. A martingale solution of the problem (2.34) is a system $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}, \bar{v}, \bar{\phi}, \bar{\eta})$, where

- (1) $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$ is a filtered probability space with a filtration $\bar{\mathbb{F}} = \{\bar{\mathcal{F}}_t\}_{t \geq 0}$.
- (2) $\bar{\eta}$ is a time homogeneous Poisson random measure on $(Z, \mathcal{B}(Z))$ over $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$ with the intensity measure ν .
- (3) $(\bar{v}, \bar{\phi}) : [0, T] \times \bar{\Omega} \rightarrow \mathcal{U}$ is a progressively measurable process with $\bar{\mathbb{P}}$ -a.e. paths

$$(\bar{v}, \bar{\phi})(\cdot, \omega) \in \mathbb{D}([0, T]; \mathcal{Y}_w) \cap L^2(0, T; \mathcal{U})$$

such that for all $t \in [0, T]$ and all $u = (w, \psi) \in \mathcal{U}$ the following identity holds $\bar{\mathbb{P}}$ -a.e.

$$\begin{aligned} (\bar{v}(t), w) &= (v_0, w) - \int_0^t \langle A_0 \bar{v} + B_0(\bar{v}, \bar{v}) - R_0(\bar{\phi}, \bar{\phi}) - g_1(s, \bar{v}, \bar{\phi}), w \rangle ds \\ &+ \int_0^t \int_Z \langle \sigma(s, \bar{v}, \bar{\phi}, z), w \rangle \bar{\eta}(ds, dz), \\ (\bar{\phi}(t), \psi) &= (\phi_0, \psi) - \int_0^t \langle A_1 \bar{\mu} + B_1(\bar{v}, \bar{\phi}), \psi \rangle ds = 0, \quad \bar{\mu} = A_1 \bar{\phi} + f(\bar{\phi}). \end{aligned} \quad (2.42)$$

In the deterministic case, the weak formulation of (2.34) was proposed and studied in [8,6,7,25,24] (see also [2,1,16]), where the existence and uniqueness results for weak and strong solutions were proved in the deterministic case.

Let us recall from [17] the following result.

Lemma 2.5. Let X, Y, I and φ be non-negative processes and Z_1 be a non-negative integrable random variable. Assume that I is non-decreasing and that there exist non-negative constants $C, \alpha_1, \beta, \gamma_1, \delta_1$ and T satisfying first

$$\int_0^T \varphi(s) ds \leq C, \quad a.s., \quad 2\beta_1 e^C \leq 1, \quad 2\delta_1 e^C \leq \alpha_1,$$

and secondly for all $t \in [0, T]$ there exists a constant $C_1 > 0$ such that

$$\begin{aligned} X(t) + \alpha_1 Y(t) &\leq Z_1 + \int_0^t \varphi(r) X(r) dr + I(t), \quad a.s., \\ \mathbb{E}I(t) &\leq \beta \mathbb{E}X(t) + \gamma_1 \int_0^t \mathbb{E}X(s) ds + \delta_1 \mathbb{E}Y(t) + C_1. \end{aligned}$$

If $X \in L^\infty([0, T] \times \Omega)$, then we have

$$\mathbb{E}[X(t) + \alpha_1 Y(t)] \leq 2 \exp(C + 2t\gamma_1 e^C)(\mathbb{E}Z + C_1), \quad t \in [0, T].$$

3. Compactness and tightness criterion

In this section, we recall from [38,40,39,13] some compactness and tightness results.

3.1. Compactness results

Let (\mathbb{M}, ρ) be a complete separable metric space. Let $\mathbb{D}([0, T]; \mathbb{M})$ be the space of all \mathbb{M} -valued càdlàg functions defined on $[0, T]$, i.e. the functions which are right continuous and have left limits at every $t \in [0, T]$. This space is endowed with the Skorokhod topology.

A sequence $(u_n) \subset \mathbb{D}([0, T]; \mathbb{M})$ converges to $u \in \mathbb{D}([0, T]; \mathbb{M})$ if and only if there exists a sequence (μ_n) of homeomorphisms of $[0, T]$ such that μ_n tends to the identity uniformly on $[0, T]$ and $u_n \circ \mu_n$ tends to u uniformly on $[0, T]$. We recall that the topology is metrizable by the following metric v_T

$$v_T(u_1, u_2) = \inf_{\mu \in \sigma_T} \left[\sup_{t \in [0, T]} \rho(u_1(t), u_2 \circ \mu(t)) + \sup_{t \in [0, T]} |t - \mu(t)| + \sup_{s \neq t} \left| \log \frac{\mu(t) - \mu(s)}{t - s} \right| \right],$$

where σ_T is the set of increasing homeomorphisms of $[0, T]$. Moreover, the space $(\mathbb{D}([0, T]; \mathbb{M}), v_T)$ is a complete metric space, see [10].

Definition 3.1. Let $u \in \mathbb{D}([0, T]; \mathbb{M})$ and let $\delta > 0$ be fixed. A modulus of u is defined by

$$\mathcal{W}_{[0, T], \mathbb{M}}(u, \delta) = \inf_{\Pi_\delta} \max_{t_i \in \tilde{\omega}} \sup_{t_i \leq s < t < t_{i+1} \leq T} \rho(u(t), u(s)), \quad (3.1)$$

where Π_δ is the set of all increasing sequences $\tilde{\omega} = \{0 = t_0 < t_1 < \dots < t_n = T\}$ with the following property

$$t_{i+1} - t_i \geq \delta, \quad i = 0, 1, \dots, n-1.$$

We recall from [39,40,13] the following criterion for the relative compactness of a subset of the space $\mathbb{D}([0, T]; \mathbb{M})$. This result is analogous to the Arzelà-Ascoli Theorem for the space of continuous functions.

Theorem 3.1. A set $X \subset \mathbb{D}([0, T]; \mathbb{M})$ is precompact if and only if it satisfies the following two conditions:

- (1) there exists a dense subset $J \subset [0, T]$ such that for every $t \in J$ the set $\{u(t), u \in X\}$ has compact closure in \mathbb{M} .
- (2) $\limsup_{\delta \rightarrow 0} \sup_{u \in X} \mathcal{W}_{[0, T], \mathbb{M}}(u, \delta) = 0$.

Proof. See [37]. \square

Now we define the following functional spaces.

$\mathbb{D}([0, T]; V_{1s}^*) =$ the space of càdlàg functions $v : [0, T] \rightarrow V_{1s}^*$
with the topology \mathfrak{T}_1 induced by the Skorokhod metric δ_T ,

$\mathbb{C}([0, T]; D(A_1)^*) =$ the space of continuous functions $\phi : [0, T] \rightarrow D(A_1)^*$
with the topology \mathfrak{T}_1^* ,

$L_w^2(0, T; V_{1s}) =$ the space $L^2(0, T; V_{1s})$ with the weak topology \mathfrak{T}_2 ,

$L_w^2(0, T; D(A_1)) =$ the space $L^2(0, T; D(A_1))$ with the weak topology \mathfrak{T}_2' ,

$L^2(0, T; H_1) =$ the space of measurable functions $v : [0, T] \rightarrow H_1$ with the topology \mathfrak{T}_3 ,

$L^2(0, T; H_2) =$ the space of measurable functions $\phi : [0, T] \rightarrow H_2$ with the topology \mathfrak{T}_3' .

Let $H_{1,w}$ denote the Hilbert space H_1 endowed with the weak topology.

Let us consider the space

$\mathbb{D}([0, T]; H_{1,w}) =$ the space of weakly càdlàg functions $v : [0, T] \rightarrow H_1$
with the weakest topology \mathfrak{T}_4 such that for all $h \in H_1$ the mappings
 $\mathbb{D}([0, T]; H_{1,w}) \ni v \mapsto (v(\cdot), h)_{H_1} \in \mathbb{D}([0, T]; \mathfrak{R})$ are continuous.

In particular, $v_n \rightarrow v$ in $\mathbb{D}([0, T]; H_{1,w})$ if and only if for all $h \in H_1$:

$$(v_n(\cdot), h)_{H_1} \rightarrow (v(\cdot), h)_{H_1} \text{ in the space } \mathbb{D}([0, T]; \mathfrak{R}).$$

Similarly we define $\mathbb{C}([0, T]; H_{1,w})$ with the topology \mathfrak{T}_4' .

We recall from [13, 39] the following compactness results, see [39] for the details of the proof.

Theorem 3.2 (Compactness Criterion for v). *Let us consider the space*

$$\mathcal{Z}_{T,1} = L_w^2(0, T; V_1) \cap L^2(0, T; H_1) \cap \mathbb{D}([0, T]; V_{1s}^*) \cap \mathbb{D}([0, T]; H_{1,w})$$

and \mathcal{T}^1 be the supremum of the corresponding topologies. Then a set $K_1 \subset \mathcal{Z}_{T,1}$ is \mathcal{T}^1 -relatively compact if the following three conditions are satisfied

- (1) $\sup_{v \in K_1} \sup_{s \in [0, T]} |v(s)|_{L^2} < \infty$,
- (2) $\sup_{v \in K_1} \int_0^T \|v(s)\|^2 ds < \infty$,
- (3) $\lim_{\delta \rightarrow 0} \sup_{v \in K_1} \mathcal{W}_{[0, T], V_{1s}^*}(v; \delta) = 0$.

Theorem 3.3 (Compactness Criterion for ϕ). *Let us consider the space*

$$\mathcal{Z}_{T,2} = L_w^2(0, T; D(A_1^{3/2})) \cap L^2(0, T; H_2) \cap \mathbb{C}([0, T]; D(A_1)^*) \cap \mathbb{C}([0, T]; H_{2,w})$$

and \mathcal{T}^2 be the supremum of the corresponding topologies. Then a set $K_2 \subset \mathcal{Z}_{T,2}$ is \mathcal{T}^2 -relatively compact if the following three conditions are satisfied

- (1) $\sup_{\phi \in K_2} \sup_{s \in [0, T]} \|\phi(s)\| < \infty,$
- (2) $\sup_{\phi \in K_2} \int_0^T |A_1^{3/2} \phi(s)|_{L^2}^2 ds < \infty,$
- (3) $\lim_{\delta \rightarrow 0} \sup_{\phi \in K_2} \sup_{s, t \in [0, T], |s-t| \leq \delta} \rho(\phi(s), \phi(s)) = 0.$

We recall from [13] that the spaces $\mathcal{Z}_{T,1}$ and $\mathcal{Z}_{T,2}$ are not Polish spaces.

3.2. The Aldous condition

Let (\mathbb{M}, ρ) be a complete, separable metric space. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a probability space with usual hypotheses. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of \mathbb{F} -adapted and \mathbb{M} -valued processes. We recall from [13,31] the following definition.

Definition 3.2. Let (X_n) be a sequence of \mathbb{M} -valued random variables. The sequence of laws of these processes form a tight sequence if and only if

$$\begin{aligned} \forall \epsilon > 0, \forall \kappa > 0, \exists \delta > 0 : \\ \sup_n \mathbb{P}\{\mathcal{W}_{[0,T],\mathbb{M}}(X_n, \delta) > \kappa\} \leq \epsilon, \end{aligned} \quad (3.2)$$

where $\mathcal{W}_{[0,T],\mathbb{M}}$ is defined in (3.1).

Definition 3.3. A sequence $(X_m)_{m \in \mathbb{N}}$ satisfies the Aldous condition in the space \mathbb{M} if and only if

$$\begin{aligned} \forall \epsilon > 0, \forall \kappa > 0, \exists \delta > 0 : \text{ such that for every sequence } (\tau_m)_{m \in \mathbb{N}} \\ \text{of } \mathbb{F} - \text{stopping times with } \tau_m \leq T, \text{ one has} \\ \sup_{m \in \mathbb{N}} \sup_{0 \leq \theta \leq \delta} \mathbb{P}\{|X_m(\tau_m + \theta) - X_m(\tau_m)|_{\mathbb{M}} \geq \kappa\} \leq \epsilon. \end{aligned} \quad (3.3)$$

The following lemma is proved in [31].

Lemma 3.4. Condition (3.2) implies condition (3.3).

Proof. See Theorem 2.2.2 of [31]. \square

The following lemma is proved in [39].

Lemma 3.5. Let $(E_1, \|\cdot\|_{E_1})$ be a separable Banach space and let $(X_m)_{m \in \mathbb{N}}$ be a sequence of E_1 -valued random variables. Assume that for every sequence $(\tau_m)_{m \in \mathbb{N}}$ of \mathbb{F} -stopping times with $\tau_m \leq T$ and for every $m \in \mathbb{N}$ and $\theta \geq 0$ the following condition is satisfied

$$\mathbb{E}[\|X_m(\tau_m + \theta) - X_m(\tau_m)\|_{E_1}^\alpha] \leq C\theta^\beta, \quad (3.4)$$

for some $\alpha, \beta > 0$ and some constant $C > 0$. Then the sequence $(X_m)_{m \in \mathbb{N}}$ satisfies the Aldous condition in the space E_1 .

Proof. See [39]. \square

3.3. Skorokhod embedding theorems

We recall from [13,14] the following Jakubowski's version of the Skorokhod Theorem due to [39].

Theorem 3.6. *Let (\mathcal{G}, τ) be a topological space such that there exists a sequence (g_m) of continuous functions $g_m : \mathcal{G} \rightarrow \mathbb{R}$ that separates points of \mathcal{G} . Let (Z_m) be a sequence of \mathcal{G} -valued random variables. Suppose that for every $\epsilon > 0$ there exists a compact subset $G_\epsilon \subset \mathcal{G}$ such that*

$$\sup_{m \in \mathbb{N}} \mathbb{P}[\{Z_m \in G_\epsilon\}] \geq 1 - \epsilon.$$

Then there exists a subsequence $(Z_{m_k})_{k \in \mathbb{N}}$, a sequence $(X_k)_{k \in \mathbb{N}}$ of \mathcal{G} -valued random variables and an \mathcal{G} -valued random variable X defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\mathcal{L}_{aw}(Z_{m_k}) = \mathcal{L}_{aw}(X_k), \quad k = 1, 2, \dots,$$

and for all $\omega \in \Omega$, we have

$$X_k(\omega) \xrightarrow{\tau} X(\omega), \quad k \rightarrow \infty.$$

We also have the following version of Skorokhod Theorem due to [39,13].

Theorem 3.7. *Let E_1, E_2 be two separable Banach spaces and let $\pi_i : E_1 \times E_2 \rightarrow E_i$, $i = 1, 2$, be the projection into E_i , i.e.*

$$E_1 \times E_2 \ni \chi = (\chi_1, \chi_2) \rightarrow \pi_i(\chi) \in E_i.$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\chi_m : \Omega \rightarrow E_1 \times E_2$, $m \in \mathbb{N}$ be a family of random variables such that the sequence $\{\mathcal{L}_{aw}(\chi_m), m \in \mathbb{N}\}$ is weakly convergent on $E_1 \times E_2$. We also assume that there exists a random variable $\rho : \Omega \rightarrow E_1$ such that $\mathcal{L}_{aw}(\pi_1 \circ \chi_m) = \mathcal{L}_{aw}(\rho)$, $\forall m \in \mathbb{N}$. Then there exists a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, a family of $E_1 \times E_2$ -valued random variables $\{\bar{\chi}_n, n \in \mathbb{N}\}$ on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ and a random variable $\chi_ : \bar{\Omega} \rightarrow E_1 \times E_2$ such that*

- (1) $\mathcal{L}_{aw}(\bar{\chi}_n) = \mathcal{L}_{aw}(\chi_m)$, $\forall m \in \mathbb{N}$;
- (2) $\bar{\chi}_m \rightarrow \chi_*$ in $E_1 \times E_2$ a.s.;
- (3) $\pi_1 \circ \bar{\chi}_m(\bar{\omega}) = \pi_1 \circ \chi_*(\bar{\omega}) \quad \forall \bar{\omega} \in \bar{\Omega}$.

We will need the following version of the Skorokhod embedding theorem proved in [38,39], see Corollary 5.3 of [38].

Theorem 3.8. *Let \mathcal{H}_1 be a separable complete metric space and let \mathcal{H}_2 be a topological space such that there exists a sequence $\{f_l\}_{l \in \mathbb{N}}$ of continuous functions $f_l : \mathcal{H}_2 \rightarrow \mathbb{R}$ separating points of \mathcal{H}_2 . Let $\mathcal{H} := \mathcal{H}_1 \times \mathcal{H}_2$ with the Tychonoff topology induced by the projections*

$$\pi_i : \mathcal{H}_1 \times \mathcal{H}_2 \rightarrow \mathcal{H}_i, \quad i = 1, 2.$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\chi_m : \Omega \rightarrow \mathcal{H}_1 \times \mathcal{H}_2$, $m \in \mathbb{N}$, be a family of random variables such that the sequence $\{\mathcal{L}(\chi_m), m \in \mathbb{N}\}$ is tight on $\mathcal{H}_1 \times \mathcal{H}_2$. We also assume that there exists a random variable $\rho : \Omega \rightarrow \mathcal{H}_1$ such that $\mathcal{L}(\pi_1 \circ \chi_m) = \mathcal{L}(\rho)$, for all $m \in \mathbb{N}$. Then there exists a subsequence $(\chi_{m_k})_{k \in \mathbb{N}}$, a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, a family of $\mathcal{H}_1 \times \mathcal{H}_2$ -valued random variables $\{\bar{\chi}_{m_k}, k \in \mathbb{N}\}$ on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, and a random variable $\chi_ : \bar{\Omega} \rightarrow \mathcal{H}_1 \times \mathcal{H}_2$ such that*

- (1) $\mathcal{L}(\chi_k) = \mathcal{L}(\chi_{m_k})$ for all $k \in \mathbb{N}$;
- (2) $\bar{\chi}_k \rightarrow \chi_*$ in $\mathcal{H}_1 \times \mathcal{H}_2$ a.s. as $k \rightarrow \infty$;
- (3) $\pi_1 \circ \bar{\chi}_k(\bar{\omega}) = \pi_1 \circ \bar{\chi}_*(\bar{\omega})$ for all $\bar{\omega} \in \bar{\Omega}$.

Proof. See [39]. \square

4. Galerkin approximations and a priori estimates

In this section, we present the Galerkin approximation and derive some a priori estimates for the generated sequence.

4.1. The approximating equations

Since the injection of $\mathcal{U} \subset \mathcal{Y}$ is compact, let $\{(w_i, \psi_i), i = 1, 2, 3, \dots\} \subset \mathcal{U}$ be an orthonormal basis of \mathcal{Y} , where $\{w_i, i = 1, 2, \dots\}$, $\{\psi_i, i = 1, 2, \dots\}$ are eigenvectors of A_0 and A_1 respectively. We set $\mathcal{U}_m = \mathcal{Y}_m = \text{span}\{(w_1, \psi_1), \dots, (w_m, \psi_m)\}$.

Hereafter, we denote by \mathcal{P}_m^1 (respectively \mathcal{P}_m^2) the L^2 -orthogonal projection from H_1 (respectively H_2) onto $V_1^m = H_{1,m} \equiv \text{span}\{w_1, w_2, \dots, w_m\}$ (respectively $V_2^m = H_{2,m} \equiv \text{span}\{\psi_1, \psi_2, \dots, \psi_m\}$).

We consider the following Galerkin approximation

$$\begin{aligned} dv_m(t) &= -\mathcal{P}_m^1 [A_0 v_m + B_0(v_m, v_m) + R_0(\phi_m, \phi_m) - g_1(s, v_m, \phi_m)] dt \\ &+ \int_Z \mathcal{P}_m^1 \sigma(t, v_m(t-), \phi_m(t-), z) \tilde{\eta}(dt, dz), \\ d\phi_m(t) &= -\mathcal{P}_m^2 [A_1 \mu_m + B_1(v_m, \phi_m)] dt, \quad \mu_m = \mathcal{P}_m^2 (A_1 \phi_m + f(\phi_m)), \end{aligned} \quad (4.1)$$

where $\mathcal{P}_m \equiv (\mathcal{P}_m^1, \mathcal{P}_m^2)$ is the orthogonal projection of \mathcal{Y} onto \mathcal{Y}_m . The system (4.1) supplemented with the initial condition $(v_m, \phi_m)(0) = \mathcal{P}_m(v_0, \phi_0)$ form a stochastic differential equation with locally Lipschitz coefficients.

Hereafter, we set

$$\tilde{B}_0(v, v) = \mathcal{P}_m^1 B_0(\chi_m^1(v), \chi_m^1(v)), \quad \tilde{R}_0(\phi, \phi) = \mathcal{P}_m^1 R_0(\chi_m^2(\phi), \phi), \quad \tilde{B}_1(v, \phi) = \mathcal{P}_m^2 B_1(\chi_m^1(v), \phi),$$

where $\chi_m^1 : H_1 \rightarrow H_1$ and $\chi_m^2 : L^2(\mathcal{M}) \rightarrow L^2(\mathcal{M})$ are defined by $\chi_m^1(v) = \Theta_m(\|v\|_{V_1^*})v$, $\chi_m^2(\phi) = \Theta_m(\|\phi\|_{D(A_1)^*})\phi$, and $\Theta_m : \mathbb{R} \rightarrow [0, 1]$ is of class \mathcal{C}^∞ such that $\Theta_m(r) = 1$ if $r \leq m$ and $\Theta_m(r) = 0$ if $r \geq m+1$.

We can check that

$$\tilde{B}_0 : H_{1,m} \times H_{1,m} \rightarrow H_{1,m}, \quad \tilde{R}_0 : H_{2,m} \times H_{2,m} \rightarrow H_{1,m}, \quad \tilde{B}_1 : H_{1,m} \times H_{2,m} \rightarrow H_{2,m}$$

are globally Lipschitz continuous. We also check $f_n(\phi) = \mathcal{P}_m^1 f(\phi)$, for $\phi \in H_{2,m}$.

We now consider the following Faedo-Galerkin approximation

$$\begin{aligned} dv_m(t) &= -[\mathcal{P}_m^1 A_0 v_m + \tilde{B}_0(v_m, v_m) + \tilde{R}_0(\phi_m, \phi_m) - \mathcal{P}_m^1 g_1(s, v_m, \phi_m)] dt \\ &+ \int_Z \mathcal{P}_m^1 \sigma(t, v_m(t-), \phi_m(t-), z) \tilde{\eta}(dt, dz), \\ d\phi_m(t) &= -[\mathcal{P}_m^2 A_1 \mu_m + \tilde{B}_1(v_m, \phi_m)] dt, \quad \mu_m = \mathcal{P}_m^2 A_1 \phi_m + f_m(\phi_m). \end{aligned} \quad (4.2)$$

Lemma 4.2. For each $m \in \mathbb{N}$, there exists a unique global, \mathbb{F} -adapted, \mathcal{Y}_m -valued process (v_m, ϕ_m) to (4.2).

Proof. The proof is similar to that in [3,13]. \square

Corollary 4.3. For each $m \in \mathbb{N}$, there exists a unique local maximal solution (v_m, ϕ_m) to (4.1).

Lemma 4.4. For every $(v, \phi) \in V_1 \times H_2$, we have

$$\lim_{m \rightarrow \infty} \|\mathcal{P}_m^1 v - v\| = 0, \quad \lim_{m \rightarrow \infty} \|\mathcal{P}_m^2 \phi - \phi\| = 0. \quad (4.3)$$

Proof. See [10]. \square

4.2. A priori estimates

Hereafter, C will denote a positive constant independent of m and which may change from one term to the next.

Lemma 4.5. The following estimate holds:

$$\mathbb{E} \sup_{t \in [0, T]} |(v_m, \phi_m)|_{\mathcal{Y}}^2 + \mathbb{E} \int_0^T (\|v_m\|^2 + \|\mu_m\|^2) ds \leq C, \quad (4.4)$$

where C is independent of m .

Proof. For each $n \geq 1$, we consider the \mathcal{F} -stopping time τ_n defined by:

$$\tau_n = \min \left(T, \inf \{ t \in [0, T]; |(v_m, \phi_m)(t)|_{\mathcal{Y}} \geq n^2 \} \right). \quad (4.5)$$

Since the process $(v_m, \phi_m)_{t \in [0, T]}$ is \mathbb{F} -adapted and right-continuous, τ_n is a stopping time. Moreover, for fixed m , the sequence $\{\tau_n; n \geq 1\}$ is increasing to T , \mathbb{P} -a.s., as n goes to infinity.

Throughout we fix $r \in [0, T]$ and $0 \leq t \leq r \wedge \tau_n$.

We apply the Itô formula to $|v_m|_{L^2}^2$ to derive that

$$\begin{aligned} |v_m(t)|_{L^2}^2 + 2 \int_0^t \|v_m(s)\|^2 ds &= |v_0|_{L^2}^2 + 2 \int_0^t \langle g_1(s, v_m(s), \phi_m(s)), v_m(s) \rangle ds \\ &\quad - 2 \int_0^t \langle R_0(\phi_m(s), \phi_m(s)), v_m(s) \rangle ds \\ &\quad + 2 \int_0^t \int_{\mathcal{Z}} (v_m(s-), \sigma(s, v_m(s), \phi_m(s), z)) \tilde{\eta}(dz, ds) + \int_0^t \int_{\mathcal{Z}} \Upsilon(s, z) \eta(dz, ds), \end{aligned} \quad (4.6)$$

where $\Upsilon(s, z)$ is given by

$$\Upsilon(s, z) = |v_m(s-) + \sigma(s, v_m(s), \phi_m(s), z)|_{L^2}^2 - |v_m(s-)|_{L^2}^2 - 2(v_m(s-), \sigma(s, v_m(s), \phi_m(s), z)). \quad (4.7)$$

Multiplying (2.34)₂ with μ_m and integrating over $[0, t]$ gives

$$\begin{aligned} \|\phi_m(t)\|^2 + 2\langle F(\phi_m(t)), 1 \rangle &+ 2 \int_0^t \|\mu_m(s)\|^2 ds = \|\phi_0\|^2 + 2\langle F(\phi_0), 1 \rangle \\ &- 2 \int_0^t \langle B_1(v_m(s), \phi_m(s)), \mu_m(s) \rangle ds. \end{aligned} \quad (4.8)$$

Now adding (4.6) and (4.8) and using the fact that

$$\langle R_0(\phi_m, \phi_m), v_m \rangle = -\langle B_1(v_m, \phi_m), A_1 \phi_m \rangle = -\langle B_1(v_m, \phi_m), \mu_m \rangle,$$

we derive that

$$\begin{aligned} \mathcal{E}(v_m, \phi_m)(t) + 2 \int_0^t (\|v_m(s)\|^2 + \|\mu_m(s)\|^2) ds &= \mathcal{E}(v_0, \phi_0) \\ + 2 \int_0^t \langle g_1(s, v_m(s), \phi_m(s)), v_m(s) \rangle ds &+ 2 \int_0^t \int_Z (v_m(s-), \sigma(s, v_m(s), \phi_m(s), z)) \tilde{\eta}(dz, ds) \\ + \int_0^t \int_Z \Upsilon(s, z) \eta(dz, ds). \end{aligned} \quad (4.9)$$

From the identity $|x|^2 - |y|^2 + |x - y|^2 = 2\langle x - y, x \rangle$, it follows that

$$\begin{aligned} \mathcal{E}(v_m(t), \phi_m(t)) + 2 \int_0^t (\|v_m(s)\|^2 + \|\mu_m(s)\|^2) ds &= \mathcal{E}(v_0, \phi_0) \\ + 2 \int_0^t \langle g_1(s, v_m(s), \phi_m(s)), v_m(s) \rangle ds &+ 2 \int_0^t \int_Z (v_m(s-), \sigma(s, v_m(s), \phi_m(s), z)) \tilde{\eta}(dz, ds) \\ + \int_0^t \int_Z |\sigma(s, v_m(s), \phi_m(s), z)|_{L^2}^2 \eta(dz, ds). \end{aligned} \quad (4.10)$$

We define the following stochastic processes

$$\begin{aligned} X(t) &= \sup_{s \in [0, t]} \mathcal{E}(v_m(s), \phi_m(s)), \quad Y(t) = 2 \int_0^t (\|v_m(s)\|^2 + \|\mu_m(s)\|^2) ds, \\ I(t) &= 2 \left| \int_0^t \int_Z (v_m(s-), \sigma(s, v_m(s), \phi_m(s), z)) \tilde{\eta}(dz, ds) \right| + \int_0^t \int_Z |\sigma(s, v_m(s), \phi_m(s), z)|_{L^2}^2 \eta(dz, ds) \\ &= \sup_{s \in [0, t]} |I_1(s)| + I_2(t), \end{aligned}$$

where

$$I_1(t) = 2 \int_0^t \int_Z (v_m(s-), \sigma(s, v_m(s), \phi_m(s), z)) \tilde{\eta}(dz, ds), \quad (4.11)$$

$$I_2(t) = \sup_{s \in [0, t]} \int_0^s \int_Z |\sigma(s, v_m(s), \phi_m(s), z)|_{L^2}^2 \eta(dz, ds).$$

Since $I_1(t)$ is a local martingale we can apply Burkholder-Davis-Gundy's inequality to derive that

$$\mathbb{E} \sup_{s \in [0, r \wedge \tau_n]} |I_1(s)| \leq C \mathbb{E} \left(\int_0^{r \wedge \tau_n} \int_Z (v_m(s-), \sigma(s, v_m(s), \phi_m(s), z))^2 \nu(dz) ds \right)^{1/2}. \quad (4.12)$$

Thanks to Hölder's and Young's inequalities we have

$$\begin{aligned} \mathbb{E} \sup_{s \in [0, t]} |I_1(s)| &\leq C \left[\epsilon \mathbb{E} \sup_{s \in [0, t]} |v_m(s)|_{L^2}^2 \right]^{1/2} \left[\epsilon^{-1} \mathbb{E} \int_0^t \int_Z |\sigma(s, v_m(s), \phi_m(s), z)|_{L^2}^2 \nu(dz) ds \right]^{1/2} \\ &\leq C \epsilon \mathbb{E} \sup_{s \in [0, t]} |v_m(s)|_{L^2}^2 + C \epsilon^{-1} \mathbb{E} \int_0^t \int_Z |\sigma(s, v_m(s), \phi_m(s), z)|_{L^2}^2 \nu(dz) ds \\ &\leq C \epsilon \mathbb{E} \sup_{s \in [0, t]} \mathcal{E}(v_m(s), \phi_m(s)) + C \epsilon^{-1} \mathbb{E} \int_0^t \int_Z |\sigma(s, v_m(s), \phi_m(s), z)|_{L^2}^2 \nu(dz) ds. \end{aligned} \quad (4.13)$$

Using (2.37), we derive that

$$\mathbb{E} \sup_{s \in [0, t]} |I_1(s)| \leq C \epsilon X(t) + C \epsilon^{-1} l_0 t + C \epsilon^{-1} \int_0^t \mathbb{E} X(s) ds. \quad (4.14)$$

Next, we will deal with the second term of $I(t)$. Taking into account that the process

$$\int_0^t \int_Z |\sigma(s, v_m(s), \phi_m(s), z)|^2 \eta(dz, ds)$$

has only positive jumps, we derive from (2.37) that

$$\begin{aligned} \mathbb{E} I_2(t) &\leq \mathbb{E} \int_0^t \int_Z |\sigma(s, v_m(s), \phi_m(s), z)|_{L^2}^2 \nu(dz) ds \\ &\leq l_0 t + l_1 \int_0^t \mathbb{E} |(v_m(s), \phi_m(s))|_{\mathcal{Y}}^2 ds \\ &\leq l_0 t + l_1 \int_0^t \mathbb{E} X(s) ds. \end{aligned} \quad (4.15)$$

We also have

$$\begin{aligned} |2\langle \mathcal{P}_m^1 g_1(s, v_m, \phi_m), v_m \rangle| &\leq 2L_1 |(v_m, \phi_m)|_{\mathcal{Y}} \|v_m\| + 2\|g_1(s, 0, 0)\|_{V_1^*} \|v_m\| \\ &\leq \frac{1}{8} \|v_m\|^2 + cL_1^2 |(v_m, \phi_m)|_{\mathcal{Y}}^2 + c\|g_1(s, 0, 0)\|_{V_1^*}^2. \end{aligned} \quad (4.16)$$

It follows from (4.9)-(4.16) that

$$\begin{aligned} \mathbb{E}\mathcal{E}(v_m, \phi_m)(t) + \mathbb{E} \int_0^t (\|v_m(s)\|^2 + \|\mu_m(s)\|^2) ds &\leq \mathbb{E}\mathcal{E}(v_0, \phi_0) + c\mathbb{E} \int_0^t \mathcal{E}(v_m, \phi_m)(s) ds \\ &+ c\mathbb{E} \int_0^t \|g_1(s, 0, 0)\|_{V_1^*}^2 ds. \end{aligned} \quad (4.17)$$

Therefore from Lemma 2.5, we derive that there exist a positive constant C such that

$$\mathbb{E}\mathcal{E}(v_m, \phi_m)(t) + \mathbb{E} \int_0^t (\|v_m(s)\|^2 + \|\mu_m(s)\|^2) ds \leq C, \quad (4.18)$$

for any $m \in \mathbb{N}$ and $t \in [0, r \wedge \tau_n], r \in [0, T]$.

We have just shown that

$$\mathbb{E} \sup_{s \in [0, t \wedge \tau_n]} \mathcal{E}(v_m, \phi_m)(t) + \mathbb{E} \int_0^t (\|v_m(s)\|^2 + \|\mu_m(s)\|^2) ds \leq C, \quad \forall t \in [0, T], \quad (4.19)$$

from which we can infer that

$$\mathbb{P}(\tau_n < t) \leq Cn^{-2}, \quad \forall t \in [0, T], \quad \forall n > 0.$$

Hence, $\lim_{n \rightarrow +\infty} \mathbb{P}(\tau_n < t) = 0$, for all $t \in [0, T]$. That is, $\tau_n \rightarrow T$ in probability. Therefore, there exists a subsequence τ_{n_k} such that $\tau_{n_k} \rightarrow T$, a.s. Since the sequence $(\tau_n)_n$ is increasing, we infer that $\tau_{n_k} \nearrow T$ a.s. Now we use Fatou's lemma and pass to the limit in (4.19) and derive that

$$\mathbb{E} \sup_{s \in [0, t]} \mathcal{E}(v_m(s), \phi_m(s)) + \mathbb{E} \int_0^t (\|v_m(s)\|^2 + \|\mu_m(s)\|^2) ds \leq C, \quad (4.20)$$

and (4.4) is proved. \square

Lemma 4.6. Any solution (v_m, ϕ_m) to (2.34) satisfies the estimate

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, T]} |(v_m, \phi_m)|_{\mathcal{Y}}^p &\leq C, \quad \mathbb{E} \left(\int_0^T \|(v_m, \phi_m)(s)\|_{\mathcal{U}}^2 ds \right)^{p/2} \leq C, \\ \mathbb{E} \left(\int_0^T |A_1 \phi_m|_{L^2}^2 ds \right)^{p/2} &\leq C, \quad \mathbb{E} \left(\int_0^T |A_1^{3/2} \phi_m|_{L^2}^2 ds \right)^{p/2} \leq C, \quad \forall p \in [1, \infty). \end{aligned} \quad (4.21)$$

Proof. To prove (4.21), we proceed as follows. By raising both sides of (4.9) to the power of $p \geq 2$, we derive that

$$\begin{aligned} & \mathbb{E} \sup_{s \in [0, t]} [\mathcal{E}(v_m(t), \phi_m(t))]^p + \mathbb{E} \left[\int_0^t (2\|v_m(s)\|^2 + \|\mu_m(s)\|^2) ds \right]^p \\ & \leq \mathbb{E}[\mathcal{E}(v_0, \phi_0)]^p + c\mathbb{E} \sup_{s \in [0, t]} |I_3(s)|^p + c\mathbb{E} \sup_{s \in [0, t]} |I_4(s)|^p + c\mathbb{E} \left[\int_0^t \|g_1(s, 0, 0)\|_{V_1^*}^2 ds \right]^p, \end{aligned} \quad (4.22)$$

where

$$I_3(t) = \int_0^t \int_Z \{ |v_m(s-) + \mathcal{P}_m^1 \sigma(s, v_m(s), \phi_m(s), t)|_{L^2}^2 - |v_m(s-)|_{L^2}^2 \} \tilde{\eta}(dz, ds), \quad (4.23)$$

$$\begin{aligned} I_4(t) &= \int_0^t \int_Z \{ |v_m(s-) + \mathcal{P}_m^1 \sigma(s, v_m(s), \phi_m(s), t)|_{L^2}^2 - |v_m(s-)|_{L^2}^2 \nu(dz) ds \\ &\quad - \int_0^t \int_Z (v_m(s-), \mathcal{P}_m^1 \sigma(s, v_m(s), \phi_m(s), z)) \nu(dz) ds \\ &\leq c \int_0^t \int_Z |\mathcal{P}_m^1 \sigma(s, v_m(s), \phi_m(s), z)|_{L^2}^2 \nu(dz) ds \\ &\leq c \int_0^t (1 + |(v_m, \phi_m)|_{\mathcal{Y}}^2) ds. \end{aligned} \quad (4.24)$$

As in [12,13], we note that

$$\begin{aligned} & \int_Z \{ |v_m(s-) + \mathcal{P}_m^1 \sigma(s, v_m(s), \phi_m(s), z)|_{L^2}^2 - |v_m(s-)|_{L^2}^2 \}^2 \nu(dz) \\ & \leq |v_m(s-)|_{L^2}^2 \int_Z |\sigma(s, v_m(s), \phi_m(s), z)|_{L^2}^2 \nu(dz) + c \int_Z |\sigma(s, v_m(s), \phi_m(s), z)|_{L^2}^4 \nu(dz) \\ & \leq c_0 + c_1 |(v_m, \phi_m)(s)|_{\mathcal{Y}}^2 + c_2 |(v_m, \phi_m)(s)|_{\mathcal{Y}}^4 \\ & \leq k_1 + k_4 |(v_m, \phi_m)(s)|_{\mathcal{Y}}^4. \end{aligned} \quad (4.25)$$

It follows that

$$\begin{aligned} & \left(\int_0^t \int_Z \left\{ |v_m(s-) + \mathcal{P}_m^1 \sigma(s, v_m(s), \phi_m(s), t)|_{L^2}^2 - |v_m(s-)|_{L^2}^2 \right\}^2 \nu(dz) ds \right)^{p/2} \\ & \leq c(k_1 T)^{p/2} + c(k_2)^{p/2} \left(\int_0^t |(v_m, \phi_m)(s)|_{\mathcal{Y}}^4 ds \right)^{p/2}. \end{aligned} \quad (4.26)$$

We derive that

$$\begin{aligned} \mathbb{E} \sup_{s \in [0, t]} |I_3(s)|^p & \leq c_p (k_1 T)^{p/2} + c_p (k_2)^{p/2} \mathbb{E} \left[\left(\int_0^t |(v_m, \phi_m)(s)|_{\mathcal{Y}}^4 ds \right)^{p/2} \right] \\ & \leq c + \frac{1}{2} \mathbb{E} \left(\sup_{s \in [0, t]} |(v_m, \phi_m)(s)|_{\mathcal{Y}}^2 \right)^p + c \mathbb{E} \left(\int_0^t |(v_m, \phi_m)(s)|_{\mathcal{Y}}^2 ds \right)^p. \end{aligned} \quad (4.27)$$

From Hölder's inequality, we have

$$\begin{aligned} \int_0^t |(v_m, \phi_m)(s)|_{\mathcal{Y}}^2 ds & \leq \left(\int_0^t |(v_m, \phi_m)(s)|_{\mathcal{Y}}^{2p} ds \right)^{1/p} \left(\int_0^t 1 ds \right)^{\frac{p-1}{p}} \\ & \leq T^{\frac{p-1}{p}} \left(\int_0^t |(v_m, \phi_m)(s)|_{\mathcal{Y}}^{2p} ds \right)^{1/p}, \end{aligned} \quad (4.28)$$

which gives

$$\left(\int_0^t |(v_m, \phi_m)(s)|_{\mathcal{Y}}^2 ds \right)^p \leq c T^{p-1} \int_0^t |(v_m, \phi_m)(s)|_{\mathcal{Y}}^{2p} ds. \quad (4.29)$$

From (4.27), (4.28), we get

$$\begin{aligned} \mathbb{E} \sup_{s \in [0, t]} |I_3(s)|^p & \leq \frac{1}{2} \mathbb{E} \left(\sup_{s \in [0, t]} |(v_m, \phi_m)(s)|_{\mathcal{Y}}^2 \right)^p + c_{p,T} \int_0^t \mathbb{E} |(v_m, \phi_m)(s)|_{\mathcal{Y}}^{2p} ds \\ & \leq \frac{1}{2} \mathbb{E} \sup_{s \in [0, t]} [\mathcal{E}(v_m(s), \phi_m(s))]^p + c_{p,T} \int_0^t \mathbb{E} [\mathcal{E}(v_m(s), \phi_m(s))]^p ds. \end{aligned} \quad (4.30)$$

From (2.37) and (4.24), we also have

$$\begin{aligned} \mathbb{E} |I_4(t)|^p & \leq c \mathbb{E} \left(\int_0^t (1 + |(v_m, \phi_m)(s)|_{\mathcal{Y}}^2) ds \right)^p \\ & \leq c_p + c_p \mathbb{E} \left(\int_0^t |(v_m, \phi_m)(s)|_{\mathcal{Y}}^2 ds \right)^p. \end{aligned} \quad (4.31)$$

It follows that

$$\begin{aligned} \mathbb{E} \sup_{s \in [0, t]} |I_2(s)|^p &\leq c_{p,T} + c_{p,T} \int_0^t |(v_m, \phi_m)(s)|_{\mathcal{Y}}^{2p} ds \\ &\leq c_{p,T} + c_{p,T} \int_0^t \mathbb{E} [\mathcal{E}(v_m(s), \phi_m(s))]^p ds. \end{aligned} \quad (4.32)$$

It follows from (4.22)-(4.32)

$$\mathbb{E} \sup_{s \in [0, t]} [\mathcal{E}(v_m(s), \phi_m(s))]^p \leq c_{p,T} + c_{p,T} \int_0^t \mathbb{E} [\mathcal{E}(v_m(s), \phi_m(s))]^p ds + c \left(\int_0^t \|g_1(t, 0, 0)\|_{V_1^*}^2 ds \right)^p, \quad (4.33)$$

and Gronwall's lemma and (4.22) give

$$\mathbb{E} \sup_{s \in [0, t]} [\mathcal{E}(v_m(s), \phi_m(s))]^p + \mathbb{E} \left[\int_0^t (2\|v_m(s)\|^2 + \|\mu_m(s)\|^2) ds \right]^p \leq C. \quad (4.34)$$

We also note that

$$\begin{aligned} |A_1 \phi_m|_{L^2} &\leq |\mu_m|_{L^2} + |f(\phi_m)|_{L^2} \\ &\leq |\mu_m|_{L^2} + c\|\phi_m\|^{k+1} + C. \end{aligned} \quad (4.35)$$

It follows that

$$\mathbb{E} \left(\int_0^T |A_1 \phi_m|_{L^2}^2 ds \right)^p \leq c \mathbb{E} \left(\int_0^T |\mu_m|_{L^2}^2 ds \right)^p + c \mathbb{E} \left(\sup_{s \in [0, T]} \|\phi_m\|^{2k+2} \right) + C \leq C. \quad (4.36)$$

Similarly, using (2.9) we can check that

$$\begin{aligned} |A_1^{3/2} \phi_m|_{L^2}^2 &\leq c\|\mu_m\|^2 + c|A_1^{1/2} f(\phi_m)|_{L^2}^2 \\ &\leq c\|\mu_m\|^2 + c\|\phi_m\|^2 + c|A_1^{1/2} \phi_m|_{L^2}^{4k+2} + |A_1 \phi_m|_{L^2}^2, \end{aligned} \quad (4.37)$$

which gives

$$\bar{\mathbb{E}} \left[\int_0^t |A_1^{3/2} \phi_m|_{L^2}^2 ds \right]^{p/2} \leq C. \quad (4.38)$$

Therefore (4.21) follows from (4.34) and (4.36). \square

5. Tightness of the laws of approximating sequences

In this section, we prove the tightness of laws of (v_m, ϕ_m) . Let us consider the space $\mathcal{Z}_T = \mathcal{Z}_{T,1} \times \mathcal{Z}_{T,2}$, where

$$\mathcal{Z}_{T,1} = L_w^2(0, T; V_1) \cap L^2(0, T; H_1) \cap \mathbb{D}([0, T]; V_{1s}^*) \cap \mathbb{D}([0, T]; H_{1,w}), \quad s > \frac{d}{2} + 1,$$

and

$$\mathcal{Z}_{T,2} = L_w^2(0, T; D(A_1^{3/2})) \cap L^2(0, T; H_2) \cap \mathbb{C}([0, T]; D(A_1)^*) \cap \mathbb{C}([0, T]; H_{2,w}).$$

For each $m \in \mathbb{N}$, the solutions (v_m, ϕ_m) of the Galerkin approximation equations define measures $\mathcal{L}(v_m, \phi_m)$ on $(\mathcal{Z}_T, \mathcal{T})$, where \mathcal{T} is the supremum of \mathcal{T}^1 and \mathcal{T}^2 . We will show that the set of measures $\mathcal{L}(v_m, \phi_m)$, $m \in \mathbb{N}$ is tight on $(\mathcal{Z}_T, \mathcal{T})$. We first recall that (v_m, ϕ_m) satisfies the following estimates.

$$\sup_m \mathbb{E} \left[\sup_{t \in [0, T]} |(v_m, \phi_m)(t)|_Y^p \right] \leq C, \quad \sup_m \mathbb{E} \left[\int_0^T \|(v_m, \phi_m)(s)\|^2 ds \right]^p \leq C, \quad \forall p \geq 1.$$

Lemma 5.1. *The set of measures $\{\mathcal{L}(v_m, \phi_m), m \in \mathbb{N}\}$ is tight on $(\mathcal{Z}_T, \mathcal{T})$.*

Proof. We will prove that for $s > \frac{d}{2} + 1$, the sequences (v_m, ϕ_m) satisfies the Aldous condition in the space $\mathcal{U}_s \equiv V_{1s} \times D(A_1^{3/2})$ for tightness. Let $(\tau_m)_{m \in \mathbb{N}}$ be a sequence of stopping times such that $0 \leq \tau_m \leq T$. \square

Lemma 5.2. *There exists a positive constant C such that the following inequality holds:*

$$\mathbb{E} \int_t^{t+\theta} \|(v_m, \phi_m)(t+\theta) - (v_m, \phi_m)(t)\|_{\mathcal{U}_s}^2 ds \leq C\theta^{1/3}. \quad (5.39)$$

Proof. We will restrict ourself in the 3D case as the proof is similar in the 2D case. For positive θ (similar can be done for negative θ) such that $t + \theta \in [0, T]$ for $t \in [0, T]$, we derive from (4.2) that

$$\begin{aligned} v_m(t+\theta) - v_m(t) = & - \int_t^{t+\theta} (\mathcal{P}_m^1 A_0 v_m + \tilde{B}_0(v_m, v_m)) ds \\ & - \int_t^{t+\theta} (\tilde{R}_0(\phi_m, \phi_m) + \mathcal{P}_m^1 g_1(s, v_m, \phi_m)) ds + \int_t^{t+\theta} \int_Z \mathcal{P}_m^1 \sigma(s, v_m, \phi_m, z) \tilde{\eta}(ds, dz), \end{aligned} \quad (5.40)$$

which gives

$$\begin{aligned} \mathbb{E} \|v_m(t+\theta) - v_m(t)\|_{V_{1s}^*} & \leq c \mathbb{E} \left(\int_t^{t+\theta} (\|\mathcal{P}_m^1 A_0 v_m\|_{V_{1s}^*} + \|\tilde{B}_0(v_m, v_m)\|_{V_{1s}^*}) ds \right) \\ & + c \mathbb{E} \left(\int_t^{t+\theta} (\|\tilde{R}_0(\phi_m, \phi_m)\|_{V_{1s}^*} + \|\mathcal{P}_m^1 g_1(s, v_m, \phi_m)\|_{V_{1s}^*}) ds \right) \\ & + c \mathbb{E} \left(\left\| \int_t^{t+\theta} \int_Z \mathcal{P}_m^1 \sigma(s, v_m, \phi_m, z) \tilde{\eta}(ds, dz) \right\|_{V_{1s}^*} \right) \\ & = K_1 + K_2 + \dots + K_5. \end{aligned} \quad (5.41)$$

Since $A_0 : V_1 \rightarrow V_1^*$, $\|A_0 v\|_{V_1^*} \leq \|v\|$ and the embedding $V_1 \subset V_{1s}^*$ is continuous, we derive from the Hölder inequality that

$$K_1 \equiv c\mathbb{E} \int_t^{t+\theta} \|A_0 v_m\|_{V_{1s}^*} ds \leq c\mathbb{E} \int_t^{t+\theta} \|v_m\| ds \leq c\theta^{1/2} \mathbb{E} \left(\int_0^T \|v_m\|^2 ds \right)^{1/2}. \quad (5.42)$$

Since $s > \frac{d}{2} + 1$, from (2.27), (2.23), we derive that

$$K_2 \equiv c\mathbb{E} \int_t^{t+\theta} \|\tilde{B}_0(v_m, v_m)\|_{V_{1s}^*} ds \leq c\mathbb{E} \int_t^{t+\theta} |v_m(s)|_{L^2}^2 ds \leq c\theta, \quad (5.43)$$

$$K_3 \equiv c\mathbb{E} \int_t^{t+\theta} \|\tilde{R}_0(\phi_m, \phi_m)\|_{V_{1s}^*} ds \leq c\mathbb{E} \int_t^{t+\theta} \|\phi_m(s)\|^2 ds \leq c\theta. \quad (5.44)$$

Using (2.37), we also have

$$\begin{aligned} K_4 &\equiv c\mathbb{E} \int_t^{t+\theta} \|\mathcal{P}_m^1 g_1(s, v_m, \phi_m)\|_{V_{1s}^*} ds \\ &\leq c\mathbb{E} \int_t^{t+\theta} \|\mathcal{P}_m^1 (g_1(s, v_m, \phi_m) - g_1(s, 0, 0))\|_{V_1^*} ds + c\mathbb{E} \int_t^{t+\theta} \|\mathcal{P}_m^1 g_1(s, 0, 0)\|_{V_1^*} ds \\ &\leq c\theta^{1/2} \mathbb{E} \left(\int_0^T |(v_m, \phi_m)|_{\mathcal{Y}}^2 ds \right)^{1/2} + c\theta^{1/2} \mathbb{E} \left(\int_0^T \|\mathcal{P}_m^1 g_1(s, 0, 0)\|_{V_1^*}^2 ds \right)^{1/2} \leq c\theta^{1/2}. \end{aligned} \quad (5.45)$$

For K_5 , we use (2.37) and the fact that $H_1 \subset V_{1s}^*$ to derive as in [38] that

$$\begin{aligned} K_5 &\equiv c\mathbb{E} \left\| \int_t^{t+\theta} \int_Z \mathcal{P}_m^1 \sigma(s, v_m, \phi_m, z) \tilde{\eta}(ds, dz) \right\|_{V_{1s}^*}^2 \\ &\leq c\mathbb{E} \left(\int_t^{t+\theta} \int_Z |\mathcal{P}_m^1 \sigma(s, v_m, \phi_m, z)|_{L^2}^2 \nu(dz) ds \right) \\ &\leq c\mathbb{E} \left(\int_t^{t+\theta} (1 + |(v_m, \phi_m)|_{\mathcal{Y}}^2) ds \right) \\ &\leq C\theta + C\theta \sup_{0 \leq t \leq T} |(v_m, \phi_m)(t)|_{\mathcal{Y}}^2 \leq C\theta. \end{aligned} \quad (5.46)$$

From (5.41)-(5.46), we derive that

$$\mathbb{E} \int_0^{t+\theta} \|v_m(t+\theta) - v_m(t)\|_{V_{1s}^*}^2 dt \leq C\theta^{1/3} \quad (5.47)$$

Following similar steps as in (5.40)-(5.47), we can also check that

$$\mathbb{E} \int_0^{t+\theta} \|\phi_m(t+\theta) - \phi_m(t)\|_{V_2^*}^2 dt \leq C\theta^{1/3} \quad (5.48)$$

and therefore, (5.39) follows from (5.47)-(5.48). \square

6. Existence of martingale solution

In this section, we prove the existence of a martingale solution. As in [13] (see also [14,28]), the Skorokhod Theorem for nonmetric spaces is used to construct a martingale solution.

6.1. Construction of new probability space and processes

We recall that from Lemma 5.1, we have shown the set of measures $\{\mathcal{L}(v_m, \phi_m), m \in \mathbb{N}\}$ is tight on $(\mathcal{Z}_{T,1} \times \mathcal{Z}_{T,2}, \mathcal{T})$. Let $\eta_m = \eta$, $m \in \mathbb{N}$. Then the set of measures $\{\mathcal{L}(\eta_m), m \in \mathbb{N}\}$ is tight on the space $M_{\mathbb{N}}([0, T] \times Z)$. Therefore the set $\{\mathcal{L}(v_m, \phi_m, \eta_m), m \in \mathbb{N}\}$ is tight on $\mathcal{Z}_T \times M_{\mathbb{N}}([0, T] \times Z)$. By Theorem 3.7, there exists a subsequence $(\eta_k)_{k \in \mathbb{N}}$, a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ and on this space, $\mathcal{Z}_T \times M_{\mathbb{N}}([0, T] \times Z)$ -valued random variables (v_*, ϕ_*, η_*) , $(\bar{v}_k, \bar{\phi}_k, \bar{\eta}_k)$, $k \in \mathbb{N}$ such that

- (a) $\mathcal{L}(\bar{v}_k, \bar{\phi}_k, \bar{\eta}_k) = \mathcal{L}(v_{m_k}, \phi_{m_k}, \eta_{m_k}), \forall k \in \mathbb{N}$,
- (b) $(\bar{v}_k, \bar{\phi}_k, \bar{\eta}_k) \rightarrow (v_*, \phi_*, \eta_*)$ in $\mathcal{Z}_T \times M_{\mathbb{N}}([0, T] \times Z)$ with probability 1 on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ as $k \rightarrow \infty$,
- (c) $\bar{\eta}_k(\bar{\omega}) = \eta_k(\bar{\omega}), \forall \bar{\omega} \in \bar{\Omega}$.

Hereafter, we denote these sequences again by $((v_m, \phi_m, \eta_m))_{m \in \mathbb{N}}$ and $((\bar{v}_m, \bar{\phi}_m, \bar{\eta}_m))_{m \in \mathbb{N}}$.

From the definition of \mathcal{Z}_T , we have $\bar{\mathbb{P}}$ -a.s.

$$\bar{v}_m \rightarrow v_* \text{ in } L_w^2(0, T; V_1) \cap L^2(0, T; H_1) \cap \mathbb{D}([0, T]; V_{1s}^*) \cap \mathbb{D}([0, T]; H_{1,w}), \quad (6.1)$$

and

$$\bar{\phi}_m \rightarrow \phi_* \text{ in } L_w^2(0, T; D(A_1^{3/2})) \cap L^2(0, T; H_2) \cap \mathbb{C}([0, T]; D(A_1)^*) \cap \mathbb{C}([0, T]; H_{2,w}). \quad (6.2)$$

6.2. Properties of the new processes and the limiting processes

We first note the following result due to Kuratowski Theorem, see [13].

Proposition 6.1. *The Borel subsets of $\mathbb{D}([0, T], H_1^m)$ are Borel subsets of $\mathcal{Z}_{T,1}$ and the Borel subsets of $\mathbb{D}([0, T], H_{2,m})$ are Borel subsets of $\mathcal{Z}_{T,2}$.*

As a corollary, we have the following results.

Corollary 6.2. *The sequence $(\bar{v}_m, \bar{\phi}_m)$ takes values in $H_{1,m} \times H_{2,m}$. Moreover, the laws of (v_m, ϕ_m) and $(\bar{v}_m, \bar{\phi}_m)$ are equal on $\mathbb{D}([0, T], H_{1,m}) \times \mathbb{C}([0, T], H_{2,m})$.*

Since the random variables (v_m, ϕ_m) and $(\bar{v}_m, \bar{\phi}_m)$ are identically distributed, it follows that

$$\sup_{m \in \mathbb{N}} \mathbb{E} \left[\sup_{s \in [0, T]} |(\bar{v}_m, \bar{\phi}_m)(s)|_{\mathcal{Y}}^{2p} \right] \leq C, \quad \sup_{m \in \mathbb{N}} \mathbb{E} \left[\int_0^T \|(\bar{v}_m, \bar{\phi}_m)(s)\|_{\mathcal{U}}^2 ds \right]^p \leq C, \quad p \geq 1. \quad (6.3)$$

From (6.3) and the Banach-Alaoglu Theorem we conclude there exists a subsequence of $(\bar{v}_m, \bar{\phi}_m)$ convergent weak star in $L^{2p}(\bar{\Omega}; L^\infty(0, T; H_1)) \times L^{2p}(\bar{\Omega}; L^\infty(0, T; H_2))$. It follows that $(v_*, \phi_*) \in (L^{2p}(\bar{\Omega}; L^\infty(0, T; H_1)) \cap L^p([0, T] \times \bar{\Omega}; V_1)) \times (L^{2p}(\bar{\Omega}; L^\infty(0, T; H_2)) \cap L^p([0, T] \times \bar{\Omega}; V_2))$, i.e.,

$$\mathbb{E} \left[\sup_{s \in [0, T]} |(v_*, \phi_*)(s)|_{\mathcal{Y}}^{2p} \right] \leq C, \quad \mathbb{E} \left[\int_0^T \|(v_*, \phi_*)(s)\|_{\mathcal{U}}^2 ds \right]^p \leq C, \quad p \geq 1. \quad (6.4)$$

6.3. Convergence of the new processes

Let $(w, \psi) \in \mathcal{U}$. For $t \in [0, T]$, we define $K_1^m(\bar{v}_m, \bar{\phi}_m, \bar{\eta}_m, w)(t)$ and $K_2^m(\bar{v}_m, \bar{\phi}_m, \bar{\eta}_m, \psi)(t)$ by:

$$\begin{aligned} K_1^m(\bar{v}_m, \bar{\phi}_m, \bar{\eta}_m, w)(t) &= \langle \bar{v}_m(0), w \rangle - \int_0^t \langle \mathcal{P}_m^1 A_0 \bar{v}_m(s), w \rangle ds - \int_0^t \langle \tilde{B}_0(\bar{v}_m(s), \bar{v}_m(s)), w \rangle ds \\ &\quad - \int_0^t \langle \tilde{R}_0(\bar{\phi}_m(s), \bar{\phi}_m(s)), w \rangle ds + \int_0^t \int_Z \langle \mathcal{P}_m^1 \sigma(s, \bar{v}_m(s), \bar{\phi}_m(s); z), w \rangle \tilde{\eta}_m(ds, dz), \\ K_2^m(\bar{v}_m, \bar{\phi}_m, \bar{\eta}_m, \psi)(t) &= \langle \bar{\phi}_m(0), \psi \rangle - \int_0^t \langle \mathcal{P}_m^2 A_1 \bar{\mu}_m(s), \psi \rangle ds - \int_0^t \langle \tilde{B}_1(\bar{v}_m(s), \bar{\phi}_m(s)), \psi \rangle ds, \end{aligned}$$

where $\bar{\mu}_m(s) = \mathcal{P}_m^2 A_1 \bar{\phi}_m(s) + f_m(\bar{\phi}_m(s))$.

For the limiting process (v_*, ϕ_*) , we also define $K_1(v_*, \phi_*, \eta_*, w)(t)$ and $K_2(v_*, \phi_*, \eta_*, \psi)(t)$ by:

$$\begin{aligned} K_1(v_*, \phi_*, \eta_*, w)(t) &= \langle v_*(0), w \rangle - \int_0^t \langle A_0 v_*(s), w \rangle ds - \int_0^t \langle B_0(v_*(s), v_*(s)), w \rangle ds \\ &\quad - \int_0^t \langle R_0(\phi_*(s), \phi_*(s)), w \rangle ds + \int_0^t \int_Z \langle \sigma(s, v_*(s), \phi_*(s); z), w \rangle \tilde{\eta}_*(ds, dz), \\ K_2(v_*, \phi_*, \eta_*, \psi)(t) &= \langle \phi_*(0), \psi \rangle - \int_0^t \langle A_1 \mu_*(s), \psi \rangle ds - \int_0^t \langle B_1(v_*(s), \phi_*(s)), \psi \rangle ds, \end{aligned}$$

where $\mu_*(s) = A_1 \phi_*(s) + f(\phi_*(s))$.

Our goal is to prove that

$$\lim_{m \rightarrow \infty} \|K_1^m(\bar{v}_m, \bar{\phi}_m, \bar{\eta}_m, w)(t) - K_1(v_*, \phi_*, \eta_*, w)(t)\|_{L^2([0, T] \times \bar{\Omega})} = 0$$

and

$$\lim_{m \rightarrow \infty} \|K_2^m(\bar{v}_m, \bar{\phi}_m, \bar{\eta}_m, \psi)(t) - K_2(v_*, \phi_*, \eta_*, \psi)(t)\|_{L^2([0,T] \times \bar{\Omega})} = 0.$$

We first note that

$$\begin{aligned} & \|K_1^m(\bar{v}_m, \bar{\phi}_m, \bar{\eta}_m, w)(t) - K_1(v_*, \phi_*, \eta_*, w)(t)\|_{L^2([0,T] \times \bar{\Omega})}^2 \\ &= \int_0^T \bar{\mathbb{E}} |K_1^m(\bar{v}_m, \bar{\phi}_m, \bar{\eta}_m, w)(t) - K_1(v_*, \phi_*, \eta_*, w)(t)|_{L^2}^2 dt, \\ & \|K_2^m(\bar{v}_m, \bar{\phi}_m, \bar{\eta}_m, \psi)(t) - K_2(v_*, \phi_*, \eta_*, \psi)(t)\|_{L^2([0,T] \times \bar{\Omega})}^2 \\ &= \int_0^T \bar{\mathbb{E}} |K_2^m(\bar{v}_m, \bar{\phi}_m, \bar{\eta}_m, \psi)(t) - K_2(v_*, \phi_*, \eta_*, \psi)(t)|_{L^2}^2 dt. \end{aligned}$$

Lemma 6.3. For all $w \in V_1$, we have

$$\begin{aligned} (a) \quad & \lim_{m \rightarrow \infty} \int_0^T \bar{\mathbb{E}} \left[\left| \int_0^t \langle \bar{v}_m(s) - v_*(s), w \rangle ds \right|^2 \right] dt = 0, \\ (b) \quad & \lim_{m \rightarrow \infty} \int_0^T \bar{\mathbb{E}} \left[|\langle \bar{v}_m(0) - v_*(0), w \rangle|^2 \right] dt = 0, \\ (c) \quad & \lim_{m \rightarrow \infty} \int_0^T \bar{\mathbb{E}} \left[\left| \int_0^t \langle \mathcal{P}_m^1 A_0 \bar{v}_m(s) - A_0 v_*(s), w \rangle ds \right|^2 \right] dt = 0, \\ (d) \quad & \lim_{m \rightarrow \infty} \int_0^T \bar{\mathbb{E}} \left[\left| \int_0^t \langle \tilde{B}_0(\bar{v}_m(s), \bar{v}_m(s)) - B_0(v_*(s), v_*(s)), w \rangle ds \right|^2 \right] dt = 0, \\ (e) \quad & \lim_{m \rightarrow \infty} \int_0^T \bar{\mathbb{E}} \left[\left| \int_0^t \langle \tilde{R}_0(\bar{\phi}_m(s), \bar{\phi}_m(s)) - R_0(\phi_*(s), \phi_*(s)), w \rangle ds \right|^2 \right] dt = 0, \\ (f) \quad & \lim_{m \rightarrow \infty} \int_0^T \bar{\mathbb{E}} \left[\left| \int_0^t \int_{\bar{\mathcal{Z}}} \langle \mathcal{P}_m^1 \sigma(s, \bar{v}_m(s), \bar{\phi}_m(s), z) - \sigma(s, v_*(s), \phi_*(s), z), w \rangle \tilde{\eta}_*(ds, dz) \right|^2 \right] dt = 0. \end{aligned}$$

Proof. The proof is similar to that given in [13,39,40], therefore we will skip some details. For (a), we note that

$$\begin{aligned} \|\langle \bar{v}_m(s) - v_*(s), w \rangle\|_{L^2([0,T] \times \bar{\Omega})}^2 &= \int_{\bar{\Omega}} \int_0^T |\langle \bar{v}_m(s) - v_*(s), w \rangle|^2 dt \bar{\mathbb{P}}(d\omega) \\ &= \bar{\mathbb{E}} \left[\int_0^T |\langle \bar{v}_m(s) - v_*(s), w \rangle|^2 dt \right]. \end{aligned}$$

Moreover, we have

$$\int_0^T |\langle \bar{v}_m(s) - v_*(s), w \rangle|^2 dt \leq c \|w\|_{V_{1s}^*}^2 \int_0^T \|\bar{v}_m(s) - v_*(s)\|_{V_{1s}^*}^2 dt. \quad (6.5)$$

From (6.1), we have $\bar{v}_m \rightarrow v_*$ in $\mathbb{D}([0, T]; V_{1s}^*)$ and from (6.3), we also have $\sup_{s \in [0, T]} |\bar{v}_m(t)|_{L^2} < \infty$, $\bar{\mathbb{P}}$ -a.s. The embedding $H_1 \hookrightarrow V_{1s}^*$ is continuous. Then by Dominated Convergence Theorem we derive that $\bar{v}_m \rightarrow v_*$ in $L^2(0, T; V_{1s}^*)$. From (6.5), we have

$$\lim_{m \rightarrow \infty} \int_0^T |\langle \bar{v}_m(s) - v_*(s), w \rangle|^2 dt = 0.$$

Moreover from (6.3), we also have

$$\begin{aligned} \bar{\mathbb{E}} \left[\left| \int_0^T |\bar{v}_m(t) - v_*(t)|_{L^2}^2 dt \right|^r \right] &\leq c \bar{\mathbb{E}} \left[\int_0^T (|\bar{v}_m(t)|_{L^2}^{2r} + |v_*(t)|_{L^2}^{2r}) dt \right] \\ &\leq c \bar{\mathbb{E}} \left[\sup_{t \in [0, T]} |\bar{v}_m(t)|_{L^2}^{2r} \right] \leq C. \end{aligned}$$

It follows from the Vitali Theorem that

$$\lim_{m \rightarrow \infty} \int_0^T \bar{\mathbb{E}} \left[\left| \int_0^t \langle \bar{v}_m(s) - v_*(s), w \rangle ds \right|^2 \right] dt = 0,$$

which proves (a).

To prove (b), we note that $\bar{v}_m \rightarrow v_*$ in $\mathbb{D}([0, T]; H_{1,w})$ $\bar{\mathbb{P}}$ -a.s. and v_* is right continuous at $t = 0$. It follows that $\langle \bar{v}_m(0) - v_*(0), w \rangle \rightarrow 0$, $\bar{\mathbb{P}}$ -a.s. From (6.3) and applying Vitali's Theorem, we derive that

$$\lim_{m \rightarrow \infty} \bar{\mathbb{E}} [|\langle \bar{v}_m(0) - v_*(0), w \rangle|^2] = 0,$$

from which we conclude that

$$\lim_{m \rightarrow \infty} \|\langle \bar{v}_m(0) - v_*(0), w \rangle\|_{L^2([0, T] \times \bar{\Omega})} = 0,$$

and (b) is proved.

To prove (c), we note that from (6.1), we have $\bar{v}_m \rightarrow v_*$ in $L_w^2([0, T]; V_{1s})$ $\bar{\mathbb{P}}$ -a.s. and from which we derive that for $\bar{\mathbb{P}}$ -a.s.

$$\lim_{m \rightarrow \infty} \int_0^t \langle \mathcal{P}_m^1 A_0 \bar{v}_m(s) - A_0 v_*(s), w \rangle ds = 0.$$

Using Hölder inequality and (6.3), we also have

$$\begin{aligned} \bar{\mathbb{E}} \left[\left| \int_0^t \langle \mathcal{P}_m^1 A_0 \bar{v}_m(s), w \rangle ds \right|^{2+r} \right] &\leq c \|w\|^{2+r} \bar{\mathbb{E}} \left[\left(\int_0^T \|\bar{v}_m(s)\| ds \right)^{2+r} \right] \\ &\leq c \bar{\mathbb{E}} \left[\left(\int_0^T \|\bar{v}_m(s)\|^2 ds \right)^{1+r/2} \right] \leq C. \end{aligned}$$

By the Vitali Theorem and the Dominated Convergence Theorem, we conclude that

$$\lim_{m \rightarrow \infty} \int_0^T \bar{\mathbb{E}} \left[\left| \int_0^t \langle \mathcal{P}_m^1 A_0 \bar{v}_m(s) - A_0 v_*(s), w \rangle ds \right|^2 \right] dt = 0,$$

and (c) is proved.

For (d), we first note that

$$\begin{aligned} &\lim_{m \rightarrow \infty} \int_0^t \langle \tilde{B}_0(\bar{v}_m(s), \bar{v}_m(s)) - B_0(v_*(s), v_*(s)), w \rangle ds \\ &= \lim_{m \rightarrow \infty} \int_0^t \langle B_0(\bar{v}_m(s), \bar{v}_m(s)) - B_0(v_*(s), v_*(s)), \mathcal{P}_m^1 w \rangle ds = 0 \quad \bar{\mathbb{P}} - a.s. \end{aligned}$$

Using Hölder inequality, we also have

$$\bar{\mathbb{E}} \left[\left| \int_0^t \langle \tilde{B}_0(\bar{v}_m(s), \bar{v}_m(s)), w \rangle ds \right|^{2+r} \right] \leq c \bar{\mathbb{E}} \left[\sup_{t \in [0, T]} |\bar{v}_m(t)|_{L^2}^{2+2r} \right] \leq C.$$

By the Vitali Theorem and the Dominated Convergence Theorem, we conclude that

$$\lim_{m \rightarrow \infty} \int_0^T \bar{\mathbb{E}} \left[\left| \int_0^t \langle \tilde{B}_0(\bar{v}_m(s), \bar{v}_m(s)) - B_0(v_*(s), v_*(s)), w \rangle ds \right|^2 \right] dt = 0,$$

and (d) is proved.

For (e), we note that

$$\begin{aligned} &\lim_{m \rightarrow \infty} \int_0^t \langle \tilde{R}_0(\bar{\phi}_m(s), \bar{\phi}_m(s)) - R_0(\phi_*(s), \phi_*(s)), w \rangle ds \\ &= \lim_{m \rightarrow \infty} \int_0^t \langle R_0(\bar{\phi}_m(s), \bar{\phi}_m(s)) - R_0(\phi_*(s), \phi_*(s)), \mathcal{P}_m^1 w \rangle ds = 0 \quad \bar{\mathbb{P}} - a.s. \end{aligned}$$

Using Hölder inequality and (2.23), we also have

$$\bar{\mathbb{E}} \left[\left| \int_0^t \langle \tilde{R}_0(\bar{\phi}_m(s), \bar{\phi}_m(s)), w \rangle ds \right|^{2+r} \right] \leq c \bar{\mathbb{E}} \left[\sup_{t \in [0, T]} \|\bar{\phi}_m(t)\|^{2+2r} \right] \leq C.$$

By the Vitali Theorem and the Dominated Convergence Theorem, we conclude that

$$\lim_{m \rightarrow \infty} \int_0^T \bar{\mathbb{E}} \left[\left| \int_0^t \langle \tilde{R}_0(\bar{\phi}_m(s), \bar{\phi}_m(s)) - R_0(\phi_*(s), \phi_*(s)), w \rangle ds \right|^2 \right] dt = 0,$$

and (d) is proved.

For the stochastic term, we note that

$$\begin{aligned} & \int_0^t \int_Z |\langle \mathcal{P}_m^1 \sigma(s, \bar{v}_m(s), \bar{\phi}_m(s), z) - \sigma(s, v_*(s), \phi_*(s), z), w \rangle|^2 d\nu(z) ds \\ & \leq \int_0^t \int_Z |\sigma(s, \bar{v}_m(s), \bar{\phi}_m(s), z) - \sigma(s, v_*(s), \phi_*(s), z)|_{\mathcal{Y}}^2 |w|_{L^2}^2 d\nu(z) ds \\ & \leq C \int_0^t |(\bar{v}_m(s), \bar{\phi}_m(s)) - (v_*(s), \phi_*(s))|_{\mathcal{Y}}^2 ds \\ & \leq C \int_0^T |(\bar{v}_m(s), \bar{\phi}_m(s)) - (v_*(s), \phi_*(s))|_{\mathcal{Y}}^2 ds. \end{aligned}$$

We recall that

$$(\bar{v}_m, \bar{\phi}_m) \rightarrow (v_*, \phi_*) \text{ in } L^2(0, T; \mathcal{Y}), \quad \bar{\mathbb{P}} - a.s.$$

It follows that for all $t \in [0, T]$, we have

$$\lim_{m \rightarrow \infty} \int_0^t \int_Z |\langle \mathcal{P}_m^1 \sigma(s, \bar{v}_m(s), \bar{\phi}_m(s), z) - \sigma(s, v_*(s), \phi_*(s), z), w \rangle|^2 d\nu(z) ds = 0.$$

We can also check that for every $t \in [0, T]$, $r \geq 1$ and $m \in \mathbb{N}$, we have

$$\begin{aligned} & \bar{\mathbb{E}} \left[\left| \int_0^t \int_Z |\langle \mathcal{P}_m^1 \sigma(s, \bar{v}_m(s), \bar{\phi}_m(s), z) - \sigma(s, v_*(s), \phi_*(s), z), w \rangle|^2 d\nu(z) ds \right|^r \right] \\ & \leq C |w|_{L^2}^{2r} \bar{\mathbb{E}} \left[\left| \int_0^t \int_Z |\sigma(s, \bar{v}_m(s), \bar{\phi}_m(s), z)|_{L^2}^2 + |\sigma(s, v_*(s), \phi_*(s), z)|_{L^2}^2 d\nu(z) ds \right|^r \right] \\ & \leq C \bar{\mathbb{E}} \left[\left| \int_0^t (2 + |(\bar{v}_m, \bar{\phi}_m)(s)|_{\mathcal{Y}}^2 + |(v_*, \phi_*)(s)|_{\mathcal{Y}}^2) ds \right|^r \right] \leq C. \end{aligned}$$

From the Vitali Theorem, it follows that

$$\lim_{m \rightarrow \infty} \bar{\mathbb{E}} \left[\left| \int_0^t \int_Z |\langle \mathcal{P}_m^1 \sigma(s, \bar{v}_m(s), \bar{\phi}_m(s), z) - \sigma(s, v_*(s), \phi_*(s), z), w \rangle|^2 d\nu(z) ds \right|^r \right] = 0.$$

Using the Dominated Convergence Theorem, we conclude that as in [13,39,40] that

$$\lim_{m \rightarrow \infty} \int_0^T \bar{\mathbb{E}} \left[\int_0^t \int_Z |\langle \mathcal{P}_m^1 \sigma(s, \bar{v}_m(s), \bar{\phi}_m(s), z) - \sigma(s, v_*(s), \phi_*(s), z), w \rangle|^2 d\nu(z) ds \right] dt = 0,$$

which proved (f). \square

Lemma 6.4. For all $\psi \in V_2$, we have

$$(a) \quad \lim_{m \rightarrow \infty} \int_0^T \bar{\mathbb{E}} \left[\left| \int_0^t \langle \bar{\phi}_m(s) - \phi_*(s), \psi \rangle ds \right|^2 \right] dt = 0,$$

$$(b) \quad \lim_{m \rightarrow \infty} \bar{\mathbb{E}} \left[|\langle \bar{\phi}_m(0) - \phi_*(0), \psi \rangle|^2 \right] = 0,$$

$$(c) \quad \lim_{m \rightarrow \infty} \int_0^T \bar{\mathbb{E}} \left[\left| \int_0^t \langle \mathcal{P}_m^2 A_1 \bar{\mu}_m(s) - A_1 \mu_*(s), \psi \rangle ds \right|^2 \right] dt = 0,$$

$$(d) \quad \lim_{m \rightarrow \infty} \int_0^T \bar{\mathbb{E}} \left[\left| \int_0^t \langle \tilde{B}_1(\bar{v}_m(s), \bar{\phi}_m(s)) - B_1(v_*(s), \phi_*(s)), \psi \rangle ds \right|^2 \right] dt = 0.$$

Proof. The proof is similar to that given in [13,39,40]. For the reader convenience, we will repeat most of the steps. For (a), we note that

$$\begin{aligned} \|\langle \bar{\phi}_m(s) - \phi_*(s), \psi \rangle\|_{L^2([0,T] \times \bar{\Omega})}^2 &= \int_{\bar{\Omega}} \int_0^T |\langle \bar{\phi}_m(s) - \phi_*(s), \psi \rangle|^2 dt \bar{\mathbb{P}}(d\omega) \\ &= \bar{\mathbb{E}} \left[\int_0^T |\langle \bar{\phi}_m(s) - \phi_*(s), \psi \rangle|^2 dt \right]. \end{aligned}$$

Moreover, we have

$$\int_0^T |\langle \bar{\phi}_m(s) - \phi_*(s), \psi \rangle|^2 dt \leq c \|\psi\|^2 \int_0^T \|\bar{\phi}_m(s) - \phi_*(s)\|^2 dt. \quad (6.6)$$

From (6.2), we have $\bar{\phi}_m \rightarrow \phi_*$ in $\mathbb{D}([0, T]; D(A_1)^*)$ and from (6.3), we also have $\sup_{s \in [0, T]} \|\bar{\phi}_m(t)\| < \infty$, $\bar{\mathbb{P}}$ -a.s.

The embedding $V_2 \hookrightarrow D(A_1)^*$ is continuous. Then by Dominated Convergence Theorem we derive that $\bar{\phi}_m \rightarrow \phi_*$ in $L^2(0, T; D(A_1)^*)$. From (6.6), we have

$$\lim_{m \rightarrow \infty} \int_0^T |\langle \bar{\phi}_m(s) - \phi_*(s), \psi \rangle|^2 dt = 0.$$

Moreover, from (6.3) we also have

$$\begin{aligned} \bar{\mathbb{E}} \left[\left| \int_0^T \|\bar{\phi}_m(t) - \phi_*(t)\|^2 dt \right|^r \right] &\leq c \bar{\mathbb{E}} \left[\int_0^T (\|\bar{\phi}_m(t)\|^{2r} + \|\phi_*(t)\|^{2r}) dt \right] \\ &\leq c \bar{\mathbb{E}} \left[\sup_{t \in [0, T]} \|\bar{\phi}_m(t)\|^{2r} \right] + C \leq C. \end{aligned}$$

It follows from the Vitali Theorem that

$$\lim_{m \rightarrow \infty} \int_0^T \bar{\mathbb{E}} \left[\left| \int_0^t \langle \bar{\phi}_m(s) - \phi_*(s), \psi \rangle ds \right|^2 \right] dt = 0,$$

which proves (a).

To prove (b), we note that $\bar{\phi}_m \rightarrow \phi_*$ in $\mathbb{D}([0, T]; V_{2,w})$ $\bar{\mathbb{P}}$ -a.s. and $\bar{\phi}_m$ is right continuous at $t = 0$. It follows that $\langle \bar{\phi}_m(0) - \phi_*(0), \psi \rangle \rightarrow 0$, $\bar{\mathbb{P}}$ -a.s. From (6.3) and applying Vitali's Theorem, we derive that

$$\lim_{m \rightarrow \infty} \bar{\mathbb{E}} [|\langle \bar{\phi}_m(0) - \psi_*(0), \psi \rangle|^2],$$

from which we conclude that

$$\lim_{m \rightarrow \infty} \|\langle \bar{\phi}_m(0) - \phi_*(0), \psi \rangle\|_{L^2([0, T] \times \bar{\Omega})} = 0,$$

and (b) is proved.

To prove (c), we note that from (6.2), we have $\bar{\phi}_m \rightarrow \phi_*$ in $L_w^2([0, T]; D(A_1^{3/2}))$ $\bar{\mathbb{P}}$ -a.s. and from which we derive the following equality for $\bar{\mathbb{P}}$ -a.s.

$$\begin{aligned} \int_0^t \langle \mathcal{P}_m^2 A_1 \bar{\mu}_m(s) - A_1 \mu_*(s), \psi \rangle ds &= \int_0^t \langle \bar{\mu}_m(s) - \mu_*(s), A_1 \psi \rangle ds \\ &= \int_0^t \langle \mathcal{P}_m^2 A_1 \bar{\phi}_m(s) - A_1 \phi_*(s), A_1 \psi \rangle ds + \int_0^t \langle f_m(\bar{\phi}_m) - f(\phi_*), A_1 \psi \rangle ds. \end{aligned}$$

As in part (c) of Lemma 6.3, we can check that

$$\lim_{m \rightarrow \infty} \int_0^T \bar{\mathbb{E}} \left[\left| \int_0^t \mathcal{P}_m^2 \langle A_1 \bar{\phi}_m(s) - A_1 \phi_*(s), A_1 \psi \rangle ds \right|^2 \right] dt = 0.$$

Let us prove that

$$\lim_{m \rightarrow \infty} \int_0^T \bar{\mathbb{E}} \left[\left| \int_0^t \langle f_m(\bar{\phi}_m) - f(\phi_*), A_1 \psi \rangle ds \right|^2 \right] dt = 0.$$

Using (2.9), (2.10), we can check that

$$\lim_{m \rightarrow \infty} \int_0^t \langle f_m(\bar{\phi}_m) - f(\phi_*), A_1 \psi \rangle ds = 0.$$

Setting $q \equiv (r+2)(k+1)$, where k is given in (2.10), it follows that

$$\begin{aligned} \bar{\mathbb{E}} \left[\left| \int_0^t \langle f_m(\bar{\phi}_m), A_1 \psi \rangle ds \right|^{r+2} \right] &\leq C + |A_1 \psi|_{L^2}^{r+2} t^{r+1} \bar{\mathbb{E}} \int_0^t \|\bar{\phi}_m\|^q ds \\ &\leq C + C \bar{\mathbb{E}} \left[\sup_{t \in [0, T]} \|\bar{\phi}_m(t)\|^q \right] \leq C. \end{aligned}$$

By the Vitali Theorem and the Dominated Convergence Theorem, we conclude that

$$\lim_{m \rightarrow \infty} \int_0^T \bar{\mathbb{E}} \left[\left| \int_0^t \langle f_m(\bar{\phi}_m) - f(\phi_*), A_1 \psi \rangle ds \right|^2 \right] dt = 0,$$

and (c) is proved.

For (d), we first note that

$$\begin{aligned} &\lim_{m \rightarrow \infty} \int_0^t \langle \tilde{B}_1(\bar{v}_m(s), \bar{\phi}_m(s)) - B_1(v_*(s), \phi_*(s)), \psi \rangle ds \\ &= \lim_{m \rightarrow \infty} \int_0^t \langle B_1(\bar{v}_m(s), \bar{\phi}_m(s)) - B_1(v_*(s), \phi_*(s)), \mathcal{P}_m^2 \psi \rangle ds = 0, \quad \bar{\mathbb{P}} - a.s. \end{aligned}$$

Using Hölder inequality and (6.3), we also have

$$\bar{\mathbb{E}} \left[\left| \int_0^t \langle \tilde{B}_1(\bar{v}_m(s), \bar{\phi}_m(s)), \psi \rangle ds \right|^{2+r} \right] \leq c \bar{\mathbb{E}} \left[\sup_{t \in [0, T]} |\bar{v}_m(t)|_{L^2}^{2+r} \|\bar{\phi}_m(t)\|^{2+r} \right] \leq C.$$

By the Vitali Theorem and the Dominated Convergence Theorem, we conclude that

$$\lim_{m \rightarrow \infty} \int_0^T \bar{\mathbb{E}} \left[\left| \int_0^t \langle \tilde{B}_1(\bar{v}_m(s), \bar{\phi}_m(s)) - B_0(v_*(s), \phi_*(s)), w \rangle ds \right|^2 \right] dt = 0,$$

and (d) is proved. \square

Theorem 6.5. *There exists a martingale solution $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}, \bar{v}, \bar{\phi}, \bar{\eta})$ of (2.34) provided that Assumption 1 given in Section 2 is satisfied.*

Proof. From Lemma 6.3, we have

$$\lim_{m \rightarrow \infty} \|\bar{v}_m(\cdot) - v_*(\cdot), w\|_{L^2([0, T] \times \bar{\Omega})} = 0 \quad (6.7)$$

and

$$\lim_{m \rightarrow \infty} \|\mathcal{K}_m^1(\bar{v}_m, \bar{\phi}_m, \bar{\eta}_m, w) - \mathcal{K}^1(v_*, \phi_*, \eta_*, w)\|_{L^2([0, T] \times \bar{\Omega})} = 0. \quad (6.8)$$

From Lemma 6.4, we also have

$$\lim_{m \rightarrow \infty} \|\langle \bar{\phi}_m(\cdot) - \phi_*(\cdot), \psi \rangle\|_{L^2([0,T] \times \bar{\Omega})} = 0 \quad (6.9)$$

and

$$\lim_{m \rightarrow \infty} \|\mathcal{K}_m^2(\bar{v}_m, \bar{\phi}_m, \bar{\eta}_m, \psi) - \mathcal{K}^2(v_*, \phi_*, \eta_*, \psi)\|_{L^2([0,T] \times \bar{\Omega})} = 0. \quad (6.10)$$

Since (v_m, ϕ_m) is the solution of Galerkin approximation (4.1), it follows that for all $t \in [0, T]$, we have for \mathbb{P} -a.s.

$$\langle v_m(t), w \rangle = \mathcal{K}_m^1(v_m, \phi_m, \eta_m, w)(t),$$

$$\langle \phi_m(t), \psi \rangle = \mathcal{K}_m^2(v_m, \phi_m, \eta_m, \psi)(t).$$

This gives

$$\int_0^T \mathbb{E}[\langle v_m(t), w \rangle - \mathcal{K}_m^1(v_m, \phi_m, \eta_m, w)(t)]^2 dt = 0,$$

$$\int_0^T \mathbb{E}[\langle \phi_m(t), \psi \rangle - \mathcal{K}_m^2(v_m, \phi_m, \eta_m, \psi)(t)]^2 dt = 0.$$

Since $\mathcal{L}(v_m, \phi_m, \eta_m) = \mathcal{L}(\bar{v}_m, \bar{\phi}_m, \bar{\eta}_m)$, it follows that

$$\int_0^T \mathbb{E}[\langle \bar{v}_m(t), w \rangle - \mathcal{K}_m^1(\bar{v}_m, \bar{\phi}_m, \bar{\eta}_m, w)(t)]^2 dt = 0,$$

$$\int_0^T \mathbb{E}[\langle \bar{\phi}_m(t), \psi \rangle - \mathcal{K}_m^2(\bar{v}_m, \bar{\phi}_m, \bar{\eta}_m, \psi)(t)]^2 dt = 0.$$

From (6.7)-(6.10), we conclude that

$$\int_0^T \mathbb{E}[\langle v_*(t), w \rangle - \mathcal{K}^1(v_*, \phi_*, \eta_*, w)(t)]^2 dt = 0,$$

$$\int_0^T \mathbb{E}[\langle \phi_*(t), \psi \rangle - \mathcal{K}^2(v_*, \phi_*, \eta_*, \psi)(t)]^2 dt = 0.$$

Therefore, for l -almost all $t \in [0, T]$ and \mathbb{P} -almost all $\omega \in \bar{\Omega}$, we derive that

$$\langle v_*(t), w \rangle = \mathcal{K}^1(v_*, \phi_*, \eta_*, w)(t),$$

$$\langle \phi_*(t), \psi \rangle = \mathcal{K}^2(v_*, \phi_*, \eta_*, \psi)(t),$$

which gives

$$\begin{aligned} (v_*(t), w) &= (v_*(0), w) - \int_0^t \langle A_0 v_* + B_0(v_*, v_*) + R_0(\phi_*, \phi_*) - g_1(s, v_*, \phi_*), w \rangle ds \\ &+ \int_0^t \int_Z \langle \sigma(s, v_*, \phi_*, z), w \rangle \bar{\eta}(dz, ds), \\ (\phi_*(t), \psi) &= (\phi_*(0), \psi) - \int_0^t \langle A_1 \mu_* + B_1(v_*, \phi_*), \psi \rangle ds = 0, \quad \mu_* = A_1 \phi_* + f(\phi_*). \end{aligned} \quad (6.11)$$

Since (v_*, ϕ_*) is $\mathcal{Z}_{T,1} \times \mathcal{Z}_{T,2}$ -valued random variable, and v_*, ϕ_* is weakly càdlàg, and weakly continuous respectively, we derive that (6.11) holds true for all $t \in [0, T]$ and all $(w, \psi) \in \mathcal{U}$. Setting $(\bar{v}, \bar{\phi}, \bar{\eta}) = (v_*, \phi_*, \eta_*)$, we obtain that $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{v}, \bar{\phi}, \bar{\eta})$ is a martingale solution to (2.34). \square

7. Pathwise uniqueness of weak solutions

In this part, we prove the pathwise uniqueness of weak solution to (2.34) in the 2D case.

Lemma 7.1. *We assume that Assumption 1 given in Section 2 is satisfied. Let $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}, \bar{v}, \bar{\phi}, \bar{\eta})$ be a martingale solution to (2.34). Let the initial data $(v_0, \phi_0) \in \mathcal{Y}$. Then for $\bar{\mathbb{P}}$ -almost all $\bar{\omega} \in \bar{\Omega}$, the trajectories of $\bar{v}(\cdot, \bar{\omega})$ is almost everywhere equal to a Càdlàg V_1 -valued function and $\bar{\phi}(\cdot, \bar{\omega})$ is almost everywhere equal to a continuous H_2 -valued function defined on $[0, T]$.*

Proof. We recall that $(\bar{v}, \bar{\phi})$ satisfies

$$\begin{aligned} \bar{v}(t) &= v_0 - \int_0^t (A_0 \bar{v} + B_0(\bar{v}, \bar{v}) + R_0(\bar{\phi}, \bar{\phi}) - g_1(s, \bar{v}, \bar{\phi})) ds \\ &+ \int_0^t \int_Z \sigma(s, \bar{v}, \bar{\phi}, z) \bar{\eta}(dz, ds), \\ \bar{\phi}(t) &= \phi_0 - \int_0^t (A_1 \bar{\mu} + B_1(\bar{v}, \bar{\phi})) ds = 0, \quad \bar{\mu} = A_1 \bar{\phi} + f(\bar{\phi}). \end{aligned} \quad (7.1)$$

We need to check that the right hand side of (7.1)₁ is V_1^* -valued and that of (7.1)₂ is V_2^* -valued.

From Lemma 4.5 and Lemma 4.6, we have (see also (6.4))

$$\bar{\mathbb{E}} \int_0^T \|A_0 \bar{v}(s)\|_{V_1^*}^2 ds \leq \bar{\mathbb{E}} \int_0^T \|\bar{v}(s)\|^2 ds < \infty. \quad (7.2)$$

From $\|B_0(\bar{v}, \bar{v})\|_{V_1^*} \leq c|\bar{v}|_{L^2}\|\bar{v}\|$, we derive from (6.4) that

$$\begin{aligned}
\bar{\mathbb{E}} \int_0^T \|B_0(\bar{v}(s), \bar{v}(s))\|_{V_1^*}^2 ds &\leq \bar{\mathbb{E}} \int_0^T |\bar{v}(s)|_{L^2}^2 \|\bar{v}(s)\|^2 ds \\
&\leq C \left[\bar{\mathbb{E}} \sup_{s \in [0, T]} |\bar{v}(s)|_{L^2}^4 \right]^{1/2} \left[\bar{\mathbb{E}} \left(\int_0^T \|\bar{v}(s)\|^2 ds \right)^2 \right]^{1/2} < \infty.
\end{aligned} \tag{7.3}$$

From $\|R_0(\bar{\phi}, \bar{\phi})\|_{V_1^*} \leq c\|\bar{\phi}\| \|A_1 \bar{\phi}\|_{L^2}$, we derive from (6.4) that

$$\begin{aligned}
\bar{\mathbb{E}} \int_0^T \|R_0(\bar{\phi}(s), \bar{\phi}(s))\|_{V_1^*}^2 ds &\leq \bar{\mathbb{E}} \int_0^T \|\bar{\phi}(s)\|^2 |A_1 \bar{\phi}(s)|_{L^2}^2 ds \\
&\leq C \left[\bar{\mathbb{E}} \sup_{s \in [0, T]} \|\bar{\phi}(s)\|^4 \right]^{1/2} \left[\bar{\mathbb{E}} \left(\int_0^T |A_1 \bar{\phi}(s)|_{L^2}^2 ds \right)^2 \right]^{1/2} < \infty.
\end{aligned} \tag{7.4}$$

Using the Itô isometry, we also have

$$\begin{aligned}
\bar{\mathbb{E}} \int_0^T \left| \int_Z \sigma(s, \bar{v}, \bar{\phi}, z) \tilde{\eta}(dz, ds) \right|_{L^2}^2 &\leq \bar{\mathbb{E}} \int_0^T \int_Z |\sigma(s, \bar{v}, \bar{\phi}, z)|_{L^2}^2 \nu(dz) ds \\
&\leq C \bar{\mathbb{E}} \int_0^T (1 + |(\bar{v}(s), \bar{\phi}(s))|_{\mathcal{Y}}^2) ds < \infty.
\end{aligned} \tag{7.5}$$

This proves that the right hand side of (7.1)₁ is V_1^* -valued.

Similarly as in (7.2) and (7.4), we also have (see (6.4))

$$\begin{aligned}
\bar{\mathbb{E}} \int_0^T \|A_1 \bar{\mu}(s)\|_{V_2^*}^2 ds &\leq C \bar{\mathbb{E}} \int_0^T \|\bar{\mu}(s)\|^2 ds < \infty, \\
\bar{\mathbb{E}} \int_0^T \|B_1(\bar{v}(s), \bar{\phi}(s))\|_{V_2^*}^2 ds &\leq C \bar{\mathbb{E}} \int_0^T |\bar{v}(s)|_{L^2}^2 |A_1 \bar{\phi}(s)|_{L^2}^2 ds \\
&\leq C \left[\bar{\mathbb{E}} \sup_{s \in [0, T]} |\bar{v}(s)|_{L^2}^4 \right]^{1/2} \left[\bar{\mathbb{E}} \left(\int_0^T |A_1 \bar{\phi}(s)|_{L^2}^2 ds \right)^2 \right]^{1/2} < \infty.
\end{aligned} \tag{7.7}$$

This proves the second part of the remark. \square

Proposition 7.2. Let $(v_0^1, \phi_0^1), (v_0^2, \phi_0^2)$ be two \mathcal{F}_0 -measurable and square integrable \mathcal{Y} -valued random variables. Let $(v_1, \phi_1), (v_2, \phi_2)$ be solutions defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{F}, \bar{\eta})$ to (2.34) corresponding to $(v_0^1, \phi_0^1), (v_0^2, \phi_0^2)$ respectively. Then there exists a constant $C > 0$ such that

$$\mathbb{E} \delta(t) |(v_1, \phi_1) - (v_2, \phi_2)|_{\mathcal{Y}}^2 \leq C \mathbb{E} |(v_0^1, \phi_0^1) - (v_0^2, \phi_0^2)|_{\mathcal{Y}}^2, \tag{7.8}$$

for all $t \in [0, T]$, where $\delta(t)$ is defined by (7.23).

Moreover, if $(v_0^1, \phi_0^1) = (v_0^2, \phi_0^2)$ almost surely, then for any $t \in [0, T]$,

$$\mathbb{P}((v_1, \phi_1)(t) = (v_2, \phi_2)(t)) = 1. \quad (7.9)$$

Proof. Let $(v_1, \phi_1), (v_2, \phi_2)$ be variational solutions to (2.34). Let $(w, \psi, \mu) = (v_1, \phi_1, \mu_1) - (v_2, \phi_2, \mu_2)$, $\bar{\mu} = \mu - \langle \mu \rangle$. Then (w, ψ) satisfies

$$\left\{ \begin{array}{l} dw + [A_0 w + B_0(v_2, w) + B_0(w, v_1)]dt + [R_0(\phi_2, \psi) + R_0(\psi, \phi_1)]dt \\ = [g_1(t, v_1, \phi_1) - g_1(t, v_2, \phi_2)]dt + \int_Z (\sigma(t, v_1, \phi_1, z) - \sigma(t, v_2, \phi_2, z))\tilde{\eta}(dz, dt), \\ \frac{d\psi}{dt} + A_1 \bar{\mu} + B_1(v_2, \psi) + B_1(w, \phi_1) = 0, \quad \mu = A_1 \psi + f(\phi_1) - f(\phi_2), \\ (w, \psi)(0) = (0, 0) \end{array} \right. \quad (7.10)$$

Reasoning as in the proof of Lemma 4.5, applying Itô's formula to $|w|_{L^2}^2$ and using (7.10)₁, we derive that

$$\begin{aligned} & |w|_{L^2}^2 + 2 \int_0^t (\|w\|^2 + b_0(w, v_1, w))ds + 2 \int_0^t \langle R_0(\phi_2, \psi) + R_0(\psi, \phi_1), w \rangle ds \\ &= 2 \int_0^t \langle g_1(t, v_1, \phi_1) - g_1(t, v_2, \phi_2), w \rangle ds \\ &+ 2 \int_0^t \int_Z (w(s-), \sigma(s, v_1(s), \phi_1(s), z) - \sigma(s, v_2(s), \phi_2(s), z))\tilde{\eta}(dz, ds) \\ &+ \int_0^t \int_Z |\sigma(s, v_1(s), \phi_1(s), z) - \sigma(s, v_2(s), \phi_2(s), z))|_{L^2}^2 \eta(ds, dz). \end{aligned} \quad (7.11)$$

Now we take the duality of (7.10)₂ and (7.10)₃ with $A_1 \bar{\mu} - \zeta A_1 \psi$ and $A_1 \psi$ respectively, where $\zeta > 0$ is small enough and will be selected later. Adding the resulting equality to (7.11), we derive that

$$\begin{aligned} & |w(t)|^2 + \|\psi(t)\|^2 + 2 \int_0^t (\|w\|^2 + \zeta |A_1 \psi|_{L^2}^2 + \|\bar{\mu}\|^2)ds = -2 \int_0^t b_0(w, v_1, w)ds \\ & - 2 \int_0^t (\langle R_0(\phi_2, \psi), w \rangle + \langle R_0(\psi, \phi_1), w \rangle + b_1(w, \phi_1, A_1 \psi) + b_1(v_2, \psi, A_1 \psi))ds \\ & \int_0^t [\zeta \langle \bar{\mu}, A_1 \psi \rangle] + 2\zeta \langle f(\phi_1) - f(\phi_2), A_1 \psi \rangle - 2 \langle f(\phi_1) - f(\phi_2), A_1 \bar{\mu} \rangle]ds \end{aligned} \quad (7.12)$$

$$\begin{aligned}
& +2 \int_0^t \langle g_1(t, v_1, \phi_1) - g_1(t, v_2, \phi_2), w \rangle ds \\
& +2 \int_0^t \int_Z (w(s-), \sigma(s, v_1(s), \phi_1(s), z) - \sigma(s, v_2(s), \phi_2(s), z)) \tilde{\eta}(dz, ds) \\
& + \int_0^t \int_Z |\sigma(s, v_1(s), \phi_1(s), z) - \sigma(s, v_2(s), \phi_2(s), z))|_{L^2}^2 \eta(dz, ds).
\end{aligned}$$

Note that

$$|b_0(w, v_1, w)| \leq \frac{1}{8} \|w\|^2 + c \|v_1\|^2 |w|_{L^2}^2, \quad (7.13)$$

$$|\langle R_0(\psi, \phi_1), w \rangle| = |b_1(w, \phi_1, A_1 \psi)| \leq \frac{1}{8} (\|w\|^2 + \zeta |A_1 \psi|_{L^2}^2) + c |w|_{L^2}^2 \|\phi_1\|^2 |A_1 \phi_1|_{L^2}^2, \quad (7.14)$$

$$|\langle R_0(\phi_2, \psi), w \rangle| = |b_1(w, \psi, A_1 \phi_2)| \leq \frac{1}{8} (\|w\|^2 + \zeta |A_1 \psi|_{L^2}^2) + c (|w|_{L^2}^2 + \|\nabla \psi\|_{L^2}^2) \|\phi_2\|^2 |\phi_2|_{H^2}^2, \quad (7.15)$$

$$\zeta |\langle f(\phi_1) - f(\phi_2), A_1 \psi \rangle| \leq \frac{\zeta}{8} |A_1 \psi|_{L^2}^2 + Q_1(|\phi_1|_{H^1}, |\phi_2|_{H^1}) \|\psi\|^2, \quad (7.16)$$

$$|\langle f(\phi_1) - f(\phi_2), A_1 \bar{\mu} \rangle| \leq \frac{1}{2} |A_1^{1/2} \bar{\mu}|_{L^2}^2 + Q_1(|\phi_1|_{H^1}, |\phi_2|_{H^1}) (|A_1 \phi_1|_{L^2}^2 + |A_1 \phi_2|_{L^2}^2) \|\psi\|^2, \quad (7.17)$$

$$|b_1(v_2, \psi, A_1 \psi)| \leq \frac{\zeta}{8} |A_1 \psi|_{L^2}^2 + c |v_2|_{L^2}^2 \|v_2\|^2 \|\psi\|^2, \quad (7.18)$$

$$\zeta |(\bar{\mu}, A_1 \psi)_{L^2}| \leq \frac{\zeta}{8} |A_1 \psi|_{L^2}^2 + c \zeta |\nabla \bar{\mu}|_{L^2}^2, \quad (7.19)$$

$$|\langle g_1(t, v_1, \phi_1) - g_1(t, v_2, \phi_2), w \rangle| \leq L_1 \|w\| |(w, \psi)|_{\mathcal{Y}} \leq \frac{1}{8} \|w\|^2 + c L_1^2 |(w, \psi)|_{\mathcal{Y}}^2, \quad (7.20)$$

$$\|\sigma(s, v_1, \phi_1) - \sigma(s, v_2, \phi_2)\|_{L^2(Z, \mu, V_1)}^2 \leq l_2^2 |(w, \psi)|_{\mathcal{Y}}^2. \quad (7.21)$$

Let

$$\mathcal{Y}_2(t) = |w(t)|_{L^2}^2 + \|\psi(t)\|^2,$$

and

$$K_1(t) = c (\|v_1\|^2 + \|\phi_1\|^2 |A_1 \phi_1|_{L^2}^2 + \|\phi_2\|^2 |A_1 \phi_2|_{L^2}^2 + |v_2|_{L^2}^2 \|v_2\|^2) \quad (7.22)$$

$$+ Q_1(|\phi_1|_{H^1}, |\phi_2|_{H^1}) (|A_1 \phi_1|_{L^2}^2 + |A_1 \phi_2|_{L^2}^2) + c L_1^2 + l_2^2,$$

$$\delta(t) = \exp \left(- \int_0^t K_1(s) ds \right). \quad (7.23)$$

Applying Itô's formula to the process $\delta(t) \mathcal{Y}_2(t)$ and using (7.12)-(7.21), we derive that

$$\mathbb{E} \delta(t) \mathcal{Y}_2(t) + \mathbb{E} \int_0^t \delta(s) (\|w\|^2 + (1 - c \zeta) \|\bar{\mu}\|^2 + \zeta |A_1 \psi|_{L^2}^2) ds \leq \mathbb{E} \mathcal{Y}_2(0) + \mathbb{E} \int_0^t \delta(s) \mathcal{Y}_2(s) ds. \quad (7.24)$$

Note that since $0 < \delta(t) \leq 1$, the expectation of the stochastic integral in (7.12) vanishes. Therefore we obtain

$$\mathbb{E}\delta(t)\mathcal{Y}_2(t) \leq \mathbb{E}\mathcal{Y}_2(0) + \mathbb{E} \int_0^t \delta(s)\mathcal{Y}_2(s)ds, \quad 0 \leq t \leq T.$$

It follows from the Gronwall lemma that there exists a constant $C > 0$ such that

$$\mathbb{E}\delta(t)\mathcal{Y}_2(t) \leq C\mathbb{E}\mathcal{Y}_2(0),$$

for any $t \in [0, T]$, which proves the first part of the Proposition. Since $\delta(t)$ is bounded and positive \mathbb{P} -a.s., we conclude that the second part of the Proposition follows from the last estimate. Note that in (7.24), we choose $\zeta > 0$ and small enough such that $1 - c\zeta > 0$. \square

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