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Homogenization and corrector result for a coupled parabolic hyperbolic system

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ABSTRACT

The article deals with the homogenization of a coupled system of two equations: a shallow water equation and the equation for the long-term dynamics of sand dunes with small parameter ϵ. The first one is a hyperbolic partial differential equation, while the second one is a parabolic partial differential equation. In previous work, we showed existence and uniqueness results and performed a general homogenization of the coupled system. Here we give a more precise homogenization. For that, we use the asymptotic expansion of the solution and the coefficients of the system. Besides, we obtain corrector results. We also extend the existence theorem of previous work by proving that the solution of the parabolic equation is bounded independently of the small parameter.

1. Introduction

We will study a system modeling the erosion phenomenon presented in [3,4], where existence and uniqueness results for the solution of the system were given. In [4] we applied a scaling and a nondimensionalization on a coupled system of a shallow water equation and an equation for sand dunes. We obtained the dimensionless system by rewriting the coupled system with respect to a small parameter ϵ, which was, for the first time, defined in [5]. The small parameter ϵ corresponds to the ratio of a 1-month tide period over a long 16-year observation period of the tide; that is, $\epsilon = 1/192$. Because of the long observation period of the tide, the equation for sand dunes presented in [5] was named the equation for "long term dynamics of dunes" of sand (LTDD).

Klainerman and Majda [6] introduced the ϵ-balanced property to prove the existence of the solution of a singular hyperbolic equation. Then we proved existence results for the coupled system by using the ϵ-balanced property of the balance law equation [4]. The solution z^ϵ of the parabolic equation was bounded by a constant C depending on ϵ. Now, by extending the ϵ-balanced property on the parabolic equation, we prove

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that z^ϵ is bounded independently of ϵ. A crucial consequence of this result concerns the homogenization. To do the homogenization, we supposed the second member of the dimensionless LTDD equation is such that $|\nabla \cdot \mathcal{C}^\epsilon(v^\epsilon)| \leq \epsilon^2 C$ [4]. So, by proving z^ϵ is bounded independently of ϵ, we can do the homogenization without any assumptions.

The rest of this article is organized as follows. In Section 2, we present the model, recall some properties, and give the main theorems for existence and homogenization. We give a proof of the existence theorem in Section 3. Section 4 is devoted to the homogenization of the coupled system. Finally, Section 5 gives a short conclusion and perspectives.

2. Model and main results

In this section, we present the model to be studied. We recall some useful properties. Finally, we give the main results.

We will study the following coupled system:

$$\begin{cases} \dfrac{\partial v^\epsilon}{\partial t} + \dfrac{1}{\epsilon^4}\displaystyle\sum_{j=1}^{2} A^j \cdot \dfrac{\partial v^\epsilon}{\partial x_j} & = \dfrac{1}{\epsilon} h(v^\epsilon) - \dfrac{1}{\epsilon^4} P(v^\epsilon, z^\epsilon), \\[2ex] \dfrac{\partial z^\epsilon}{\partial t} - \dfrac{1}{\epsilon^2}\nabla \cdot [\mathcal{A}^\epsilon \nabla z^\epsilon] & = \dfrac{1}{\epsilon^2}\nabla \cdot C^\epsilon, \end{cases} \tag{1}$$

with $v^\epsilon = \begin{pmatrix} m^\epsilon \\ q_1^\epsilon \\ q_2^\epsilon \end{pmatrix}$,

$$A^1(v^\epsilon) = \begin{pmatrix} 0 & a^1 & 0 \\ -\dfrac{a^1 {q_1^\epsilon}^2}{(m^\epsilon + b^1)^2} + c^1(m^\epsilon + b^1) & \dfrac{2a^1 q_1}{(m^\epsilon + b^1)} & 0 \\ -\dfrac{a^1 q_1 q_2}{(m^\epsilon + b^1)^2} & \dfrac{a^1 q_2}{(m^\epsilon + b^1)} & \dfrac{a^1 q_1}{(m^\epsilon + b^1)} \end{pmatrix},$$

$$A^2(v^\epsilon) = \begin{pmatrix} 0 & 0 & a^1 \\ -\dfrac{a^1 q_1 q_2}{(m^\epsilon + b^1)^2} & \dfrac{a^1 q_2}{(m^\epsilon + b^1)} & \dfrac{a^1 q_1}{(m^\epsilon + b^1)} \\ -\dfrac{a^1 {q_2^\epsilon}^2}{(m^\epsilon + b^1)^2} + c^1(m^\epsilon + b^1) & 0 & \dfrac{2a^1 q_2}{(m^\epsilon + b^1)} \end{pmatrix},$$

$$h(v^\epsilon) = -\frac{f}{\epsilon} {v^\epsilon}^{\perp} + \frac{1}{\epsilon} H(v^\epsilon), \text{ where } H(v^\epsilon) = \begin{pmatrix} 0 \\ -\dfrac{k q_1}{m^\epsilon + b^1} \\ -\dfrac{k q_2}{m^\epsilon + b^1} \end{pmatrix},$$

$$P(v^\epsilon, z^\epsilon) = \begin{pmatrix} 0 \\ d^1(m + b^1)\dfrac{\partial z^\epsilon}{\partial x} \\ d^1(m + b^1)\dfrac{\partial z^\epsilon}{\partial y} \end{pmatrix},$$

$$\mathcal{A}^\epsilon = \frac{a(1 - b\epsilon m^\epsilon)|q^\epsilon|^3}{|m^\epsilon + b^1|^3},$$

and

$$\mathcal{C}^\epsilon = \frac{c(1-b\epsilon m^\epsilon)|q^\epsilon|^2 q^\epsilon}{(m^\epsilon + b^1)^3}.$$

The dimensionless shallow water equation,

$$\frac{\partial v^\epsilon}{\partial t} + \frac{1}{\epsilon^4}\sum_{j=1}^{2} A^j \cdot \frac{\partial v^\epsilon}{\partial x_j} = \frac{1}{\epsilon} h(v^\epsilon) - \frac{1}{\epsilon^4} P(v^\epsilon, z^\epsilon), \tag{2}$$

is a first-order hyperbolic system of the balance law, and the dimensionless LTDD equation,

$$\frac{\partial z^\epsilon}{\partial t} - \frac{1}{\epsilon^2}\nabla \cdot [\mathcal{A}^\epsilon \nabla z^\epsilon] = \frac{1}{\epsilon^2}\nabla \cdot C^\epsilon, \tag{3}$$

is a parabolic equation.

These two equations become singular or degenerate when ϵ takes particular values. For example, when $\epsilon = 0$, we have a singular case, and when their coefficients equal 0 for some values of ϵ, we have a degenerate case.

The dimensionless shallow water equation is symmetrizable in Friedrich sense, since there is a symmetric positive definite matrix B^0 such that we can write it in the following form:

$$B^0 \frac{\partial v^\epsilon}{\partial t} + \frac{1}{\epsilon^4}\sum_{j=1}^{2} B^j \cdot \frac{\partial v^\epsilon}{\partial x_j} = \frac{1}{\epsilon}\tilde{h}(v^\epsilon) - \frac{1}{\epsilon^4}\tilde{p}(v^\epsilon, z^\epsilon), \tag{4}$$

where

$$B^0 = \begin{pmatrix} \frac{1}{a^1}[\frac{a^1 q^{\epsilon\ 2}}{(m^\epsilon + b^1)^2} + c^1(m^\epsilon + b^1)] & -\frac{q_1^\epsilon}{(m^\epsilon + b^1)} & -\frac{q_2^\epsilon}{(m^\epsilon + b^1)} \\ -\frac{q_1^\epsilon}{(m^\epsilon + b^1)} & 1 & 0 \\ -\frac{q_2^\epsilon}{(m^\epsilon + b^1)} & 0 & 1 \end{pmatrix},$$

B^1 and B^2 are symmetric matrices given, respectively, by

$$\begin{pmatrix} \frac{q_1^\epsilon}{m^\epsilon + b^1}[\frac{a^1 q^{\epsilon\ 2}}{(m^\epsilon + b^1)^2} - c^1(m^\epsilon + b^1)] & -\frac{a^1 q_1^{\epsilon\ 2}}{(m^\epsilon + b^1)^2} + c^1(m^\epsilon + b^1) & -\frac{a^1 q_1^\epsilon q_2^\epsilon}{(m^\epsilon + b^1)^2} \\ -\frac{a^1 q_1^{\epsilon\ 2}}{(m^\epsilon + b^1)^2} + c^1(m^\epsilon + b^1) & -\frac{a^1 q_1^\epsilon}{(m^\epsilon + b^1)} & 0 \\ -\frac{a^1 q_1^\epsilon q_2^\epsilon}{(m^\epsilon + b^1)^2} & 0 & -\frac{a^1 q_1^\epsilon}{(m^\epsilon + b^1)} \end{pmatrix}$$

and

$$\begin{pmatrix} \frac{q_2^\epsilon}{m^\epsilon + b^1}[\frac{a^1 q^{\epsilon\ 2}}{(m^\epsilon + b^1)^2} - c^1(m^\epsilon + b^1)] & -\frac{a^1 q_1^\epsilon q_2^\epsilon}{(m^\epsilon + b^1)^2} & -\frac{a^1 q_2^{\epsilon\ 2}}{(m^\epsilon + b^1)^2} + c^1(m^\epsilon + b^1) \\ -\frac{a^1 q_1^\epsilon q_2^\epsilon}{(m^\epsilon + b^1)^2} & -\frac{a^1 q_2^\epsilon}{(m^\epsilon + b^1)} & 0 \\ -\frac{a^1 q_2^{\epsilon\ 2}}{(m^\epsilon + b^1)^2} + c^1(m^\epsilon + b^1) & 0 & -\frac{a^1 q_2^\epsilon}{(m^\epsilon + b^1)} \end{pmatrix},$$

$$\tilde{h}(v^\epsilon) = B^0 \cdot h(v^\epsilon) = \begin{pmatrix} \dfrac{kq^{\epsilon\ 2}}{(m^\epsilon + b^1)^2} \\ -\dfrac{kq_1^\epsilon}{m^\epsilon + b^1} + fq_2^\epsilon \\ -\dfrac{kq_2^\epsilon}{m^\epsilon + b^1} - fq_1^\epsilon \end{pmatrix},$$

and

$$\tilde{p}(v^\epsilon, z^\epsilon) = B^0 \cdot P(v^\epsilon, z^\epsilon).$$

Further in this section, we will give the main theorems for existence and homogenization.

We now recall some necessary proprieties in [4,6] before presenting the main results.

We consider a parameter $d(\epsilon)$ such that $\lim_{\epsilon \to 0} d(\epsilon) = +\infty$, and a weighted norm corresponding to a splitting vector $v := {}^t(v_1, v_2)$, $v_1 \in \mathbb{R}$, $v_2 \in \mathbb{R}^2$:

$$\|v\|_\epsilon^2 = d(\epsilon)^2 |v_1|^2 + \|v_2\|^2,$$

where $|\cdot|$ and $\|\cdot\|$ stand for the usual euclidean norm in $\mathbb{R}$ and $\mathbb{R}^2$, respectively.

For a scalar function $r(v)$,

$$\|r(v)\|_\epsilon^2 = \|r(v)\|^2.$$

For the two-dimensional functional vector $(r_1(v), r_2(v))$, we have

$$\|(r_1(v), r_2(v))\|_\epsilon^2 = \|(r_1(v), r_2(v))\|^2.$$

Let D be the matrix associated with this splitting of v:

$$D = \begin{pmatrix} D_{11}, D_{12} \\ D_{21}, D_{22} \end{pmatrix}.$$

Then we can define the norms with D as follows:

$$\|D\|_\epsilon = |D_{11}| + d(\epsilon)|D_{12}| + d(\epsilon)^{-1}|D_{21}| + |D_{22}|,$$
$$[D]_\epsilon = d(\epsilon)^{-2}|D_{11}| + d(\epsilon)^{-1}(|D_{12}| + |D_{21}|) + |D_{22}|.$$

We can deduce the following inequalities for any splitting vector v defined as above with $C > 0$ constant:

$$\|Dv\|_\epsilon \leq C\|D\|_\epsilon \cdot \|v\|_\epsilon,$$
$$|(Dv,\ v)| \leq C[D]_\epsilon \|v\|_\epsilon \cdot \|v\|_\epsilon.$$

The associated Sobolev norm is given by

$$\|v\|_{s,\epsilon}^2 = d(\epsilon)^2 \|v_1\|_s^2 + \|v_2\|_s^2$$

for integers s such that $s \geq s_0 + 1$, where $s_0 = 1$ and $\|\cdot\|_s$ stands for the usual Sobolev norm.

For the functional space $C([0,T], H^s(\mathbb{T}^2, \mathbb{R}^3)) \cap C^1([0,T], H^{s-1}(\mathbb{T}^2, \mathbb{R}^3))$, we consider the following norms:

$$|||v|||_{\epsilon,T} = \sup_{t\in[0,T]} |||v(t)|||_{\epsilon}$$

and

$$|||v(t)|||_{\epsilon} = \|v(t)\|_{s,\epsilon} + \left\|\frac{dv}{dt}(t)\right\|_{s-1,\epsilon}.$$

We now give an extended definition of the structural conditions called the ϵ-balanced property.

Definition 2.1. For $s \geq s_0 + 1$, the system (1) is said to be ϵ-balanced around v^0, a given fixed vector, if the following structural conditions are satisfied:

There exists $\delta > 0$ independent of ϵ such that

$$\theta_{1,s}(A^j, v^0, \delta),\ \theta_{1,s}(h, v^0, \delta),\ \theta_{1,s}(P, v^0, \delta) < \infty \quad \text{for } j = 1, 2, \tag{5}$$

$$\gamma_{1,1}(B^j, v^0, \delta) < \infty \quad \text{for } j = 0, 1, 2, \tag{6}$$

$$\theta_{1,s}(\mathcal{A}^\epsilon, v^0, \delta), \theta_{1,s}(\mathcal{C}^\epsilon, v^0, \delta) < \infty, \tag{7}$$

where

$$\begin{aligned}\theta_{1,s}(\cdot, v^0, \delta) &= \theta_{1,s}(\cdot)\\ &= \max_{\epsilon\to 0}[\sum_{1\leq|s_1|+|s_2|\leq s} d(\epsilon)^{-|s_1|} \max_{\|p-v^0\|_\epsilon\leq\delta} \left\|D_{p_1}^{s_1} D_{p_2}^{s_2} \cdot (p_1, p_2)\right\|_\epsilon]\end{aligned} \tag{8}$$

and

$$\begin{aligned}\gamma_{1,s}(\cdot, v^0, \delta) &= \gamma_{1,s}(\cdot)\\ &= \max_{\epsilon\to 0}[\sum_{1\leq|s_1|+|s_2|\leq s} d(\epsilon)^{-|s_1|} \max_{\|p-v^0\|_\epsilon\leq\delta} [D_{p_1}^{s_1} D_{p_2}^{s_2} \cdot (p_1, p_2)]_\epsilon].\end{aligned} \tag{9}$$

Another property concerns the choice of the initial value condition:

$$\begin{cases} v^\epsilon(t=0, x) &= v_0(x) + \tilde{v}_0^\epsilon(x),\\ z^\epsilon(0, x) &= z^0(x),\end{cases} \tag{10}$$

such that

$$\begin{cases} \text{I.} & v_0 = (v_0^I, v_0^{II}) \in H^s(\mathbb{T}^2),\ s \geq s_0 + 1,\\ \text{II.} & v_0^I \text{ constant}, \left\|v_0^{II}\right\|_s + \left\|\displaystyle\sum_{j=1}^{2} A^j(v_0, \epsilon)\frac{\partial v_0}{\partial x_j}\right\|_{s-1,\epsilon} \leq K,\\ \text{III.} & \left\|\displaystyle\sum_{j=1}^{2} A^j(v_0, \epsilon)\frac{\partial \tilde{v}_0^\epsilon}{\partial x_j}\right\|_{s-1,\epsilon} \leq K,\\ \text{IV.} & \|\tilde{v}_0^\epsilon\|_{s,\epsilon} \leq \delta',\\ \text{V.} & z^0 \in H^p(\mathbb{T}^2),\ p \geq 1,\ \left\|z^0\right\|_{s,\epsilon} \leq \Delta_z,\end{cases} \tag{IC}$$

where $\delta' > 0$ is considered small enough and is to be chosen later, and Δ_z is a constant.

We now give the theorem.

Theorem 2.1. *Consider the Cauchy problem given by* (1)–(10) *with initial value satisfying* (IC)*. Then there exists* $T > 0$ *not depending on* ϵ *such that for any* ϵ *close to 0, the Cauchy problem given by* (1)–(10) *has a unique solution:* $(\bar{v}^\epsilon, \bar{z}^\epsilon) \in C([0,T], H^s(\mathbb{T}^2, \mathbb{R}^3)) \cap C^1([0,T], H^{s-1}(\mathbb{T}^2, \mathbb{R}^3)) \times L^\infty([0,T], H^p(\mathbb{T}^2))$*. In addition, the following estimates hold:*

$$
\begin{gathered}
||\bar{z}^\epsilon(t)||_{p,\epsilon} \le \tilde{\gamma} + \Delta_z, \\
\left\| \frac{\partial \bar{z}^\epsilon}{\partial t} \right\|_{p,\epsilon} \le \tilde{\gamma}', \\
\left\| \bar{v}^\epsilon(t) - v^0 \right\|_{s,\epsilon} + \left\| \frac{d\bar{v}^\epsilon(t)}{dt} \right\|_{s-1,\epsilon} \le \Delta
\end{gathered}
$$

for any $t \in [0,T]$*, where* Δ*,* $\tilde{\gamma}$*, and* $\tilde{\gamma}'$ *are constants independent of* ϵ*.*

The following paragraphs concern the homogenization of (1). We start by considering a new variable: $w^\epsilon = \begin{pmatrix} v^\epsilon \\ z^\epsilon \end{pmatrix}$.

The system (1) becomes

$$
\frac{\partial w^\epsilon}{\partial t} + \frac{1}{\epsilon^4} \sum_{j=1}^{2} L^j \cdot \frac{\partial w^\epsilon}{\partial x_j} - \frac{1}{\epsilon^2} \sum_{j=1}^{2} M^j \cdot \frac{\partial w^\epsilon}{\partial x_j} - \frac{1}{\epsilon^2} \sum_{j=1}^{2} N^j \cdot \frac{\partial^2 w^\epsilon}{\partial x_j^2} = \frac{1}{\epsilon} E(w^\epsilon), \tag{11}
$$

with

$$
L^1 = \left[\begin{array}{c|c} & 0 \\ A^1 & d^1(m^\epsilon + b^1) \\ & 0 \\ \hline & \\ 0 & 0 \end{array} \right], \quad L^2 = \left[\begin{array}{c|c} & 0 \\ A^2 & 0 \\ & d^1(m^\epsilon + b^1) \\ \hline & \\ 0 & 0 \end{array} \right],
$$

$$
M^j = \left[\begin{array}{c|c} 0 & 0 \\ \hline & \\ C^j & \partial_{x_j} \mathcal{A}^\epsilon \end{array} \right], \quad N^j = \left[\begin{array}{c|c} 0 & 0 \\ \hline & \\ 0 & \mathcal{A}^\epsilon \end{array} \right] \text{ for } j = 1, 2,
$$

$$
E(w^\epsilon) = \begin{pmatrix} h(v^\epsilon) \\ \hline \\ 0 \end{pmatrix}.
$$

We consider the Ansatz

$$
w^\epsilon(t,x) = w(t, \frac{t}{\epsilon}, \frac{t}{\epsilon^4}, x) = \bar{w}^{0,\epsilon} + \epsilon \bar{w}^{1,\epsilon} + \epsilon^2 \bar{w}^{2,\epsilon} + \epsilon^3 \bar{w}^{3,\epsilon} + \epsilon^4 \bar{w}^{4,\epsilon} \tag{Antz1}
$$

and the two scales $\theta = \dfrac{t}{\epsilon}$ and $\tau = \dfrac{t}{\epsilon^4}$.

Introducing (Antz1) in the matrix coefficients of the system (11) yields the following:

$$
\begin{aligned}
L^j &= L_0^j + \epsilon L_1^j + o(\epsilon), \ j = 1, 2, \\
M^j &= M_0^j + \epsilon M_1^j + o(\epsilon), \ j = 1, 2,
\end{aligned} \tag{Antz2}
$$

$$N^j = N_0^j + \epsilon N_1^j + o(\epsilon),\ j = 1, 2,$$
$$E = E_0 + \epsilon E_1 + o(\epsilon).$$

By Theorem 2.1, since the solution (v^ϵ, z^ϵ) is bounded by a constant not depending on ϵ, we can prove three-scale convergence of the solution to a profile that is the solution of a limit equation of (11).

We have the following theorem.

Theorem 2.2. *Consider the Cauchy problem given by* (1)–(10) *with initial value satisfying (IC). The solutions w^ϵ of* (11) *three-scale converge to a profile $\bar{w}^0$ belonging in the space $L^\infty([0,T), L^\infty_\#([0,1], L^\infty_\#([0,1], H^l(\mathbb{T}^2, \mathbb{R}^4))))$, with $l = \min\{p, s\}$ and $\bar{w}^0$ the solution of the equation*

$$\frac{\partial \bar{w}_0}{\partial \tau} + \sum_{j=1}^{2} L_0^j \cdot \frac{\partial \bar{w}_0}{\partial x_j} = 0. \tag{12}$$

The following theorem gives the corrector result.

Theorem 2.3. *Consider the Cauchy problem given by* (1)–(10) *with initial value satisfying (IC). If the solution w^ϵ satisfies the assumptions* (Antz1)*, then we have that $K^{1,\epsilon} = \dfrac{w^\epsilon - \bar{w}_0^\epsilon}{\epsilon}$ three-scale converges to $\bar{w}_1 \in L^\infty([0,T), L^\infty_\#([0,1], L^\infty_\#([0,1], H^l(\mathbb{T}^2, \mathbb{R}^4))))$, the solution of the equation*

$$\frac{\partial \bar{w}_1}{\partial \tau} + \sum_{j=1}^{2} L_0^j \cdot \frac{\partial \bar{w}_1}{\partial x_j} + \sum_{j=1}^{2} L_1^j \cdot \frac{\partial \bar{w}_0}{\partial x_j} = 0, \tag{13}$$

$K^{2,\epsilon} = \dfrac{K^{1,\epsilon} - \bar{w}_1^\epsilon}{\epsilon}$ three-scale converges to $\bar{w}_2 \in L^\infty([0,T), L^\infty_\#([0,1], L^\infty_\#([0,1], H^l(\mathbb{T}^2, \mathbb{R}^4))))$, the solution of the equation

$$\begin{aligned}\frac{\partial \bar{w}_2}{\partial \tau} &+ \sum_{j=1}^{2} L_0^j \cdot \frac{\partial \bar{w}_2}{\partial x_j} + \sum_{j=1}^{2} L_1^j \cdot \frac{\partial \bar{w}_1}{\partial x_j} \\ &- \sum_{j=1}^{2} M_0^j \cdot \frac{\partial \bar{w}_0}{\partial x_j} - \sum_{j=1}^{2} N_0^j \cdot \frac{\partial^2 \bar{w}_0}{\partial x_j^2} = 0,\end{aligned} \tag{14}$$

$K^{3,\epsilon} = \dfrac{K^{2,\epsilon} - \bar{w}_2^\epsilon}{\epsilon}$ three-scale converges to $\bar{w}_3 \in L^\infty([0,T), L^\infty_\#([0,1], L^\infty_\#([0,1], H^l(\mathbb{T}^2, \mathbb{R}^4))))$, the solution of the equation

$$\begin{aligned}\frac{\partial \bar{w}_3}{\partial \tau} &+ \sum_{j=1}^{2} L_0^j \cdot \frac{\partial \bar{w}_3}{\partial x_j} + \sum_{j=1}^{2} L_1^j \cdot \frac{\partial \bar{w}_2}{\partial x_j} \\ &- \sum_{j=1}^{2} M_0^j \cdot \frac{\partial \bar{w}_1}{\partial x_j} - \sum_{j=1}^{2} N_0^j \cdot \frac{\partial^2 \bar{w}_1}{\partial x_j^2} \\ &+ \frac{\partial \bar{w}_0}{\partial \theta} - \sum_{j=1}^{2} M_1^j \cdot \frac{\partial \bar{w}_0}{\partial x_j} - \sum_{j=1}^{2} N_1^j \cdot \frac{\partial^2 \bar{w}_0}{\partial x_j^2} - E_0 = 0,\end{aligned} \tag{15}$$

and $K^{4,\epsilon} = \dfrac{K^{3,\epsilon} - \bar{w}_3^\epsilon}{\epsilon}$ three-scale converges to $\bar{w}_4 \in L^\infty([0,T), L^\infty_\#([0,1], L^\infty_\#([0,1], H^l(\mathbb{T}^2, \mathbb{R}^4))))$, the solution of the equation

$$\begin{aligned} \frac{\partial \bar{w}_4}{\partial \tau} + \sum_{j=1}^{2} L_0^j \cdot \frac{\partial \bar{w}_4}{\partial x_j} + \sum_{j=1}^{2} L_1^j \cdot \frac{\partial \bar{w}_3}{\partial x_j} \\ - \sum_{j=1}^{2} M_0^j \cdot \frac{\partial \bar{w}_2}{\partial x_j} - \sum_{j=1}^{2} N_0^j \cdot \frac{\partial^2 \bar{w}_2}{\partial x_j^2} \\ + \frac{\partial \bar{w}_0}{\partial t} + \frac{\partial \bar{w}_1}{\partial \theta} - \sum_{j=1}^{2} M_1^j \cdot \frac{\partial \bar{w}_1}{\partial x_j} - \sum_{j=1}^{2} N_1^j \cdot \frac{\partial^2 \bar{w}_1}{\partial x_j^2} + E_1 = 0. \end{aligned} \tag{16}$$

3. Estimates

In this section, we give the proof of Theorem 2.1. Theorem 2.1 is different from the existence theorem given in [4] only in the estimates for the solution $\bar{z}^\epsilon$ of the LTDD equation. So, we only have to prove the estimates for $\bar{z}^\epsilon$.

We give first the following remarks and properties.

Remark 3.1. $A^j(v^\epsilon(t,x))$, $j = 1,2$, $B^j(v^\epsilon(t,x))$, $j = 0,1,2$, $\tilde{h}(v^\epsilon(t,x))$, $\tilde{p}(v^\epsilon(t,x))$, $\mathcal{A}^\epsilon(v^\epsilon(t,x))$, and $\mathcal{C}^\epsilon(v^\epsilon(t,x))$ are C^s-differentiable with respect to $v^\epsilon(t,x) \in \mathbb{R}^3$ and are valued functions in $B(\mathbb{R}^3)$ or $\mathbb{R}^3$ for v^ϵ belonging to a set included in $H^s([0,T) \times \mathbb{T}^2, \mathbb{R}^3)$.

In [4], the ϵ-balanced property of the system (2) was proved. Hence, we recall it as a remark.

Remark 3.2.

- In [4], to prove that $\theta_{1,s}(P(v^\epsilon, z^\epsilon, \epsilon), v_0, \delta) < \infty$, we showed that it is quite possible to find $d^\star(\epsilon)$ such that

$$\lim_{\epsilon \to 0} d^\star(\epsilon) \left\| D_v^\alpha P(v^\epsilon, z^\epsilon, \epsilon) \right\| = 0, \; |\alpha| \geq 0. \tag{17}$$

- The system (2) is ϵ-balanced around an initial condition v^0 satisfying (IC).

Proposition 3.1. *The following estimations hold:*

$$\theta_{1,s}(\mathcal{A}^\epsilon, v^0, \delta) < \tilde{K}, \tag{18}$$

$$\theta_{1,s}(\mathcal{C}^\epsilon, v^0, \delta) < \tilde{K}. \tag{19}$$

Proof.

$$\begin{aligned} \theta_{1,s}(\mathcal{C}^\epsilon, v^0, \delta) &= \max_{\epsilon \to 0}[\sum_{1 \leq |s_1| + |s_2| \leq s} d(\epsilon)^{-|s_1|} \max_{\|v - v^0\|_\epsilon \leq \delta} \left\| D_{v_1}^{s_1} D_{v_2}^{s_2} \mathcal{C}^\epsilon(v_1, v_2, \epsilon) \right\|_\epsilon], \\ &= \max_{\epsilon \to 0}[\sum_{1 \leq |s_1| + |s_2| \leq s} \frac{d(\epsilon)^{-|s_1|}}{\epsilon^2} \max_{\|v - v^0\|_\epsilon \leq \delta} \left\| D_{v_1}^{s_1} D_{v_2}^{s_2} \mathcal{C}^\epsilon(v_1, v_2) \right\|_\epsilon], \\ &= \max_{\epsilon \to 0}[\sum_{1 \leq |s_1| + |s_2| \leq s} \frac{d(\epsilon)^{-|s_1|}}{\epsilon^2} \max_{\|v - v^0\|_\epsilon \leq \delta} \left\| D_{v_1}^{s_1} D_{v_2}^{s_2} \mathcal{C}^\epsilon(v_1, v_2) \right\|]. \end{aligned}$$

Since $\mathcal{C}^\epsilon(v)$ is s-differentiable with respect to v and since $|s_1| \geq 1$,

$$\max_{\epsilon\to 0}[\sum_{1\le|s_1|+|s_2|\le s}\frac{d(\epsilon)^{-|s_1|}}{\epsilon^2}\max_{\|v-v^0\|_\epsilon\le\delta}\left\|D^{s_1}_{v_1}D^{s_2}_{v_2}\mathcal{C}^\epsilon(v_1,v_2)\right\|]<\infty,$$

$$\theta_{1,s}(\mathcal{C}^\epsilon,v^0,\delta)<\infty.$$

Finally we set $\theta_{1,s}(\mathcal{C}^\epsilon,v^0,\delta)<\tilde{K}$.

We can use the same argument for $\mathcal{A}^\epsilon(v,\epsilon)$. Indeed, $\mathcal{A}^\epsilon(v,\epsilon)$ is a scalar function, s-differentiable with respect to v. Hence, we can set $\theta_{1,s}(\mathcal{A}^\epsilon,v^0,\delta)<\tilde{K}$. □

As in [4], we define the function spaces $B^\epsilon_T(v_0)=B^\epsilon_T(v_0,\delta,\Delta)$, a subset of

$$C([0,T],H^s(\mathbb{T}^2,\mathbb{R}^3))\cap C^1([0,T],H^{s-1}(\mathbb{T}^2,\mathbb{R}^3)),$$

with $s=s_0+1=2$, such that

$$\begin{cases}\|v-v_0\|_\epsilon\le\delta,\\ |||v-\hat{v}_0|||_{\epsilon,T}\le\Delta,\ \Delta\ge\delta>0,\end{cases}\tag{20}$$

with $\hat{v}^I_0={v_0}^I$ and $\hat{v}^{II}_0=0$.

Remark 3.3. In [4], by using existence results of [6,7], we proved the existence of the solution (v^ϵ,z^ϵ) of the system given by (1)–(10) when the initial value satisfies (IC). With v^ϵ belongs in $B^\epsilon_T(v_0)$ and satisfies the estimate $\left\|v^\epsilon(t)-v^0\right\|_{s,\epsilon}+\left\|\dfrac{dv^\epsilon(t)}{dt}\right\|_{s-1,\epsilon}\le\Delta$. Thus, we need only to prove the estimates for z^ϵ and $\left\|\dfrac{\partial(D^\alpha z^\epsilon)}{\partial t}\right\|_{0,\epsilon}$ for any $v\in B^\epsilon_T(v_0)$.

We will proceed in stages: first we will prove two propositions given below and then we will give a proof of the estimates for z^ϵ and $\left\|\dfrac{\partial(D^\alpha z^\epsilon)}{\partial t}\right\|_{0,\epsilon}$.

Proposition 3.2. *For $v\in B^\epsilon_T(v_0,\delta,\Delta)$, the following estimation holds:*

$$\|\nabla\cdot\mathcal{C}(v,\epsilon)\|_{0,\epsilon}\le C\tilde{K}\Delta.\tag{21}$$

Proof.

$$\begin{aligned}\|\nabla\cdot\mathcal{C}(v,\epsilon)\|_\epsilon&=\left\|\sum_{j=1}^2\frac{\partial\mathcal{C}_j}{\partial x_j}(v,\epsilon)\right\|_\epsilon,\\
&\le\sum_{j=1}^2\left\|\frac{\partial\mathcal{C}_j}{\partial x_j}(v,\epsilon)\right\|_\epsilon,\\
&\le\sum_{j=1}^2\left\|D_{v_1}\mathcal{C}_j\frac{\partial v_1}{\partial x_j}+D_{v_2}\mathcal{C}_j\frac{\partial v_2}{\partial x_j}\right\|_\epsilon,\\
&\le\sum_{j=1}^2\left\|(d(\epsilon)^{-1}D_{v_1}\mathcal{C}_j+D_{v_2}\mathcal{C}_j)(d(\epsilon)\frac{\partial v_1}{\partial x_j}+\frac{\partial v_2}{\partial x_j})\right\|_\epsilon,\\
&\le C\sum_{j=1}^2(d(\epsilon)^{-1}\|D_{v_1}\mathcal{C}_j\|_\epsilon+\|D_{v_2}\mathcal{C}_j\|_\epsilon)\|\nabla v\|_\epsilon,\end{aligned}$$

$$\leq C \max_{\epsilon \longrightarrow 0} \sum_{j=1}^{2} \sup_{\|v-v^0\|_\epsilon \leq \delta} (d(\epsilon)^{-1} \|D_{v_1}\mathcal{C}_j\|_\epsilon + \|D_{v_2}\mathcal{C}_j\|_\epsilon) \|\nabla v\|_\epsilon ,$$
$$\leq C\theta_{1,1}(\mathcal{C}^\epsilon, v^0, \delta) \|\nabla v\|_\epsilon .$$

Using Proposition 3.1, we obtain

$$\|\nabla \cdot \mathcal{C}(v,\epsilon)\|_\epsilon \leq C\tilde{K} \|\nabla v\|_\epsilon .$$

Integrating the last inequality in $\mathbb{T}^2$ yields

$$\begin{aligned}\|\nabla \cdot \mathcal{C}(v,\epsilon)\|_{0,\epsilon} &\leq C\tilde{K} \|\nabla v\|_{0,\epsilon} \\ &\leq C\tilde{K} \|\nabla v\|_{s-1,\epsilon} .\end{aligned}$$

Using (20), we obtain

$$\|\nabla \cdot \mathcal{C}(v,\epsilon)\|_{0,\epsilon} \leq C\tilde{K}\Delta. \quad \square$$

Proposition 3.3. *For ϵ small enough, the following estimate holds:*

$$\theta_{1,s}(\mathcal{A}^\epsilon(v,\epsilon)\nabla z^\epsilon, v^0, \delta) \leq \tilde{K}. \tag{22}$$

Proof. For ϵ small enough, we have

$$\begin{aligned}\left\|D_{v_1}^{s_1} D_{v_2}^{s_2} \mathcal{A}^\epsilon(v_1,v_2)\nabla z^\epsilon\right\|_\epsilon &\leq \frac{1}{\epsilon^2} \left\|D_{v_1}^{s_1} D_{v_2}^{s_2} (v_1+b^1)\nabla z^\epsilon\right\|_\epsilon , \\ \frac{1}{\epsilon^2} \left\|D_{v_1}^{s_1} D_{v_2}^{s_2} \mathcal{A}^\epsilon(v_1,v_2)\nabla z^\epsilon\right\|_\epsilon &\leq \frac{1}{\epsilon^4} \left\|D_{v_1}^{s_1} D_{v_2}^{s_2} (v_1+b^1)\nabla z^\epsilon\right\|_\epsilon , \\ \left\|D_{v_1}^{s_1} D_{v_2}^{s_2} \mathcal{A}^\epsilon(v_1,v_2,\epsilon)\nabla z^\epsilon\right\|_\epsilon &\leq \left\|D_{v_1}^{s_1} D_{v_2}^{s_2} P(v,z^\epsilon,\epsilon)\right\|_\epsilon .\end{aligned}$$

Using (17), we obtain

$$\max_{\epsilon\to 0}[\sum_{1\leq|s_1|+|s_2|\leq s} d(\epsilon)^{-|s_1|} \max_{\|v-v^0\|_\epsilon\leq\delta} \left\|D_{v_1}^{s_1} D_{v_2}^{s_2} \mathcal{A}^\epsilon(v_1,v_2,\epsilon)\nabla z^\epsilon\right\|_\epsilon] \leq$$
$$\theta_{1,s}(P(v,z^\epsilon,\epsilon), v^0, \delta) \leq \tilde{K}.$$

Then $\theta_{1,s}(\mathcal{A}^\epsilon(v,\epsilon)\nabla z^\epsilon, v^0, \delta) \leq \tilde{K}$. $\square$

We can now give the proof of estimates for z^ϵ and $\left\|\dfrac{\partial(D^\alpha z^\epsilon)}{\partial t}\right\|_{0,\epsilon}$ of the existence theorem (Theorem 2.1).

Proof. Multiplying (3) by z^ϵ and integrating by parts the resulting equation in $\mathbb{T}^2$, we get

$$\frac{1}{2}\frac{d}{dt} \|z^\epsilon\|_{0,\epsilon}^2 + \frac{1}{\epsilon^2}\int_{\mathbb{T}^2} \mathcal{A}^\epsilon(v)(\nabla z^\epsilon)^2 = \frac{1}{\epsilon^2}\int_{\mathbb{T}^2} z^\epsilon \nabla \cdot \mathcal{C}^\epsilon(v). \tag{23}$$

For any $\epsilon > 0$ small enough, we have $\mathcal{A}^\epsilon(v^\epsilon) \geq 0$, and then $\dfrac{1}{\epsilon^2}\displaystyle\int_{\mathbb{T}^2} \mathcal{A}^\epsilon(v)(\nabla z^\epsilon)^2 \geq 0$. In addition, using the Cauchy-Schwartz inequality in the second member of (23), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|z^\epsilon\right\|_{0,\epsilon}^2 &\leq \left\|\nabla\cdot\mathcal{C}(v,\epsilon)\right\|_{0,\epsilon}\left\|z^\epsilon\right\|_{0,\epsilon},\\ &\leq C\tilde{K}\Delta\left\|z^\epsilon\right\|_{0,\epsilon}.\end{aligned}$$

Setting $\tilde{C}=C\tilde{K}\Delta$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|z^\epsilon\right\|_{0,\epsilon}^2 &\leq \tilde{C}\left\|z^\epsilon\right\|_{0,\epsilon},\\ \frac{d}{dt}\left\|z^\epsilon\right\|_{0,\epsilon} &\leq \tilde{C}.\end{aligned}$$

If we consider z^ϵ as a scalar function of v, we have $\left\|z^\epsilon\right\|_{0,\epsilon}=\left\|z^\epsilon\right\|_{L^2(\mathbb{T}^2)}$. Then integrating the last inequality in $[0,r]$, $r\leq T$, we have

$$\begin{aligned}\int_0^r \frac{d}{dt}\left\|z^\epsilon\right\|_{L^2(\mathbb{T}^2)}dt &\leq \int_0^s \tilde{C}dt,\\ \left\|z^\epsilon(r)\right\|_{L^2(\mathbb{T}^2)} &\leq \tilde{C}r+\left\|z^\epsilon(0)\right\|_{L^2(\mathbb{T}^2)}\ \forall\, r\in[0,T], \qquad (24)\\ \max_{r\in[0,T]}\left\|z^\epsilon(r)\right\|_{L^2(\mathbb{T}^2)} &\leq \tilde{C}T+\left\|z^\epsilon(0)\right\|_{L^2(\mathbb{T}^2)}.\end{aligned}$$

Differentiating (3) at order α with respect to x, with $1\leq|\alpha|\leq p$, and multiplying the resulting equation by $2D^\alpha$, we obtain

$$\begin{aligned}\int_{\mathbb{T}^2} 2D^\alpha\frac{\partial(D^\alpha z^\epsilon)}{\partial t}-\int_{\mathbb{T}^2} 2D^\alpha z^\epsilon D^\alpha\nabla\cdot[\mathcal{A}^\epsilon\nabla z^\epsilon] &= \int_{\mathbb{T}^2} 2D^\alpha z^\epsilon D^\alpha\nabla\cdot C^\epsilon,\\ \frac{d\left\|D^\alpha z^\epsilon\right\|_{0,\epsilon}^2}{dt} &\leq 2\int_{\mathbb{T}^2}\left\|D^\alpha z^\epsilon D^\alpha\nabla\cdot[\mathcal{A}^\epsilon\nabla z^\epsilon]\right\|_\epsilon+2\int_{\mathbb{T}^2}\left\|D^\alpha z^\epsilon D^\alpha\nabla\cdot C^\epsilon\right\|_\epsilon,\\ \frac{d\left\|D^\alpha z^\epsilon\right\|_{0,\epsilon}^2}{dt} &\leq C\left\|D^\alpha z^\epsilon\right\|_{0,\epsilon}\left\|D^\alpha\nabla\cdot[\mathcal{A}^\epsilon\nabla z^\epsilon]\right\|_{0,\epsilon}+C\left\|D^\alpha z^\epsilon\right\|_{0,\epsilon}\left\|D^\alpha\nabla\cdot C^\epsilon\right\|_{0,\epsilon}.\end{aligned}$$

Using Lemma (A.3) in [6], we obtain

$$\frac{d\left\|D^\alpha z^\epsilon\right\|_{0,\epsilon}^2}{dt}\leq C\left\|D^\alpha z^\epsilon\right\|_{0,\epsilon}\Delta^{p+1}(\theta_{1,p}(\mathcal{A}^\epsilon\nabla z^\epsilon,v_0,\delta)+\theta_{1,p}(C^\epsilon,v_0,\delta)).$$

Using (22) and (19), we have

$$\frac{d\left\|D^\alpha z^\epsilon\right\|_{0,\epsilon}^2}{dt}\leq C\tilde{K}\Delta^{p+1}\left\|D^\alpha z^\epsilon\right\|_{0,\epsilon}.$$

We set $\tilde{C}_p=C\tilde{K}\Delta^{p+1}$ and obtain

$$\frac{d\left\|D^\alpha z^\epsilon\right\|_{0,\epsilon}}{dt}\leq\tilde{C}_p.$$

Integrating the last inequality in $[0,t]$, with $t\in\ [0,T]$, we have

$$
\begin{aligned}
\|D^\alpha z^\epsilon\|_{0,\epsilon}\big|_0^t &\le \tilde{C}_p t,\\
\|D^\alpha z^\epsilon(t)\|_{0,\epsilon} &\le \tilde{C}_p t + \|D^\alpha z^\epsilon(0)\|_{0,\epsilon}\,,\\
\sum_{1\le|\alpha|\le p} \|D^\alpha z^\epsilon(t)\|_{0,\epsilon} &\le p\tilde{C}_p t + \sum_{1\le|\alpha|\le p} \|D^\alpha z^\epsilon(0)\|_{0,\epsilon}\,.
\end{aligned}
$$

Summing the last inequality with (24), we obtain

$$
\begin{aligned}
\|z^\epsilon(t)\|_{0,\epsilon} + \sum_{1\le|\alpha|\le p} \|D^\alpha z^\epsilon(t)\|_{0,\epsilon} &\le (p\tilde{C}_p + \tilde{C})t + \|z_0^\epsilon\|_{0,\epsilon} + \sum_{1\le|\alpha|\le p} \|D^\alpha z_0^\epsilon\|_{0,\epsilon}\,,\\
\sum_{|\alpha|\le p} \|D^\alpha z^\epsilon(t)\|_{0,\epsilon} &\le (p\tilde{C}_p + \tilde{C})t + \sum_{|\alpha|\le s} \|D^\alpha z_0^\epsilon\|_{0,\epsilon}\,,\\
\|z^\epsilon(t)\|_{p,\epsilon} &\le (p\tilde{C}_p + \tilde{C})t + \|z_0^\epsilon\|_{p,\epsilon}\,,\\
\max_{t\in[0,T]} \|z^\epsilon(t)\|_{p,\epsilon} &\le (p\tilde{C}_p + \tilde{C})T + \Delta_z.
\end{aligned}
$$

We can set $\tilde{\gamma} = (p\tilde{C}_p + \tilde{C})T$.

Differentiating (3) at order α with respect to x, with $|\alpha| \le p$, we multiply the resulting equation by $\dfrac{\partial(D^\alpha z^\epsilon)}{\partial t}$. Then integrating that new equation over the torus, we obtain

$$
\begin{aligned}
\int_{\mathbb{T}^2} \left\|\frac{\partial(D^\alpha z^\epsilon)}{\partial t}\right\|_\epsilon - \int_{\mathbb{T}^2} \frac{\partial(D^\alpha z^\epsilon)}{\partial t} D^\alpha\nabla\cdot[\mathcal{A}^\epsilon\nabla z^\epsilon] &= \int_{\mathbb{T}^2} \frac{\partial(D^\alpha z^\epsilon)}{\partial t} D^\alpha\nabla\cdot C^\epsilon,\\
\left\|\frac{\partial(D^\alpha z^\epsilon)}{\partial t}\right\|_{0,\epsilon}^2 &\le C(\theta_{1,p}(\mathcal{A}^\epsilon\nabla z^\epsilon, v_0,\delta) + \theta_{1,p}(\nabla\cdot C^\epsilon, v_0,\delta)),\\
\left\|\frac{\partial(D^\alpha z^\epsilon)}{\partial t}\right\|_{0,\epsilon}^2 &\le C\tilde{K}\Delta^{p+1}\left\|\frac{\partial(D^\alpha z^\epsilon)}{\partial t}\right\|_{0,\epsilon}.
\end{aligned}
$$

Then we have

$$
\left\|\frac{\partial(D^\alpha z^\epsilon)}{\partial t}\right\|_{0,\epsilon} \le \tilde{C}_p \ \forall\, t\in[0,T],
$$

and summing the last inequality on $|\alpha| \le p$, we obtain

$$
\left\|\frac{\partial z^\epsilon}{\partial t}\right\|_{p,\epsilon} \le p\tilde{C}_p \ \forall\, t\in[0,T].
$$

We now set $\tilde{\gamma}' = p\tilde{C}_p$, and we have $\left\|\dfrac{\partial z^\epsilon}{\partial t}\right\|_{p,\epsilon} \le \tilde{\gamma}' \ \forall\, t\in[0,T]$. $\square$

4. Homogenization

This section is devoted to the homogenization of the system (1). In [4], we showed that, since $\bar{v}^\epsilon$ is bounded independently of ϵ and under the assumption that $\bar{z}^\epsilon$ is bounded independently of ϵ, the solution of (1) two-scale converges to a profile:

$$
\begin{cases}
\bar{v}^\epsilon \overset{2-s}{\rightharpoonup} \bar{v} \in L^\infty([0,T), L^\infty_{per}(\mathbb{R}, L^2(\mathbb{T}^2,\mathbb{R}^3))),\\
\bar{z}^\epsilon \overset{2-s}{\rightharpoonup} \bar{z} \in L^\infty([0,T), L^\infty_{per}(\mathbb{R}, L^2(\mathbb{T}^2))).
\end{cases}
$$

That profile is the solution of the system

$$\begin{cases} \bar{B}^1 \dfrac{\partial \bar{v}}{\partial x_1} + \bar{B}^2 \dfrac{\partial \bar{v}}{\partial x_2} = -\bar{p}(\bar{v}, \bar{z}) & \text{on } [0,T) \times \mathbb{R} \times \mathbb{T}^2, \\ -\nabla \cdot (\tilde{\mathcal{A}} \nabla \bar{z}) = \nabla \cdot \tilde{\mathcal{C}} & \text{on } [0,T) \times \mathbb{R} \times \mathbb{T}^2, \end{cases} \tag{25}$$

with $B^1(t, \frac{t}{\epsilon}, x) \overset{2-s}{\rightharpoonup} \bar{B}^1(t,\theta,x)$, $B^2(t, \frac{t}{\epsilon}, x) \overset{2-s}{\rightharpoonup} \bar{B}^2(t,\theta,x)$, $\tilde{p}(v^\epsilon, z^\epsilon) \overset{2-s}{\rightharpoonup} \bar{p}(\bar{v}, \bar{z})$, $\mathcal{A}^\epsilon(\bar{v}^\epsilon) = \mathcal{A}(t, \dfrac{t}{\epsilon}, x) \overset{2-s}{\rightharpoonup} \tilde{\mathcal{A}}(t,\theta,x)$, and $\mathcal{C}^\epsilon(\bar{v}^\epsilon) = \mathcal{C}(t, \dfrac{t}{\epsilon}, x) \overset{2-s}{\rightharpoonup} \tilde{\mathcal{C}}(t,\theta,x)$.

In this work, we do not need any hypothesis for the homogenization of (1), since by Theorem 2.1 the solution is bounded independently of ϵ. But as indicated in [4], the two equations do not have the same scale for $\dfrac{1}{\epsilon}$. So globally, the two-scale convergence of (1) yields the two-scale convergence of the equation with the biggest order for $\dfrac{1}{\epsilon}$. Therefore, in this work we give a more precise homogenization of (1) in Theorems 2.2 and 2.3.

In [4], we used the two-scale convergence introduced in [1,8], but in this work will we use a multiscale convergence presented in [2]. Particularly, we will use the three-scale convergence.

Let us first recall some properties of two-scale and three-scale convergence.

Definition 4.1. Let (w^ϵ) in $L^\infty([0,T), H^l(\mathbb{T}^2, \mathbb{R}^4))$, $l \geq 1$, be a sequence of functions.

It two-scale converges to $\bar{w} \in L^\infty([0,T), L^\infty_\#(\mathbb{R}, H^l(\mathbb{T}^2, \mathbb{R}^4)))$ if for every $\psi \in C([0,T], C_\#(\mathbb{R}, H^l(\mathbb{T}^2, \mathbb{R}^4)))$ we have

$$\lim_{\epsilon \to 0} \int_{\mathbb{T}^2} \int_0^T w^\epsilon(t,x) \psi(t, \frac{t}{\epsilon}, x) dt dx = \int_{\mathbb{T}^2} \int_0^T \int_0^1 \bar{w}(t,\theta,x) \psi(t,\theta,x) d\theta dt dx.$$

It three-scale converges to $\bar{w} \in L^\infty([0,T), L^\infty_\#(\mathbb{R}, L^\infty_\#(\mathbb{R}, H^l(\mathbb{T}^2, \mathbb{R}^4))))$ if for every $\psi \in C([0,T], C_\#(\mathbb{R}, C_\#(\mathbb{R}, H^l(\mathbb{T}^2, \mathbb{R}^4))))$ we have

$$\lim_{\epsilon \to 0} \int_{\mathbb{T}^2} \int_0^T w^\epsilon(t,x) \psi(t, \frac{t}{\epsilon}, \frac{t}{\epsilon^4}, x) dt dx = \int_{\mathbb{T}^2} \int_0^T \int_0^1 \int_0^1 \bar{w}(t,\theta,x) \psi(t,\theta,\tau,x) d\tau d\theta dt dx.$$

We can give the following theorem stated in [1,8].

Theorem 4.1. *If a sequence of functions (w^ϵ) is bounded in $L^\infty([0,T), H^l(\mathbb{T}^2, \mathbb{R}^4))$, there exists a subsequence still denoted (w^ϵ) and a function $\bar{w} \in L^\infty([0,T), L^\infty_\#([0,1], H^l(\mathbb{T}^2, \mathbb{R}^4)))$ (or $L^\infty([0,T), L^\infty_\#([0,1], L^\infty_\#([0,1], H^l(\mathbb{T}^2, \mathbb{R}^4))))$) such that w^ϵ two-scale converges (or three-scale converges) to $\bar{w}$.*

We can now start the proof of Theorem 2.2.

Proof. Multiplying (11) by a test function $\psi^\epsilon(t, \dfrac{t}{\epsilon}, \dfrac{t}{\epsilon^4}, x) = \psi^\epsilon(t,x) \in \mathbb{R}^4$, regular enough, with $\mathrm{supp}(\psi^\epsilon) \subset [0,T) \times \mathbb{T}^2$ and $\psi(t,\theta,\tau,x)$ is 1-periodic in θ and τ. Integrating the last multiplied equation over $[0,T) \times \mathbb{T}^2$ yields

$$\int_{\mathbb{T}^2} \int_0^T \frac{\partial w^\epsilon}{\partial t} \psi^\epsilon + \frac{1}{\epsilon^4} \sum_{j=1}^2 \int_{\mathbb{T}^2} \int_0^T L^j \cdot \frac{\partial w^\epsilon}{\partial x_j} \psi^\epsilon - \frac{1}{\epsilon^2} \sum_{j=1}^2 \int_{\mathbb{T}^2} \int_0^T M^j \cdot \frac{\partial w^\epsilon}{\partial x_j} \psi^\epsilon$$

$$-\frac{1}{\epsilon^2}\sum_{j=1}^{2}\int_{\mathbb{T}^2}\int_0^T N^j\cdot\frac{\partial^2 w^\epsilon}{\partial x_j^2}\psi^\epsilon-\frac{1}{\epsilon}\int_{\mathbb{T}^2}\int_0^T E(w^\epsilon)\psi^\epsilon=0,$$

$$-\int_{\mathbb{T}^2}\int_0^T w^\epsilon\frac{\partial\psi^\epsilon}{\partial t}+\frac{1}{\epsilon^4}\sum_{j=1}^{2}-\int_{\mathbb{T}^2}\int_0^T w^\epsilon\cdot\frac{\partial(L^j\psi^\epsilon)}{\partial x_j}+\frac{1}{\epsilon^2}\sum_{j=1}^{2}\int_{\mathbb{T}^2}\int_0^T w^\epsilon\cdot\frac{\partial(M^j\psi^\epsilon)}{\partial x_j}$$

$$-\frac{1}{\epsilon^2}\sum_{j=1}^{2}\int_{\mathbb{T}^2}\int_0^T w^\epsilon\cdot\frac{\partial^2(N^j\psi^\epsilon)}{\partial x_j^2}-\frac{1}{\epsilon}\int_{\mathbb{T}^2}\int_0^T E(w^\epsilon)\psi^\epsilon-\int_{\mathbb{T}^2} w^\epsilon(0,x)\psi^\epsilon(0,0,0,x)=0.$$

Using the chain rule $\frac{\partial\psi^\epsilon}{\partial t}=\left(\frac{\partial\psi}{\partial t}\right)^\epsilon+\frac{1}{\epsilon}\left(\frac{\partial\psi}{\partial\theta}\right)^\epsilon+\frac{1}{\epsilon^4}\left(\frac{\partial\psi}{\partial\tau}\right)^\epsilon$, we have

$$-\frac{1}{\epsilon^4}\int_{\mathbb{T}^2}\int_0^T\left(w^\epsilon\left(\frac{\partial\psi}{\partial\tau}\right)^\epsilon+\sum_{j=1}^{2}w^\epsilon\cdot\frac{\partial(L^j\psi^\epsilon)}{\partial x_j}\right)$$
$$+\frac{1}{\epsilon^2}\int_{\mathbb{T}^2}\int_0^T\left(\sum_{j=1}^{2}w^\epsilon\cdot\frac{\partial(M^j\psi^\epsilon)}{\partial x_j}-\sum_{j=1}^{2}w^\epsilon\cdot\frac{\partial^2(N^j\psi^\epsilon)}{\partial x_j^2}\right)$$
$$+\frac{1}{\epsilon}\int_{\mathbb{T}^2}\int_0^T\left(w^\epsilon\left(\frac{\partial\psi}{\partial\theta}\right)^\epsilon-E(w^\epsilon)\psi^\epsilon\right)$$
$$-\int_{\mathbb{T}^2}\left(\int_0^T w^\epsilon\left(\frac{\partial\psi}{\partial t}\right)^\epsilon+w^\epsilon(0,x)\psi^\epsilon(0,0,0,x)\right)=0.$$

Multiplying the last equation by ϵ^4 and passing to the three-scale convergence, we obtain

$$\int_{\mathbb{T}^2}\int_0^T\int_0^1\int_0^1\left(\bar{w}_0\frac{\partial\psi}{\partial\tau}+\sum_{j=1}^{2}\bar{w}_0\cdot\frac{\partial(L_0^j\psi)}{\partial x_j}\right)d\tau d\theta dt dx=0.$$

Finally, we have the equation

$$\frac{\partial\bar{w}_0}{\partial\tau}+\sum_{j=1}^{2}L_0^j\cdot\frac{\partial\bar{w}_0}{\partial x_j}=0.$$

In the following, the above steps applied to (11) are called the "scale convergence process," and the following steps applied to (11) and (12) are called the "α-scale convergence process."

Differentiating (11) at order $|\alpha|$, $1\le|\alpha|\le l$, we obtain

$$\frac{\partial D^\alpha w^\epsilon}{\partial t}+\frac{1}{\epsilon^4}\sum_{j=1}^{2}D^\alpha L^j\cdot\frac{\partial w^\epsilon}{\partial x_j}+\frac{1}{\epsilon^4}\sum_{j=1}^{2}L^j\cdot\frac{\partial D^\alpha w^\epsilon}{\partial x_j}$$
$$-\frac{1}{\epsilon^2}\sum_{j=1}^{2}D^\alpha M^j\cdot\frac{\partial w^\epsilon}{\partial x_j}-\frac{1}{\epsilon^2}\sum_{j=1}^{2}M^j\cdot\frac{\partial D^\alpha w^\epsilon}{\partial x_j}-\frac{1}{\epsilon^2}\sum_{j=1}^{2}D^\alpha N^j\cdot\frac{\partial^2 w^\epsilon}{\partial x_j^2}$$
$$-\frac{1}{\epsilon^2}\sum_{j=1}^{2}N^j\cdot\frac{\partial^2 D^\alpha w^\epsilon}{\partial x_j^2}=\frac{1}{\epsilon}D^\alpha E(w^\epsilon).\tag{26}$$

Multiplying (26) by $\psi^\epsilon(t, \frac{t}{\epsilon}, \frac{t}{\epsilon^4}, x) = \psi^\epsilon(t,x)$, integrating the resulting equation by parts in $[0,T) \times \mathbb{T}^2$, and finally passing to the three-scale limit, we obtain

$$\frac{\partial \overline{D^\alpha w^\epsilon}}{\partial t} + \sum_{j=1}^{2} D^\alpha L_0^j \cdot \frac{\partial \bar{w}^0}{\partial x_j} + \sum_{j=1}^{2} L_0^j \cdot \frac{\partial \overline{D^\alpha w^\epsilon}}{\partial x_j} = 0$$

since $D^\alpha w^\epsilon$ three-scale converges to $\overline{D^\alpha w^\epsilon}$.

We now need to prove that $D^\alpha \bar{w}^0 = \overline{D^\alpha w^\epsilon}$. For that consider the homogenized equation for $\bar{w}^0$.

Differentiating (12) at order $|\alpha| \leq l$ yields

$$\frac{\partial D^\alpha \bar{w}^0}{\partial t} + \sum_{j=1}^{2} D^\alpha L_0^j \cdot \frac{\partial \bar{w}^0}{\partial x_j} + \sum_{j=1}^{2} L_0^j \cdot \frac{\partial D^\alpha \bar{w}^0}{\partial x_j} = 0. \tag{27}$$

In addition, $\bar{w}_0$ is a unique solution of (12), and then $D^\alpha \bar{w}^0$ is a unique solution of (27). Then $D^\alpha \bar{w}^0 = \overline{D^\alpha w^\epsilon}$. $\square$

We now give the proof of Theorem 2.3.

Proof. Replacing (Antz2) in (11), we have

$$\begin{aligned}
&\frac{\partial w^\epsilon}{\partial t} + \frac{1}{\epsilon^4} \sum_{k=0}^{1} \epsilon^k \sum_{j=1}^{2} L_k^j \cdot \frac{\partial w^\epsilon}{\partial x_j} - \frac{1}{\epsilon^2} \sum_{k=0}^{1} \epsilon^k \sum_{j=1}^{2} M_k^j \cdot \frac{\partial w^\epsilon}{\partial x_j} \\
&\qquad - \frac{1}{\epsilon^2} \sum_{k=0}^{1} \epsilon^k \sum_{j=1}^{2} N_k^j \cdot \frac{\partial^2 w^\epsilon}{\partial x_j^2} - \frac{1}{\epsilon} \sum_{k=0}^{1} \epsilon^k E_k = 0, \\
&\frac{\partial w^\epsilon}{\partial t} + \frac{1}{\epsilon^4} \sum_{j=1}^{2} L_0^j \cdot \frac{\partial w^\epsilon}{\partial x_j} + \frac{1}{\epsilon^3} \sum_{j=1}^{2} L_1^j \cdot \frac{\partial w^\epsilon}{\partial x_j} \\
&\qquad - \frac{1}{\epsilon^2} \left(\sum_{j=1}^{2} M_0^j \cdot \frac{\partial w^\epsilon}{\partial x_j} + \sum_{j=1}^{2} N_0^j \cdot \frac{\partial^2 w^\epsilon}{\partial x_j^2} \right) \\
&- \frac{1}{\epsilon} \left(\sum_{j=1}^{2} M_1^j \cdot \frac{\partial w^\epsilon}{\partial x_j} + \sum_{j=1}^{2} N_1^j \cdot \frac{\partial^2 w^\epsilon}{\partial x_j^2} + E_0) \right) - E_1 = 0.
\end{aligned} \tag{28}$$

Replacing the chain rule $\frac{\partial \bar{w}_0^\epsilon}{\partial t} = \left(\frac{\partial \bar{w}_0}{\partial t} \right)^\epsilon + \frac{1}{\epsilon} \left(\frac{\partial \bar{w}_0}{\partial \theta} \right)^\epsilon + \frac{1}{\epsilon^4} \left(\frac{\partial \bar{w}_0}{\partial \tau} \right)^\epsilon$ in (12), we have

$$\frac{\partial \bar{w}_0^\epsilon}{\partial t} + \sum_{j=1}^{2} L_{0,\epsilon}^j \cdot \frac{\partial \bar{w}_0^\epsilon}{\partial x_j} - \left(\frac{\partial \bar{w}_0}{\partial t} \right)^\epsilon - \frac{1}{\epsilon} \left(\frac{\partial \bar{w}_0}{\partial \theta} \right)^\epsilon = 0. \tag{29}$$

Subtracting (29) from (28) yields

$$\begin{aligned}
&\frac{\partial (w^\epsilon - \bar{w}_0^\epsilon)}{\partial t} + \frac{1}{\epsilon^4} \sum_{j=1}^{2} L_0^j \cdot \frac{\partial (w^\epsilon - \bar{w}_0^\epsilon)}{\partial x_j} + \frac{1}{\epsilon^3} \sum_{j=1}^{2} L_1^j \cdot \frac{\partial w^\epsilon}{\partial x_j} \\
&\qquad - \frac{1}{\epsilon^2} \left(\sum_{j=1}^{2} M_0^j \cdot \frac{\partial w^\epsilon}{\partial x_j} + \sum_{j=1}^{2} N_0^j \cdot \frac{\partial^2 w^\epsilon}{\partial x_j^2} \right)
\end{aligned}$$

$$-\frac{1}{\epsilon}\left(-\left(\frac{\partial\bar{w}_0}{\partial\theta}\right)^{\epsilon}+\sum_{j=1}^{2}M_1^j\cdot\frac{\partial w^\epsilon}{\partial x_j}+\sum_{j=1}^{2}N_1^j\cdot\frac{\partial^2 w^\epsilon}{\partial x_j^2}+E_0)\right)$$
$$+\left(\frac{\partial\bar{w}_0}{\partial t}\right)^{\epsilon}-E_1=0,$$

$$\epsilon\frac{\partial K^{1,\epsilon}}{\partial t}+\frac{1}{\epsilon^3}\sum_{j=1}^{2}L_0^j\cdot\frac{\partial K^{1,\epsilon}}{\partial x_j}+\frac{1}{\epsilon^3}\sum_{j=1}^{2}L_1^j\cdot\frac{\partial w^\epsilon}{\partial x_j}$$
$$-\frac{1}{\epsilon^2}\left(\sum_{j=1}^{2}M_0^j\cdot\frac{\partial w^\epsilon}{\partial x_j}+\sum_{j=1}^{2}N_0^j\cdot\frac{\partial^2 w^\epsilon}{\partial x_j^2}\right)$$
$$-\frac{1}{\epsilon}\left(-\left(\frac{\partial\bar{w}_0}{\partial\theta}\right)^{\epsilon}+\sum_{j=1}^{2}M_1^j\cdot\frac{\partial w^\epsilon}{\partial x_j}+\sum_{j=1}^{2}N_1^j\cdot\frac{\partial^2 w^\epsilon}{\partial x_j^2}+E_0)\right)$$
$$+\left(\frac{\partial\bar{w}_0}{\partial t}\right)^{\epsilon}-E_1=0.$$

Then $\dfrac{w^\epsilon-\bar{w}_0^\epsilon}{\epsilon}=K^{1,\epsilon}=\bar{w}_1^\epsilon+\epsilon\bar{w}_2^\epsilon+\epsilon^2\bar{w}_3^\epsilon+\epsilon^3\bar{w}_4^\epsilon$ is a solution of

$$\frac{\partial K^{1,\epsilon}}{\partial t}+\frac{1}{\epsilon^4}\sum_{j=1}^{2}L_0^j\cdot\frac{\partial K^{1,\epsilon}}{\partial x_j}+\frac{1}{\epsilon^4}\sum_{j=1}^{2}L_1^j\cdot\frac{\partial w^\epsilon}{\partial x_j}$$
$$-\frac{1}{\epsilon^3}\left(\sum_{j=1}^{2}M_0^j\cdot\frac{\partial w^\epsilon}{\partial x_j}+\sum_{j=1}^{2}N_0^j\cdot\frac{\partial^2 w^\epsilon}{\partial x_j^2}\right)$$
$$-\frac{1}{\epsilon^2}\left(-\left(\frac{\partial\bar{w}_0}{\partial\theta}\right)^{\epsilon}+\sum_{j=1}^{2}M_1^j\cdot\frac{\partial w^\epsilon}{\partial x_j}+\sum_{j=1}^{2}N_1^j\cdot\frac{\partial^2 w^\epsilon}{\partial x_j^2}+E_0)\right)$$
$$+\frac{1}{\epsilon}\left(\left(\frac{\partial\bar{w}_0}{\partial t}\right)^{\epsilon}-E_1\right)=0. \tag{30}$$

By using the same properties and assumptions as in Theorem 2.1, we can prove that the system (30) has a unique solution in a space B_T^ϵ satisfying estimates as in Theorem 2.1.

We now apply the scale convergence process to (30).

Multiplying (30) by a test functions $\psi^\epsilon(t,\frac{t}{\epsilon},\frac{t}{\epsilon^4},x)=\psi^\epsilon(t,x)\in\mathbb{R}^4$ regular enough, with $\mathrm{supp}(\psi^\epsilon)\subset[0,T)\times\mathbb{T}^2$ and $\theta,\ \tau\mapsto\psi(t,\theta,\tau,x)$ is 1-periodic in θ and τ. Then, integrating that new equation by parts, and using the chain rule, we obtain

$$-\frac{1}{\epsilon^3}\int_{\mathbb{T}^2}\int_0^T\left(K^{1,\epsilon}\left(\frac{\partial\psi}{\partial\tau}\right)^{\epsilon}+\sum_{j=1}^{2}K^{1,\epsilon}\cdot\frac{\partial L_{0,\epsilon}^j\psi^\epsilon}{\partial x_j}+\sum_{j=1}^{2}w^\epsilon\cdot\frac{\partial L_{1,\epsilon}^j\psi^\epsilon}{\partial x_j}\right)$$
$$+\frac{1}{\epsilon^2}\int_{\mathbb{T}^2}\int_0^T\left(\sum_{j=1}^{2}w^\epsilon\cdot\frac{\partial M_{0,\epsilon}^j\psi^\epsilon}{\partial x_j}-\sum_{j=1}^{2}w^\epsilon\cdot\frac{\partial^2 N_{0,\epsilon}^j\psi^\epsilon}{\partial x_j^2}\right)$$

$$+\frac{1}{\epsilon}\int_{\mathbb{T}^2}\int_0^T\left(\left(\frac{\partial \bar{w}_0}{\partial \theta}\right)^{\epsilon}\psi^{\epsilon}+\sum_{j=1}^{2} w^{\epsilon}\cdot\frac{\partial M^j_{1,\epsilon}\psi^{\epsilon}}{\partial x_j}-\sum_{j=1}^{2} w^{\epsilon}\cdot\frac{\partial^2 N^j_{1,\epsilon}\psi^{\epsilon}}{\partial x_j^2}+E_{0,\epsilon}\psi^{\epsilon}\right)$$
$$+\left(\int_{\mathbb{T}^2}\int_0^T\left(\left(\frac{\partial \bar{w}_0}{\partial t}\right)^{\epsilon}\psi^{\epsilon}-E_{1,\epsilon}\psi^{\epsilon}-K^{1,\epsilon}\left(\frac{\partial \psi}{\partial \theta}\right)^{\epsilon}\right)-\int_{\mathbb{T}^2}K^{1,\epsilon}(0,x)\psi^{\epsilon}(0,0,0,x)\right)=0.$$

Multiplying the last equation by ϵ^3 and using three-scale convergence, we have

$$\int_{\mathbb{T}^2}\int_0^T\int_0^1\int_0^1\left(\bar{w}_1\frac{\partial \psi}{\partial \tau}+\sum_{j=1}^{2}\bar{w}_1\cdot\frac{\partial L^j_0\psi}{\partial x_j}+\sum_{j=1}^{2}\bar{w}_0\cdot\frac{\partial L^j_1\psi}{\partial x_j}\right)=0.$$

Finally, we obtain

$$\frac{\partial \bar{w}_1}{\partial \tau}+\sum_{j=1}^{2}L^j_0\cdot\frac{\partial \bar{w}_1}{\partial x_j}+\sum_{j=1}^{2}L^j_1\cdot\frac{\partial \bar{w}_0}{\partial x_j}=0.$$

Hence, $K^{1,\epsilon}$ three-scale converges to $\bar{w}_1$ in the space $L^{\infty}([0,T),L^{\infty}_{\#}([0,1],L^{\infty}_{\#}([0,1],L^2(\mathbb{T}^2,\mathbb{R}^4))))$. Now it remains to prove the three-scale convergence for all differentials at order $|\alpha|\le l$ of $K^{1,\epsilon}$. Applying the α-scale convergence process to (30) and (13), we obtain $D^{\alpha}K^{1,\epsilon}$ three-scale converges to $D^{\alpha}\bar{w}_1$ in the space $L^{\infty}([0,T),L^{\infty}_{\#}([0,1],L^{\infty}_{\#}([0,1],L^2(\mathbb{T}^2,\mathbb{R}^4))))$. Then $K^{1,\epsilon}$ three-scale converges to $\bar{w}_1$ in the space $L^{\infty}([0,T),L^{\infty}_{\#}([0,1],L^{\infty}_{\#}([0,1],H^l(\mathbb{T}^2,\mathbb{R}^4))))$.

We can do the same work for $\dfrac{K^{1,\epsilon}-\bar{w}_1^{\epsilon}}{\epsilon}=K^{2,\epsilon}=\bar{w}_2^{\epsilon}+\epsilon\bar{w}_3^{\epsilon}+\epsilon^2\bar{w}_4^{\epsilon}$, which is the solution of the following system:

$$\begin{aligned}&\frac{\partial K^{2,\epsilon}}{\partial t}+\frac{1}{\epsilon^4}\sum_{j=1}^{2}L^j_0\cdot\frac{\partial K^{2,\epsilon}}{\partial x_j}+\frac{1}{\epsilon^4}\sum_{j=1}^{2}L^j_1\cdot\frac{\partial K^{1,\epsilon}}{\partial x_j}\\&\qquad-\frac{1}{\epsilon^4}\left(\sum_{j=1}^{2}M^j_0\cdot\frac{\partial w^{\epsilon}}{\partial x_j}+\sum_{j=1}^{2}N^j_0\cdot\frac{\partial^2 w^{\epsilon}}{\partial x_j^2}\right)\\&-\frac{1}{\epsilon^3}\left(-\left(\frac{\partial \bar{w}_0}{\partial \theta}\right)^{\epsilon}+\sum_{j=1}^{2}M^j_1\cdot\frac{\partial w^{\epsilon}}{\partial x_j}+\sum_{j=1}^{2}N^j_1\cdot\frac{\partial^2 w^{\epsilon}}{\partial x_j^2}+E_{0,\epsilon})\right)\\&+\frac{1}{\epsilon^2}\left(\left(\frac{\partial \bar{w}_0}{\partial t}\right)^{\epsilon}-E_{1,\epsilon}+\left(\frac{\partial \bar{w}_1}{\partial \theta}\right)^{\epsilon}\right)+\frac{1}{\epsilon}\left(\frac{\partial \bar{w}_1}{\partial t}\right)^{\epsilon}=0.\end{aligned}\tag{31}$$

As above, by using the same properties and assumptions as in Theorem 2.1, we can prove that the system (31) has a unique solution in a space B^{ϵ}_T satisfying estimates as in Theorem 2.1.

Applying the scale convergence process to (31), we find that $\bar{w}_2$ is the solution of the following equation:

$$\begin{aligned}&\frac{\partial \bar{w}_2}{\partial \tau}+\sum_{j=1}^{2}L^j_0\cdot\frac{\partial \bar{w}_2}{\partial x_j}+\sum_{j=1}^{2}L^j_1\cdot\frac{\partial \bar{w}_1}{\partial x_j}\\&\qquad-\sum_{j=1}^{2}M^j_0\cdot\frac{\partial \bar{w}_0}{\partial x_j}-\sum_{j=1}^{2}N^j_0\cdot\frac{\partial^2 \bar{w}_0}{\partial x_j^2}=0.\end{aligned}$$

Applying the α-scale convergence process to (14) and (31), we obtain $D^\alpha K^{2,\epsilon}$ three-scale converges to $D^\alpha \bar{w}_2$ in the space $L^\infty([0,T), L^\infty_\#([0,1], L^\infty_\#([0,1], L^2(\mathbb{T}^2, \mathbb{R}^4))))$. So, $K^{2,\epsilon}$ three-scale converges to $\bar{w}_2$ in the space $L^\infty([0,T), L^\infty_\#([0,1], L^\infty_\#([0,1], H^l(\mathbb{T}^2, \mathbb{R}^4))))$.

We do the same work for $\dfrac{K^{2,\epsilon} - \bar{w}_2^\epsilon}{\epsilon} = K^{3,\epsilon} = \bar{w}_3^\epsilon + \epsilon \bar{w}_4^\epsilon$, and find $K^{3,\epsilon}$ is the solution of the following system:

$$\begin{aligned}
&\frac{\partial K^{3,\epsilon}}{\partial t} + \frac{1}{\epsilon^4}\sum_{j=1}^{2} L_0^j \cdot \frac{\partial K^{3,\epsilon}}{\partial x_j} + \frac{1}{\epsilon^4}\sum_{j=1}^{2} L_1^j \cdot \frac{\partial K^{2,\epsilon}}{\partial x_j} \\
&\qquad - \frac{1}{\epsilon^4}\left(\sum_{j=1}^{2} M_0^j \cdot \frac{\partial K^{1,\epsilon}}{\partial x_j} + \sum_{j=1}^{2} N_0^j \cdot \frac{\partial^2 K^{1,\epsilon}}{\partial x_j^2}\right) \\
&- \frac{1}{\epsilon^4}\left(-\left(\frac{\partial \bar{w}_0}{\partial \theta}\right)^\epsilon + \sum_{j=1}^{2} M_1^j \cdot \frac{\partial w^\epsilon}{\partial x_j} + \sum_{j=1}^{2} N_1^j \cdot \frac{\partial^2 w^\epsilon}{\partial x_j^2} + E_{0,\epsilon}\right) \\
&\qquad + \frac{1}{\epsilon^3}\left(\left(\frac{\partial \bar{w}_0}{\partial t}\right)^\epsilon - E_{1,\epsilon} + \left(\frac{\partial \bar{w}_1}{\partial \theta}\right)^\epsilon\right) \\
&\qquad + \frac{1}{\epsilon^2}\left(\left(\frac{\partial \bar{w}_0}{\partial t}\right)^\epsilon + \left(\frac{\partial \bar{w}_2}{\partial \theta}\right)^\epsilon\right) + \frac{1}{\epsilon}\left(\frac{\partial \bar{w}_2}{\partial t}\right)^\epsilon = 0.
\end{aligned} \tag{32}$$

As previously, we can prove that the system (32) has a unique solution in a space B_T^ϵ satisfying estimates as in Theorem 2.1. Applying the scale convergence process to (32), we show that $\bar{w}_3$ is the solution of the following equation:

$$\begin{aligned}
&\frac{\partial \bar{w}_3}{\partial \tau} + \sum_{j=1}^{2} L_0^j \cdot \frac{\partial \bar{w}_3}{\partial x_j} + \sum_{j=1}^{2} L_1^j \cdot \frac{\partial \bar{w}_2}{\partial x_j} \\
&\qquad - \sum_{j=1}^{2} M_0^j \cdot \frac{\partial \bar{w}_1}{\partial x_j} - \sum_{j=1}^{2} N_0^j \cdot \frac{\partial^2 \bar{w}_1}{\partial x_j^2} \\
&+ \frac{\partial \bar{w}_0}{\partial \theta} - \sum_{j=1}^{2} M_1^j \cdot \frac{\partial \bar{w}_0}{\partial x_j} - \sum_{j=1}^{2} N_1^j \cdot \frac{\partial^2 \bar{w}_0}{\partial x_j^2} - E_0 = 0.
\end{aligned}$$

As above, we apply the α-scale convergence process to (15) and (32) and obtain $D^\alpha K^{3,\epsilon}$ three-scale converges to $D^\alpha \bar{w}_3$ in the space $L^\infty([0,T), L^\infty_\#([0,1], L^\infty_\#([0,1], L^2(\mathbb{T}^2, \mathbb{R}^4))))$. Then $K^{3,\epsilon}$ three-scale converges to $\bar{w}_3$ in the space $L^\infty([0,T), L^\infty_\#([0,1], L^\infty_\#([0,1], H^l(\mathbb{T}^2, \mathbb{R}^4))))$.

Now $\dfrac{K^{3,\epsilon} - \bar{w}_3^\epsilon}{\epsilon} = K^{4,\epsilon} = \bar{w}_4^\epsilon$ is the solution of the following system:

$$\begin{aligned}
&\frac{\partial K^{4,\epsilon}}{\partial t} + \frac{1}{\epsilon^4}\left(\sum_{j=1}^{2} L_0^j \cdot \frac{\partial K^{4,\epsilon}}{\partial x_j} + \sum_{j=1}^{2} L_1^j \cdot \frac{\partial K^{3,\epsilon}}{\partial x_j} - \sum_{j=1}^{2} M_0^j \cdot \frac{\partial K^{2,\epsilon}}{\partial x_j} - \sum_{j=1}^{2} N_0^j \cdot \frac{\partial^2 K^{2,\epsilon}}{\partial x_j^2}\right) \\
&\quad - \frac{1}{\epsilon^4}\left(-\left(\frac{\partial \bar{w}_0}{\partial t}\right)^\epsilon - \left(\frac{\partial \bar{w}_1}{\partial \theta}\right)^\epsilon + \sum_{j=1}^{2} M_1^j \cdot \frac{\partial K^{1,\epsilon}}{\partial x_j} + \sum_{j=1}^{2} N_1^j \cdot \frac{\partial^2 K^{1,\epsilon}}{\partial x_j^2} + E_{1,\epsilon}\right) \\
&\quad + \frac{1}{\epsilon^3}\left(\left(\frac{\partial \bar{w}_0}{\partial t}\right)^\epsilon + \left(\frac{\partial \bar{w}_2}{\partial \theta}\right)^\epsilon\right) + \frac{1}{\epsilon^2}\left(\left(\frac{\partial \bar{w}_2}{\partial t}\right)^\epsilon + \left(\frac{\partial \bar{w}_3}{\partial \theta}\right)^\epsilon\right) + \frac{1}{\epsilon}\left(\frac{\partial \bar{w}_3}{\partial t}\right)^\epsilon = 0.
\end{aligned} \tag{33}$$

We can prove that the system (33) has a unique solution in a space B_T^ϵ satisfying estimates as in Theorem 2.1. Applying the scale convergence process to (33), we show that $\bar{w}_4$ is the solution of the following equation:

$$\begin{aligned}\frac{\partial \bar{w}_4}{\partial \tau} + \sum_{j=1}^{2} L_0^j \cdot \frac{\partial \bar{w}_4}{\partial x_j} + \sum_{j=1}^{2} L_1^j \cdot \frac{\partial \bar{w}_3}{\partial x_j} \\ - \sum_{j=1}^{2} M_0^j \cdot \frac{\partial \bar{w}_2}{\partial x_j} - \sum_{j=1}^{2} N_0^j \cdot \frac{\partial^2 \bar{w}_2}{\partial x_j^2} \\ + \frac{\partial \bar{w}_0}{\partial t} + \frac{\partial \bar{w}_1}{\partial \theta} - \sum_{j=1}^{2} M_1^j \cdot \frac{\partial \bar{w}_1}{\partial x_j} - \sum_{j=1}^{2} N_1^j \cdot \frac{\partial^2 \bar{w}_1}{\partial x_j^2} + E_1 = 0.\end{aligned}$$

We do the same work as above, and apply the α-scale convergence process to (16) and (33); we obtain $D^\alpha K^{4,\epsilon}$ three-scale converges to $D^\alpha \bar{w}_4$ in the space $L^\infty([0,T), L^\infty_\#([0,1], L^\infty_\#([0,1], L^2(\mathbb{T}^2, \mathbb{R}^4))))$. Then $K^{4,\epsilon}$ three-scale converges to $\bar{w}_4$ in the space $L^\infty([0,T), L^\infty_\#([0,1], L^\infty_\#([0,1], H^l(\mathbb{T}^2, \mathbb{R}^4))))$. $\square$

5. Conclusion

In this work, we have extended the existence and uniqueness and the homogenization results given in [4]. That is, we have shown the solution of the LTDD equation, z^ϵ, is bounded independently of the small parameter ϵ. Also, we have given homogenization and corrector results for the coupled system.

In subsequent work, we will perform numerical studies on both systems: the coupled system and the homogenized one. Precisely, we will compare the solutions of these systems.

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