

# Defect-Deferred Correction Method Based on a Subgrid Artificial Viscosity Modeling

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## Abstract

An alternative first step approximation based on subgrid artificial viscosity modeling (SAV) is proposed for defect-deferred correction method (DDC) for incompressible Navier-Stokes equation at high Reynolds number. This new approach not only preserves all qualifications of the conventional artificial viscosity (AV) based DDC, such as unconditional stability, high order of accuracy and so on, it has also shown its superiority over choosing AV approximation in the predictor step. Both theory and computational results presented in this paper illustrate that this alternative approach indeed increases the efficiency of the DDC method.

*Keywords:* high Reynolds number, defect-correction, deferred-correction, subgrid artificial viscosity, variational multiscale

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## 1. Introduction

In this report, for the pair of unknown velocity  $\mathbf{u}$  and pressure  $\mathbf{p}$ , we consider the incompressible Navier-Stokes equation (NSE) (1.1) at high Reynolds number ( $Re^{-1} \propto \nu$ ). According to Kolmogorov's K41 Theory [1], as Reynolds number increases required computational cost raises prohibitively high. Attempting to solve the problem directly with an affordable computational cost (on a much coarser mesh than required) usually causes related linear systems to converge too slowly, or even if they converge within a reasonable time frame, their results are far from being realistic.

$$\begin{aligned} \mathbf{u}_t - \nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \mathbf{p} &= \mathbf{f}, \\ \nabla \cdot \mathbf{u} &= 0. \end{aligned} \quad (1.1)$$

Various techniques including the defect correction have been introduced to mitigate this issue, see [2, 3, 5]. Also a recent work combining this correction approach with a deferred correction (see e.g. [6, 7, 8, 9]) for an increased temporal accuracy have been proposed in [10]. Methods on both of these papers are based on a predictor-corrector scheme: As a predictor step, an approximation is found by a computationally very attractive artificial viscosity (AV) approximation with a backward-Euler time discretization (eq. 1.2), and as for the corrector step, the affect of the AV is subtracted via previously found predictor step approximation (eq. 1.5). In particular, see the following scheme for the artificial viscosity based defect-deferred correction method (AV-DDC) presented in [10]:

$$\begin{aligned} \left( \frac{u_1^{h,n+1} - u_1^{h,n}}{k}, v^h \right) + (\nu + h)(\nabla u_1^{h,n+1}, \nabla v^h) + b^*(u_1^{h,n+1}, u_1^{h,n+1}, v^h) \\ - (p_1^{h,n+1}, \nabla \cdot v^h) = (f(t_{n+1}), v^h) \end{aligned} \quad (1.2)$$

where  $b^*(\cdot, \cdot, \cdot)$  is defined below.

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AV-DDC beside being an efficient method, it has been successfully applied to various problems including two-domain convection-dominated convection diffusion problem and nonlinearly-coupled fluid-fluid interaction, see [11, 12]. In all of these papers, it has been shown to be an unconditionally-stable, high-accuracy regularization technique (a second order in time and space). It is also parallelizable for a faster result since only data transfer required for the correction step is the AV solutions on the current and previous time steps. Therefore, one can easily run the scheme in parallel as long as AV approximation marches only two time steps earlier than the correction steps.

On the other hand, the accuracy of the correction step approximation is strongly dependent on the accuracy of the predictor step, in general. Especially, for AV-DDC methods, accuracy of the correction step is lifted by an order of 1 due to the multiplication of  $h$  in the laplacian of the AV approximation, see the last term of the equation (1.5). Also AV approximation is known to be too dissipative (in all scales) so that it cannot capture turbulent characteristics of the flow and results in a fully-laminar flow, e.g. see [10]. Therefore, replacing the predictor AV step with a less dissipative and high-accuracy approximation fosters the overall accuracy of the first step approximation and, in consequence, correction step approximation produces better solutions. In this report, AV approximation in the first step will be replaced with a projection-based subgrid artificial viscosity method (SAV) to further increase the accuracy of the correction step approximation, see e.g. [13, 14, 15, 16, 17] for SAV and its inspiration source variational multiscale methods (VMS).

Hence, the replacement of (1.2) with (1.3) for the predictor step is proposed, and the new defect-deferred correction based on SAV is abbreviated to SAV-DDC (1.3-1.5). In contrast to commonly used coupled (implicit) form of SAV in the literature this replacement decouples the projection step from the NSE for computational efficiency, further discussed below. Although this decoupling comes with an extra  $O(\Delta t)$  error, decoupled SAV still meets with our expectations from the predictor step approximation since it only has to be first order of accuracy as in AV approximation.

$$\begin{aligned} & \left( \frac{u_1^{h,n+1} - u_1^{h,n}}{k}, v^h \right) + (\nu + h)(\nabla u_1^{h,n+1}, \nabla v^h) + b^*(u_1^{h,n+1}, u_1^{h,n+1}, v^h) \\ & - (p_1^{h,n+1}, \nabla \cdot v^h) = (f(t_{n+1}), v^h) + h(\mathbb{G}_1^{\mathbb{H},n}, \nabla v^h), \end{aligned} \quad (1.3)$$

$$(\mathbb{G}_1^{\mathbb{H},n} - \nabla u_1^{h,n}, \mathbb{L}^H) = 0. \quad (1.4)$$

$$\begin{aligned} & \left( \frac{u_2^{h,n+1} - u_2^{h,n}}{k}, v^h \right) + (\nu + h)(\nabla u_2^{h,n+1}, \nabla v^h) + b^*(u_2^{h,n+1}, u_2^{h,n+1}, v^h) \\ & - (p_2^{h,n+1}, \nabla \cdot v^h) = \left( \frac{f(t_{n+1}) + f(t_n)}{2}, v^h \right) + \frac{\nu}{2} k \left( \nabla \left( \frac{u_1^{h,n+1} - u_1^{h,n}}{k} \right), v^h \right) \\ & + \frac{1}{2} b^*(u_1^{h,n+1}, u_1^{h,n+1}, v^h) - \frac{1}{2} b^*(u_1^{h,n}, u_1^{h,n}, v^h) + h(\nabla u_1^{h,n+1}, \nabla v^h), \end{aligned} \quad (1.5)$$

The equation (1.4) means that  $\mathbb{G}_1^{\mathbb{H},n}$  is the projection of  $\nabla u_1^{h,n}$  on a coarse mesh. Consider

$$(\nu + h)(\nabla u_1^{h,n+1}, \nabla v^h) - h(\mathbb{G}_1^{\mathbb{H},n}, \nabla v^h) = \nu(\nabla u_1^{h,n+1}, \nabla v^h) + h(\nabla u_1^{h,n+1} - \mathbb{G}_1^{\mathbb{H},n}, \nabla v^h).$$

Roughly ignoring the difference in the corresponding time levels, the last term corresponds to the gradient of the small scales that would disappear upon the projection onto the given coarse mesh. Therefore, we infer that the dissipative affect of the artificial viscosity is only introduced on small scales and it acts solely indirectly on large scales, see [13] for details. This distinction results in resolving large eddies (the ultimate goal of most practitioners) with a much higher accuracy with SAV than AV approximation, which acts on all scales regardless of their sizes.

Next, we comment on SAV and its various treatments. Implicit treatment of the VMS requires solving for the projection of the gradient in a coupled way. An efficient implementation of VMS for fully-implicit treatment is discussed in [13] with a requirement that  $L^H$  space has to be a discontinuous finite element space with an  $L^2$  orthogonal basis. In this setting small modifications of the existing code will be enough to solve the system fully-coupled. On the other hand, explicit treatment on the projection term decouples this terms from the original system. It only requires taking the projection of the previously-found velocity field each time. A detailed comparison of these two approaches has been studied in [18].

The next section presents the notation and some preliminaries. It is followed by the analysis of the first step approximation (SAV approximation) in terms of stability and accuracy. The theoretical findings for the second (correction) step approximation is studied in the fourth section. Finally, the paper is concluded with the computations section.

## 2. Mathematical Preliminaries and Notations

The norm  $\|\cdot\|$  denotes the usual  $L^2(\Omega)$  norm induced by the usual  $L^2$  inner-product, denoted by  $(\cdot, \cdot)$ . The space that velocity belongs to is

$$X = H_0^1(\Omega)^d = \{v \in L^2(\Omega)^d : \nabla v \in L^2(\Omega)^{dx d} \text{ and } v = 0 \text{ on } \partial\Omega\}.$$

with the norm  $\|v\|_X = \|\nabla v\|$ . The space dual to  $X$  is equipped with the norm

$$\|\cdot\|_{-1} = \sup_{v \in X} \frac{(\cdot, v)}{\|\nabla v\|}.$$

The space where the pressure is sought is

$$Q = L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_{\Omega} q(x) dx = 0\}.$$

$V$  is the space of weakly divergence-free functions given by

$$X \supset V = \{v \in X : (\nabla \cdot v, q) = 0, \forall q \in Q\}.$$

For measurable functions, we define

$$\|\cdot\|_{L^p(0,T;X)} = \left( \int_0^T \|\cdot\|_X^p dt \right)^{\frac{1}{p}}, 1 \leq p < \infty,$$

and

$$\|\cdot\|_{L^\infty(0,T;X)} = \operatorname{ess\,sup}_{0 \leq t \leq T} \|\cdot\|_X.$$

We assume that the velocity-pressure finite element spaces  $X^h \subset X$  and  $Q^h \subset Q$  are conforming, have typical approximation properties of finite element spaces, and satisfy the discrete inf-sup, or  $LBB^h$ , condition

$$\inf_{q^h \in Q^h} \sup_{v^h \in X^h} \frac{(q^h, \nabla \cdot v^h)}{\|\nabla v^h\| \|q^h\|} \geq \beta^h > 0, \quad (2.1)$$

where  $\beta^h$  is bounded away from zero uniformly in  $h$ . Examples of such spaces can be found in [19]. We consider commonly-used Taylor-Hood finite elements  $X^h \subset X$ ,  $Q^h \subset Q$  to be spaces of continuous piecewise polynomials of degree  $m$  and  $m-1$ , respectively, with  $m \geq 2$ . The space of discretely divergence-free functions is defined as follows

$$V^h = \{v^h \in X^h : (q^h, \nabla \cdot v^h) = 0, \forall q^h \in Q^h\}.$$

In the theoretical analysis the following Modified Stokes Projection will be used.

**Definition 2.1 (Modified Stokes Projection).** *The Stokes projection operator  $P_S : (X, Q) \rightarrow (X^h, Q^h)$ ,  $P_S(v, q) = (\tilde{v}, \tilde{q})$ , satisfies*

$$\begin{aligned} (\nu + h)(\nabla(v - \tilde{v}), \nabla \phi^h) - (q - \tilde{q}, \nabla \cdot \zeta^h) &= 0, \\ (\nabla \cdot (v - \tilde{v}), \zeta^h) &= 0, \end{aligned} \quad (2.2)$$

for any  $\phi^h \in V^h, \zeta^h \in Q^h$ .

In  $(V^h, Q^h)$  this formulation means: given  $(v, q) \in (X, Q)$ , find  $\tilde{v} \in V^h$  satisfying

$$(\nu + h)(\nabla(v - \tilde{v}), \nabla\phi^h) - (q - \zeta^h, \nabla \cdot \phi^h) = 0, \quad (2.3)$$

for any  $\phi^h \in V^h, \zeta^h \in Q^h$ .

Define the explicitly skew-symmetrized trilinear form

$$b^*(u, v, w) := \frac{1}{2}(u \cdot \nabla v, w) - \frac{1}{2}(u \cdot \nabla w, v).$$

The following estimate is easy to prove (see, e.g., [19]): there exists a constant  $C = C(\Omega)$  such that

$$|b^*(u, v, w)| \leq C(\Omega) \|\nabla u\| \|\nabla v\| \|\nabla w\|. \quad (2.4)$$

The proofs will require the sharper bound on the nonlinearity. This upper bound is improvable in  $\mathbb{R}^2$ .

**Lemma 2.2 (The sharper bound on the nonlinear term).** *Let  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$ . For all  $u, v, w \in X$*

$$|b^*(u, v, w)| \leq C(\Omega) \sqrt{\|u\| \|\nabla u\|} \|\nabla v\| \|\nabla w\|.$$

**Proof 2.1.** See [19].

We will also need the following inequalities: for any  $u \in V$

$$\inf_{v \in V^h} \|\nabla(u - v)\| \leq C(\Omega) \inf_{v \in X^h} \|\nabla(u - v)\|, \quad (2.5)$$

$$\inf_{v \in V^h} \|u - v\| \leq C(\Omega) \inf_{v \in X^h} \|\nabla(u - v)\|, \quad (2.6)$$

The proof of (2.5) can be found, e.g., in [19], and (2.6) follows from the Poincare-Friedrich's inequality and (2.5).

Also assume the inverse inequality holds; there exists a constant  $C$  independent of  $h$ , such that

$$\|\nabla v\| \leq Ch^{-1} \|v\|, \quad \forall v \in X^h. \quad (2.7)$$

The number of time steps is  $N := \frac{T}{k}$ .

Also, the error decomposition is as follows:

$$\begin{aligned} e_\ell^i &= u^i - u_\ell^{h,i} = u^i - \tilde{u}^i + \tilde{u}^i - u_\ell^{h,i} = \eta_\ell^i - \psi_\ell^{h,i}, \\ \text{where } \tilde{u}^i &\in V^h \text{ is some projection of } u^i \text{ onto } V^h, \\ \text{and } \eta_\ell^i &= u^i - \tilde{u}^i, \psi_\ell^{h,i} = u_\ell^{h,i} - \tilde{u}^i, \psi_\ell^{h,i} \in V^h, \forall i, \forall \ell = 1, 2. \end{aligned} \quad (2.8)$$

The  $L^2$  projection is defined in the usual way.

**Definition 2.3.** *The  $L^2$  projection  $P^H$  of a given function  $\mathbb{L}$  onto the finite element space  $L^H$  is the solution of the following : find  $\bar{\mathbb{L}} = P^H \mathbb{L} \in L^H$  such that*

$$(\mathbb{L} - P^H \mathbb{L}, S_H) = 0, \quad (2.9)$$

for all  $S_H \in L^H$ .

Hence, we get

$$\|I - P^H\| \leq 1, \quad (2.10)$$

$$\|(I - P^H)\mathbb{L}\| \leq CH^k \|\mathbb{L}\|_{k+1}, \quad (2.11)$$

for all  $\mathbb{L} \in (L(\Omega_i))^{d \times d} \cap (H^{k+1}(\Omega_i))^{d \times d}$ .

For the following Gronwall's lemma, see, e.g. [20].

**Lemma 2.4.** Let  $k, A$ , and  $a_\mu, b_\mu, c_\mu, \Gamma_\mu$ , for integers  $\mu \geq 0$ , be nonnegative numbers such that:

$$a_n + k \sum_{\mu=0}^n b_\mu \leq k \sum_{\mu=0}^n \Gamma_\mu a_\mu + k \sum_{\mu=0}^n c_\mu + A \text{ for } n \geq 0.$$

Suppose that  $k\Gamma_\mu < 1$  for all  $\mu$ , and set  $\sigma_\mu = (1 - k\Gamma_\mu)^{-1}$ . Then

$$a_n + k \sum_{\mu=0}^n b_\mu \leq e^{k \sum_{\mu=0}^n \sigma_\mu \Gamma_\mu} \cdot [k \sum_{\mu=0}^n c_\mu + A].$$

### 3. Stability and Error Estimates of the First Step Approximation

The unconditional stability and error estimate of the first step (SAV) approximation  $u_1^h$  are proven in this section. Also, an error estimate of its time derivative  $\frac{e_1^{n+1} - e_1^n}{k}$  is presented.

Hence, the formulation (1.3) results in an  $O(h^m + H^m h + k)$  accurate, unconditionally stable solution to Navier-Stokes equations.

We start by stating stability and error estimate of the modified Stokes Projection, which we use as the approximation  $\tilde{u}^0$  to the initial velocity  $u_0$ .

**Proposition 3.1 (Stability of the Stokes projection).** Let  $v, \tilde{v}$  satisfy (2.3). The following bound holds for the dimensions,  $d=2,3$ ,

$$(\nu + h) \|\nabla \tilde{v}\|^2 \leq 2(\nu + h) \|\nabla v\|^2 + 2d(\nu + h)^{-1} \inf_{q^h \in Q^h} \|p - q^h\|^2, \quad (3.1)$$

**Proposition 3.2. (Error estimate for Stokes Projection).** Suppose the discrete inf-sup condition (2.1) holds. Then the error in Stokes Projection satisfies for a constant  $C$ , independent of  $h$  and  $\nu$ , that

$$\begin{aligned} (\nu + h) \|\nabla(u - \tilde{u})\|^2 &\leq C[(\nu + h) \inf_{v^h \in V^h} \|\nabla(u - v^h)\|^2 \\ &\quad + (\nu + h)^{-1} \inf_{q^h \in Q^h} \|p - q^h\|^2]. \end{aligned} \quad (3.2)$$

**Proof 3.1.** See [5] for the proof.

**Lemma 3.3 (Stability of the first step approximation).** Let  $u_1^{h,i}$  satisfy the equation (1.3). Let  $f \in L^2(0, T; H^{-1}(\Omega))$ . Then for  $n = 0, \dots, N-1$ ,

$$\begin{aligned} &\|u_1^{h,n+1}\|^2 + h \|\nabla u_1^{h,n+1}\|^2 + \nu k \sum_{i=0}^{n+1} \|\nabla u_1^{h,i}\|^2 \\ &+ h k \sum_{i=0}^{n+1} (\|\nabla u_1^{h,i+1} - \mathbb{G}_1^{\mathbb{H},i}\|^2 + \|\nabla u_1^{h,i} - \mathbb{G}_1^{\mathbb{H},i}\|^2) \\ &\leq \|u^{s,0}\|^2 + h \|\nabla u^{s,0}\|^2 + \frac{1}{\nu} k \sum_{i=0}^{n+1} \|f(t_i)\|_{-1}^2. \end{aligned}$$

**Proof 3.2.** Taking  $v^h = u_1^{h,n+1} \in V^h$  in the equation (1.3), and then applying Cauchy-Schwarz and Young's inequalities give:

$$\begin{aligned} &\frac{1}{2k} (\|u_1^{h,n+1}\|^2 - \|u_1^{h,n}\|^2) + (\nu + h) \|\nabla u_1^{h,n+1}\|^2 - h (\mathbb{G}_1^{\mathbb{H},n}, u_1^{h,n+1}) \\ &\leq (f(t_{n+1}), u_1^{h,n+1}). \end{aligned} \quad (3.3)$$

Also considering the fact that  $(\nabla u_1^{h,n} - \mathbb{G}_1^{\mathbb{H},n}, \mathbb{G}_1^{\mathbb{H},n}) = 0$ , one can easily show

$$\|\nabla u_1^{h,n} - \mathbb{G}_1^{\mathbb{H},n}\|^2 = \|\nabla u_1^{h,n}\|^2 - \|\mathbb{G}_1^{\mathbb{H},n}\|^2.$$

The last equality and some algebraic manipulations give

$$\begin{aligned} & (\nu + h)\|\nabla u_1^{h,n+1}\|^2 - h(\mathbb{G}_1^{\mathbb{H},n}, \nabla u_1^{h,n+1}) \\ &= \nu\|\nabla u_1^{h,n+1}\|^2 + \frac{h}{2}(\|\nabla u_1^{h,n+1} - \mathbb{G}_1^{\mathbb{H},n}\|^2 + 2(\mathbb{G}_1^{\mathbb{H},n}, \nabla u_1^{h,n+1}) - \|\mathbb{G}_1^{\mathbb{H},n}\|^2) \\ & \quad - h(\mathbb{G}_1^{\mathbb{H},n}, \nabla u_1^{h,n+1}) + \frac{h}{2}(\|\nabla u_1^{h,n+1}\|^2 - \|\nabla u_1^{h,n}\|^2) + \frac{h}{2}\|\nabla u_1^{h,n}\|^2 \\ &= \nu\|\nabla u_1^{h,n+1}\|^2 + \frac{h}{2}\|\nabla u_1^{h,n+1} - \mathbb{G}_1^{\mathbb{H},n}\|^2 + \frac{h}{2}\|\nabla u_1^{h,n} - \mathbb{G}_1^{\mathbb{H},n}\|^2 \\ & \quad + \frac{h}{2}(\|\nabla u_1^{h,n+1}\|^2 - \|\nabla u_1^{h,n}\|^2). \end{aligned} \quad (3.4)$$

The definition of the dual norm with the regularity assumption on the forcing function followed by Cauchy-Schwarz and Young's inequalities produces

$$(f(t_{n+1}), u_1^{h,n+1}) \leq \frac{1}{2\nu}\|f(t_{n+1})\|_{-1}^2 + \frac{\nu}{2}\|u_1^{h,n+1}\|^2. \quad (3.5)$$

Substituting (3.4) and (3.5) in (3.3), we get

$$\begin{aligned} & \frac{1}{2k}(\|u_1^{h,n+1}\|^2 - \|u_1^{h,n}\|^2) + \frac{\nu}{2}\|\nabla u_1^{h,n+1}\|^2 \\ & + \frac{h}{2}\|\nabla u_1^{h,n+1} - \mathbb{G}_1^{\mathbb{H},n}\|^2 + \frac{h}{2}\|\nabla u_1^{h,n} - \mathbb{G}_1^{\mathbb{H},n}\|^2 \\ & + \frac{h}{2}(\|\nabla u_1^{h,n+1}\|^2 - \|\nabla u_1^{h,n}\|^2) \leq \frac{1}{2\nu}\|f(t_{n+1})\|_{-1}^2 \end{aligned}$$

Multiplying both sides by  $2k$  and summing over all time levels, the desired result can be found.

**Definition 3.4.** Let

$$\begin{aligned} C_u &:= \|u(x, t)\|_{L^\infty(0, T; L^\infty(\Omega))}, \\ C_{\nabla u} &:= \|\nabla u(x, t)\|_{L^\infty(0, T; L^\infty(\Omega))}, \end{aligned}$$

and introduce  $\tilde{C}$ , satisfying

$$\inf_{v \in V^h} \|\nabla(u - v)\| \leq C_1 \inf_{v \in X^h} \|\nabla(u - v)\| \leq C_2 h^m \|u\|_{H^{m+1}} \leq \tilde{C} h^m \quad (3.6)$$

Also, define  $\bar{C} := 1728C^4(\Omega)$ , where  $C(\Omega)$  is from Lemma 2.2.

**Theorem 3.5 (Error estimate of the first step approximation).** Let  $f \in L^2(0, T; H^{-1})$ , let  $u_1^h$  satisfy (1.3),

$$\begin{aligned} k &\leq \frac{\nu + h}{18 + 4C_u^2 + 2(\nu + h)C_{\nabla u} + 2\bar{C}\tilde{C}^4(\nu + h)^{-2}h^{4m}}, \\ u &\in L^2(0, T; H^{m+1}(\Omega)) \cap L^\infty(0, T; L^\infty(\Omega)), \nabla u \in L^\infty(0, T; L^\infty(\Omega)), \\ u_t &\in L^2(0, T; H^{m+1}(\Omega)), u_{tt} \in L^2(0, T; L^2(\Omega)), p \in L^2(0, T; H^m(\Omega)). \end{aligned}$$

Then, there is a constant  $C = C(\Omega, T, u, p, f, \nu + h)$ , such that

$$\max_{1 \leq i \leq N} \|u(t_i) - u_1^{h,i}\| + \left( k \sum_{i=1}^{n+1} (\nu + h) \|\nabla(u(t_i) - u_1^{h,i})\|^2 \right)^{1/2} \leq C(h^m + H^m h + k)$$

**Proof 3.3.** The first step approximation of this report and that of [10] differ only in their treatment of the diffusion terms. Therefore, we follow the same idea and use the same bounds for most of the terms but for the viscosity. The following equation is the analogue of the equation (4.10) of [5].

$$\begin{aligned}
& \left( \frac{\eta_1^{n+1} - \eta_1^n}{k}, \psi_1^{h,n+1} \right) - \left( \frac{\psi_1^{h,n+1} - \psi_1^{h,n}}{k}, \psi_1^{h,n+1} \right) \\
& + (\nu + h)(\nabla \eta_1^{n+1}, \nabla \psi_1^{h,n+1}) - (\nu + h)\|\nabla \psi_1^{h,n+1}\|^2 \\
& + b^*(\eta_1^{n+1}, u(t_{n+1}), \psi_1^{h,n+1}) - b^*(\psi_1^{h,n+1}, u(t_{n+1}), \psi_1^{h,n+1}) \\
& + b^*(u_1^{h,n+1}, \eta_1^{n+1}, \psi_1^{h,n+1}) - (p(t_{n+1}) - q^{h,n+1}, \nabla \cdot \psi_1^{h,n+1}) \\
& = h(\nabla u(t_{n+1}) - \mathbb{G}_1^{\mathbb{H},n}, \nabla \psi_1^{h,n+1}) - k(\rho^{n+1}, \psi_1^{h,n+1}).
\end{aligned} \tag{3.7}$$

The equation (1.3) states that  $\mathbb{G}_1^{\mathbb{H},n} = P^H \nabla u^{h,n}$  where  $P^H$  is the  $L^2$ -orthogonal projection defined by (2.9). Hence, utilizing Cauchy-Schwarz and Young's inequality,

$$\begin{aligned}
& h(\mathbb{G}_1^{\mathbb{H},n} - \nabla u(t^{n+1}), \nabla \psi_1^{h,n+1})_{\Omega_1} \\
& \leq (P^H \nabla(u_1^{h,n} - u(t^n)), \nabla \psi_1^{h,n+1})_{\Omega_1} - ((I - P^H) \nabla u(t^n), \nabla \psi_1^{h,n+1})_{\Omega_1} \\
& \quad - (\nabla(u(t^{n+1}) - u(t^n)), \nabla \psi_1^{h,n+1})_{\Omega_1} \\
& \leq \frac{h^2}{4\epsilon(\nu + h)} \left( \|P^H \nabla \eta_1^n\|^2 + \|P^H \nabla \psi_1^{h,n}\|^2 \right. \\
& \quad \left. + \|(I - P^H) \nabla u(t^n)\|^2 + \|\nabla(u(t^{n+1}) - u(t^n))\|^2 \right) \\
& \quad + \epsilon(\nu + h)\|\nabla \psi_1^{h,n+1}\|^2.
\end{aligned} \tag{3.8}$$

Taylor remainder formula is used along with (2.9), (2.10) and inverse inequality to get

$$\begin{aligned}
& h(\mathbb{G}_1^{\mathbb{H},n} - \nabla u(t^{n+1}), \nabla \psi_1^{h,n+1})_{\Omega_1} \\
& \leq \frac{h^2}{4\epsilon(\nu + h)} \left( \|\nabla \eta_1^n\|^2 + h^{-2}\|\psi_1^{h,n}\|^2 + H^{2m}\|u(t^n)\|_{m+1}^2 \right. \\
& \quad \left. + k^2\|u_t\|_{L^\infty(t^n, t^{n+1}; H^1(\Omega))}^2 \right) + \epsilon(\nu + h)\|\nabla \psi_1^{h,n+1}\|^2.
\end{aligned} \tag{3.9}$$

Apply the Cauchy-Schwarz and Young's inequalities to (3.7) to obtain

$$\begin{aligned}
& \frac{\|\psi_1^{h,n+1}\|^2 - \|\psi_1^{h,n}\|^2}{2k} + (\nu + h)\|\nabla \psi_1^{h,n+1}\|^2 \\
& \leq 6\epsilon(\nu + h)\|\nabla \psi_1^{h,n+1}\|^2 + \frac{1}{4\epsilon(\nu + h)} \left\| \frac{\eta_1^{n+1} - \eta_1^n}{k} \right\|_{-1}^2 \\
& + |b^*(\eta_1^{n+1}, u(t_{n+1}), \psi_1^{h,n+1})| + |b^*(\psi_1^{h,n+1}, u(t_{n+1}), \psi_1^{h,n+1})| + |b^*(u_1^{h,n+1}, \eta_1^{n+1}, \psi_1^{h,n+1})| \\
& + \frac{h^2}{4\epsilon(\nu + h)} \left( \|\nabla \eta_1^n\|^2 + h^{-2}\|\psi_1^{h,n}\|^2 + H^{2m}\|u(t^n)\|_{m+1}^2 + k^2\|u_t\|_{L^\infty(t^n, t^{n+1}; H^1(\Omega))}^2 \right) \\
& + \frac{d}{4\epsilon(\nu + h)} \inf_{q^h \in Q^h} \|p(t_{n+1}) - q^{h,n+1}\|^2 + \frac{(\nu + h)}{4\epsilon} \|\nabla \eta_1^{n+1}\|^2 \frac{1}{4\epsilon(\nu + h)} k^2 \|\rho^{n+1}\|_{-1}^2.
\end{aligned} \tag{3.10}$$



Take  $\epsilon = \frac{1}{18}$  in (3.10). Using the bounds (3.12)-(3.16) of [5], we obtain

$$\begin{aligned}
 & \frac{\|\psi_1^{h,n+1}\|^2 - \|\psi_1^{h,n}\|^2}{2k} + \frac{\nu + h}{2} \|\nabla \psi_1^{h,n+1}\|^2 \\
 & \leq \frac{C}{\nu + h} \left\| \frac{\eta_1^{n+1} - \eta_1^n}{k} \right\|_{-1}^2 + C(\nu + h) \|\nabla \eta_1^{n+1}\|^2 + \frac{C}{\nu + h} \inf_{q^h \in Q^h} \|p(t_{n+1}) - q^{h,n+1}\|^2 \\
 & \quad + \frac{9h^2}{2(\nu + h)} \left( \|\nabla \eta_1^n\|^2 + h^{-2} \|\psi_1^{h,n}\|^2 + H^{2m} \|u(t^n)\|_{m+1}^2 + k^2 \|u_t\|_{L^\infty(t^n, t^{n+1}; H^1(\Omega))}^2 \right) \\
 & \quad + \frac{C}{\nu + h} k^2 \|\rho^{n+1}\|_{-1}^2 + \frac{C}{\nu + h} (\|\nabla \eta_1^{n+1}\|^2 + \|\nabla \eta_1^{n+1}\|^4) \\
 & \quad + \left( \frac{1}{2} C_{\nabla u} + \frac{C_u^2}{\nu + h} + \frac{\bar{C}}{(\nu + h)^3} \|\nabla \eta_1^{n+1}\|^4 \right) \|\psi_1^{h,n+1}\|^2.
 \end{aligned} \tag{3.11}$$

Following the same steps as in the reference [5], we obtain

$$\begin{aligned}
 & \|u(t_{n+1}) - u_1^{h,n+1}\|^2 + k \sum_{i=0}^n (\nu + h) \|\nabla (u(t_{n+1}) - u_1^{h,n+1})\|^2 \\
 & \leq \frac{C}{\nu + h} [h^{2m} + h^2 H^{2m} + k^2], \\
 & \quad \text{where } C = C(\Omega, T, u, p, f).
 \end{aligned} \tag{3.12}$$

This proves the theorem.

The following lemma will be used in the proof of Theorem 3.7.

**Lemma 3.6.** Let  $f \in L^2(0, T; H^{-1}(\Omega))$ . Let  $\psi^{h,0}$  and  $\psi^{h,1}$  be the modified Stokes projections of the initial velocities. Suppose  $m \geq 2$  and

$$k < \frac{4(\nu + h)}{13(4(\nu + h)C_{\nabla u} + 3C_u^2)}.$$

Then, there is a constant  $C = C(\Omega, T, u, p, f, \nu + h)$ , such that

$$\left\| \frac{\psi^{h,1} - \psi^{h,0}}{k} \right\|^2 + \frac{13}{2} (\nu + h) k \left\| \nabla \frac{\psi^{h,1} - \psi^{h,0}}{k} \right\|^2 \leq C(kh^{2m} + h^2 + k^2 + k^2 h^{2m-3} + H^{2m}) \tag{3.13}$$

**Proof 3.4.** For the same argument of the above proof, we follow the same idea and use the same bounds of [10] for most of the terms on the right hand side of

$$\begin{aligned}
 & \left\| \frac{\psi^{h,1} - \psi^{h,0}}{k} \right\|^2 + k(\nu + h) \left\| \nabla \frac{\psi^{h,1} - \psi^{h,0}}{k} \right\|^2 \\
 & + b^*(u(t_1), u(t_1), \frac{\psi^{h,1} - \psi^{h,0}}{k}) - b^*(u_1^{h,1}, u_1^{h,1}, \frac{\psi^{h,1} - \psi^{h,0}}{k}) \\
 & \quad - k \left( \frac{p^1 - p^0}{k} - q, \nabla \cdot \frac{\psi^{h,1} - \psi^{h,0}}{k} \right) \\
 & = h(\nabla u(t_1) - \mathbb{G}_1^{\mathbb{H},0}, \nabla \frac{\psi^{h,1} - \psi^{h,0}}{k}) + (\rho^1, \frac{\psi^{h,1} - \psi^{h,0}}{k}) + \left( \frac{\eta^1 - \eta^0}{k}, \frac{\psi^{h,1} - \psi^{h,0}}{k} \right) \\
 & \quad + k(\nu + h) \left( \nabla \frac{\eta^1 - \eta^0}{k}, \nabla \frac{\psi^{h,1} - \psi^{h,0}}{k} \right)
 \end{aligned} \tag{3.14}$$

The only difference between the last equation and the equation (3.8) of [10] is on the first term of the right hand side. Thus, we bound this term next.

The equation (1.3) states that  $\mathbb{G}^{\mathbb{H},0} = P^H \nabla u(t_0)$  is the  $L^2$ -orthogonal projection of the initial value. Hence, utilizing Cauchy-Schwarz and Young's inequality gives

$$\begin{aligned}
 & h(\nabla u(t_1) - \mathbb{G}_1^{\mathbb{H},0}, \nabla \frac{\psi^{h,1} - \psi^{h,0}}{k}) \\
 &= hk(\nabla \frac{u(t_1) - u(t_0)}{k}, \nabla \frac{\psi^{h,1} - \psi^{h,0}}{k}) + h(\nabla u(t_0) - \mathbb{G}_1^{\mathbb{H},0}, \nabla \frac{\psi^{h,1} - \psi^{h,0}}{k}) \\
 &\leq k\mu^*(\nu + h) \|\nabla \frac{\psi^{h,1} - \psi^{h,0}}{k}\|^2 + \frac{kh^2}{4\mu^*(\nu + h)} \|\nabla \frac{u(t_1) - u(t_0)}{k}\|^2 \\
 &+ C\|(I - P^H)\nabla u(t_0)\|^2 + \mu h^2 \|\nabla \frac{\psi^{h,1} - \psi^{h,0}}{k}\|^2.
 \end{aligned} \tag{3.15}$$

Taylor remainder formula is used along with (2.9), (2.10) and inverse inequality to get

$$\begin{aligned}
 & h(\nabla u(t_1) - \mathbb{G}_1^{\mathbb{H},0}, \nabla \frac{\psi^{h,1} - \psi^{h,0}}{k}) \\
 &\leq k\mu^*(\nu + h) \|\nabla \frac{\psi^{h,1} - \psi^{h,0}}{k}\|^2 + \frac{kh^2}{4\mu^*(\nu + h)} \|u_t\|_{L^\infty(t^0, t^1; H^1(\Omega))}^2 \\
 &+ CH^{2m} \|u(t_0)\|_{m+1}^2 + \mu \|\frac{\psi^{h,1} - \psi^{h,0}}{k}\|^2.
 \end{aligned} \tag{3.16}$$

Using corresponding bounds from [10], we have

$$\begin{aligned}
 & (1 - 12\mu - (\frac{C_{\nabla u}}{2} + \frac{C_u^2}{16(\nu + h)\mu^*})k) \|\frac{\psi^{h,1} - \psi^{h,0}}{k}\|^2 \\
 &+ (1 - 4\mu^*)(\nu + h)k \|\nabla \frac{\psi^{h,1} - \psi^{h,0}}{k}\|^2 \\
 &\leq \frac{dk}{4\mu^*(\nu + h)} \inf_{q \in Q^h} \|\frac{p^1 - p^0}{k} - q\|^2 + \frac{k^2}{4\mu} \|\rho^1\|^2 + \frac{1}{4\mu} \|\frac{\eta^1 - \eta^0}{k}\|^2 \\
 &+ \frac{k(\nu + h)}{4\mu^*} \|\nabla \frac{\eta^1 - \eta^0}{k}\|^2 + \frac{Ch^{-2}}{4\mu} \|\nabla \psi^{h,0}\|^2 + \frac{C_u^2}{4\mu} \|\nabla \psi^{h,0}\|^2 + \frac{C_u^2 h^{-2}}{4\mu} \|\psi^{h,0}\|^2 \\
 &+ \frac{Ch^{-3}}{4\mu} \|\psi^{h,1}\|^2 \|\nabla \psi^{h,0}\|^2 + \frac{Ch^{-2}}{4\mu} \|\nabla \eta^1\|^2 \|\nabla \psi^{h,0}\|^2 \\
 &+ \frac{Ch^{-2}}{2\mu} \|\nabla \eta^1\|^2 + Ch^{-4} \|\psi^{h,1}\|^2 \|\nabla \eta^1\|^2 + Ch^{-2} \|\nabla \eta^1\|^4 \\
 &+ \frac{kh^2}{4\mu^*(\nu + h)} \|u_t\|_{L^\infty(t^0, t^1; H^1(\Omega))}^2 + CH^{2m} \|u(t_0)\|_{m+1}^2.
 \end{aligned} \tag{3.17}$$

Use the approximation properties of  $X^h, Q^h$  and take  $\mu = 1/13$  and  $\mu^* = 1/8$ , from the regularity assumption of  $u$ , we obtain

$$\begin{aligned}
 & (\frac{1}{13} - (\frac{C_{\nabla u}}{2} + \frac{3C_u^2}{8(\nu + h)})k) \|\frac{\psi^{h,1} - \psi^{h,0}}{k}\|^2 + \frac{1}{2}(\nu + h)k \|\nabla \frac{\psi^{h,1} - \psi^{h,0}}{k}\|^2 \\
 &\leq C(h^{2m-2} + h^2 + k^2 + k^2 h^{2m-3} + H^{2m}).
 \end{aligned} \tag{3.18}$$

The statement of the lemma follows from the last inequality.

**Theorem 3.7.** Assume the assumptions given for the Lemma 3.6 and Theorem 3.5 be satisfied.

Let  $k \leq \min\{\frac{\nu+h}{2CC_{\nabla u}(\nu+h)+2CC_u^2}, C(\nu + h)^{\frac{5}{3}}, C(\nu + h)^3\}$

Then

$$\|\frac{e_1^{n+1} - e_1^n}{k}\|^2 + k \sum_{i=0}^n (\nu + h) \|\nabla \frac{e_1^{i+1} - e_1^i}{k}\|^2 \leq C[h^{2m} + h^2 + k^2]$$

**Proof 3.5.** We start with the proof of the bound for  $\|\frac{\psi_1^{h,n+1}-\psi_1^{h,n}}{k}\|$ . Subtracting (1.3) from the weak formulation of NSE, we obtain for  $n \geq 1$

$$\begin{aligned} & (\frac{e_1^{n+1}-e_1^n}{k}, v) + (\nu + h)(\nabla e_1^{n+1}, \nabla v) \\ & + b^*(e_1^{n+1}, u(t_{n+1}), v) + b^*(u_1^{h,n+1}, e_1^{n+1}, v) \\ & - ((p(t_{n+1}) - p_1^{h,n+1}), \nabla \cdot v) = h(\nabla u(t_{n+1}) - \mathbb{G}_1^{\mathbb{H},n}, \nabla v) - k(\rho^{n+1}, v), \\ & \text{where } k\rho^{n+1} = u_t(t_{n+1}) - \frac{u(t_{n+1}) - u(t_n)}{k}. \end{aligned} \quad (3.19)$$

Take  $v = \frac{\psi_1^{h,n+1}-\psi_1^{h,n}}{k} =: s^{h,n+1} \in V^h$  in (3.19). And then, rewrite (3.19) at the previous time level and choose the same  $v = s^{h,n+1} \in V^h$ . Subtracting these two equations and using the Taylor's Theorem to simplify the last term on the right-hand side. We get

$$\begin{aligned} & k(\frac{\eta_1^{n+1} - 2\eta_1^n + \eta_1^{n-1}}{k^2}, s^{h,n+1}) - (s^{h,n+1} - s^{h,n}, s^{h,n+1}) \\ & + (\nu + h)k(\nabla(\frac{\eta_1^{n+1} - \eta_1^n}{k}), \nabla s^{h,n+1}) - (\nu + h)k\|\nabla s^{h,n+1}\|^2 \\ & + b^*(e_1^{n+1}, u(t_{n+1}), s^{h,n+1}) + b^*(u_1^{h,n+1}, e_1^{n+1}, s^{h,n+1}) \\ & - b^*(e_1^n, u(t_n), s^{h,n+1}) - b^*(u_1^{h,n}, e_1^n, s^{h,n+1}) \\ & - k(\frac{(p(t_{n+1}) - p_1^{h,n+1}) - (p(t_n) - p_1^{h,n}))}{k}, \nabla \cdot s^{h,n+1}) \\ & = hk(\nabla \frac{u(t_{n+1}) - u(t_n)}{k} - \frac{\mathbb{G}_1^{\mathbb{H},n} - \mathbb{G}_1^{\mathbb{H},n-1}}{k}, \nabla s^{h,n+1}) - Ck^2(\rho_t^{n+1}, s^{h,n+1}), \\ & \text{where } \rho_t^{n+1} = u_{ttt}(t_{n+\theta}) \text{ for some } \theta \in [0, 1]. \end{aligned} \quad (3.20)$$

Applying Cauchy-Schwarz and Young's inequalities

$$\begin{aligned} & hk(\nabla \frac{u(t_{n+1}) - u(t_n)}{k} - \frac{\mathbb{G}_1^{\mathbb{H},n} - \mathbb{G}_1^{\mathbb{H},n-1}}{k}, \nabla s^{h,n+1}) \\ & \leq \frac{Ch^2 \cdot k}{\nu + h} \|\nabla \frac{u(t_{n+1}) - u(t_n)}{k}\|^2 + \frac{Ch^2 \cdot k}{\nu + h} \|\frac{\mathbb{G}_1^{\mathbb{H},n} - \mathbb{G}_1^{\mathbb{H},n-1}}{k}\|^2 + \epsilon k(\nu + h) \|\nabla s^{h,n+1}\|^2 \end{aligned} \quad (3.21)$$

By the properties of the projection, error decomposition and the inverse inequality, the following can be found

$$\begin{aligned} & \|\frac{\mathbb{G}_1^{\mathbb{H},n} - \mathbb{G}_1^{\mathbb{H},n-1}}{k}\|^2 = \|P^H \nabla \frac{u_1^{h,n} - u_1^{h,n-1}}{k}\|^2 \\ & \leq \|\nabla \frac{u_1^{h,n} - u_1^{h,n-1}}{k}\|^2 \leq \|\nabla \frac{u(t_n) - u(t_{n-1})}{k}\|^2 + \|\nabla(\frac{\eta_1^n - \eta_1^{n-1}}{k})\|^2 + h^{-2} \|s^{h,n}\|^2 \end{aligned} \quad (3.22)$$

Applying the Cauchy-Schwarz and Young's inequalities to (3.20) and using the bounds (5.4)-(5.10) of the

technical report [26] and the last two bounds give

$$\begin{aligned}
& \frac{\|s^{h,n+1}\|^2 - \|s^{h,n}\|^2}{2} + (\nu + h)k \|\nabla s^{h,n+1}\|^2 \\
& \leq 13\epsilon(\nu + h)k \|\nabla s^{h,n+1}\|^2 \\
& + \frac{C}{\nu + h}k \left\| \frac{\eta_1^{n+1} - 2\eta_1^n + \eta_1^{n-1}}{k^2} \right\|_{-1}^2 + C(\nu + h)k \left\| \nabla \left( \frac{\eta_1^{n+1} - \eta_1^n}{k} \right) \right\|^2 \\
& + \frac{C}{\nu + h}k \inf_{q^h \in Q^h} \left\| \frac{p(t_{n+1}) - p(t_n)}{k} - \frac{q^{h,n+1} - q^{h,n}}{k} \right\|^2 \\
& + \frac{C}{\nu + h}k \left[ \left\| \nabla \left( \frac{\eta_1^{n+1} - \eta_1^n}{k} \right) \right\|^2 + \|\nabla e_1^n\|^2 + \left\| \nabla \left( \frac{\eta_1^{n+1} - \eta_1^n}{k} \right) \right\|^2 \|\nabla e_1^n\|^2 \right] \\
& + Ck \|e_1^n\|^2 \|\nabla e_1^n\|^2 + Ck \left\| \nabla \left( \frac{\eta_1^{n+1} - \eta_1^n}{k} \right) \right\|^4 + \frac{C}{\nu + h}k \cdot k^2 \|\rho_t^{n+1}\|_{-1}^2 \\
& + \frac{C}{\nu + h}k \cdot h^2 \left( \left\| \nabla \left( \frac{u(t_{n+1}) - u(t_n)}{k} \right) \right\|^2 + \left\| \nabla \frac{u(t_n) - u(t_{n-1})}{k} \right\|^2 + \left\| \nabla \left( \frac{\eta_1^n - \eta_1^{n-1}}{k} \right) \right\|^2 \right) \\
& + \frac{C}{\nu + h}k \|s^{h,n}\|^2 + C(C_{\nabla u} + \frac{C_u^2}{\nu + h} + \frac{1}{(\nu + h)^3} \|\nabla e_1^n\|^4)k \|s^{h,n+1}\|^2.
\end{aligned} \tag{3.23}$$

Following similar steps as in the references [10] and [26] concludes the proof.

#### 4. Stability and Error Estimate of Correction Step Approximation

The correction step approximation presented here is identically same with that of the reference paper [10]. Only differences are on their first step approximations. Therefore, the stability and accuracy analysis will be the same with the reference model up to the point when the first step approximation comes into play. For this reason, we are going to copy results from this paper up to some point, and then continue proving our theorem statements from there on.

Theoretical findings below illustrate that the formulation (1.5) produces  $O(h^2 + k^2)$  accurate, unconditionally stable correction step approximation to the time-dependent Navier-Stokes equations.

**Theorem 4.1 (Stability of the Correction Step Approximation).** Assume  $f \in L^2(0, T; H^{-1}(\Omega))$ . Let  $u_1^h, u_2^h$  satisfy (1.3) and (1.5), respectively. Then for  $n=0, \dots, N-1$ ,

$$\begin{aligned}
& \|u_2^{h,n+1}\|^2 + 5h^2\nu^{-1}(\nu + h)^{-1}\|u_1^{h,n+1}\|^2 \\
& + 5h^3\nu^{-1}(\nu + h)^{-1}k \sum_{i=0}^{n+1} (\|\nabla u_1^{h,i+1} - \mathbb{G}_1^{\mathbb{H},i}\|^2 + \|\nabla u_1^{h,i} - \mathbb{G}_1^{\mathbb{H},i}\|^2) + k \sum_{i=1}^{n+1} (\nu + h) \|\nabla u_2^{h,i}\|^2 \\
& \leq C[\|u_0^s\|^2 + (\nu + h)^{-1}k \sum_{i=1}^{n+1} \|f(t_i)\|_{-1}^2].
\end{aligned}$$

**Proof 4.1.** From the inequality (4.4) in [10], we have

$$\begin{aligned}
 & \frac{1}{2k} (\|u_2^{h,n+1}\|^2 - \|u_2^{h,n}\|^2) + \frac{1}{2}(\nu + h) \|\nabla u_2^{h,n+1}\|^2 \\
 & \leq \frac{5}{2(\nu + h)} \left\| \frac{f(t_{n+1}) - f(t_n)}{2} \right\|_{-1}^2 \\
 & + \frac{5\nu^2 k^2}{4(\nu + h)} C_{\nabla u_t}^2 + \frac{5\nu^2 k}{4(\nu + h)^2} k(\nu + h) \left\| \nabla \left( \frac{e_1^{n+1} - e_1^n}{k} \right) \right\|^2 \\
 & + \frac{5h^2}{2\nu(\nu + h)} \nu \|\nabla u_1^{h,n+1}\|^2 \\
 & + \frac{5}{4\nu(\nu + h)^2} (\nu + h) k \left\| \nabla \left( \frac{e_1^{h,n+1} - e_1^{h,n}}{k} \right) \right\|^2 [\nu k \|\nabla u_1^{h,n+1}\|^2 + \nu k \|\nabla u_1^{h,n}\|^2] \\
 & + \frac{5}{4\nu(\nu + h)} k C_{\nabla u_t}^2 [\nu k \|\nabla u_1^{h,n+1}\|^2 + \nu k \|\nabla u_1^{h,n}\|^2].
 \end{aligned} \tag{4.1}$$

Multiplying the last inequality by  $2k$ , summing over all time levels, and following the Lemma 3.3 and Theorem 3.7 give

$$\begin{aligned}
 & \|u_2^{h,n+1}\|^2 + k \sum_{i=1}^{n+1} (\nu + h) \|\nabla u_2^{h,i}\|^2 \\
 & \leq \|u_0^s\|^2 + \frac{5}{(\nu + h)} k \sum_{i=1}^{n+1} \left\| \frac{f(t_i) - f(t_{i-1})}{2} \right\|_{-1}^2 \\
 & + \frac{5\nu^2 k^3}{2(\nu + h)} C_{\nabla u_t}^2 + \frac{5\nu^2 k^2}{2(\nu + h)^2} C(h^{2m} + h^2 + k^2) \\
 & + \frac{5h^2}{\nu(\nu + h)} \left[ \|u_0^s\|^2 + h \|\nabla u^{s,0}\|^2 - \|u_1^{h,n+1}\|^2 \right] \\
 & - hk \sum_{i=0}^{n+1} (\|\nabla u_1^{h,i+1} - \mathbb{G}_1^{h,i}\|^2 + \|\nabla u_1^{h,i} - \mathbb{G}_1^{h,i}\|^2) + \frac{1}{\nu + h} k \sum_{i=1}^{n+1} \|f(t_i)\|_{-1}^2 \\
 & + \frac{5}{2\nu(\nu + h)} \left( \frac{h^{2m} + h^2 + k^2}{\nu + h} + k^2 C_{\nabla u_t}^2 \right) \left[ 2\|u_0^s\|^2 + 2h \|\nabla u^{s,0}\|^2 \right. \\
 & \left. + \frac{1}{\nu + h} k \sum_{i=1}^{n+1} \|f(t_i)\|_{-1}^2 + \frac{1}{\nu + h} k \sum_{i=1}^n \|f(t_i)\|_{-1}^2 \right].
 \end{aligned} \tag{4.2}$$

We have the following inequality upon some algebraic manipulation:

$$\begin{aligned}
 & \|u_2^{h,n+1}\|^2 + \frac{5h^2}{\nu(\nu + h)} \|u_1^{h,n+1}\|^2 + \sum_{i=1}^{n+1} (\nu + h) \|\nabla u_2^{h,i}\|^2 \\
 & + \frac{5h^3}{\nu(\nu + h)} k \sum_{i=0}^{n+1} (\|\nabla u_1^{h,i+1} - \mathbb{G}_1^{h,i}\|^2 + \|\nabla u_1^{h,i} - \mathbb{G}_1^{h,i}\|^2) \\
 & \leq \|u_0^s\|^2 + \frac{5}{(\nu + h)} k \sum_{i=1}^{n+1} \left\| \frac{f(t_i) - f(t_{i-1})}{2} \right\|_{-1}^2 \\
 & + \frac{5\nu^2 k^3}{2(\nu + h)} C_{\nabla u_t}^2 + \frac{5\nu^2 k^2}{2(\nu + h)^2} C(h^{2m} + h^2 + k^2) \\
 & + C(\|u_0^s\|^2 + h \|\nabla u^{s,0}\|^2 + \frac{1}{\nu + h} k \sum_{i=1}^{n+1} \|f(t_i)\|_{-1}^2).
 \end{aligned} \tag{4.3}$$

The last inequality proves the theorem.

Theorem 4.1 together with the Proposition 3.1 proves the unconditional stability of  $u_2^{h,i}$  for any  $i \geq 0$ .

**Theorem 4.2 (Error Estimate of Correction Step Approximation).** *Let the assumptions of Theorem 3.7 be satisfied. Let*

$$k < \frac{\nu + h}{(\nu + h)C_{\nabla u} + 2C_u^2 + (\nu + h)Ch^{m-1} + 2Ch^{2m}}.$$

Then there exists a constant  $C = C(\Omega, T, u, p, f, (\nu + h)^{-1})$ , such that

$$\begin{aligned} \max_{1 \leq i \leq N} \|u(t_i) - u_2^{h,i}\| + (k \sum_{i=0}^n (\nu + h) \|\nabla(u(t_i) - u_2^{h,i})\|^2)^{1/2} \\ \leq C(h^m + h^2 + k^2 + hk). \end{aligned}$$

**Proof 4.2.** From the inequality (4.17) in [10], we have

$$\begin{aligned} & \|\psi_2^{h,n+1}\|^2 + (\nu + h)k \sum_{i=0}^n \|\nabla \psi_2^{h,i+1}\|^2 \\ & \leq \frac{C}{\nu + h} k \sum_{i=0}^n \left[ \inf_{q^h \in Q^h} \left\| \frac{p^{h,i+1} + p^{h,i}}{2} - q^{h,i+1} \right\|^2 \right. \\ & \quad k^2 \left\| \nabla \left( \frac{e_1^{i+1} - e_1^{i+1}}{k} \right) \right\|^2 + h^2 \|\nabla e_1^{i+1}\|^2 + k^4 + \left\| \frac{\eta_2^{i+1} - \eta_2^i}{k} \right\|_{-1}^2 \\ & \quad + \|\nabla \eta_2^{i+1}\|^2 + k^2 \|\nabla e_1^{i+1}\|^2 + \|\nabla \eta_2^{i+1}\|^4 \\ & \quad + k \left\| \nabla \left( \frac{e_1^{i+1} - e_1^{i+1}}{k} \right) \right\|^2 (k \|\nabla e_1^{i+1}\|^2 + k \|\nabla e_1^i\|^2) \\ & \quad + k \sum_{i=0}^n \|\psi_2^{h,i+1}\|^2 \left[ \frac{C_{\nabla u}}{2} + \frac{2C_u^2}{(\nu + h)} + \frac{1}{2} \|\nabla \eta_2^{i+1}\| \right. \\ & \quad \left. \left. + \frac{2}{\nu + h} \|\nabla \eta_2^{i+1}\|^2 \right] + \|\psi_2^{h,0}\|^2 \right] \end{aligned} \quad (4.4)$$

Take  $\tilde{u}^i$  in the error decomposition (2.8) to be the  $L^2$ -projection onto  $V^h$ , for  $i \geq 1$ . Take  $\tilde{u}^0$  to be  $u_0^*$ . This gives  $\psi_2^{h,0} = 0$  and  $e_1^0 = \eta_2^0$ . Also it follows from the Proposition 3.2 that  $\|\eta_2^0\| \leq Ch^m$ ; under the assumption of the theorem applying the discrete Gronwall's lemma 2.4 and using bounds in theorems 3.5, 3.7, give

$$\begin{aligned} & \|\psi_2^{h,n+1}\|^2 + (\nu + h)k \sum_{i=0}^n \|\nabla \psi_2^{h,i+1}\|^2 \\ & \leq \frac{C}{\nu + h} k \sum_{i=0}^n \left[ \inf_{q^h \in Q^h} \left\| \frac{p^{h,i+1} + p^{h,i}}{2} - q^{h,i+1} \right\|^2 \right. \\ & \quad + \frac{k^2}{\nu + h} (h^2 + k^2) + \frac{h^2}{\nu + h} (h^2 + H^{2m}h^2 + k^2) + k^4 \\ & \quad + \left\| \frac{\eta_2^{i+1} - \eta_2^i}{k} \right\|_{-1}^2 + \|\nabla \eta_2^{i+1}\|^2 + \|\nabla \eta_2^{i+1}\|^4 \\ & \quad \left. + \frac{1}{(\nu + h)^2} (h^2 + k^2) (h^2 + H^{2m}h^2 + k^2) \right] + Ch^{2m} \end{aligned} \quad (4.5)$$

Using the approximation properties of  $X^h, Q^h$ , it follows from (2.5), (2.6) and (4.5) that

$$\begin{aligned}
& \|\psi_2^{h,n+1}\|^2 + (\nu + h)k \sum_{i=0}^n \|\nabla \psi_2^{h,i+1}\|^2 \\
& \leq \frac{C}{(\nu + h)^2} (h^{2m} + H^{2m}(h^4 + h^2 k^2) + h^4 + k^4 + h^2 k^2).
\end{aligned} \tag{4.6}$$

Using the error decomposition and triangle inequality with (4.6), we obtain

$$\begin{aligned}
& \|e_2^{h,n+1}\| + ((\nu + h)k \sum_{i=0}^n \|\nabla e_2^{h,i+1}\|^2)^{\frac{1}{2}} \\
& \leq \frac{C}{(\nu + h)} (h^m + H^m(h^2 + hk) + h^2 + k^2 + hk).
\end{aligned} \tag{4.7}$$

This proves the Theorem statement.

Therefore, the correction step approximation  $u_2^h$  lifts the accuracy of an order of  $h$  in space and of  $k$  in time, compared to the first step approximation  $u_1^h$ .

Some computational results will be given next.

## 5. Computational Tests

We perform one quantitative and one qualitative comparison test of SAV-DDC and AV-DDC models. Computational results with both tests not only support the theoretical findings of this paper but also illustrate superiority of SAV-DDC over AV-DDC.

Firstly, consider a manufactured true solution of NSE in  $\Omega = [0, 1]^2$  given by

$$\begin{aligned}
u_1(x, y, t) &= e^{-t} \cos(2\pi(y - t)), \\
u_2(x, y, t) &= e^{-t} \sin(2\pi(x - t)), \\
p(x, y, t) &= 0.
\end{aligned}$$

The forcing function  $f(x, y, t)$ , the initial condition  $u(x, y, 0)$  and non-homogeneous boundary conditions are computed to comply with the given exact solution. Computations are ended at the final time  $T = 1$ . The computations have been performed using the Taylor-Hood finite element space (P2/P1) for velocity and pressure pair, and also piecewise linear finite element space (P1) for the large scale space on the same mesh instead of piecewise quadratic finite element space (P2) on a different coarse mesh, see [13].

In particular, the exact solution is a rotational flow that moves along the line  $y = x$  with a maximum velocity of 1 in each direction. Therefore, we choose the time step size smaller than the mesh size,  $\Delta t = h/8$ ; a possible analogue of the well-known CFL condition [22]. Also the additional viscosities in each case has been chosen equal to the time step size, and all these quantities have been refined together to observe convergence rates of the models.

The convergence rates in Table 2 verify Theorems (3.5) and (4.2); the first step approximations produces first order of accuracy while the correction step approximation gives a second order of accuracy.

Comparing the first step approximations of each model, we observe that the convergence rates in the first step of AV-DDC has an asymptotic behaviour while that of SAV-DDC directly produces first order of accuracy with a better error estimate. On the other hand, defect-deferred correction methods rely mostly on the accuracy of the first step approximations. Therefore we can clearly conclude that employing SAV on the first step of defect-deferred correction methods contributes the overall accuracy of the correction step approximation. Also the computational results below show this expectation has been met.

Tables 3 and 4 illustrates the accuracy and convergence results for small viscosities. As  $\nu$  decreases the convergence rates improve slower. This is a typical phenomenon for defect correction methods, see [10, 21].

For the first( $i=1$ ) and the correction( $i=2$ ) step approximations, define errors by:

$$\|e_i\|_{L^2} = \|u_i - u^h\|_{L^2(0,T;L^2(\Omega))},$$

$$\|e_i\|_{H^1} = \|u_i - u^h\|_{L^2(0,T;H^1(\Omega))}.$$

1/h	First Step				Correction Step			
	$\ e_1\ _{L^2}$	CR	$\ e_1\ _{H^1}$	CR	$\ e_2\ _{L^2}$	CR	$\ e_2\ _{H^1}$	CR
4	0.119674	-	1.24694	-	0.0555588	-	0.802727	-
8	0.0714521	0.74	0.719949	0.79	0.0215584	1.37	0.301951	1.41
16	0.0399497	0.84	0.412976	0.80	0.00749371	1.52	0.114142	1.40
32	0.021289	0.91	0.226708	0.87	0.00231485	1.69	0.0385353	1.57
64	0.0110232	0.95	0.120044	0.92	0.000655197	1.82	0.0116292	1.73

Table 1: Errors and Convergence Rates(CR) with AV-DDC,  $\nu = 0.01$ .

1/h	First Step				Correction Step			
	$\ e_1\ _{L^2}$	CR	$\ e_1\ _{H^1}$	CR	$\ e_2\ _{L^2}$	CR	$\ e_2\ _{H^1}$	CR
4	0.0751673	-	0.875352	-	0.0396217	-	0.689005	-
8	0.0353812	1.09	0.390454	1.16	0.011821	1.74	0.209921	1.71
16	0.0171605	1.04	0.189928	1.04	0.00341711	1.79	0.0649275	1.69
32	0.00843965	1.02	0.0928222	1.03	0.000938785	1.86	0.0184928	1.81
64	0.00418593	1.01	0.0459234	1.02	0.000249308	1.91	0.00500629	1.89

Table 2: Errors and Convergence Rates(CR) with SAV-DDC,  $\nu = 0.01$ .

1/h	First Step				Correction Step			
	$\ e_1\ _{L^2}$	CR	$\ e_1\ _{H^1}$	CR	$\ e_2\ _{L^2}$	CR	$\ e_2\ _{H^1}$	CR
4	0.1335320	-	1.397030	-	0.0721421	-	1.00334	-
8	0.0822746	0.70	0.876335	0.67	0.0329968	1.13	0.515188	0.96
16	0.0476722	0.79	0.566698	0.63	0.0139666	1.24	0.319917	0.69
32	0.0262944	0.86	0.367175	0.63	0.00533722	1.39	0.213709	0.58
64	0.0140016	0.91	0.239464	0.62	0.00191136	1.48	0.144893	0.56
128	0.0072796	0.94	0.157301	0.61	0.000666623	1.52	0.0958644	0.60

Table 3: Errors and Convergence Rates(CR) with AV-DDC,  $\nu = 0.0001$ .

1/h	First Step				Correction Step			
	$\ e_1\ _{L^2}$	CR	$\ e_1\ _{H^1}$	CR	$\ e_2\ _{L^2}$	CR	$\ e_2\ _{H^1}$	CR
4	0.0899498	-	1.01711	-	0.0548386	-	0.853157	-
8	0.0450233	1.00	0.563964	0.85	0.0215061	1.35	0.400645	1.09
16	0.0226471	0.99	0.389837	0.53	0.00866109	1.31	0.290310	0.46
32	0.0112428	1.01	0.260555	0.58	0.00321679	1.43	0.201103	0.53
64	0.00558191	1.01	0.171782	0.60	0.00113210	1.51	0.133269	0.59
128	0.00278042	1.01	0.109830	0.65	0.000379425	1.58	0.0823622	0.69

Table 4: Errors and Convergence Rates(CR) with SAV-DDC,  $\nu = 0.0001$ .

For the qualitative testing, flow past a forward-backward facing step is considered. A  $40 \times 10$  rectangular domain is used as the channel, and a  $1 \times 1$  step is placed at the bottom of the channel, 5 units in. No-slip boundary conditions are strongly enforced on the walls of the channel and on the step, while parabolic inflow with maximum inlet 1 is introduced on the inflow boundary. Also on the outflow, 'do nothing' boundary



condition is weakly enforced. The initial condition is set to be parabolic flow across the channel, and there is no external forcing,  $f = 0$ . Viscosity  $\nu = 1/600$  is chosen in particular. For this setup the expected behavior is recirculating vortex formations behind the step and their detachment, see [23, 24, 25].

This comparison test is performed on the same coarse mesh (the smallest  $h=0.125$ ) for both methods, and choose additional viscosity is equal to the time step size  $\Delta t = 0.05$ . Computations have been ended at the final time  $T = 40$ .

Figures 1-2 illustrate both method produces stable results. On the other hand, AV-DDC is too dissipative to capture vortex detachment, i.e. eddies which should detach and evolve remain attached and attain steady state, while SAV-DDC is able to reliably met with expectations of the problem setup and replicates the behavior of the flow given in the reference papers [23, 24, 25]. This test clearly shows that SAV-DDC is not over-dissipative as AV-DDC is, and hence, is able to capture turbulent characteristics of the flow better than AV-DDC.

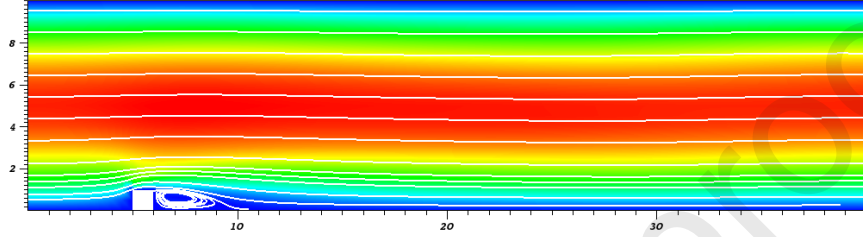


Figure 1: AV-DDC

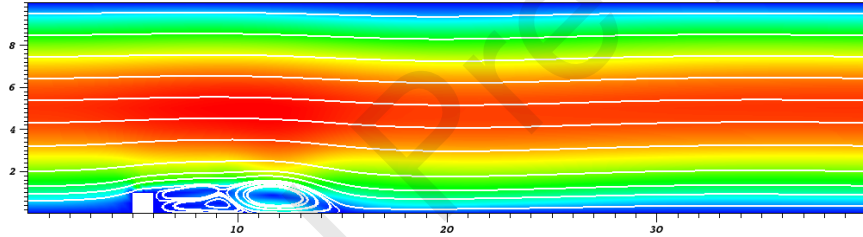


Figure 2: SAV-DDC

Although the correction step approximations are computed with the same weak formulation, the first step approximation plays a great role in how accurate results they will give and how well the flow will be resolved.

## 6. Conclusion

The method presented here replaces the artificial viscosity approximation step of the defect-deferred correction method with an alternative to a projection-based subgrid artificial viscosity approximation. This alternative approach has both theoretically and computationally shown its superiority over conventional artificial viscosity approximation based defect-deferred correction method.

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