JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **217,** 72–94 (1998)
ARTICLE NO. AY975702

Global L^∞ Estimates for a Class of Reaction–Diffusion Systems

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Submitted by Konstantin A. Lurie

Received November 8, 1996

We show that *a priori* L^1 estimates (dissipativity) imply L^∞ estimates (dissipativity) for a class of reaction–diffusion systems. This generalizes the results obtained in biological models such as unmixed bioreactors, plug-flow models, and Lotka–Volterra systems. © 1998 Academic Press

Key Words: Sobolev inequalities; *a priori* estimates; reaction diffusion systems; evolution operators.

1. INTRODUCTION

The problem to be considered in this paper is the system

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = \mathscr{A}_i(t,x,D)u_i + f_i(t,x,u) & t>0, \quad x\in\Omega, \quad i=1,\dots,m. \\ \mathscr{B}_i(x,D)u_i = v_i^0 & \text{on } \partial\Omega, \quad t>0 \\ u_i(0,x) = u_i^0(x) & \text{in } \Omega \end{cases} \tag{1.1}$$

where $u=(u_1,\dots,u_m)$, Ω is a bounded open set in R^N with smooth boundary $\partial\Omega$, and $\mathscr{A}_i(t,x,D)$s are linear elliptic operators, and $\mathscr{B}_i$s are regular elliptic boundary operators. In this general form, (1.1) represents many important reaction–diffusion models in ecology, biology, chemistry, etc.

A basic problem in the study of such a parabolic partial differential system is to determine conditions under which the solutions exist globally. From the standard theory of evolution equations, the local existence result

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0022-247X/98 $25.00

follows easily if the system is regular parabolic (e.g., see [13, 21, 25]). Using a similar argument as that of the theory of O.D.E (see [16]), one can show easily that the solutions exist globally if they do not blow up in finite time. Therefore, the question of global existence boils down to the problem of *a priori* estimates of the supremum norms of solutions, and showing that these norms are finite for all time.

On the other hand, if the system (1.1) is autonomous then it can be interpreted as generating a semiflow in the spirit of [16, 23], and thus allow us to apply the theory of dynamical systems to study the dynamics or asymptotic behavior of the solutions. In many cases, satisfactory implications would be obtained only for those systems that generate dissipative semiflows on appropriate Banach spaces which are usually the spaces (or their products) of nonnegative continuous functions with supremum norms. Additional compactness properties are also very often needed (see [15]). Once again, uniform L^∞ (or even Hölder) estimates of solutions play an important role to initiate such a theory.

There are many ways to obtain desired L^∞ estimates for solutions of parabolic systems of the form (1.1). Classical techniques and results can be found in [21] where a systematic treatment for general parabolic systems from linear to nonlinear was developed using level sets technique of De Giorgi. Since then, many more estimates have been obtained by adapting this idea to different settings (see [8] and references therein). However, most of these results are concerned with the local regularity of the solutions and therefore only give local estimates for L^∞ norms which are not well suited to the analysis of asymptotic behavior of solutions. Also, due to the nature of this technique, the estimates for the L^∞ norms usually require *a priori* L^∞ estimates for the solutions not only on the base domain Ω, but also on the lateral boundary $(0,\infty) \times \partial\Omega$. The latter seems not to be an easy task for many boundary conditions involving boundary derivatives. Even more, it is very possible that such estimates may not be uniform in time.

For some models, it is easy to establish dissipativity or L^∞ boundedness via standard comparison theorems or invariance principles based on the maximum principle, but this technique requires severe restrictions on the structure of the systems. One can look for the results and methods in this direction in, e.g., [22, 24, 31].

In this paper, we present a new result on the L^∞ and Hölder norms of the solutions of (1.1). Essentially, we show that the L^∞ and Hölder estimates (or dissipativity) can be obtained in terms of *a priori* L^1 estimates. This type of result is quite suitable for reaction–diffusion systems encountered in biology. Because, in many cases, the components u_i of the solutions are nonnegative functions, and therefore by a simple integration over the domain one can obtain an ordinary differential equation (or differential inequality) for the spatial averages of u_is and derive the estimates for their L^1 norms from this simpler system.

The L^∞ estimates which implies only global existence results were derived by using a Moser-type iterative method in the works of Alikakos (see [2, 1]) *for one equation* with homogeneous Neumann boundary condition and restricted structure (specifically, he considered equations whose diffusion terms are Laplacian and the reaction terms are linear). In [26], Rothe devised an alternative technique using a "feedback" argument to obtain similar estimates. Alikakos's technique was reworked and combined with an induction argument in [7, 19] to obtain the dissipativeness of the semiflows generated by some ecological models. These authors then applied this estimate to systems of Lotka−Volterra type whose reaction terms satisfy the so-called food pyramid condition or its related versions.

Here we consider semilinear parabolic systems satisfying a general structure and boundary conditions [see (i)−(iii) of Section 3]. The technique in this paper works directly with the whole system, and therefore allows us to drop the food pyramid condition on the nonlinearities. Actually, our use of the theory of evolution operators makes the proof simpler than those of the aforementioned techniques. The idea of the present paper was applied in a recent work [11] to an autonomous parabolic system, with less general structure, modelling competition in bioreactors.

Our system must satisfy more restrictive structure conditions than those in [21], but it is still general enough to cover many important models in applications. However, we should point out that the results here can be extended to quasilinear parabolic systems by using different techniques (see [9, 10, 12]). Since our primary interest here is to apply the results to the systems frequently encountered in applications, we will not pursue such a general setting, but rather restrict ourself to the case of semilinear systems with nonlinearities independent of gradients.

In Section 2, we present L^p estimates, which may be of independent interest, for general quasilinear parabolic systems. We then restrict ourself to a class of semilinear parabolic systems in Section 3, and show how to translate the L^p results in Section 2 into the L^∞ results to obtain the global existence and uniform boundedness of the solutions. Finally, in Section 4, we consider some models in mathematical biology such as bioreactors, plug-flow models, and Lotka−Volterra systems, and present two methods to obtain *a priori* L^1 estimates. One method bases on the technique of using principal eigenfunction as in [11], and another exploits the situation that the associated system of O.D.Es possesses certain Liapunov function. This generalizes some results in [1, 11].

2. L^p ESTIMATES

In this section, general conditions are described to ensure that one can obtain L^p estimates for p arbitrarily large if *a priori* L^1 estimates are assumed. We consider the following general reaction−diffusion systems:

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = \mathscr{A}_i u_i + f_i(t, x, u, Du_i) & t > 0, \quad x \in \Omega, \quad i = 1, \dots, m \\ \mathscr{B}_i u_i = v_i^0 & \text{on } \partial\Omega, \quad t > 0 \\ u_i(0, x) = u_i^0(x) & \text{in } \Omega, \end{cases} \tag{2.1}$$

where $u = (u_1, \dots, u_m)$, Ω is a bounded open set in R^N with smooth boundary $\partial\Omega$, and

$$\begin{cases} \mathscr{A}_i v = D_k\big(a_k^i(t, x, v, Dv)\big) & t > 0, \quad x \in \Omega, \quad i = 1, \dots, m \\ \mathscr{B}_i v = \dfrac{\partial v}{\partial \mathscr{N}_i} + b_i(t, x, v) & \text{on } \partial\Omega, \quad t > 0 \\ \dfrac{\partial v}{\partial \mathscr{N}_i} = a_k^i(t, x, v, Dv) \circ n_k(x). \end{cases} \tag{2.2}$$

with $n_k(x)$ being the cosine of the angle formed by the outward normal vector $n(x)$ of $\partial\Omega$ and the x_k axis. Here, we follow the convention that repeated indices will be summed from 1 to N. We assume that

(A) The differential operators $\mathscr{A}_i$ are uniformly elliptic. That is, there exist positive constants ν_0, μ_1, μ_2, and δ such that for any $(t, x, u, p) \in R^+ \times \Omega \times R^m \times R^N$ and $i = 1, \dots, m$,

$$a_k^i(t, x, u, p) p_k \geq \nu_0 \|p\|^2 - \mu_1 |u|^\delta - \mu_2, \tag{2.3}$$

with $0 \leq \delta < 2 + 2/N$.

(B) b_is are continuous functions in their variables. In addition, there exist positive constants ν_1, ν_2 such that

$$b_i(t, x, u) u \geq -\nu_1 |u|^2 - \nu_2$$

for all $(t, x) \in R^+ \times \partial\Omega$ and $u \in R$. Note that (A) above implies that $\partial / \partial \mathscr{N}_i$ are regular oblique derivative boundary operators.

Concerning the boundary and initial conditions, we assume that v_i^0, u_i^0 are bounded continuous functions on $R_+ \times \partial\Omega$ and Ω, respectively. We also denote $v_0 = (v_1^0, \dots, v_m^0)$ and $u_0 = (u_1^0, \dots, u_m^0)$.

Remark 2.1. We should remark that all of the results in this paper still hold for problems with Dirichlet or mixed boundary conditions. The techniques can be modified easily and even become simpler since we can transform the nonhomogeneous Dirichlet boundary conditions into homogeneous ones by subtracting the boundary data from the solutions. In this case, all the boundary integrals appearing in the proofs will disappear.

To obtain the L^p estimates from L^1 estimates we need to impose the following *growth conditions* on the nonlinearities f_i of (2.1)

(F) There exists positive constants $c_1, c_2, c_3, \alpha, \sigma$ such that $0 \le \alpha < (N+2)/(N+1)$ and $0 \le \sigma < (N+2)/N$ if $N > 2$, and $\alpha, \sigma > 0$ if $N \le 2$, and for $k = 1, \ldots, m$

$$|f_k(t,x,u,\zeta)| \le c_1 \sum_{i=1}^{m} |u_i|^\sigma + c_2 |\zeta|^\alpha + c_3 \tag{2.4}$$

for all $(t,x,u,\zeta) \in R_+ \times \Omega \times R^m \times R^N$. Remember that $u = (u_1, \ldots, u_m)$.

An easy consequence of the Young inequality implies that, for all $p > 0$,

$$\sum_{i=1}^{m} |f_i(t,x,u,\zeta_i) u_i^p| \le k_1 \sum_{i=1}^{m} |u_i|^{\sigma+p} + k_2 \sum_{i=1}^{m} |\zeta_i|^\alpha |u_i|^p + k_3 \tag{2.5}$$

for some positive constants k_1, k_2, k_3 independent of t, x, u, ζ_is.

Remark 2.2. We could allow all the constants in the hypotheses (A), (B), and (F) to depend on x, t and belong to some weighted Lebesgue spaces. Our proofs still work in this case by using weighted Sobolev space inequalities developed in [10]. Moreover, in many applications, special forms of some f_is may directly give L^∞ bounds for the corresponding components of the solutions via comparison principles. This would relax the restrictions on the growth rates of these components in the condition (2.4).

Our structure assumptions allow us to apply the standard theory of quasilinear parabolic systems in divergence form (e.g., see [13, 21]) to assert the following local existence result.

PROPOSITION 2.3. *Assume* (A), (B), *and* (F), *there is a positive number* $\tau(u_0)$ *such that there exists a unique solution for* (2.1) *on the maximal interval of existence* $(0, \tau(u^0))$.

We will need the following easy consequence of the Nirenberg–Gagliardo inequality [$\|\bullet\|_p$ denotes the L^p-norm in $L^p(\Omega)$].

LEMMA 2.4. *Let* $p \in [1,2]$ *and* $r \in [\, p, (2n+2)/n)$. *Then, for any given* $\varepsilon > 0$ *there exist positive constants* $C(\varepsilon)$, q *depending only on* p, r *such* *that*

$$\int_{\Omega} |u|^r \, dx \leq \varepsilon \left(\int_{\Omega} |Du|^2 \, dx + \|u\|_p^2 \right) + C(\varepsilon) \|u\|_p^q, \tag{2.6}$$

for any $u \in W^{1,2}(\Omega)$. *Actually we have* $q = 2r(1-\tau)/(2-r\tau)$ *with* $\tau = 2^*(r-p)/r(2^*-p)$ *and* $2^* = 2n/(n-2)$.

We also have, for $r \in [1,2]$,

$$\int_{\partial\Omega} |u|^r \, d\sigma \leq \varepsilon \int_{\Omega} |Du|^2 \, dx + C(\varepsilon)\|u\|_1^2 + C(\varepsilon). \tag{2.7}$$

We are now ready to prove

THEOREM 2.5. *Suppose that there exists a positive function* $C_1(t, v^0, u^0)$ *such that*

$$\|u_i(t, \bullet)\|_1 \leq C_1(t, v^0, u^0), \qquad \text{for all } t \in \big(0, \tau(u^0)\big). \tag{2.8}$$

then for any $p \geq 1$ *there exists a positive function* $C_p(t, v^0, u^0)$ *such that*

$$\|u_i(t, \bullet)\|_p \leq C_p(t, v^0, u^0), \qquad \text{for all } t \in \big(0, \tau(u^0)\big). \tag{2.9}$$

In addition, if there is a number K_1 *independent of initial data such that*

$$\limsup_{t \to \tau(u^0)} \|u_i(t, \bullet)\|_1 \leq K_1, \tag{2.10}$$

then there exists a number K_p *independent of initial data such that*

$$\limsup_{t \to \tau(u^0)} \|u_i(t, \bullet)\|_p \leq K_p. \tag{2.11}$$

Proof. We shall prove by induction. Let us assume that (2.9) holds for some $p \geq 1$ (it holds for $p = 1$). Consider the equation for u_i. Multiply the equation by $u_i|u_i|^{2p-2}$ and integrate to get

$$\begin{aligned} \int_{\Omega} u_i |u_i|^{2p-2} \frac{\partial u_i}{\partial t} \, dx &= \int_{\Omega} \mathscr{A}_i(u_i) u_i |u_i|^{2p-2} \, dx \\ &\quad + \int_{\Omega} f(t, x, u, Du_i) u_i |u_i|^{2p-2} \, dx. \end{aligned} \tag{2.12}$$

Put $w_i = |u_i|^p$. Integrating by parts, and using the boundary conditions and the Young inequality in a standard way, we obtain

$$
\begin{aligned}
&\int_\Omega \mathscr{A}_i(u_i)u_i|u_i|^{2p-2}\,dx \\
&\quad = \int_{\partial\Omega} u_i|u_i|^{2p-2}\frac{\partial u_i}{\partial \mathscr{N}_i}\,d\sigma - (2p-1)\int_\Omega a_k^i(\cdots)D_k u_i|u_i|^{2p-2}\,dx \\
&\quad \le \int_{\partial\Omega} v_i^0 u_i|u_i|^{2p-2}\,d\sigma + \nu_1\int_{\partial\Omega}|u_i|^{2p}\,d\sigma + \nu_2\int_{\partial\Omega}|u_i|^{2p-2}\,d\sigma \\
&\qquad - \frac{\nu_0(2p-1)}{p^2}\int_\Omega |Dw_i|^2\,dx + C_1(p)\int_\Omega |w_i|^{(2p-2+\delta)/p}\,dx + C_2(p).
\end{aligned}
$$

Using these estimates in (2.12), summing over i, and taking into account (2.5) of (F) and the boundedness of v_i^0s we find

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \sum_{i=1}^m w_i^2\,dx \\
&\quad \le -2\nu_0\int_\Omega \sum_{i=1}^m |Dw_i|^2\,dx + k_0\int_{\partial\Omega}\sum_{i=1}^m\left\{w_i^2 + w_i^{2-2/p} + w_i^{2-1/p}\right\}d\sigma \\
&\qquad + k_1\int_\Omega \sum_{i=1} w_i^s\,dx + k_2\int_\Omega \sum_{i=1}^m |Du_i|^\alpha |u_i|^{2p-1}\,dx + k_3, \qquad (2.13)
\end{aligned}
$$

where the constants k_i depend also on p. Here, we have used the Young inequality to combine the powers of w_i to $s = \max\{(2p-2+\delta)/p, (2p-1+\sigma)/p\} < 2 + 2/N$ (for all $p \ge 1$). We are now going to estimate the terms on the rhs of (2.13). Using (2.7), we can majorize the boundary integrals by

$$
\begin{aligned}
&\int_{\partial\Omega}\left\{w_i^2 + w_i^{2-2/p} + w_i^{2-1/p}\right\}d\sigma \\
&\qquad \le \varepsilon\int_\Omega |Dw_i|^2\,dx + C(\varepsilon)\left(\int_\Omega w_i\,dx\right)^2 + C(\varepsilon). \qquad (2.14)
\end{aligned}
$$

For the third term on the rhs of (2.13) we apply (2.6) to see that

$$
\int_\Omega w_i^s\,dx \le \varepsilon\left(\int_\Omega |Dw_i|^2\,dx + \left(\int_\Omega w_i\,dx\right)^2\right) + K(\varepsilon)\left(\int_\Omega w_i\,dx\right)^q \qquad (2.15)
$$

for some positive constants $q, K(\varepsilon)$. As for the fourth term, we write

$$\int_\Omega |Du_i|^\alpha |u_i|^{2p-1}\,dx = p\int_\Omega |Dw_i|^\alpha |u_i|^{p(2-\alpha)+\alpha-1}\,dx$$
$$\le \varepsilon\int_\Omega |Dw_i|^2\,dx + C(\varepsilon,p)\int_\Omega |u_i|^\beta\,dx, \quad (2.16)$$

where $\beta = 2(p(2-\alpha)+\alpha-1)/(2-\alpha) = 2p + 2(\alpha-1)/(2-\alpha)$. We then write $|u_i|^\beta = w_i^\eta$, $\eta = \beta/p$. From the fact that $0 \le \alpha < (N+2)/(N+1)$ we easily see that $\eta < 2 + 2/N$. Therefore, we can estimate the integrals of w_i^η by using the Nirenberg–Gagliardo inequality as in (2.15). Putting these estimates together and choosing ε small enough, we obtain from (2.13) that

$$\frac{d}{dt}\int_\Omega \sum_{i=1}^m w_i^2\,dx$$
$$\le -\nu_0\int_\Omega \sum_{i=1}^m |Dw_i|^2\,dx + l_1\sum_{i=1}^m\left(\int_\Omega w_i\,dx\right)^2 + l_2\sum_{i=1}^m\left(\int_\Omega w_i\,dx\right)^q + k_3.$$

Applying the Nirenberg–Gagliardo inequality again in the form

$$\int_\Omega w_i^2\,dx \le \left(\int_\Omega |Dw_i|^2\,dx + \left(\int_\Omega w_i\,dx\right)^2\right) + C\left(\int_\Omega w_i\,dx\right)^r,$$

where $r, C > 0$, we get

$$\frac{d}{dt}\int_\Omega \sum_{i=1}^m w_i^2\,dx$$
$$\le -\nu_0\int_\Omega \sum_{i=1}^m w_i^2\,dx$$
$$+\sum_{i=1}^m\left\{l_1\left(\int_\Omega w_i\,dx\right)^2 + l_2\left(\int_\Omega w_i\,dx\right)^q + l_3\left(\int_\Omega w_i\,dx\right)^r\right\} + k_3.$$

The asserted estimates now follow by applying the induction hypotheses, noting that $\|u_i(t,\bullet)\|_p^p = \int_\Omega w_i$ and $\int_\Omega w_i^2 = \|u_i(t,\bullet)\|_{2p}^{2p}$, and integrating the last inequality. ∎

3. L^∞ ESTIMATES

In this section, we shall show that L^p estimates for large p can be translated into L^∞-estimates provided that the system satisfied more restrictive structure conditions. Since our primary interest here is to apply our results to the systems frequently encountered in applications we will restrict ourselves to the case of *semilinear systems* with nonlinearities not depending on gradients. On the other hand, we prefer to base our consideration on the easily accessible and well established theory of nonautonomous evolution equations with operators having constant domains rather than using more recent results, e.g., by Amann (see [3]), to allow the boundary conditions to depend also on t (but, see also [10, 12]).

Let us consider the following semilinear parabolic system:

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = \mathscr{A}_i(t, x, D)u_i + f_i(t, x, u) & t > 0, \quad x \in \Omega, \quad i = 1, \dots, m \\ \mathscr{B}_i(x, D)u_i = 0 & \text{on } \partial\Omega, \quad t > 0 \\ u_i(0, x) = u_i^0(x) & \text{in } \Omega, \end{cases} \tag{3.1}$$

where

$$\begin{cases} \mathscr{A}_i(t, x, D)u = d_i(t) D_k\big(a_{kl}^i(x) D_l u\big) + a_k^i(t, x) D_k u, \\ \qquad t > 0, \quad x \in \Omega, \quad i = 1, \dots, m \\ \mathscr{B}_i(x, D)u = \dfrac{\partial u}{\partial \mathscr{N}_i} + r_i(x) u \\ \qquad \text{on } \partial\Omega, \quad t > 0 \\ \dfrac{\partial u}{\partial \mathscr{N}_i} = a_{kl}^i(x) D_k u \cdot n_l(x). \end{cases} \tag{3.2}$$

We impose the following smoothness conditions:

(i) $d_i \in C^\theta(R_+)$, $a_{kl}^i \in C^{1+\theta}(\Omega)$, $a_k^i \in C^\theta(R_+ \times \Omega)$ for some $\theta > 0$ and r_is are continuous functions on $\partial\Omega$.

(ii) (Ellipticity) There are positive constants $\lambda, \Lambda, \alpha, \beta$ such that for all $(x, \zeta, t) \in \Omega \times R^N \times R_+$

$$\lambda|\zeta|^2 \le a_{kl}^i(x)\zeta_k\zeta_l \le \Lambda|\zeta|^2 \quad \text{and} \quad \alpha \le d_i(t) \le \beta.$$

(iii) (Growth condition) There exists positive constants c_1, c_2, σ such that $0 \le \sigma < (N+2)/N$ if $N > 2$ and $\sigma > 0$ if $N \le 2$, and

$$|f_i(t, x, u)| \le c_1 \sum_{i=1}^{m} |u_i|^\sigma + c_2, \tag{3.3}$$

for all $(t, x, u) \in R_+ \times \Omega \times R^m$ [see also the remark after condition (F) of Section 2].

PROPOSITION 3.1. *Assuming* (*i*)–(*iii*), *the results of Theorem* 2.5 *hold for the system* (3.1).

Proof. We simply set $\bar{f}_i(t, x, u, \zeta) = a_k^i(t, x)\zeta_k + f_i(t, x, u)$ and observe that the $\bar{f}_i$s satisfy the growth condition (F). ∎

We can regard our problem in larger class of measurable functions L^p, $1 < p < \infty$. Let $\mathscr{X} = L^p(\Omega)$ and $A_i(t)$ be the realization of $(\mathscr{A}_i, \mathscr{B}_i)$ in $\mathscr{X}$. That is,

$$\begin{cases} \operatorname{dom}(A_i(t)) = W^{2,p}_{\mathscr{B}_i}(\Omega) = \{v \in W^{2,p}(\Omega) : \mathscr{B}_i(v) = 0\} \\ A_i(t)v = \mathscr{A}_i(t, x, D)v. \end{cases}$$

Let $u = (u_1, \dots, u_m)$ and $u_0 = (u_1^0, \dots, u_m^0)$. We can write abstractly our system as

$$\begin{cases} \dfrac{\partial u}{\partial t} = A(t)u + F(t, u) \\ u(0) = u_0 \end{cases} \qquad \text{in } X = \mathscr{X} \times \cdots \times \mathscr{X} \ (m \text{ times}), \tag{3.4}$$

where $A(t) = \operatorname{diag}\{A_i(t)\}$ and $F(t, u) = (f_1(t, x, u), \dots, f_m(t, x, u))$. Under the smoothness assumptions (i)–(iii), we easily see that $A(t)$ is a family of closed linear operators on X and satisfies all the conditions in [13] to ensure the existence of the *evolution operators*

$$U(t, \tau) \in \mathscr{L}(X) \qquad 0 \le \tau \le t < \infty. \tag{3.5}$$

So, the solution of (3.4) can be represented in the form

$$u(t) = U(t, 0)(u_0) + \int_0^t U(t, s)F(s, u(s))\, ds. \tag{3.6}$$

We have the following estimate concerning the operator $U(t, s)$: There exist positive numbers ω, C_γ such that for any $0 \le \gamma \le 1$ and $0 \le s < t < \infty$ [see (16.38) of [13]]

$$\|A^\gamma(t)U(t, s)\|_{\mathscr{L}(X)} \le \frac{C_\gamma e^{-\omega(t-s)}}{(t-s)^\gamma}. \tag{3.7}$$

Remark 3.2. We notice that (3.7) comes from Theorem 14.1 and (13.18), (13.19) in [13] which still hold if the function $\eta(\mu)$ in Lemma 13.1 [see (13.9)] in the reference is bounded in μ. This is satisfied here because of the uniform ellipticity conditions in (ii).

We are now ready to show that

THEOREM 3.3. *Suppose that (i)–(iii) hold and there exists a positive function $C_1(t, v_0, u_0)$ such that*

$$\|u_i(t,\bullet)\|_1 \le C_1(t, v_0, u_0) \qquad 0 \le t < \tau(u_0), \tag{3.8}$$

then the solution exists for all time $[\tau(u_0) = \infty]$. Moreover, if there is a finite number K_1 independent of initial data such that

$$\limsup_{t \to \tau(u^0)} \|u_i(t,\bullet)\|_1 \le K_1, \tag{3.9}$$

then there exists a finite number K_∞ independent of initial data such that

$$\limsup_{t \to \infty} \|u_i(t,\bullet)\|_\infty \le K_\infty. \tag{3.10}$$

Proof. Apply $A^\gamma(t)$ to both sides of (3.6) to have

$$A^\gamma(t)u(t) = A^\gamma(t)U(t,0)(u_0) + \int_0^t A^\gamma(t)U(t,s)F(s,u(s))\,ds.$$

From the result of the previous section and the polynomial growth condition on f_is, we can find a positive continuous function C_p such that

$$\|F(t,u(t))\|_p \le C_p(t, v_0, u_0), \qquad \forall t \in (0, \tau(u_0)).$$

Then, for any positive $0 < t < \tau(u_0)$, we have

$$\begin{aligned}
&\|A^\gamma(t)u(t)\|_p \\
&\quad \le \|A^\gamma(t)U(t,0)(u_0)\|_p + \int_0^t \|A^\gamma(t)U(t,s)\|_{\mathscr{L}(X)} \|F(s,u(s))\|_p\,ds \\
&\quad \le C_\gamma t^{-\gamma} e^{-\omega t}\|u_0\|_p + \int_0^t C_\gamma (t-s)^{-\gamma} e^{-\omega(t-s)} \|F(s,u(s))\|_p\,ds \\
&\quad \le C_\gamma t^{-\gamma} e^{-\omega t}\|u_0\|_p + \max_{0 \le \tau \le t} C_p(\tau, v_0, u_0) \\
&\qquad \times \int_0^t C_\gamma (t-s)^{-\gamma} e^{-\omega(t-s)}\,ds \\
&\quad \le C_\gamma t^{-\gamma} e^{-\omega t}\|u_0\|_p + \max_{0 \le \tau \le t} C_p(\tau, v_0, u_0) \int_0^\infty C_\gamma r^{-\gamma} e^{-\omega r}\,dr.
\end{aligned} \tag{3.11}$$

Because of the uniform ellipticity condition (ii) of the operator $A(t)$, we see that

$$\sup_{0<t,s<\infty} \|A(t)A^{-1}(s)\|_{\mathscr{L}(X)} < \infty.$$

So, we can obtain from the above estimates that

$$\|A^\gamma(t_0)u(t)\|_p \le CC_\gamma t^{-\gamma}e^{-\omega t}\|u_0\|_p + \overline{C}_p(t, v_0, u_0)\int_0^\infty C_\gamma r^{-\gamma}e^{-\omega r}\,dr \tag{3.12}$$

for some fixed positive constants t_0 and some continuous function $\overline{C}_p$. Consider the space $Y^\gamma \equiv D(A^\gamma(t_0))$ with the graph norm $\|u\|_{Y^\gamma} = \|A^\gamma(t_0)u\|_p$. We choose p such that $N/2p < \gamma < 1$ and note the imbedding $Y^\gamma \to C^\nu$, $0 \le \nu < 2\gamma - N/p$. This imbedding and (3.12) show that

$$\|u(t)\|_{C^\nu} \le C_\infty(t, v_0, u_0)$$

for $t \ge 1$. To bound the uniform norm of $u(t)$ for $t \in [0,1]$ we note that $\|U(t,0)\|_{\mathscr{L}(\mathscr{C}_m)}$ is uniformly bounded on $[0,1]$ $[\mathscr{C}_m \equiv \Pi_1^m C(\Omega)]$. The integral term in (3.6) can be estimated in the space Y^γ exactly as before. It follows immediately from this and Theorem 3.3.4 of [16] that $u(t)$ is defined for all $t \ge 0$.

To obtain (3.10) assuming (3.9), we see that there is an $\eta = \eta(u_0)$ and a positive constant c, independent of u_0, such that

$$\|F(s, u(s))\|_p \le \begin{cases} C_p(v_0, u_0) & \text{for } 0 \le s \le \eta, \\ c & \text{for } \eta < s < \tau(u_0). \end{cases}$$

Then by splitting the integral term in (3.11) into integrals on $(0, \eta)$ and $(\eta, \tau(u_0))$, we obtain, similarly, the following:

$$\|A^\gamma(t)u(t)\|_p \le C_\gamma t^{-\gamma}e^{-\omega t}\|u_0\|_p + C_p(v_0, u_0)C_\gamma \eta t^{-\gamma}e^{-\omega t} + c\int_0^\infty C_\gamma r^{-\gamma}e^{-\omega r}\,dr.$$

This obviously gives (3.10). Our proof is complete. ∎

Remark 3.4. It is also possible to consider nonhomogeneous boundary conditions; that is, $\mathscr{B}_i(x, D)u = v_i^0(x,t)$ with $v_i^0 \not\equiv 0$. In this case, we consider the temporal variable t as a parameter and let $\{u_{i*}^t\}_{t>0}$ be the family of unique solution of the BVP

$$\begin{cases} D_k\big(a_{kl}^i(t,x)D_l u\big) + a_k^i(t,x)D_k u = 0 & \text{in } \Omega \\ \mathscr{B}_i(x,D)u = v_i^0(x,t) & \text{on } \partial\Omega. \end{cases} \tag{3.13}$$

Let $u = (u_1, \ldots, u_m)$ and $u_*^t = (u_{1*}^t, \ldots, u_{m*}^t)$. Set $w = u - u_*$. By replacing u by w and u_0 by $u_0 - u_*^t$ in the above argument we conclude that (3.10) holds for w. If we assume that the smoothness conditions (i) and (ii) are uniform with respect to the variable t then, from the Schauder estimates for elliptic equations (e.g., see [14]), we can see that $\|u_*^t\|_\infty$ and $\|u_*^t\|_{C^\nu}$ are bounded uniformly with respect to t. Therefore, the estimates on w imply similar estimates on u.

4. APPLICATIONS

In this section, we consider some models in mathematical biology and show that *a priori* L^1 uniform boundedness of the solutions can be obtained by using a technique developed in our earlier work [11]. These models satisfy the structure conditions (i)–(iii) of Section 3, and therefore Theorem 3.3 can be applied to assert the L^∞ uniform boundedness and global existence of the solutions.

4.1. *General unmixed bioreactors*

The bioreactor is assumed to occupy a domain Ω in N-dimensional space (usually, $N = 3$) which contains growth medium in which nutrient and microorganisms are suspended. It is an open system in the sense that fresh nutrient is supplied from an external reservoir while growth medium, including unused nutrient and organisms, is removed. In unmixed bioreactor, the nutrient and organisms are allowed to diffuse in the environment. The equations satisfied by nutrient S and the m microbial populations u_i, $1 \le i \le m$, will take the following general form:

$$\begin{cases} \dfrac{\partial S}{\partial t} = \mathscr{A}_0(t,x,D)S + f_0(t,x,S,u) \\ \dfrac{\partial u_i}{\partial t} = \mathscr{A}_i(t,x,D)u_i + f_i(t,x,S,u) & t > 0, \quad x \in \Omega, \end{cases} \tag{4.1}$$

with the boundary conditions and initial conditions

$$\begin{cases} \mathscr{B}_0(x, D)S = S^0(x) \\ \mathscr{B}_i(x, D)u_i = u_i^0(x) \qquad t > 0, \quad x \in \partial\Omega, \end{cases} \tag{4.2}$$

$$S(x,0) = S_0(x) \geq 0, \quad u_i(x,0) = u_{i0}(x) \geq 0, \qquad x \in \Omega. \tag{4.3}$$

In mathematical biology literature, the following special forms were often encountered:

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_0(t)\Delta S + f_0(t, x, S, u) \\ \dfrac{\partial u_i}{\partial t} = d_i(t)\Delta u_i + f_i(t, x, S, u) \qquad t > 0, \quad x \in \Omega \end{cases} \tag{4.4}$$

with the Robin boundary conditions

$$\begin{cases} \dfrac{\partial S}{\partial n} + r_0(x)S = S^0(x) \\ \dfrac{\partial u_i}{\partial n} + r_i(x)u_i = u_i^0(x) \qquad t > 0, \quad x \in \partial\Omega, \end{cases} \tag{4.5}$$

and initial conditions as in (4.3). This model has been extensively studied in [18, 17, 30], where the diffusion coefficients were assumed to be identical and independent of time. In [4, 20, 27], the studies of steady state problems had been carried out under the assumption that d_is are different *constants* but there is a *common set of boundary conditions for nutrient and organisms*. In these papers, the reaction terms in (4.4) were assumed to take the following special forms:

$$\begin{aligned} f_0(t, x, u, S) &= -\sum_{i=1}^{m} c_i u_i F_i(S), \\ f_i(t, x, u, S) &= u_i\big(F_i(S) - k_i\big), \end{aligned} \tag{4.6}$$

where $F_i\colon R_+ \to R_+$ is continuously differentiable and $F_i(0) = 0$, c_is, k_is are positive constants. In [11], we considered the system (4.4), (4.5) assuming d_is are independent of time and general form of the f_is. Inspired by (4.6), we assumed that f_is satisfy the following *cancellation condition*:

(C1) There exist positive constants h_i and real constants k, c such that

$$\sum_{i=0}^{m} h_i f_i(t, x, u) \leq k \sum_{i=0}^{m} h_i u_i + c \qquad \text{for all } u \in R^{m+1}. \tag{4.7}$$

Then we were able to obtain the uniform L^1 boundedness of the solutions of (4.4). Note that $k = 0$ if the reaction terms are given in the form (4.6).

Remark 4.1. For systems of the form (4.4), (C1) can be generalized a bit by hypothesizing the existence of functions h_i: $R_+ \to R_+$ of class C^2 satisfying $h_i'' \geq 0$ and $h_i(0) = 0$ and

$$\sum_{i=0}^{m} h_i'(u_i) f_i(u) \leq k \sum_{i=0}^{m} h_i(u_i) + c \quad \text{and} \quad \sum_{i=0}^{m} u_i \leq c_1 \sum_{i=0}^{m} h_i(u_i),$$

where k, c satisfy the same hypotheses as in (F2) above and $c_1 > 0$. However, if $v_i^0 \neq 0$ for some i, then we have to assume that $h_i'(u_i)$ is bounded for $u_i \geq 0$.

We would like to describe here a general outline of the technique developed in [11] for the more general system (4.1). We assume the following.

(C2) There exist a positive function ϕ in $\overline{\Omega}$ and positive constants μ, C such that for $i = 0, \ldots, m$ we have

$$\int_\Omega \mathscr{A}_i(t, x, D) u_i \phi \, dx \leq -\mu \int_\Omega u_i \phi \, dx + C. \tag{4.8}$$

Then we have the following L^1 estimate for the solutions of (4.1).

PROPOSITION 4.2. *Suppose that the system* (4.1), (4.2), *and* (4.3) *satisfies the conditions* $(C1)$ *and* $(C2)$, *and that* $\alpha = \mu - k > 0$. *There exist positive constants* C_1, C_2 *independent of* S_0, u_{i0} *such that* (we denote S, S_0 by u_0, u_{00})

$$\sum_{i=0}^{m} \|u_i(t, \bullet)\|_1 \leq C_1(1 - e^{-(\mu - k)t}) + C_2 \sum_{i=0}^{m} \|u_{i0}\|_1 e^{-(\mu - k)t}. \tag{4.9}$$

Proof. Multiply the equation for u_i by $h_i \phi$ and integrate over Ω to obtain

$$\frac{d}{dt} \int_\Omega h_i u_i \phi \, dx = h_i \int_\Omega \mathscr{A}_i(t, x, D) u_i \phi \, dx + \int_\Omega h_i \phi f_i(t, x, u) \, dx. \tag{4.10}$$

Now set $H(u) = \sum_{i=0}^m h_i u_i$, put (4.8) into (4.10), and add the result inequalities to get

$$\frac{d}{dt}\int_\Omega H(u)\phi\,dx \le -\mu\int_\Omega H(u)\phi\,dx + \int_\Omega \phi \sum_{i=0}^m h_i f_i(t,x,u)\,dx + C$$

$$\le -\alpha\int_\Omega H(u)\phi\,dx + c + C,$$

where we have also used (4.7) and the fact that $\alpha = \mu - k > 0$. This gives

$$\int_\Omega H(u)\phi\,dx \le e^{-\alpha t}\int_\Omega H(u)\phi\,dx + \frac{C+c}{\alpha}(1 - e^{-\alpha t}).$$

As $\phi(x) > 0$ in $\overline{\Omega}$, (4.9) follows. ∎

As we mentioned earlier, this L^1 boundedness and Theorem 3.3 implies

THEOREM 4.3. *Given the assumptions of Proposition* 4.2 *and* (i)–(iii) *of Section* 3, *the solutions of* (4.1), (4.2), *and* (4.3) *exist globally and are uniformly bounded.*

Remark 4.4. In [11], the function ϕ and constant μ in the condition (C2) were taken to be the principal eigenfunction and eigenvalue of the problem

$$\begin{cases} -\Delta\phi = \mu\phi \\ \dfrac{\partial\phi}{\partial n} + \bar{r}(x)\phi = 0, \end{cases}$$

where $\bar{r}(x) = \min_i\{r_i(x)\}$. We assumed that $\bar{r}(x)$ is nonnegative and does not vanish identically on $\partial\Omega$. This ensures that $\mu > 0$. So that, Theorem 4.3 generalizes some results in [11] and allows d_i to depend on time. We should also remark that one can very often obtain L^∞ estimate for nutrient S via simple uses of comparison principles, and therefore allow the F_is in (4.6) to grow arbitrarily in S.

4.2. *Plug-flow models*

Consider a tubular culture vessel of unit length in which m microbial populations $u_1, \ldots, u_m$ compete for the nutrient S. The symbols denote the concentrations per unit volume of these quantities. Assume that the vessel, occupying the cylinder $\Omega \equiv \omega \times [0,1]$ in R^3, is fed with growth medium at $\omega \times \{0\}$ due to a constant flow velocity v of the fluid in the vessel along the axis of the cylinder. This external feed contains nutrient at concentration $S^0(x)$. Medium, nutrient, and organisms exit the vessel at

$\omega \times \{1\}$ with velocity v. Nutrient S and organisms u_i are assumed to diffuse in the vessel with constant diffusivities d_0 and d_i, respectively. The system describing the concentrations S, and u_i is then given below

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_0 \Delta S - v \cdot \nabla S + f_0(t, x, S, u) \\ \dfrac{\partial u_i}{\partial t} = d_i \Delta u_i - v \cdot \nabla u_i + f_i(t, x, S, u) \qquad t > 0, \quad x \in \Omega \end{cases} \tag{4.11}$$

with the boundary conditions and initial conditions

$$\begin{cases} d_0 \dfrac{\partial S}{\partial n} + r_0(x) S = S^0(x) \\ d_i \dfrac{\partial u_i}{\partial n} + r_i(x) u_i = u_i^0(x) \qquad t > 0, \quad x \in \partial\Omega \end{cases} \tag{4.12}$$

$$S(x, 0) = S_0(x) \geq 0, \quad u_i(x, 0) = u_{i0}(x) \geq 0, \quad x \in \omega \times [0, 1]. \tag{4.13}$$

The functions f_i describe the consuming rate of the ith organism, as in the case of bioreactor considered in Section 4.1, and satisfy the cancellation condition (C1).

This type of model has been investigated by several authors (see [6, 28, 29]) and has been often assumed that the tubular vessel is very thin so that the systems can be formulated on one-dimensional intervals. Moreover, the diffusivity constants were again assumed to be the same for nutrient and organisms so that the global existence and uniform boundedness of the solutions can be easily derived by using comparison techniques. Here, we allow the vessel to be a three-dimensional domain and the d_is to be different.

We are going to apply the general procedure described in Section 4.1 to find conditions which ensure the L^1 estimates for the solutions of (4.11), (4.12). However, we are able to obtain the uniform L^1 (and then L^∞) boundedness of the solutions only in the following two cases.

(a) d_i are arbitrary, but the velocity v is small enough.

(b) The velocity v is arbitrary, but d_i are slightly different.

What it means by small and slightly different will be described below. We consider these two cases separately. The techniques differ only in the way we choose the functions ϕ described in condition (C2) of Section 4.1.

Case (a). We take ϕ to be the eigenfunction ϕ of the eigenvalue problem

$$\begin{cases} -\Delta\phi = \lambda\phi \\ \dfrac{\partial\phi}{\partial n} + \bar{r}(x)\phi = 0, \end{cases} \tag{4.14}$$

where $\bar{r} \geq 0$ will be determined later so that $\phi > 0$ on $\overline{\Omega}$. To check the condition (C2) we need to consider the integral (again we denote S, S_0 by u_0, u_{00})

$$\begin{aligned} \int_\Omega (d_i\Delta u_i - v\cdot\nabla u_i)\phi\,dx &= \int_\Omega (d_i\Delta\phi + v\cdot\nabla\phi)u_i\,dx \\ &\quad + d_i\int_{\partial\Omega}\left(\frac{\partial u_i}{\partial n}\phi - \frac{\partial\phi}{\partial n}u_i - \frac{v\cdot n}{d_i}u_i\phi\right)d\sigma \\ &\leq \left(-\lambda d_i + \sup_\Omega \frac{|v\cdot\nabla\phi|}{|\phi|}\right)\int_\Omega u_i\phi\,dx \\ &\quad + d_i\int_{\partial\Omega}\left(u_{i0} - ru_i + \bar{r}u_i - \frac{v\cdot n}{d_i}u_i\right)\phi\,d\sigma. \end{aligned}$$

Since u_i, ϕ are nonnegative, we see that (4.8) of (C2) will be verified if we impose the following condition.

(PL1) There exists a *nonzero* function $\bar{r}(x)$ given on $\partial\Omega$ such that

$$0 \leq \bar{r}(x) \leq r_i(x) + \frac{v\cdot n}{d_i}, \qquad \forall i, \quad \forall x\in\partial\Omega, \tag{4.15}$$

and let v be such that [k is the real number in (C1)]

$$\sup_\Omega \frac{|v\cdot\nabla\phi|}{|\phi|} < \lambda\min_i d_i - k. \tag{4.16}$$

Remark 4.5. In [28], the functions r_i are as follows

$$r_i(x) \equiv \begin{cases} -\dfrac{v\cdot n}{d_i} & x\in\omega\times\{0\} \\ 0 & x\in\partial\Omega\setminus\omega\times\{0\}. \end{cases} \tag{4.17}$$

It is easy to see that (4.15) is satisfied in this case if we take (note that $v\cdot n > 0$ on $\omega\times\{1\}$ and $v\cdot n \geq 0$ on $\partial\omega\times[0,1]$)

$$\bar{r}(x) \equiv \begin{cases} 0 & x\in\omega\times\{0\}\cup\partial\omega\times[0,1], \\ \dfrac{v\cdot n}{\max_i d_i} & x\in\omega\times\{1\}. \end{cases}$$

So, we have the following result.

THEOREM 4.6. *Assume the condition* $(PL1)$. *The solutions of* (4.11), (4.12), *and* (4.13) *exist globally and are uniformly bounded.*

Proof. The maximum principle and (4.15) imply that $\phi > 0$ on $\overline{\Omega}$. We then choose μ in (C2) to be $\min_i \lambda d_i - \sup_\Omega\{|v \cdot \nabla\phi|/\phi\}$. Our result follows from Proposition 4.2 and Theorem 3.3 of Section 3. ∎

Case (b). We consider the principal eigenfunction and eigenvalue ϕ, λ of the eigenvalue problem

$$\begin{cases} -\delta\Delta\phi - v \cdot \nabla\phi = \lambda\phi \\ \dfrac{\partial\phi}{\partial n} + \bar{r}(x)\phi = 0, \end{cases} \tag{4.18}$$

where δ is the average of d_i. Again, we have

$$\begin{aligned} \int_\Omega (d_i\Delta u_i - v \cdot \nabla u_i)\phi\,dx &= \int_\Omega (d_i\Delta\phi + v \cdot \nabla\phi)u_i\,dx \\ &\quad + d_i\int_{\partial\Omega}\left(\frac{\partial u_i}{\partial n}\phi - \frac{\partial\phi}{\partial n}u_i - \frac{v \cdot n}{d_i}u_i\phi\right)d\sigma \\ &= -\lambda\int_\Omega u_i\phi\,dx + (d_i - \delta)\int_\Omega u_i\Delta\phi\,dx \\ &\quad + d_i\int_{\partial\Omega}\left(u_{i0} - ru_i + \bar{r}u_i - \frac{v \cdot n}{d_i}u_i\right)\phi\,d\sigma. \end{aligned}$$

So, if

$$\bar{r}(x) \le r_i(x) + \frac{v \cdot n}{d_i}, \qquad \forall i, \quad \forall x \in \partial\Omega,$$

then

$$\begin{aligned} &\int_\Omega (d_i\Delta u_i - v \cdot \nabla u_i)\phi\,dx \\ &\qquad \le \left(|d_i - \delta|\sup_\Omega \frac{|\Delta\phi|}{|\phi|} - \lambda\right)\int_\Omega u_i\phi\,dx + d_i\int_{\partial\Omega} u_{i0}\phi\,d\sigma. \end{aligned}$$

Hence, we will assume that

(PL2) There exist a *nonzero* function $\bar{r}(x)$ given on $\partial\Omega$ such that [see also the remark after (PL1)]

$$0 \le \bar{r}(x) \le r_i(x) + \frac{v \cdot n}{d_i}, \qquad \forall i, \quad \forall x \in \partial\Omega, \tag{4.19}$$

and a positive constant δ such that for ϕ, λ as in (4.18) we have

$$\max_i \left\{ |d_i - \delta| \sup_\Omega \frac{|\Delta \phi|}{|\phi|} \right\} < \lambda - k. \tag{4.20}$$

Remark 4.7. Since $\bar{r}(x) \geq 0$ on $\partial\Omega$, Theorem 4.3 in [3] shows that (4.18) always has a smallest eigenvalue λ_0 which is positive, and its corresponding eigenfunction ϕ is positive on $\overline{\Omega}$ if $\bar{r}(x)$ is not identically zero.

Similarly as in case (a) we have

THEOREM 4.8. *Assume the condition* $(PL2)$. *The solutions of* (4.11), (4.12), *and* (4.13) *exist globally and are uniformly bounded.*

4.3. *Lotka–Volterra systems*

The technique presented above relies essentially on the existence of the positive function ϕ and the positive constant μ of (C1). As we have seen earlier, they are usually the principal eigenfunction and eigenvalue of a relative eigenvalue problem (EVP) whose boundary condition closely relates to those of the system. This method seems to be unapplicable for systems with Neumann boundary conditions because for the EVPs (4.14), (4.18) with Neumann boundary condition $\partial\phi/\partial n = 0$ on $\partial\Omega$, zero is always an eigenvalue with constant eigenfunction, and there may not be a positive eigenvalue with positive eigenfunction (but see [5]).

However, if the system of ODEs associated to the system of PDEs possesses a Liapunov function that satisfies some additional convexity properties then we will be able to obtain such an L^1 estimate. This technique had been used in [1, 7] to treat systems of Lotka–Volterra type. These authors also obtained L^∞ estimates assuming that the nonlinearities f_i satisfy the so called *food pyramid condition* (see [1]) or its generalized versions (see [7]). Roughly speaking, this condition requires that the growth of the jth component of the solution be bounded in terms of the magnitudes of the ith components for $i < j$. Consequently, they can use an induction argument starting with an L^∞ estimate for the first component of solutions, and thus reduce the *a priori* estimate problem to that of one equation.

Our method presented in Sections 2 and 3 works directly with the whole system, and therefore relaxes this food pyramid condition which somewhat restricts the range of applicability. We shall only present here a generalization of some results in [1]. Similar results can be obtained for the systems considered in [7].

Consider the system of m equations

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = d_i(t)\Delta u_i + f_i(t,u) & t > 0, \quad x \in \Omega \\ \dfrac{\partial u_i}{\partial n} = 0 & \text{on } \partial\Omega, \quad t > 0 \\ u_i(0,x) = u_i^0(x) \geq 0 & \text{in } \Omega. \end{cases} \tag{4.21}$$

The associated system of ordinary differential equations is

$$\frac{du_i}{dt} = f_i(t,u), \qquad i = 1,\dots,m. \tag{4.22}$$

We assume that this system satisfies:

(L1) There exists a Liapunov function $E(z_1,\dots,z_m)$ for (4.22) on $\{(z_1,\dots,z_m),\ z_i \geq 0\}$.

(L2) The Hessian $(\partial^2 E/\partial z_i \partial z_j)$ is positive semidefinite and $(d_i(t)\partial^2 E/\partial z_i \partial z_j)$ is positive definite for all $t > 0$.

(L3) If $z = (z_1,\dots,z_m)$ then $\lim_{z\to\infty} E(z) = +\infty$.

The following lemma shows that E gives rise to a Liapunov function for (4.21). The proof is exactly that of Lemma 4.1 of [1] (the time dependency of d_i and f_i does not cause any change).

LEMMA 4.9. *Assume* (*L*1) *and* (*L*2). *Then the functional V defined by*

$$V(u)(t) = V(u_1,\dots,u_m)(t) = \int_\Omega E(u_1(t,x),\dots,u_m(t,x))\,dx$$

is a Liapunov function for (4.21) *on* $\{(u_1,\dots,u_m): u_i(t,x) \geq 0,\ (t,x) \in R_+ \times \Omega\}$.

Now let $u(t,x)$ be a solution of (4.21). Since V is a Liapunov function for (4.21)

$$\int_\Omega E(u_1(t,x),\dots,u_m(t,x))\,dx \leq \int_\Omega E\big(u_1^0(x),\dots,u_m^0(x)\big)\,dx.$$

Furthermore, (L2) implies that E is convex so that Jensen's inequality (for vectors) gives

$$E\left(\int_\Omega u_1(t,x)\,dx,\dots,\int_\Omega u_m(t,x)\,dx\right) \leq \int_\Omega E\big(u_1^0(x),\dots,u_m^0(x)\big)\,dx.$$

Finally, (L3) implies that

$$\int_\Omega u_i(t,x)\,dx \le C\left(u_1^0,\ldots,u_m^0\right)$$

for some positive function C. We have shown the L^1 boundedness of the solution. We then conclude similarly as before

THEOREM 4.10. *Assume* $(L1)$–$(L3)$. *Then the solutions of* (4.21) *exist globally and have* L^∞ *norm bounded in terms of that of initial data*.

Remark 4.11. In [1], pp. 215–216, one can find examples of Lotka–Volterra systems having convex Liapunov functions that satisfy (L1)–(L3).

ACKNOWLEDGMENTS

The author would like to thank Professors H. Kuiper, B. Nicolaenco, and H. L. Smith for helpful discussions.

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