

# A Mean Ergodic Theorem for Nonlinear Semigroups which are Asymptotically Nonexpansive in the Intermediate Sense

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Let  $X$  be a uniformly convex Banach space such that its dual  $X^*$  has the Kadec–Klee property,  $C$  a bounded closed convex subset of  $X$ , and  $\mathcal{T} = \{T(t) : t \in [0, \infty)\}$  a one-parameter nonlinear semigroup which is asymptotically nonexpansive in the intermediate sense. We show that every continuous almost orbit of  $\mathcal{T}$  is weakly almost convergent to a common fixed point of  $\mathcal{T}$ . A discrete version of our main result is also included. © 2000 Academic Press

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## 1. INTRODUCTION

This paper is devoted to a mean ergodic theorem for nonlinear semigroups and mappings which are asymptotically nonexpansive in the intermediate sense. The first results of this kind for nonexpansive semigroups



were established by J.-B. Baillon and H. Brezis [3] (see also [2]), N. Hirano and W. Takahashi [14], and S. Reich [24–26]. See also, [12, 13, 18, 20, 21, 23, 27–29, 31, 32], and especially the very nice survey by R. E. Bruck [8]. An up-to-date bibliography can be found in [10]. Our main theorem (Theorem 4.1) improves upon a result proved by J. García Falset and the authors (Theorem 4.5 in [10]) for asymptotically nonexpansive semigroups of Lipschitzian mappings. We also include a new mean ergodic theorem for the discrete case (Theorem 5.3). The Zarantonello inequality [33] (rediscovered by J.-B. Baillon in [1]) and the  $\gamma$ -method due to R. E. Bruck [6, 8], are basic tools in the proofs of almost all mean ergodic theorems for asymptotically nonexpansive mappings. The main problem in the proofs of our results is that we cannot use this method directly because our operators are not Lipschitzian. To overcome this difficulty we need a long sequence of auxiliary lemmas and theorems (see Section 3). Only the first three lemmas are similar to those proved by H. Oka [22] for the iterates of a mapping which is asymptotically nonexpansive in the intermediate sense (see also [15]).

## 2. PRELIMINARIES

Throughout this paper  $X$  is assumed to be a uniformly convex real Banach space,  $C$  a nonempty bounded closed convex subset of  $X$ , and  $T$  a mapping from  $C$  into itself. We denote by  $\text{co } M$  and by  $\text{clco } M$  the convex hull and the closed convex hull of  $M \subset X$ , respectively, and  $\omega_w(\{x_n\})$  stands for the set of all weak subsequential limits of a bounded sequence  $\{x_n\}$  in  $X$ . The closed ball centered at  $0 \in X$  and of radius  $r > 0$  is denoted by  $B_r$ . We also put

$$\Delta^{n-1} = \left\{ \lambda = (\lambda_1, \dots, \lambda_n) : \lambda_i \geq 0, i = 1, 2, \dots, n, \sum_{i=1}^n \lambda_i = 1 \right\}.$$

Let  $\mathcal{T} = \{T(t) : t \in [0, \infty)\}$  be a family of self-mappings of  $C$ . The family  $\mathcal{T}$  is said to be a nonlinear semigroup which is asymptotically nonexpansive in the intermediate sense if the following conditions are satisfied:

- (i)  $T(t) : C \rightarrow C$  is continuous for each  $t \in [0, \infty)$ ;
- (ii)  $T(s + t)x = T(s)T(t)x$  for all  $s, t \in [0, \infty)$  and  $x \in C$ ;
- (iii)  $T(0) = I$ ;
- (iv) The inequality

$$(*) \quad \limsup_{t \rightarrow \infty} \sup_{x, y \in C} (\|T(t)x - T(t)y\| - \|x - y\|) \leq 0$$

holds;

(v)  $T(t)x$  is continuous in  $t \in [0, \infty)$  for each  $x \in C$ .

The notion of a mapping which is asymptotically nonexpansive in the intermediate sense was introduced in [9].

The set of common fixed points of the semigroup  $\mathcal{F}$  is denoted by  $F(\mathcal{F})$ .

We say that a function  $u: [0, \infty) \rightarrow C$  is an almost orbit of  $\mathcal{F}$  [5] if

$$\lim_{s \rightarrow \infty} \left( \sup_{t \in [0, \infty)} \|u(t+s) - T(t)u(s)\| \right) = 0.$$

Let  $\omega_w(u)$  denote the set of all weak subsequential limits of  $\{u(t)\}_{t \in [0, \infty)}$  as  $t \rightarrow \infty$ .

Finally, recall that a Banach space is said to have the Kadec–Klee property (KK-property, for short) [11] if whenever  $w\text{-}\lim_n x_n = x$  with  $\|x_n\| \rightarrow \|x\|$ , it follows that  $\lim_n x_n = x$  strongly.

### 3. AUXILIARY LEMMAS AND THEOREMS

The common assumptions in the sequence of lemmas and theorems in this section are as follows:  $X$  is a uniformly convex Banach space,  $C$  is a bounded closed convex subset of  $X$ ,  $\mathcal{F} = \{T(t) : t \in [0, \infty)\}$  is a one-parameter semigroup of self-mappings of  $C$  which is asymptotically nonexpansive in the intermediate sense, and  $u$  is an almost orbit of  $\mathcal{F}$ .

The first three lemmas and their proofs are analogous to the lemmas given in [22] (see also [15] for a correct proof of Lemma 3.1) and therefore we omit their proofs here. Let us only mention that the convex approximation property of  $X \times X$  (where  $X$  is a uniformly convex Banach space) [7] plays a crucial role in these proofs.

LEMMA 3.1. *For  $\epsilon > 0$  there exist  $t_\epsilon > 0$  and  $\delta_{2, \epsilon} > 0$  such that if  $t \geq t_\epsilon$ ,  $z_1, z_2 \in C$ , and if  $\|z_1 - z_2\| - \|T(t)z_1 - T(t)z_2\| \leq \delta_{2, \epsilon}$ , then*

$$\|T(t)(\lambda_1 z_1 + \lambda_2 z_2) - \lambda_1 T(t)z_1 - \lambda_2 T(t)z_2\| < \epsilon$$

for all  $\lambda = (\lambda_1, \lambda_2) \in \Delta^1$ .

LEMMA 3.2. *For  $\epsilon > 0$  and for each integer  $n \geq 2$  there exist  $t_\epsilon > 0$  and  $\delta_{n, \epsilon} > 0$  where  $t_\epsilon$  is independent of  $n$ , such that if  $t \geq t_\epsilon$ ,  $z_1, z_2, \dots, z_n \in C$ , and if  $\|z_i - z_j\| - \|T(t)z_i - T(t)z_j\| \leq \delta_{n, \epsilon}$  for  $1 \leq i, j \leq n$ , then*

$$\left\| T(t) \left( \sum_{i=1}^n \lambda_i z_i \right) - \sum_{i=1}^n \lambda_i T(t)z_i \right\| < \epsilon$$

for all  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \Delta^{n-1}$ .

LEMMA 3.3. For  $\epsilon > 0$  and for each integer  $n \geq 2$  there exist  $t_\epsilon > 0$  and  $\delta_\epsilon > 0$  where both  $t_\epsilon$  and  $\delta_\epsilon$  are independent of  $n$ , such that if  $t \geq t_\epsilon$ ,  $z_1, z_2, \dots, z_n \in C$ , and if  $\|z_i - z_j\| - \|T(t)z_i - T(t)z_j\| \leq \delta_\epsilon$  for  $1 \leq i, j \leq n$ , then

$$\left\| T(t) \left( \sum_{i=1}^n \lambda_i z_i \right) - \sum_{i=1}^n \lambda_i T(t) z_i \right\| < \epsilon$$

for all  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \Delta^{n-1}$ .

Now we are ready to prove a theorem about the limit behavior (in the norm) of almost orbits.

THEOREM 3.1. Let  $X$  be a uniformly convex Banach space and  $\mathcal{T}$  a semigroup of self-mappings of a bounded closed convex subset  $C$  of  $X$  which is asymptotically nonexpansive in the intermediate sense. Suppose that  $u_1$  and  $u_2$  are almost orbits of  $\mathcal{T}$ ,  $f_1, f_2 \in F(\mathcal{T})$ , and  $0 \leq \alpha \leq 1$ . Then the limits

$$\lim_{t \rightarrow \infty} \|u_1(t) - u_2(t)\|,$$

$$\lim_{t \rightarrow \infty} \|u_1(t) - f_1\|,$$

and

$$\lim_{t \rightarrow \infty} \|\alpha u_1(t) + (1 - \alpha)f_1 - f_2\|$$

exist.

*Proof.* We have

$$\begin{aligned} & \|u_1(t+s) - u_2(t+s)\| \\ & \leq \|u_1(t+s) - T(t)u_1(s)\| + \|T(t)u_2(s) - u_2(t+s)\| \\ & \quad + \|T(t)u_1(s) - T(t)u_2(s)\| \\ & \leq \sup_{t' \in [0, \infty)} \|u_1(t'+s) - T(t')u_1(s)\| \\ & \quad + \sup_{t' \in [0, \infty)} \|u_2(t'+s) - T(t')u_2(s)\| \\ & \quad + [\|T(t)u_1(s) - T(t)u_2(s)\| - \|u_1(s) - u_2(s)\|] \\ & \quad + \|u_1(s) - u_2(s)\|. \end{aligned}$$

Therefore, by (\*), we get

$$\begin{aligned}
& \limsup_{t \rightarrow \infty} \|u_1(t) - u_2(t)\| \\
&= \limsup_{t \rightarrow \infty} \|u_1(t+s) - u_2(t+s)\| \\
&\leq \sup_{t' \in [0, \infty)} \|u_1(t'+s) - T(t')u_1(s)\| \\
&\quad + \sup_{t' \in [0, \infty)} \|u_2(t'+s) - T(t')u_2(s)\| \\
&\quad + \limsup_{t \rightarrow \infty} \sup_{x, y \in C} (\|T(t)x - T(t)y\| - \|x - y\|) \\
&\quad + \|u_1(s) - u_2(s)\| \\
&\leq \sup_{t' \in [0, \infty)} \|u_1(t'+s) - T(t')u_1(s)\| \\
&\quad + \sup_{t' \in [0, \infty)} \|u_2(t'+s) - T(t')u_2(s)\| \\
&\quad + \|u_1(s) - u_2(s)\|
\end{aligned}$$

for each  $s \in [0, \infty)$ . Next, applying the definition of an almost orbit we obtain

$$\limsup_{t \rightarrow \infty} \|u_1(t) - u_2(t)\| \leq \liminf_{s \rightarrow \infty} \|u_1(s) - u_2(s)\|.$$

Since the constant function  $u_3: [0, \infty) \rightarrow C$  defined by

$$u_3(t) = f_1, \quad t \geq 0,$$

is an almost orbit of  $\mathcal{F}$ , it follows from the above that

$$\lim_{t \rightarrow \infty} \|u_1(t) - f_1\|$$

exists. Moreover, if  $f_1, f_2 \in F(\mathcal{F})$  and  $0 \leq \alpha \leq 1$ , then

$$\begin{aligned}
& \|\alpha u_1(t+s) + (1-\alpha)f_1 - f_2\| \\
&\leq \alpha \|u_1(t+s) - T(t)u_1(s)\| \\
&\quad + \|\alpha T(t)u_1(s) + (1-\alpha)f_1 - T(t)(\alpha u_1(s) + (1-\alpha)f_1)\| \\
&\quad + \|T(t)(\alpha u_1(s) + (1-\alpha)f_1) - f_2\| \\
&\quad - \|\alpha u_1(s) + (1-\alpha)f_1 - f_2\| + \|\alpha u_1(s) + (1-\alpha)f_1 - f_2\|
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha \sup_{t' \in [0, \infty)} \|u_1(t' + s) - T(t')u_1(s)\| \\
&\quad + \|\alpha T(t)u_1(s) + (1 - \alpha)f_1 - T(t)(\alpha u_1(s) + (1 - \alpha)f_1)\| \\
&\quad + \|T(t)(\alpha u_1(s) + (1 - \alpha)f_1) - f_2\| \\
&\quad - \|\alpha u_1(s) + (1 - \alpha)f_1 - f_2\| + \|\alpha u_1(s) + (1 - \alpha)f_1 - f_2\|.
\end{aligned} \tag{3.1}$$

For a given  $\epsilon > 0$ , let  $\delta_{2, \epsilon} > 0$  be taken from Lemma 3.1. There exists  $s_0$  such that for  $s > s_0$  and for an arbitrary  $t$ ,

$$\|u_1(s) - f_1\| - \|T(t)u_1(s) - f_1\| \leq \delta_{2, \epsilon}.$$

Indeed,

$$\begin{aligned}
&\|u_1(s) - f_1\| - \|T(t)u_1(s) - f_1\| \\
&\leq \|u_1(s) - f_1\| - \|u_1(s + t) - f_1\| + \|T(t)u_1(s) - u_1(t + s)\| \\
&\leq \|u_1(s) - f_1\| - \|u_1(s + t) - f_1\| \\
&\quad + \sup_{t' \in [0, \infty)} \|T(t')u_1(s) - u_1(t' + s)\|
\end{aligned}$$

and it is sufficient to apply the definition of an almost orbit and the fact that

$$\lim_{s \rightarrow \infty} \|u_1(s) - f_1\|$$

exists. Returning to (3.1), we observe that for  $s > s_0$  and  $t > t_\epsilon$  (see Lemma 3.1) we have

$$\begin{aligned}
\|\alpha u_1(t + s) + (1 - \alpha)f_1 - f_2\| &\leq \alpha \sup_{t' \in [0, \infty)} \|u_1(t' + s) - T(t')u_1(s)\| \\
&\quad + \|T(t)(\alpha u_1(s) + (1 - \alpha)f_1) - f_2\| \\
&\quad - \|\alpha u_1(s) + (1 - \alpha)f_1 - f_2\| \\
&\quad + \|\alpha u_1(s) + (1 - \alpha)f_1 - f_2\| + \epsilon.
\end{aligned}$$

Hence, by (\*), we get

$$\begin{aligned}
&\limsup_{t \rightarrow \infty} \|\alpha u_1(t) + (1 - \alpha)f_1 - f_2\| \\
&= \limsup_{t \rightarrow \infty} \|\alpha u_1(t + s) + (1 - \alpha)f_1 - f_2\| \\
&\leq \alpha \sup_{t' \in [0, \infty)} \|u_1(t' + s) - T(t')u_1(s)\| \\
&\quad + \|\alpha u_1(s) + (1 - \alpha)f_1 - f_2\| + \epsilon
\end{aligned}$$

for  $s > s_0$ . Therefore directly from the definition of an almost orbit we obtain

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \|\alpha u_1(t) + (1 - \alpha)f_1 - f_2\| \\ & \leq \liminf_{s \rightarrow \infty} \|\alpha u_1(s) + (1 - \alpha)f_1 - f_2\| + \epsilon, \end{aligned}$$

and since  $\epsilon > 0$  is arbitrary we see that

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \|\alpha u_1(t) + (1 - \alpha)f_1 - f_2\| \\ & \leq \liminf_{s \rightarrow \infty} \|\alpha u_1(s) + (1 - \alpha)f_1 - f_2\|, \end{aligned}$$

which yields the claimed result.  $\blacksquare$

To prove the next theorem we need the following lemma also.

LEMMA 3.4. *Let  $X$  be a uniformly convex Banach space and  $\mathcal{T}$  a semi-group of self-mappings of a bounded closed convex subset  $C$  of  $X$  which is asymptotically nonexpansive in the intermediate sense. Suppose that  $u$  is a continuous almost orbit of  $\mathcal{T}$ . Then  $u$  is uniformly continuous on  $[0, \infty)$ .*

*Proof.* Let  $\epsilon > 0$  be arbitrary. One can choose  $s_0 > 0$  such that for all  $u, v \in C$  and  $s \geq s_0$ ,

$$\|T(s)u - T(s)v\| - \|u - v\| < \frac{\epsilon}{4}$$

and

$$\sup_{t \in [0, \infty)} \|u(t + s) - T(t)u(s)\| < \frac{\epsilon}{4}.$$

Next, since  $u$  is uniformly continuous on  $[0, 2s_0 + 1]$  there is a positive  $\delta < 1$  such that

$$\|u(s') - u(s)\| < \frac{\epsilon}{4}$$

for  $s, s' \in [0, 2s_0 + 1]$  with  $|s - s'| < \delta$ . Now suppose  $s, s' \geq 0$  are such that  $0 < s' - s < \delta$ . If  $s \leq 2s_0$ , then  $s' \leq 2s_0 + 1$  and hence

$$\|u(s') - u(s)\| < \frac{\epsilon}{4} < \epsilon.$$

If  $s > 2s_0$ , then

$$\begin{aligned}
\|u(s') - u(s)\| &\leq \|u(s') - T(s - s_0)u(s' - s + s_0)\| \\
&\quad + \|T(s - s_0)u(s' - s + s_0) - T(s - s_0)u(s_0)\| \\
&\quad + \|T(s - s_0)u(s_0) - u(s)\| \\
&< \frac{\epsilon}{4} + \|u(s' - s + s_0) - u(s_0)\| + \frac{\epsilon}{4} + \frac{\epsilon}{4} < \epsilon.
\end{aligned}$$

■

Now we prove an inequality which will play a role similar to that of the  $\gamma$ -inequality in the Bruck method (see also Lemma 2.6 in [31]).

**THEOREM 3.2.** *Let  $X$  be a uniformly convex Banach space and  $\mathcal{T}$  a semigroup of self-mappings of a bounded closed convex subset  $C$  of  $X$  which is asymptotically nonexpansive in the intermediate sense. Suppose that  $u$  is a continuous almost orbit of  $\mathcal{T}$ . For any  $\epsilon > 0$  and  $t > 0$  there exists  $R_{\epsilon,t} > 0$  such that for all  $h \geq R_{\epsilon,t}$ ,  $r \geq R_{\epsilon,t}$ , and  $t > 0$ ,*

$$\left\| T(h) \left( \frac{1}{t} \int_0^t u(r + \tau) d\tau \right) - \frac{1}{t} \int_0^t T(h)u(r + \tau) d\tau \right\| < \epsilon.$$

In particular, for each  $t > 0$  there exists  $r_t > 0$  such that

$$\left\| T(h) \left( \frac{1}{t} \int_0^t u(r + \tau) d\tau \right) - \frac{1}{t} \int_0^t T(h)u(r + \tau) d\tau \right\| < \frac{1}{t} \quad (3.2)$$

for all  $h, r \geq r_t$ .

*Proof.* By Lemma 3.4, for a given  $\epsilon > 0$  there exists

$$0 < \delta < \delta_{\epsilon/6}$$

( $\delta_{\epsilon/6}$  is taken from Lemma 3.3) such that  $\|u(\tau') - u(\tau)\| < \frac{\epsilon}{6}$  if  $|\tau' - \tau| < \delta$ . Let  $0 = \tau_0 < \tau_1 < \dots < \tau_{n_t} = t$  be a partition of  $[0, t]$  with  $n_t \geq 2$  and  $\Delta\tau_i = \tau_i - \tau_{i-1} < \delta$  for  $i = 1, 2, \dots, n_t$ . Then

$$\begin{aligned}
&\left\| \frac{1}{t} \int_0^t u(r + \tau) d\tau - \frac{1}{t} \sum_{i=1}^{n_t} u(r + \tau_i) \Delta\tau_i \right\| \\
&\leq \frac{1}{t} \sum_{i=1}^{n_t} \int_{\tau_{i-1}}^{\tau_i} \|u(r + \tau) - u(r + \tau_i)\| d\tau < \frac{\epsilon}{6}.
\end{aligned}$$

Since the semigroup  $\mathcal{T}$  is assumed to be asymptotically nonexpansive in the intermediate sense, there exists  $R'_\epsilon$  such that

$$\|T(h)v - T(h)w\| < \|v - w\| + \frac{\epsilon}{6}$$

for every  $v, w \in C$  and  $h \geq R'_\epsilon$ . Hence

$$\begin{aligned} & \left\| \left( \frac{1}{t} \sum_{i=1}^{n_i} T(h)(u(r + \tau_i)) \Delta\tau_i \right) - \frac{1}{t} \int_0^t T(h)u(r + \tau) d\tau \right\| \\ & \leq \frac{1}{t} \sum_{i=1}^{n_i} \int_{\tau_{i-1}}^{\tau_i} \|T(h)u(r + \tau_i) - T(h)u(r + \tau)\| d\tau \\ & < \frac{1}{t} \sum_{i=1}^{n_i} \int_{\tau_{i-1}}^{\tau_i} \left( \|u(r + \tau_i) - u(r + \tau)\| + \frac{\epsilon}{6} \right) d\tau < \frac{\epsilon}{3} \end{aligned}$$

for  $h \geq R'_\epsilon$ . Since  $u$  is an almost orbit, there exists  $\tilde{R}_{\epsilon,t} > 0$  such that

$$\sup_{h \geq 0} \|u(r + h) - T(h)u(r)\| < \frac{\delta_{\epsilon/3}}{3}$$

and (see Theorem 3.1)

$$\begin{aligned} & \sup_{h \geq 0} \left| \|u(r + \tau_j) - u(r + \tau_i)\| - \|u(r + h + \tau_j) - u(r + h + \tau_i)\| \right| \\ & < \frac{\delta_{\epsilon/3}}{3} \end{aligned}$$

for  $r \geq \tilde{R}_{\epsilon,t}$  and  $1 \leq i, j \leq n_i$ . It then follows that

$$\begin{aligned} & \left\| u(r + \tau_j) - u(r + \tau_i) \right\| - \left\| T(h)u(r + \tau_j) - T(h)u(r + \tau_i) \right\| \\ & \leq \left\| u(r + \tau_j) - u(r + \tau_i) \right\| - \left\| u(r + h + \tau_j) - u(r + h + \tau_i) \right\| \\ & \quad + \left\| u(r + h + \tau_j) - T(h)u(r + \tau_j) \right\| \\ & \quad + \left\| u(r + h + \tau_i) - T(h)u(r + \tau_i) \right\| \\ & < \delta_{\epsilon/3} \end{aligned}$$

for  $r \geq \tilde{R}_{\epsilon,t}$  and  $0 < i, j \leq n_t$ . This in turn implies (see Lemma 3.3—now we denote  $t_{\epsilon/3}$  by  $R''_\epsilon$ )

$$\left\| T(h) \left( \frac{1}{t} \sum_{i=1}^{n_t} u(r + \tau_i) \Delta \tau_i \right) - \left( \frac{1}{t} \sum_{i=1}^{n_t} T(h)(u(r + \tau_i)) \Delta \tau_i \right) \right\| < \frac{\epsilon}{3}$$

for  $r \geq \tilde{R}_{\epsilon,t}$  and  $h \geq R''_\epsilon$ . Next, we have

$$\begin{aligned} & \left\| T(h) \left( \frac{1}{t} \int_0^t u(r + \tau) d\tau \right) - T(h) \left( \frac{1}{t} \sum_{i=1}^{n_t} u(r + \tau_i) \Delta \tau_i \right) \right\| \\ & \leq \left\| \left( \frac{1}{t} \int_0^t u(r + \tau) d\tau \right) - \left( \frac{1}{t} \sum_{i=1}^{n_t} u(r + \tau_i) \Delta \tau_i \right) \right\| + \frac{\epsilon}{6} \\ & < \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{3} \end{aligned}$$

for  $h \geq R'_\epsilon$ . Finally, we get

$$\begin{aligned} & \left\| T(h) \left( \frac{1}{t} \int_0^t u(r + \tau) d\tau \right) - \frac{1}{t} \int_0^t T(h)u(r + \tau) d\tau \right\| \\ & \leq \left\| T(h) \left( \frac{1}{t} \int_0^t u(r + \tau) d\tau \right) - T(h) \left( \frac{1}{t} \sum_{i=1}^{n_t} u(r + \tau_i) \Delta \tau_i \right) \right\| \\ & \quad + \left\| T(h) \left( \frac{1}{t} \sum_{i=1}^{n_t} u(r + \tau_i) \Delta \tau_i \right) \right. \\ & \quad \left. - \left( \frac{1}{t} \sum_{i=1}^{n_t} T(h)(u(r + \tau_i)) \Delta \tau_i \right) \right\| \\ & \quad + \left\| \left( \frac{1}{t} \sum_{i=1}^{n_t} T(h)(u(r + \tau_i)) \Delta \tau_i \right) - \frac{1}{t} \int_0^t T(h)u(r + \tau) d\tau \right\| \\ & < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

for all  $h \geq R_{\epsilon,t}$  and  $r \geq R_{\epsilon,t}$ , where  $R_{\epsilon,t} = \max(R'_\epsilon, R''_\epsilon, \tilde{R}_{\epsilon,t})$ . To get the second inequality it is sufficient to set

$$r_t = R_{1/t,t}.$$

■

*Remark 3.1.* Without loss of generality (see the proof of Theorem 3.2) we will assume in the following that

$$\lim_{t \rightarrow \infty} r_t = \infty.$$

Let us observe that in place of  $r_t$  we can take arbitrary  $r'_t \geq r_t$ .

The theorem just established allows us to prove the next four lemmas which are crucial in the proof of our main theorem. The common assumptions in this sequence of lemmas are as follows:  $X$  is a uniformly convex Banach space,  $C$  is a nonempty bounded closed convex subset of  $X$ ,  $\mathcal{T}$  is a one-parameter semigroup of self-mappings of  $C$  which is asymptotically nonexpansive in the intermediate sense,  $u$  is a continuous almost orbit of  $\mathcal{T}$ , and  $\{r_t\}_{t > 0}$  is the family of positive real numbers appearing in Theorem 3.2.

**LEMMA 3.5.** *For each  $f \in F(\mathcal{T})$  and any family  $\{r'_t\}$  of real numbers such that  $r'_t \geq r_t$  for  $t > 0$ ,*

$$\lim_{t \rightarrow \infty} \left\| \frac{1}{t} \int_0^t u(r'_t + \tau) d\tau - f \right\|$$

*exists and*

$$\lim_{t \rightarrow \infty} \left\| \frac{1}{t} \int_0^t u(r'_t + \tau) d\tau - f \right\| = \lim_{t \rightarrow \infty} \left\| \frac{1}{t} \int_0^t u(r_t + \tau) d\tau - f \right\|.$$

*Proof.* We have the identity ([27])

$$\frac{1}{t} \int_0^t u(\tau + h) d\tau = \frac{1}{t} \int_0^t \left( \frac{1}{p} \int_0^p u(\tau + \eta + h) d\eta \right) d\tau + g(t, p, h), \quad (3.3)$$

where

$$g(t, p, h) = \frac{1}{tp} \int_0^p (p - \eta) [u(\eta + h) - u(\eta + h + t)] d\eta$$

for  $t, p > 0$  and  $h \geq 0$ . This implies that

$$\begin{aligned} \frac{1}{t+s} \int_0^{t+s} u(r'_{t+s} + \tau) d\tau &= \frac{1}{t+s} \int_0^{t+s} \left( \frac{1}{s} \int_0^s u(\tau + \eta + r'_{t+s}) d\eta \right) d\tau \\ &\quad + g(t+s, s, r'_{t+s}) \end{aligned}$$

for  $t, s > 0$ . Assume now that

$$t > r'_s.$$

If

$$M = \sup_{x \in C} \|x\|,$$

then

$$\|g(t+s, s, r'_{t+s})\| \leq \frac{2M}{s(t+s)} \int_0^s (s-\eta) d\eta = \frac{sM}{t+s} \xrightarrow{t \rightarrow \infty} 0.$$

Next, we observe that

$$\begin{aligned} & \left\| \frac{1}{t+s} \int_0^{t+s} \left( \frac{1}{s} \int_0^s u(\tau + \eta + r'_{t+s}) d\eta \right) d\tau - f \right\| \\ & \leq \left\| \frac{1}{t+s} \int_0^{r'_s} \left[ \frac{1}{s} \int_0^s (u(\tau + \eta + r'_{t+s}) - f) d\eta \right] d\tau \right\| \\ & \quad + \left\| \frac{1}{t+s} \int_{r'_s}^{t+s} \left[ \frac{1}{s} \int_0^s (u(\tau + \eta + r'_{t+s}) - f) d\eta \right] d\tau \right\| \\ & = I_1 + I_2. \end{aligned}$$

One can easily see that

$$\begin{aligned} I_1 &= \left\| \frac{1}{t+s} \int_0^{r'_s} \left[ \frac{1}{s} \int_0^s (u(\tau + \eta + r'_{t+s}) - f) d\eta \right] d\tau \right\| \\ &\leq \frac{2Mr'_s}{t+s} \xrightarrow{t \rightarrow \infty} 0. \end{aligned}$$

For  $\tau \geq r'_s$ , set

$$h_{t,s} = r'_{t+s} - r'_s + \tau.$$

Consequently,

$$h_{t,s} \geq r'_{t+s}$$

and

$$h_{t,s} \geq r'_{t+s} \xrightarrow{t \rightarrow \infty} \infty, \quad h_{t,s} \geq r'_{t+s} \xrightarrow{s \rightarrow \infty} \infty \quad (3.4)$$

uniformly in  $s$  (in  $t$ , respectively). Fix  $\epsilon > 0$ . By (\*) and (3.4), there exists  $\bar{s} > 0$  such that for all  $s > \bar{s}$  and for every  $v, w \in C$ ,

$$\frac{1}{s} < \frac{\epsilon}{2} \quad (3.5)$$

and

$$\|T(h_{t,s})v - T(h_{t,s})w\| < \|v - w\| + \frac{\epsilon}{2}. \quad (3.6)$$

Now we fix  $s > \tilde{s}$  for a moment. By (3.4) there exists  $\tilde{t} > 0$  such that

$$h_{t,s} > r'_s$$

for all  $t > \tilde{t}$ . Therefore, by (3.2), (3.5), and (3.6),

$$\begin{aligned} & \left\| \frac{1}{s} \int_0^s (u(\tau + \eta + r'_{t+s}) - f) d\eta \right\| \\ & \leq \left\| \frac{1}{s} \int_0^s u(h_{t,s} + r'_s + \eta) d\eta - \left( \frac{1}{s} \int_0^s T(h_{t,s})u(r'_s + \eta) d\eta \right) \right\| \\ & \quad + \left\| \frac{1}{s} \int_0^s T(h_{t,s})u(r'_s + \eta) d\eta - T(h_{t,s}) \left( \frac{1}{s} \int_0^s u(r'_s + \eta) d\eta \right) \right\| \\ & \quad + \left\| T(h_{t,s}) \left( \frac{1}{s} \int_0^s u(r'_s + \eta) d\eta \right) - f \right\| \\ & \leq \frac{1}{s} \int_0^s \|u(h_{t,s} + r'_s + \eta) - T(h_{t,s})u(r'_s + \eta)\| d\eta \\ & \quad + \left\| \frac{1}{s} \int_0^s T(h_{t,s})u(r'_s + \eta) d\eta - T(h_{t,s}) \left( \frac{1}{s} \int_0^s u(r'_s + \eta) d\eta \right) \right\| \\ & \quad + \left\| \left( \frac{1}{s} \int_0^s u(r'_s + \eta) d\eta \right) - f \right\| + \frac{\epsilon}{2} \\ & \leq \frac{1}{s} \int_0^s \|u(h_{t,s} + r'_s + \eta) - T(h_{t,s})u(r'_s + \eta)\| d\eta + \frac{1}{s} \\ & \quad + \left\| \left( \frac{1}{s} \int_0^s u(r'_s + \eta) d\eta \right) - f \right\| + \frac{\epsilon}{2} \\ & \leq \frac{1}{s} \int_0^s \|u(h_{t,s} + r'_s + \eta) - T(h_{t,s})u(r'_s + \eta)\| d\eta \\ & \quad + \left\| \left( \frac{1}{s} \int_0^s u(r'_s + \eta) d\eta \right) - f \right\| + \epsilon \\ & \leq \sup_{h \geq 0, r \geq r'_s} \|u(h + r) - T(h)u(r)\| \\ & \quad + \left\| \left( \frac{1}{s} \int_0^s u(r'_s + \eta) d\eta \right) - f \right\| + \epsilon \end{aligned}$$

for  $\tau \geq r'_s$  and for all  $t > \tilde{t}$ . Hence we see that

$$\begin{aligned} \limsup_{t \rightarrow \infty} I_2 &= \limsup_{t \rightarrow \infty} \left\| \frac{1}{t+s} \int_{r'_s}^{t+s} \left[ \frac{1}{s} \int_0^s (u(\tau + \eta + r'_{t+s}) - f) d\eta \right] d\tau \right\| \\ &\leq \sup_{h \geq 0, r \geq r'_s} \|u(h+r) - T(h)u(r)\| \\ &\quad + \left\| \left( \frac{1}{s} \int_0^s u(r'_s + \eta) d\eta \right) - f \right\| + \epsilon \end{aligned}$$

for all  $s > \tilde{s}$ . Consequently, for all  $s > \tilde{s}$  we have

$$\begin{aligned} \limsup_{t \rightarrow \infty} &\left\| \frac{1}{t} \int_0^t u(r'_t + \tau) d\tau - f \right\| \\ &= \limsup_{t \rightarrow \infty} \left\| \frac{1}{t+s} \int_0^{t+s} u(r'_{t+s} + \tau) d\tau - f \right\| \\ &\leq \sup_{h \geq 0, r \geq r'_s} \|u(h+r) - T(h)u(r)\| \\ &\quad + \left\| \left( \frac{1}{s} \int_0^s u(r'_s + \eta) d\eta \right) - f \right\| + \epsilon. \end{aligned}$$

Now, it is sufficient to observe that the definition of an almost orbit implies that

$$\sup_{h \geq 0, r \geq r'_s} \|u(h+r) - T(h)u(r)\| \xrightarrow{s \rightarrow \infty} 0$$

and therefore

$$\limsup_{t \rightarrow \infty} \left\| \frac{1}{t} \int_0^t u(r'_t + \tau) d\tau - f \right\| \leq \liminf_{s \rightarrow \infty} \left\| \left( \frac{1}{s} \int_0^s u(r'_s + \eta) d\eta \right) - f \right\| + \epsilon.$$

Since  $\epsilon > 0$  is arbitrary, this concludes the proof of Lemma 3.5.  $\blacksquare$

**COROLLARY 3.1.** *For each  $f \in F(\mathcal{F})$  and every  $\epsilon > 0$  there exists  $t_0 > 0$  such that*

$$\left\| \left\| \frac{1}{t} \int_0^t u(\eta + r'_t) d\eta - f \right\| - \lim_{l \rightarrow \infty} \left\| \frac{1}{l} \int_0^l u(\eta + r_l) d\eta - f \right\| \right\| < \epsilon$$

for all  $t > t_0$  and all families  $\{r'_t\}$  of real numbers with  $r'_t \geq r_t$  for  $t > 0$ .

**LEMMA 3.6.** *For every  $h > 0$  and  $r \geq 0$ ,*

$$\lim_{t \rightarrow \infty} \left\| \frac{1}{t} \int_0^t T(h)u(r + \tau) d\tau - \frac{1}{t} \int_0^t u(r + \tau) d\tau \right\| = 0.$$

*Proof.* Let

$$M = \sup_{x \in C} \|x\|.$$

For fixed  $h > 0$ ,  $r \geq 0$ , and  $t > h$  we have

$$\begin{aligned}
 & \left\| \frac{1}{t} \int_0^t T(h)u(r + \tau) d\tau - \frac{1}{t} \int_0^t u(r + \tau) d\tau \right\| \\
 & \leq \frac{1}{t} \int_0^t \|T(h)u(r + \tau) - u(h + r + \tau)\| d\tau \\
 & \quad + \frac{1}{t} \left\| \int_0^t u(h + r + \tau) d\tau - \int_0^t u(r + \tau) d\tau \right\| \\
 & \leq \frac{1}{t} \int_0^t \left( \sup_{\tilde{h} \geq 0} \|T(\tilde{h})u(r + \tau) - u(\tilde{h} + r + \tau)\| \right) d\tau \\
 & \quad + \frac{1}{t} \left\| \int_0^h [u(t + r + \tau) - u(r + \tau)] d\tau \right\| \\
 & \leq \frac{1}{t} \int_0^t \left( \sup_{\tilde{h} \geq 0} \|T(\tilde{h})u(r + \tau) - u(\tilde{h} + r + \tau)\| \right) d\tau + \frac{2hM}{t} \xrightarrow{t \rightarrow \infty} 0
 \end{aligned}$$

and the proof is complete.  $\blacksquare$

LEMMA 3.7. For  $h > 0$  and any family  $\{r'_i\}$  of real numbers such that  $r'_i \geq r_i$  for  $t > 0$ ,

$$\lim_{t \rightarrow \infty} \left\| \frac{1}{t} \int_0^t T(h)u(r'_i + \tau) d\tau - \frac{1}{t} \int_0^t u(r'_i + \tau) d\tau \right\| = 0.$$

*Proof.* Indeed, we have

$$\begin{aligned}
 & \left\| \frac{1}{t} \int_0^t T(h)u(r'_i + \tau) d\tau - \frac{1}{t} \int_0^t u(r'_i + \tau) d\tau \right\| \\
 & \leq \frac{1}{t} \int_0^t \|T(h)u(r'_i + \tau) - u(h + r'_i + \tau)\| d\tau \\
 & \quad + \frac{1}{t} \left\| \int_0^t u(h + r'_i + \tau) d\tau - \int_0^t u(r'_i + \tau) d\tau \right\| \\
 & \leq \frac{1}{t} \int_0^t \left( \sup_{\tilde{h} \geq 0} \|T(\tilde{h})u(r'_i + \tau) - u(\tilde{h} + r'_i + \tau)\| \right) d\tau \\
 & \quad + \frac{1}{t} \left\| \int_0^h [u(t + r'_i + \tau) - u(r'_i + \tau)] d\tau \right\| \\
 & \leq \frac{1}{t} \int_0^t \left( \sup_{\tilde{h} \geq 0} \|T(\tilde{h})u(r'_i + \tau) - u(\tilde{h} + r'_i + \tau)\| \right) d\tau + \frac{2hM}{t} \xrightarrow{t \rightarrow \infty} 0.
 \end{aligned}$$

$\blacksquare$

LEMMA 3.8. For each  $f_1, f_2 \in F(\mathcal{F})$  and  $0 \leq \alpha \leq 1$ ,

$$\lim_{t \rightarrow \infty} \left\| \alpha \frac{1}{t} \int_0^t u(r'_t + \tau) d\tau + (1 - \alpha)f_1 - f_2 \right\|$$

exists for every family  $\{r'_t\}$  of real numbers such that  $r'_t \geq r_t$  for  $t \geq 0$ .

*Proof.* We will apply once again the following identity ([27]):

$$\frac{1}{t} \int_0^t u(\tau + h) d\tau = \frac{1}{t} \int_0^t \left( \frac{1}{p} \int_0^p u(\tau + \eta + h) d\eta \right) d\tau + g(t, p, h),$$

where

$$g(t, p, h) = \frac{1}{tp} \int_0^p (p - \eta) [u(\eta + h) - u(\eta + h + t)] d\eta$$

for  $t, p > 0$  and  $h \geq 0$ . Using this identity we see that

$$\begin{aligned} & \frac{1}{t+s} \int_0^{t+s} u(\tau + r'_{t+s}) d\tau \\ &= \frac{1}{t+s} \int_0^{t+s} \left( \frac{1}{s} \int_0^s u(\tau + \eta + r'_{t+s}) d\eta \right) d\tau + g(t+s, s, r'_{t+s}). \end{aligned}$$

Assume that

$$t > r'_s.$$

Setting

$$M = \sup_{x \in C} \|x\|,$$

we get

$$\|g(t+s, s, r'_{t+s})\| \leq \frac{2M}{s(t+s)} \int_0^s (s - \eta) d\eta = \frac{sM}{t+s} \xrightarrow{t \rightarrow \infty} 0.$$

Next, we have

$$\begin{aligned} & \left\| \alpha \frac{1}{t+s} \int_0^{t+s} \left( \frac{1}{s} \int_0^s u(\tau + \eta + r'_{t+s}) d\eta \right) d\tau + (1 - \alpha)f_1 - f_2 \right\| \\ & \leq \left\| \frac{1}{t+s} \int_0^{r'_s} \left( \frac{1}{s} \int_0^s [\alpha u(\tau + \eta + r'_{t+s}) + (1 - \alpha)f_1 - f_2] d\eta \right) d\tau \right\| \\ & \quad + \left\| \frac{1}{t+s} \int_{r'_s}^{t+s} \left( \frac{1}{s} \int_0^s [\alpha u(\tau + \eta + r'_{t+s}) + (1 - \alpha)f_1 - f_2] d\eta \right) d\tau \right\| \\ & = I_1 + I_2. \end{aligned}$$

Let us note that

$$I_1 \leq \frac{r'_s}{t+s} (M + \|f_1\| + \|f_2\|) \xrightarrow{t \rightarrow \infty} 0.$$

Assume now that  $\tau \geq r'_s$  and set

$$h_{t,s} = r'_{t+s} - r'_s + \tau.$$

Then

$$h_{t,s} \geq r'_{t+s}$$

and therefore

$$h_{t,s} \geq r'_{t+s} \xrightarrow{t \rightarrow \infty} \infty, \quad h_{t,s} \geq r'_{t+s} \xrightarrow{s \rightarrow \infty} \infty \quad (3.7)$$

uniformly in  $s$  (in  $t$ , respectively). Next we define for  $s > 0$ ,

$$\Psi(s) = \sup\{\|u(h+r) - T(h)u(r)\| : h \geq 0, r \geq r_s\}.$$

It is obvious that

$$\Psi(s) \xrightarrow{s \rightarrow \infty} 0. \quad (3.8)$$

Let us fix  $\epsilon > 0$  and let  $0 < \delta < \min(\delta_{\epsilon/2}, \frac{\epsilon}{3})$ , where  $\delta_{\epsilon/2}$  is as in Lemma 3.3. By Corollary 3.1 there exists  $\tilde{s}$  such that for all  $s > \tilde{s}$ ,

$$\left\| \left\| \frac{1}{s} \int_0^s u(\eta + r'_s) d\eta - f_1 \right\| - \lim_{l \rightarrow \infty} \left\| \frac{1}{l} \int_0^l u(\eta + r_l) d\eta - f_1 \right\| \right\| < \frac{\delta}{3} \quad (3.9)$$

and

$$\left\| \left\| \frac{1}{s} \int_0^s u(h_{t,s} + \eta + r'_s) d\eta - f_1 \right\| - \lim_{l \rightarrow \infty} \left\| \frac{1}{l} \int_0^l u(\eta + r_l) d\eta - f_1 \right\| \right\| < \frac{\delta}{3}. \quad (3.10)$$

Next, without loss of generality, by (3.8) we can also assume that for  $s > \tilde{s}$ ,

$$\frac{1}{s} + \Psi(s) < \frac{\delta}{3}. \quad (3.11)$$

Fix  $s > \tilde{s}$  for a moment. By (\*) and (3.7) there exists  $\tilde{t} > 0$  such that

$$h_{t,s} > r'_s$$

and

$$\|T(h_{t,s})v - T(h_{t,s})w\| < \|v - w\| + \frac{\epsilon}{3}$$

for every  $v, w \in C$  and  $t > \tilde{t}$ . Consequently, by (3.9), (3.10), and (3.11) for this  $s$  and each  $t > \tilde{t}$  we obtain

$$\begin{aligned}
& \left\| \frac{1}{s} \int_0^s u(\eta + r'_s) d\eta - f_1 \right\| - \left\| T(h_{t,s}) \left( \frac{1}{s} \int_0^s u(\eta + r'_s) d\eta \right) - T(h_{t,s}) f_1 \right\| \\
& \leq \left\| \frac{1}{s} \int_0^s u(\eta + r'_s) d\eta - f_1 \right\| - \lim_{l \rightarrow \infty} \left\| \frac{1}{l} \int_0^l u(\eta + r_l) d\eta - f_1 \right\| \\
& \quad + \left\| T(h_{t,s}) \left( \frac{1}{s} \int_0^s u(\eta + r'_s) d\eta \right) - \frac{1}{s} \int_0^s T(h_{t,s}) u(\eta + r'_s) d\eta \right\| \\
& \quad + \frac{1}{s} \int_0^s \| T(h_{t,s}) u(\eta + r'_s) - u(h_{t,s} + \eta + r'_s) \| d\eta \\
& \quad + \lim_{l \rightarrow \infty} \left\| \frac{1}{l} \int_0^l u(\eta + r_l) d\eta - f_1 \right\| - \left\| \frac{1}{s} \int_0^s u(h_{t,s} + \eta + r'_s) d\eta - f_1 \right\| \\
& \leq \frac{\delta}{3} + \frac{1}{s} + \Psi(s) + \frac{\delta}{3} < \delta < \delta_{\epsilon/2}.
\end{aligned}$$

Next, applying Theorem 3.2, Lemma 3.3, and the above conditions and inequalities, we arrive at

$$\begin{aligned}
& \left\| \alpha \frac{1}{s} \int_0^s u(\tau + \eta + r'_{t+s}) d\eta + (1 - \alpha) f_1 - f_2 \right\| \\
& = \left\| \alpha \frac{1}{s} \int_0^s u(h_{t,s} + \eta + r'_s) d\eta + (1 - \alpha) f_1 - f_2 \right\| \\
& \leq \alpha \frac{1}{s} \int_0^s \| u(h_{t,s} + \eta + r'_s) - T(h_{t,s}) u(\eta + r'_s) \| d\eta \\
& \quad + \alpha \left\| \frac{1}{s} \int_0^s T(h_{t,s}) u(\eta + r'_s) d\eta - T(h_{t,s}) \left( \frac{1}{s} \int_0^s u(\eta + r'_s) d\eta \right) \right\| \\
& \quad + \left\| \alpha T(h_{t,s}) \left( \frac{1}{s} \int_0^s u(\eta + r'_s) d\eta \right) + (1 - \alpha) T(h_{t,s}) f_1 \right. \\
& \quad \quad \left. - T(h_{t,s}) \left( \alpha \frac{1}{s} \int_0^s u(\eta + r'_s) d\eta + (1 - \alpha) f_1 \right) \right\| \\
& \quad + \left\| T(h_{t,s}) \left( \alpha \frac{1}{s} \int_0^s u(\eta + r'_s) d\eta + (1 - \alpha) f_1 \right) - f_2 \right\| \\
& \leq \alpha \Psi(s) + \alpha \frac{1}{s} + \frac{\epsilon}{2} + \left\| \left( \alpha \frac{1}{s} \int_0^s u(\eta + r'_s) d\eta + (1 - \alpha) f_1 \right) - f_2 \right\| \\
& \quad + \frac{\epsilon}{2}.
\end{aligned}$$

It then follows that

$$\begin{aligned} \limsup_{t \rightarrow \infty} I_2 &= \limsup_{t \rightarrow \infty} \left\| \frac{1}{t+s} \int_{r'_s}^{t+s} \left( \frac{1}{s} \int_0^s [\alpha u(\tau + \eta + r'_{t+s}) \right. \right. \\ &\quad \left. \left. + (1 - \alpha)f_1 - f_2] d\eta \right) d\tau \right\| \\ &\leq \alpha \Psi(s) + \alpha \frac{1}{s} + \epsilon + \left\| \alpha \frac{1}{s} \int_0^s u(\eta + r'_s) d\eta + (1 - \alpha)f_1 - f_2 \right\| \end{aligned}$$

for  $s > \bar{s}$  and hence

$$\begin{aligned} \limsup_{t \rightarrow \infty} \left\| \alpha \frac{1}{t} \int_0^t u(\eta + r'_t) d\eta + (1 - \alpha)f_1 - f_2 \right\| \\ \leq \liminf_{s \rightarrow \infty} \left\| \alpha \frac{1}{s} \int_0^s u(\eta + r'_s) d\eta + (1 - \alpha)f_1 - f_2 \right\| + \epsilon. \end{aligned}$$

Since  $\epsilon > 0$  is arbitrary, we obtain

$$\begin{aligned} \limsup_{t \rightarrow \infty} \left\| \alpha \frac{1}{t} \int_0^t u(\eta + r'_t) d\eta + (1 - \alpha)f_1 - f_2 \right\| \\ \leq \liminf_{s \rightarrow \infty} \left\| \alpha \frac{1}{s} \int_0^s u(\eta + r'_s) d\eta + (1 - \alpha)f_1 - f_2 \right\|, \end{aligned}$$

which concludes the proof of Lemma 3.8. ■

Directly from Lemma 3.8 we arrive at the following corollary.

**COROLLARY 3.2.** *For each  $f_1, f_2 \in F(\mathcal{F})$  and  $0 \leq \alpha \leq 1$ ,*

$$\begin{aligned} \lim_{t \rightarrow \infty} \left\| \alpha \frac{1}{t} \int_0^t u(r'_t + \tau) d\tau + (1 - \alpha)f_1 - f_2 \right\| \\ = \lim_{t \rightarrow \infty} \left\| \alpha \frac{1}{t} \int_0^t u(r_t + \tau) d\tau + (1 - \alpha)f_1 - f_2 \right\| \end{aligned}$$

for all families  $\{r'_t\}$  of real numbers such that  $r'_t \geq r_t$  for  $t > 0$ .

Finally, we recall that F. E. Browder [4] proved the demiclosedness principle for nonexpansive mappings in uniformly convex Banach spaces. We now extend Browder's principle to nonlinear semigroups which are asymptotically nonexpansive in the intermediate sense (see also [27] and [30]).

**THEOREM 3.3.** *Let  $X$  be a uniformly convex Banach space,  $C$  a closed bounded convex subset of  $X$ , and  $\mathcal{T}$  a semigroup of self-mappings of  $C$  which*

is asymptotically nonexpansive in the intermediate sense. If  $\{x_\beta\}_{\beta \in \Lambda}$  is a net in  $C$  converging weakly to  $x$  and if

$$\lim_{t \rightarrow \infty} \left( \limsup_{\beta \in \Lambda} \|x_\beta - T(t)x_\beta\| \right) = 0,$$

then  $T(t)x = x$  for all  $t \in [0, \infty)$  (in other words,  $x \in F(\mathcal{T})$ ).

*Proof.* We shall show that

$$\lim_{t \rightarrow \infty} \|T(t)x - x\| = 0.$$

We choose  $t_{\epsilon/5}$  and  $\delta_{\epsilon/5}$  as in Lemma 3.3. By (\*) and by our hypothesis there exists  $t_1(\epsilon)$  such that if  $t \geq t_1(\epsilon)$  then

$$\|T(t)u - T(t)v\| - \|u - v\| < \frac{\epsilon}{5}$$

for all  $u, v \in C$  and

$$\limsup_{\beta \in \Lambda} \|x_\beta - T(t)x_\beta\| < \frac{1}{2}\delta_{\epsilon/5}.$$

Hence there exists  $\beta_{\epsilon, t} \in \Lambda$  such that

$$\|x_{\beta_{\epsilon, t}} - T(t)x_{\beta_{\epsilon, t}}\| < \frac{1}{2}\delta_{\epsilon/5}$$

for  $\beta \geq \beta_{\epsilon, t}$ . Put  $\epsilon' = \min(\frac{1}{2}\delta_{\epsilon/5}, \frac{\epsilon}{5})$ . Now we can also take  $t_{\epsilon'} > 0$ . Let  $t_2(\epsilon) = \max(t_{\epsilon/5}, t_1(\epsilon), t_{\epsilon'})$  and let  $t \geq t_2(\epsilon)$ . Since  $x \in \text{clco}\{x_\beta : \beta \geq \beta_{\epsilon, t}\}$ , there exists a sequence

$$\left\{ \sum_{i=1}^{l_n} \lambda_{n,i} x_{\beta(n,i)} \right\} \in \text{co}\{x_\beta : \beta \geq \beta_{\epsilon, t}\}$$

such that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{l_n} \lambda_{n,i} x_{\beta(n,i)} = x.$$

Since

$$\begin{aligned} & \|x_{\beta(n,i)} - x_{\beta(n,j)}\| - \|T(t)x_{\beta(n,i)} - T(t)x_{\beta(n,j)}\| \\ & \leq \|x_{\beta(n,i)} - T(t)x_{\beta(n,i)}\| + \|x_{\beta(n,j)} - T(t)x_{\beta(n,j)}\| \leq \delta_{\epsilon/5} \end{aligned}$$

for  $1 \leq i, j \leq l_n$ , Lemma 3.3 implies that

$$\left\| T(t) \left( \sum_{i=1}^{l_n} \lambda_{n,i} x_{\beta(n,i)} \right) - \sum_{i=1}^{l_n} \lambda_{n,i} T(t)x_{\beta(n,i)} \right\| < \frac{\epsilon}{5}.$$

There is also  $N_{t, \epsilon} \geq 1$  such that

$$\left\| \sum_{i=1}^{l_n} \lambda_{n,i} x_{\beta(n,i)} - x \right\| < \frac{\epsilon}{5}$$

for all  $n \geq N_{t, \epsilon}$ . Since  $x \in C$ , the combination of the above inequalities gives

$$\begin{aligned} \|T(t)x - x\| &\leq \left\| T(t)x - T(t) \left( \sum_{i=1}^{l_n} \lambda_{n,i} x_{\beta(n,i)} \right) \right\| \\ &\quad + \left\| T(t) \left( \sum_{i=1}^{l_n} \lambda_{n,i} x_{\beta(n,i)} \right) - \sum_{i=1}^{l_n} \lambda_{n,i} T(t)x_{\beta(n,i)} \right\| \\ &\quad + \left\| \sum_{i=1}^{l_n} \lambda_{n,i} (T(t)x_{\beta(n,i)} - x_{\beta(n,i)}) \right\| \\ &\quad + \left\| \sum_{i=1}^{l_n} \lambda_{n,i} x_{\beta(n,i)} - x \right\| < \epsilon \end{aligned}$$

whenever  $n \geq N_{t, \epsilon}$  and  $t \geq t_2(\epsilon)$ . This shows that

$$\|T(t)x - x\| < \epsilon$$

for  $t \geq t_2(\epsilon)$  and therefore

$$\lim_{t \rightarrow \infty} T(t)x = x.$$

By continuity of the semigroup  $\mathcal{T}$  we get

$$T(s)x = T(s) \left( \lim_{t \rightarrow \infty} T(t)x \right) = \lim_{t \rightarrow \infty} T(s)T(t)x = \lim_{t \rightarrow \infty} T(s+t)x = x$$

for each  $s \geq 0$ . ■

#### 4. MAIN RESULT

Our main result stems from the following lemma proved in [10]. The idea of this lemma and its proof are due to J. García Falset.

LEMMA 4.1. *Let  $X$  be a uniformly convex Banach space such that its dual  $X^*$  has the KK-property. Suppose  $\{x_n\}$  is a bounded sequence such that*

$$\lim_{n \rightarrow \infty} \|\alpha x_n + (1 - \alpha)f_1 - f_2\|$$

*exists for all  $\alpha \in [0, 1]$  and  $f_1, f_2 \in \omega_w(\{x_n\})$ . Then  $\omega_w(\{x_n\})$  is a singleton.*

Let  $C$  be a nonempty bounded subset of a uniformly convex Banach space  $X$ . Following Lorentz [19], we say that a continuous function  $u: [0, \infty) \rightarrow C$  is weakly almost convergent to some element  $y \in X$  if

$$w\text{-}\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t u(h + \tau) d\tau = y$$

uniformly in  $h \geq 0$  (cf. [26]).

**THEOREM 4.1.** *Let  $X$  be a uniformly convex Banach space such that  $X^*$  has the KK-property,  $C$  a bounded closed convex subset of  $X$ , and  $\mathcal{T} = \{T(t) : t \in [0, \infty)\}$  a one-parameter semigroup of self-mappings of  $C$  which is asymptotically nonexpansive in the intermediate sense. Then every continuous almost orbit  $u$  of  $\mathcal{T}$  is weakly almost convergent to some  $y \in F(\mathcal{T})$ .*

*Proof.* First, by Lemma 3.7 and Theorems 3.2 and 3.3 each weak subsequential limit of  $\{\frac{1}{t} \int_0^t u(r'_i + \tau) d\tau\}$  is a fixed point of  $\mathcal{T}$ . Next, by Corollary 3.2 and Lemma 4.1 we see that  $\{\frac{1}{t} \int_0^t u(r'_i + \tau) d\tau\}$  is weakly convergent to some  $y \in F(\mathcal{T})$  which is independent of the choice of  $\{r'_i\}$ . Therefore

$$w\text{-}\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t u(r_i + h + \tau) d\tau = y$$

uniformly in  $h \geq 0$ . To conclude the proof it is sufficient to apply the decomposition

$$\begin{aligned} & \frac{1}{s} \int_0^s u(h + \tau) d\tau \\ &= \frac{1}{s} \left\{ \int_0^{r_i} u(h + \tau) d\tau + \int_{r_i}^{at+r_i} u(h + \tau) d\tau + \int_{at+r_i}^s u(h + \tau) d\tau \right\} \\ &= \frac{1}{s} \left\{ \int_0^{r_i} u(h + \tau) d\tau + \sum_{j=0}^{a-1} \int_0^t u(r_i + jt + h + \tau) d\tau \right. \\ & \qquad \qquad \qquad \left. + \int_{at+r_i}^s u(h + \tau) d\tau \right\}, \end{aligned}$$

where  $s = at + r_i + b$ ,  $0 \leq b < t$ . ■

5. THE DISCRETE CASE

In this section we present an analog of Theorem 4.1 for the case of a single mapping. First we recall a few definitions.

Let  $C$  be a nonempty subset of a Banach space  $X$ . If  $T: C \rightarrow C$  is continuous and

$$\limsup_{n \rightarrow \infty} \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0,$$

then  $T$  is said to be asymptotically nonexpansive in the intermediate sense [9].

If  $T$  is a self-mapping of  $C$ , then  $F(T)$  will always denote the set of fixed points of  $T$ .

Recall that if  $X$  is a uniformly convex Banach space and  $T$  is a self-mapping of a bounded closed convex subset  $C$  of  $X$  which is asymptotically nonexpansive in the intermediate sense, then  $F(T) \neq \emptyset$  ([16]).

A sequence  $\{x_m\}_{m \geq 0}$  in  $C$  is called an almost orbit of a mapping  $T: C \rightarrow C$  if

$$\lim_{m \rightarrow \infty} \left[ \sup_{k \geq 0} \|x_{m+k} - T^k x_m\| \right] = 0$$

[5].

A sequence  $\{x_m\}_{m \geq 0}$  in a Banach space  $X$  is said to be weakly almost convergent to  $x \in X$  if  $\frac{1}{n} \sum_{i=0}^{n-1} x_{i+k}$  converges weakly, as  $n \rightarrow \infty$ , to  $x$ , uniformly in  $k$  ([5, 19, 25]).

To prove a mean ergodic theorem for a mapping which is asymptotically nonexpansive in the intermediate sense we need to modify in a suitable way all the lemmas and theorems from Sections 3 and 4. Moreover, in place of (3.3) we have to use the equality ([17])

$$\frac{1}{n} \sum_{i=0}^{n-1} x_{i+k} = \frac{1}{n} \sum_{i=0}^{n-1} \left( \frac{1}{p} \sum_{j=0}^{p-1} x_{i+j+k} \right) + g(n, p, k),$$

where

$$g(n, p, k) = \frac{1}{np} \sum_{j=1}^{p-1} (p-j)(x_{j+k-1} - x_{j+k+n-1}).$$

As an example we will present the proof of an analog of Theorem 3.2. First we recall two known lemmas.

LEMMA 5.1 ([15, 22]). *For each  $\epsilon > 0$  there exist  $N_\epsilon \geq 1$  and  $\delta_{2,\epsilon} > 0$  such that if  $k \geq N_\epsilon$ ,  $z_1, z_2 \in C$ , and  $\|z_1 - z_2\| - \|T^k z_1 - T^k z_2\| \leq \delta_{2,\epsilon}$ , then*

$$\|T^k(\lambda_1 z_1 + \lambda_2 z_2) - \lambda_1 T^k z_1 - \lambda_2 T^k z_2\| < \epsilon$$

for all  $\lambda = (\lambda_1, \lambda_2) \in \Delta^1$ .

LEMMA 5.2 ([22]). *For each  $\epsilon > 0$  and for each integer  $n \geq 2$  there exist  $N_\epsilon > 0$  and  $\delta_\epsilon > 0$ , where both  $N_\epsilon$  and  $\delta_\epsilon$  are independent of  $n$ , such that if  $k \geq N_\epsilon$ ,  $z_1, z_2, \dots, z_n \in C$ , and  $\|z_i - z_j\| - \|T^k z_i - T^k z_j\| \leq \delta_\epsilon$  for  $1 \leq i, j \leq n$ , then*

$$\left\| T^k \left( \sum_{i=1}^n \lambda_i z_i \right) - \sum_{i=1}^n \lambda_i T^k z_i \right\| < \epsilon$$

for all  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \Delta^{n-1}$ .

We will also apply the following theorem.

THEOREM 5.1 ([22]). *Let  $X$  be a uniformly convex Banach space and  $T$  a self-mapping of a bounded closed convex subset  $C$  of  $X$  which is asymptotically nonexpansive in the intermediate sense. Suppose that  $\{x_m\}_{m \geq 0}$  and  $\{y_m\}_{m \geq 0}$  are almost orbits of  $T$ ,  $f_1, f_2 \in F(T)$ , and  $0 \leq \alpha \leq 1$ . Then the limits*

$$\lim_{m \rightarrow \infty} \|x_m - y_m\|,$$

$$\lim_{m \rightarrow \infty} \|x_m - f_1\|,$$

and

$$\lim_{m \rightarrow \infty} \|\alpha x_m + (1 - \alpha)f_1 - f_2\|$$

exist.

Now we are ready to prove the following discrete analog of Theorem 3.2.

THEOREM 5.2. *Let  $X$  be a uniformly convex Banach space and  $T$  a self-mapping of a bounded closed convex subset  $C$  of  $X$  which is asymptotically nonexpansive in the intermediate sense. Suppose that  $\{x_m\}_{m \geq 0}$  is an almost orbit of  $T$ . Then for any  $\epsilon > 0$  and  $n \geq 1$  there exists  $M_{\epsilon,n} \geq 1$  such that for all  $k \geq M_{\epsilon,n}$  and  $m \geq M_{\epsilon,n}$ ,*

$$\left\| T^k \left( \frac{1}{n} \sum_{i=0}^{n-1} x_{i+m} \right) - \frac{1}{n} \sum_{i=0}^{n-1} T^k x_{i+m} \right\| < \epsilon.$$

In particular, for each  $n \geq 1$  there exists  $m_n \geq 1$  such that

$$\left\| T^k \left( \frac{1}{n} \sum_{i=0}^{n-1} x_{i+m} \right) - \frac{1}{n} \sum_{i=0}^{n-1} T^k x_{i+m} \right\| < \frac{1}{n}$$

for all  $k, m \geq m_n$ .

*Proof.* Let  $\epsilon > 0$ . Since  $\{x_m\}_{m \geq 0}$  is an almost orbit, there exists  $\tilde{m}_{\epsilon, n} \geq 1$  such that

$$\sup_{k \geq 0} \|x_{m+k} - T^k x_m\| < \frac{\delta_\epsilon}{3}$$

and (see Theorem 5.1)

$$\sup_{k \geq 0} \left| \|x_{m+j} - x_{m+i}\| - \|x_{m+j+k} - x_{m+i+k}\| \right| < \frac{\delta_\epsilon}{3}$$

for  $m \geq \tilde{m}_{\epsilon, n}$  and  $0 \leq i, j \leq n - 1$ . It then follows that

$$\begin{aligned} & \|x_{m+j} - x_{m+i}\| - \|T^k x_{m+j} - T^k x_{m+i}\| \\ & \leq \|x_{m+j} - x_{m+i}\| - \|x_{m+j+k} - x_{m+i+k}\| \\ & \quad + \|x_{m+j+k} - T^k x_{m+j}\| + \|x_{m+i+k} - T^k x_{m+i}\| \\ & < \delta_\epsilon \end{aligned}$$

for  $m \geq \tilde{m}_{\epsilon, n}$  and  $0 < i, j \leq n - 1$ . This implies (see Lemma 5.2—now we denote  $N_\epsilon$  by  $m_\epsilon$ ) that

$$\left\| T^k \left( \frac{1}{n} \sum_{i=0}^{n-1} x_{m+i} \right) - \left( \frac{1}{n} \sum_{i=0}^{n-1} T^k x_{m+i} \right) \right\| < \epsilon$$

for  $m \geq \tilde{m}_{\epsilon, n}$  and  $k \geq m_\epsilon$ . Hence we get

$$\left\| T^k \left( \frac{1}{n} \sum_{i=0}^{n-1} x_{m+i} \right) - \left( \frac{1}{n} \sum_{i=0}^{n-1} T^k x_{m+i} \right) \right\| < \epsilon$$

for all  $k \geq M_{\epsilon, n}$  and  $m \geq M_{\epsilon, n}$ , where  $M_{\epsilon, n} = \max(m_\epsilon, \tilde{m}_{\epsilon, n})$ . To obtain the second inequality it is sufficient to set

$$m_n = M_{1/n, n}.$$

■

We conclude this paper by stating our discrete analog of Theorem 4.1. We intend to present its complete proof elsewhere.

**THEOREM 5.3.** *Let  $X$  be a uniformly convex Banach space such that  $X^*$  has the KK-property,  $C$  a bounded closed convex subset of  $X$ , and  $T: C \rightarrow C$  a mapping which is asymptotically nonexpansive in the intermediate sense. Then every almost orbit  $\{x_m\}_{m \geq 0}$  of  $T$  is weakly almost convergent to some  $y \in F(T)$ .*

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