

Generalized Coherent Pairs

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A pair of quasi-definite moment functionals $\{u_0, u_1\}$ is a generalized coherent pair if monic orthogonal polynomials $\{P_n(x)\}_{n=0}^\infty$ and $\{R_n(x)\}_{n=0}^\infty$ relative to u_0 and u_1 , respectively, satisfy a relation

$$R_n(x) = \frac{1}{n+1} P'_{n+1}(x) - \frac{\sigma_n}{n} P'_n(x) - \frac{\tau_{n-1}}{n-1} P'_{n-1}(x), \quad n \geq 2,$$

where σ_n and τ_n are arbitrary constants, which may be zero. If $\{u_0, u_1\}$ is a generalized coherent pair, then u_0 and u_1 must be semiclassical. We find conditions under which either u_0 or u_1 is classical. In such a case, we also determine the types of the “companion” moment functionals. Also some illustrating examples and two ways of generating generalized coherent pairs are given. We also discuss the corresponding Sobolev orthogonal polynomials. © 2001 Academic Press

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1. INTRODUCTION

Concerning the problem of evaluating the Fourier coefficients in the Fourier expansion of functions by polynomials orthogonal with respect to a Sobolev inner product

$$\phi_\lambda(f, g) := \int_{-\infty}^{\infty} f(x)g(x) d\mu_0(x) + \lambda \int_{-\infty}^{\infty} f'(x)g'(x) d\mu_1(x) \quad (1.1)$$



(where $d\mu_0$ and $d\mu_1$ are positive Borel measures with finite moments), Iserles *et al.* [9] introduced the concept of coherency and symmetric coherency: $\{d\mu_0, d\mu_1\}$ is coherent (resp., symmetrically coherent) if there are constants $a_n \neq 0$ ($n \geq 1$) (resp., and $d\mu_0, d\mu_1$ are symmetric) such that $\{a_{n+1}P'_{n+1} - a_n P'_n\}_{n=0}^\infty$ ($a_0 = 1$) (resp., $a_{n+1}P'_{n+1} - a_{n-1}P'_{n-1}$) is orthogonal with respect to $d\mu_1(x)$. Here, $\{P_n(x)\}_{n=0}^\infty$ are the monic orthogonal polynomials with respect to $d\mu_0$. Let $\{R_n(x)\}_{n=0}^\infty$ be the monic orthogonal polynomials with respect to $d\mu_1$. Then $\{d\mu_0, d\mu_1\}$ is coherent (resp., symmetrically coherent) if and only if there are constants $\sigma_n \neq 0$ (resp., $\tau_n \neq 0$) such that

$$R_n(x) = \frac{1}{n+1}P'_{n+1}(x) - \frac{\sigma_n}{n}P'_n(x), \quad n \geq 1$$

$$\left(\text{resp., } R_n(x) = \frac{1}{n+1}P'_{n+1}(x) - \frac{\tau_{n-1}}{n-1}P'_{n-1}(x), \quad n \geq 2 \right).$$

After the work by Iserles *et al.* [9], there have been many works [4, 18–21, 24–27] on coherency from different points of view even allowing $d\mu_0$ and $d\mu_1$ to be signed or even complex valued measures. In particular, de Bruin and Meijer [4] introduced the concept of generalized coherent pair: a pair of positive Borel measures $\{d\mu_0, d\mu_1\}$ with finite moments is a generalized coherent pair if there are constants B_n and C_n such that $C_n \leq 0$ and

$$R_n(x) = \frac{1}{n+1}P'_{n+1}(x) + B_n P'_n(x) + C_n P'_{n-1}(x), \quad n \geq 1$$

and studied some properties of the zeros of the corresponding Sobolev orthogonal polynomials. We will call such a kind of generalized coherent pairs a 3-term coherent pair taking into account that 3 terms are involved in the right hand side.

In this work, we will study the generalized coherency in a more general setting by using the formal approach to orthogonality via moment functionals as was done in [20, 21] for usual coherency.

In Section 2, we collect basic definitions, notations, and lemmas that we will use later. In Section 3, we analyze the semiclassical character of generalized coherent pairs $\{u_0, u_1\}$ (see Definition 3.1) following ideas in [24]. In Section 4, we characterize the types of “companion” moment functionals when either u_0 or u_1 is classical and give ways of generating generalized coherent pairs together with some examples.

In Section 5, we consider Sobolev orthogonal polynomials relative to $\phi_\lambda(\cdot, \cdot)$ in (1.1) and give an efficient way of computing Sobolev–Fourier coefficients when $\{d\mu_0, d\mu_1\}$ is a generalized coherent pair, which extends the result in [9].

2. PRELIMINARIES

Let \mathcal{P} be the linear space of all polynomials in one variable with complex coefficients. We denote the degree of a polynomial $P(x)$ by $\deg(P)$ with the convention that $\deg(0) = -1$. A polynomial system (PS) is a sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ with $\deg(P_n) = n$, $n \geq 0$.

A linear functional u on \mathcal{P} is said to be a moment functional and we denote its action on a polynomial $\phi(x)$ by $\langle u, \phi \rangle$. We say that a moment functional u is quasi-definite (positive-definite, respectively) if its moments $a_n := \langle u, x^n \rangle$, $n \geq 0$, satisfy the Hamburger condition

$$\Delta_n(u) := \det[a_{i+j}]_{i,j=0}^n \neq 0 \quad (\Delta_n(u) > 0, \text{ respectively}), \quad n \geq 0.$$

DEFINITION 2.1. A PS $\{P_n(x)\}_{n=0}^{\infty}$ is called an orthogonal polynomial system (OPS) if there is a moment functional u such that

$$\langle u, P_m P_n \rangle = p_n \delta_{mn}, \quad m, n \geq 0,$$

where p_n are non-zero constants. In this case, we call $\{P_n(x)\}_{n=0}^{\infty}$ an OPS relative to u and u an orthogonalizing moment functional of $\{P_n(x)\}_{n=0}^{\infty}$. A moment functional u is quasi-definite if and only if there is an OPS $\{P_n(x)\}_{n=0}^{\infty}$ relative to u (see [6]). Moreover, in this case, each $P_n(x)$ is uniquely determined up to a non-zero constant factor.

For a moment functional u , a polynomial $\phi(x)$, and a constant c , we define moment functionals u' , ϕu , and $(x - c)^{-1}u$ by

$$\begin{aligned} \langle u', p \rangle &= -\langle u, p' \rangle; \\ \langle \phi u, p \rangle &= \langle u, \phi p \rangle; \\ \langle (x - c)^{-1}u, p \rangle &= \left\langle u, \frac{p(x) - p(c)}{x - c} \right\rangle, \quad p \in \mathcal{P}. \end{aligned}$$

For a PS $\{P_n(x)\}_{n=0}^{\infty}$ the dual basis of $\{P_n(x)\}_{n=0}^{\infty}$ is the sequence $\{u_n\}_{n=0}^{\infty}$ of moment functionals defined by the relation

$$\langle u_m, P_n \rangle = \delta_{mn}, \quad m \text{ and } n \geq 0.$$

In particular, u_0 is said to be the canonical moment functional of $\{P_n(x)\}_{n=0}^{\infty}$. If $\{P_n(x)\}_{n=0}^{\infty}$ is a monic OPS(MOPS), then $\{P_n(x)\}_{n=0}^{\infty}$ must be orthogonal relative to u_0 and

$$u_n = \frac{P_n(x)}{p_n} u_0, \quad n \geq 0.$$

DEFINITION 2.2 [22]. A quasi-definite moment functional u is said to be semiclassical if u satisfies a Pearson-type functional equation

$$(\varphi u)' = \psi u, \quad (2.1)$$

for some polynomials $\varphi(x)$ and $\psi(x)$ with $(\varphi, \psi) \neq (0, 0)$. Then quasi-definiteness of u implies $\deg(\varphi) \geq 0$ and $\deg(\psi) \geq 1$.

For a semiclassical moment functional u , we introduce

$$s := \min\{\max(\deg(\varphi) - 2, \deg(\psi) - 1)\}$$

the class number of u , where the minimum is taken over all pairs $(\varphi, \psi) \neq (0, 0)$ of polynomials satisfying (2.1). Note that (see [22]) if u is a semiclassical moment functional of class s satisfying

$$(\varphi u)' = \psi u \quad \text{and} \quad (\tilde{\varphi} u)' = \tilde{\psi} u$$

and if $s = \max(\deg(\varphi) - 2, \deg(\psi) - 1)$, then φ divides $\tilde{\varphi}$. In particular, a semiclassical moment functional of class 0 is called a classical moment functional.

Classical moment functionals can be characterized in many other ways. For an MOPS $\{P_n(x)\}_{n=0}^\infty$ relative to u , the following statements are all equivalent:

(i) $\{P_n(x)\}_{n=0}^\infty$ is a classical OPS, that is, $(\varphi u)' = \psi u$ for some polynomial φ and ψ with $0 \leq \deg(\varphi) \leq 2$ and $\deg(\psi) = 1$;

(ii) (Hahn [8]) $\{Q_n(x) := \frac{1}{n+1}P'_{n+1}(x)\}_{n=0}^\infty$ is also an MOPS. Then $\{Q_n(x)\}_{n=0}^\infty$ is orthogonal relative to $\tilde{u} = \varphi u$ satisfying

$$(\varphi \tilde{u})' = (\psi + \varphi')\tilde{u}; \quad (2.2)$$

(iii) (Bochner [2]). There are polynomials φ and ψ with $0 \leq \deg(\varphi) \leq 2$ and $\deg(\psi) = 1$ such that

$$\varphi(x)P_n''(x) + \psi(x)P_n'(x) = \left(\frac{1}{2}n(n-1)\varphi''(x) + n\psi'(x)\right)P_n(x),$$

$$n \geq 0; \quad (2.3)$$

(iv) (Marcellán *et al.* [16]). There are constants r_n and s_n such that

$$P_n(x) = Q_n(x) + r_n Q_{n-1}(x) + s_n Q_{n-2}(x), \quad n \geq 2,$$

where $Q_n(x) = \frac{1}{n+1}P'_{n+1}(x)$, $n \geq 0$.

It is well known that there are essentially four distinct classical OPSs, up to a linear change of variable [2, 14]:

- (i) Hermite polynomials $\{H_n\}_{n=0}^\infty$: $\varphi(x) = 1$, $\psi(x) = -2x$;
- (ii) Laguerre polynomials $\{L_n^{(\alpha)}\}_{n=0}^\infty$: $\varphi(x) = x$, $\psi(x) = -x + \alpha + 1$ ($\alpha \notin \{-1, -2, \dots\}$);
- (iii) Bessel polynomials $\{B_n^{(\alpha)}\}_{n=0}^\infty$: $\varphi(x) = x^2$, $\psi(x) = (\alpha + 2)x + 2$ ($\alpha \notin \{-2, -3, \dots\}$);
- (iv) Jacobi polynomials $\{P_n^{(\alpha, \beta)}\}_{n=0}^\infty$: $\varphi(x) = 1 - x^2$, $\psi(x) = \beta - \alpha - (\alpha + \beta + 2)x$ ($\alpha, \beta, \alpha + \beta + 1 \notin \{-1, -2, \dots\}$).

We denote by $u_H, u_L^{(\alpha)}, u_B^{(\alpha)}$, and $u_J^{(\alpha, \beta)}$ the orthogonalizing moment functionals for Hermite, Laguerre, Bessel, and Jacobi polynomials as above.

For an MOPS $\{P_n(x)\}_{n=0}^\infty$ relative to u and complex numbers ξ and c , let $\{P_n^*(\xi; x)\}_{n=0}^\infty$, $\{P_n^{(1)}(x)\}_{n=0}^\infty$, and $\{P_n(c; x)\}_{n=0}^\infty$ be the monic kernel polynomials, the monic numerator polynomials (also called the associated polynomials of first kind (see [6])), and the monic co-recursive polynomials of $\{P_n(x)\}_{n=0}^\infty$, respectively. Then

$$P_n^*(\xi; x) = \frac{\langle u, P_n^2 \rangle}{P_n(\xi)} \sum_{k=0}^n \frac{P_k(x) P_k(\xi)}{\langle u, P_k^2 \rangle}, \quad n \geq 0 \quad [6];$$

$$P_n(x) = P_n^*(\xi; x) - \frac{P_{n-1}(\xi)}{P_n(\xi)} \frac{\langle u, P_n^2 \rangle}{\langle u, P_{n-1}^2 \rangle} P_{n-1}^*(\xi; x), \quad n \geq 1 \quad [12];$$

(2.4)

$$P_n(c; x) = P_n(x) - c P_{n-1}^{(1)}(x), \quad n \geq 1 \quad [5].$$

(2.5)

It is well known (see [6, Theorem 7.1, p. 36]) that for a quasi-definite moment functional u with MOPS $\{P_n(x)\}_{n=0}^\infty$ and a complex number ξ , $(x - \xi)u$ is also quasi-definite if and only if $P_n(\xi) \neq 0$, $n \geq 1$, and then the MOPS relative to $(x - \xi)u$ is $\{P_n^*(\xi; x)\}_{n=0}^\infty$. Moreover (see [12, Theorem 3.6]), if u is semiclassical of class s satisfying (2.1), then $(x - \xi)u$ is also semiclassical of class

$$\begin{cases} s - 1 & \text{if } \varphi(\xi) = \psi(\xi) = 0 \\ s & \text{if } \varphi(\xi) = 0 \text{ and } \psi(\xi) \neq 0 \\ s + 1 & \text{if } \varphi(\xi) \neq 0. \end{cases}$$

Conversely if $(x - \xi)u$ is semiclassical of class s , then u is semiclassical of class $s - 1$ or s or $s + 1$.

PROPOSITION 2.3. *Let $\{P_n(x)\}_{n=0}^\infty$ and $\{Q_n(x)\}_{n=0}^\infty$ be the MOPSs relative to u and v , respectively. Then $\{Q_n(x)\}_{n=0}^\infty = \{P_n^*(\xi; x)\}_{n=0}^\infty$ for some complex number ξ if and only if there are complex numbers $\alpha_n (n \geq 1)$ such that $\alpha_1 \neq 0$ and*

$$P_n(x) = Q_n(x) - \alpha_n Q_{n-1}(x), \quad n \geq 0 \quad (Q_{-1}(x) = 0, \alpha_0 \text{ arbitrary}).$$

In this case $\alpha_n \neq 0, n \geq 1$ (cf. (2.4)), $P_n(\xi) \neq 0, n \geq 1$, and $(x - \xi)u = v$. Moreover, if u (resp. v) is positive-definite and ξ is real, then $Q_n(x)$ (resp. $P_n(x)$), $n \geq 1$, has n real simple zeros, which interlace with the zeros of $P_n(x)$ and $P_{n+1}(x)$ (resp. $Q_n(x)$ and $Q_{n+1}(x)$).

Proof. See [12, Theorem 3.2, Theorem 3.3, and Theorem 3.4. ■

In Proposition 2.3, we have (see Theorem 2 in [20] or Theorem 4.2 in [12])

$$\xi = \frac{\tilde{c}_1}{\alpha_1} + \alpha_2 + \tilde{b}_1$$

and

$$\alpha_n = \frac{P_{n-1}(\xi)}{P_n(\xi)} \frac{\langle u, P_n^2 \rangle}{\langle u, P_{n-1}^2 \rangle} = \frac{Q_n(c; \xi)}{Q_{n-1}(c; \xi)} \neq 0, \quad n \geq 1, \quad (2.6)$$

where $Q_2(x) = (x - \tilde{b}_1)Q_1(x) - \tilde{c}_1Q_0(x)$ and $c = \xi - \alpha_1 + Q_1(0)$.

LEMMA 2.4. *Let u and v be moment functionals and $a, b (a \neq b)$ complex numbers. Then*

- (i) $(x - a)u = v$ if and only if $u = (x - a)^{-1}v + \langle u, 1 \rangle \delta(x - a)$;
- (ii) $(x - a)(x - b)u = v$ if and only if $u = (x - a)^{-1}(x - b)^{-1}v + \frac{1}{a-b}[\langle u, x - b \rangle \delta(x - a) - \langle u, x - a \rangle \delta(x - b)]$, where $\delta(x - a), p(x) \rangle = p(a)$ for every $p \in \mathcal{P}$.
- (iii) $(x - a)^2u = v$ if and only if $u = (x - a)^{-2}v + \langle u, 1 \rangle \delta(x - a) - \langle u, x - a \rangle \delta'(x - a)$.

Proof. (i) Assume $(x - a)u = v$. Then

$$\langle (x - a)^{-1}v + \langle u, 1 \rangle \delta(x - a), (x - a)^n \rangle = \langle u, (x - a)^n \rangle, \quad n \geq 0$$

so that $u = (x - a)^{-1}v + \langle u, 1 \rangle \delta(x - a)$. The converse is trivial. Now (ii) and (iii) follow by repeated applications of (i). ■

LEMMA 2.5. *Let v be a quasi-definite moment functional with MOPS $\{Q_n(x)\}_{n=0}^\infty$, and a, ξ complex numbers. Then the moment functional $u = (x$*

$-\xi)^{-1}v + a\delta(x - \xi)$ is quasi-definite if and only if

$$aQ_n(\xi) + \langle v, 1 \rangle Q_{n-1}^{(1)}(\xi) \neq 0, \quad n \geq 0.$$

Proof. See Theorem 1.1 in [23] and (4.12) in [12]. ■

LEMMA 2.6. Let u be a quasi-definite moment functional with MOPS $\{P_n(x)\}_{n=0}^\infty$. Then for another moment functional v , $\langle v, P_n \rangle = 0$, $n > k (\geq 0)$, and $\langle v, P_k \rangle \neq 0$ if and only if $v = \pi_k(x)u$ for some polynomial $\pi_k(x)$ of degree k .

Proof. See Lemma 2.2 in [13].

3. GENERALIZED COHERENT PAIRS

Let u_0 and u_1 be two quasi-definite moment functionals with corresponding MOPSs $\{P_n(x)\}_{n=0}^\infty$ and $\{R_n(x)\}_{n=0}^\infty$, respectively, satisfying three-term recurrence relations

$$P_{n+1} = (x - b_n)P_n - c_n P_{n-1}, \quad n \geq 0 \text{ and } \langle u_0, P_n^2 \rangle = p_n, \quad n \geq 0; \quad (3.1)$$

$$R_{n+1} = (x - \beta_n)R_n - \gamma_n R_{n-1}, \quad n \geq 0 \text{ and } \langle u_1, R_n^2 \rangle = r_n, \quad n \geq 0, \quad (3.2)$$

where $c_n \neq 0$ and $\gamma_n \neq 0$.

DEFINITION 3.1. $\{u_0, u_1\}$ is a generalized coherent pair if there exist complex numbers $\{\sigma_n\}_{n=1}^\infty$ and $\{\tau_n\}_{n=1}^\infty$ such that

$$R_n = Q_n - \sigma_n Q_{n-1} - \tau_{n-1} Q_{n-2}, \quad n \geq 0, \quad (3.3)$$

where $Q_{-1} = Q_{-2} = 0$, $Q_n = \frac{1}{n+1}P'_{n+1}$, $n \geq 0$, and $\sigma_0 = \tau_{-1} = \tau_0 = 0$.

In particular, if $\sigma_n \neq 0$ for some $n \geq 1$ and $\tau_n = 0$, $n \geq 1$ (resp. $\tau_n \neq 0$ for some $n \geq 1$), then we call $\{u_0, u_1\}$ a 2-term (resp. 3-term) coherent pair.

In these cases, we call u_1 (resp. u_0) a “companion” of u_0 (resp. u_1).

Following [9], we also call $\{u_0, u_1\}$ a coherent pair if $\sigma_n \neq 0$ and $\tau_n = 0$, $n \geq 1$ and a symmetrically coherent pair if u_0 and u_1 are symmetric and $\sigma_n = 0$, $\tau_n \neq 0$, $n \geq 1$. Iserles *et al.* [9] proved that $\{u_0, u_1\}$ is a coherent pair if and only if there are non-zero complex numbers a_n , $n \geq 1$, such that

$$a_m a_n \langle u_1, P'_m P'_n \rangle \quad (m \text{ and } n \geq 1)$$

depend only on $\min(m, n)$. However, this fact is no longer true if we allow some σ_n to be 0. Note that if $\sigma_n = \tau_n = 0$, $n \geq 1$, then (3.3) implies by Hahn's theorem that both u_0 and u_1 are classical moment functionals of the same type and $u_1 = \varphi u_0$ for some polynomial $\varphi(x)$ with $0 \leq \deg(\varphi) \leq 2$.

In the following, we always assume that $\{u_0, u_1\}$ is a generalized coherent pair unless stated otherwise.

LEMMA 3.2 (cf. [19, Proposition 8.1]). *We have*

$$nP_n u_0 = p_n(G_n u_1)', \quad n \geq 1, \quad (3.4)$$

where

$$G_n = \frac{\tau_n}{r_{n+1}} R_{n+1} + \frac{\sigma_n}{r_n} R_n - \frac{1}{r_{n-1}} R_{n-1}, \quad n \geq 0, \quad (3.5)$$

so that $n - 1 \leq \deg(G_n) \leq n + 1$.

Proof. Let $u_n^{(0)}$, $\tilde{u}_n^{(0)}$, and $u_n^{(1)}$, $n \geq 0$ be the dual bases of $\{P_n(x)\}_{n=0}^\infty$, $\{Q_n(x)\}_{n=0}^\infty$, and $\{R_n(x)\}_{n=0}^\infty$, respectively. Then, it is easy to see that

$$\tilde{u}_n^{(0)} = u_n^{(1)} - \sigma_{n+1} u_{n+1}^{(1)} - \tau_{n+1} u_{n+2}^{(1)} = -G_{n+1} u_1 \quad (n \geq 0).$$

Hence,

$$(\tilde{u}_n^{(0)})' = -(n+1)u_{n+1}^{(0)} = -(n+1)\frac{1}{p_{n+1}}P_{n+1}u_0, \quad n \geq 0.$$

Therefore, we have the result. ■

THEOREM 3.3. *Both u_0 and u_1 are semiclassical (of class ≤ 6 for u_0 and of class ≤ 2 for u_1) satisfying*

$$(\rho_i u_i)' = \eta_i u_i, \quad i = 0, 1, \quad (3.6)$$

where

$$\rho_1 = 2p_1 P_2 G_1 - p_2 P_1 G_2, \quad \eta_1 = 2p_1 P_2' G_1 - p_2 P_1' G_2, \quad (3.7)$$

$$\rho_0 = \rho_1 H, \quad \eta_0 = 2\rho_1 H' + \nu H, \quad (3.8)$$

$$H = G_1 G_2' - G_1' G_2, \quad \nu = p_2 P_1 G_2' - 2p_1 P_2 G_1'. \quad (3.9)$$

Moreover,

$$np_1 p_2 P_n H = p_n(\rho_1 G_n' + \nu G_n), \quad n \geq 1. \quad (3.10)$$

Proof. Set $n = 1$ and 2 in (3.4). Then

$$P_1 u_0 = p_1 G'_1 u_1 + p_1 G_1 u'_1, \quad (3.11)$$

$$2P_2 u_0 = p_2 G'_2 u_1 + p_2 G_2 u'_1. \quad (3.12)$$

Eliminating u_0 , u_1 , and u'_1 from (3.11) and (3.12) give (3.6) for $i = 1$ and

$$\rho_1 u_0 = p_1 p_2 H u_1, \quad (3.13)$$

$$\nu u_0 = p_1 p_2 H u'_1. \quad (3.14)$$

Hence $(\rho_0 u_0)' = (\rho_1 H u_0)' = p_1 p_2 (H^2 u_1)' = \eta_0 u_0$ by (3.8), (3.13), and (3.14), which gives (3.6) for $i = 0$.

By (3.4) and (3.13), we have

$$n p_1 p_2 P_n H u_1 = n P_n \rho_1 u_0 = p_n (\rho_1 G'_n + \nu G_n) u_1$$

since $\rho_1 u'_1 = \nu u_1$ so that (3.10) follows. It is now easy to see that $H = (\tau_1 \tau_2 / r_2 r_3) x^4 + \text{lower degree terms}$ so that $\deg(H) \leq 4$ and

$$\deg(H) = \begin{cases} 4 & \text{if } \tau_1 \tau_2 \neq 0 \\ 3 & \text{if } \tau_1 = 0, \sigma_1 \tau_2 \neq 0 \\ 2 & \text{if (i) } \sigma_1 = \tau_1 = 0, \tau_2 \neq 0 \text{ or (ii) } \tau_1 \neq 0, \tau_2 = 0, \\ & \sigma_1 \sigma_2 + \tau_1 \neq 0 \text{ or (iii) } \tau_1 = \tau_2 = 0, \sigma_1 \sigma_2 \neq 0 \\ 1 & \text{if (i) } \tau_1 \neq 0, \tau_2 = \sigma_1 \sigma_2 + \tau_1 = 0 \text{ or} \\ & \text{(ii) } \sigma_1 = \tau_1 = \tau_2 = 0, \sigma_2 \neq 0 \\ 0 & \text{if } \sigma_2 = \tau_1 = \tau_2 = 0. \end{cases} \quad (3.15)$$

Hence $H \neq 0$ so that $0 \leq \deg(H) \leq 4$, $0 \leq \deg(\rho_1) \leq 4$, $0 \leq \deg(\rho_0) \leq 8$, and $0 \leq \deg(\nu) \leq 3$, by (3.7), (3.8), and (3.9). Hence u_0 and u_1 are semiclassical of class ≤ 6 and ≤ 2 , respectively, and so $1 \leq \deg(\eta_1) \leq 3$, $1 \leq \deg(\eta_0) \leq 7$. ■

Meijer [24] proved that if $\{u_0, u_1\}$ is coherent or symmetrically coherent, then either u_0 or u_1 must be classical with some extra relation between u_0 and u_1 .

We say a quasi-definite moment functional u with MOPS $\{P_n(x)\}_{n=0}^\infty$ is strongly classical if there is another MOPS $\{S_n(x)\}_{n=0}^\infty$ relative to w such that $P_n(x) = \frac{1}{n+1} S'_{n+1}(x)$, $n \geq 0$. Then u and w must be classical moment functionals of the same type satisfying

$$(\varphi u)' = \psi u, \quad (\varphi w)' = (\psi - \varphi') w, \quad \text{and} \quad \varphi w = u.$$

Classical moment functionals $u_j^{(\alpha, \beta)}(\alpha, \beta, \alpha + \beta \neq 0, -1, -2, \dots)$, $u_B^{(\alpha)}(\alpha \neq 0, -1, -2, \dots)$, $u_L^{(\alpha)}(\alpha \neq 0, -1, -2, \dots)$, and u_H are strongly classical.

In our more general case, both u_0 and u_1 may not be classical but we have:

THEOREM 3.4. *Assume that either u_0 is classical or u_1 is strongly classical.*

(i) *If $\tau_k = 0$ for some $k \geq 1$, then $\tau_n = 0$ for all $n \geq 1$.*

(ii) *If $\sigma_j = 0$ for some $j \geq 1$ and $\tau_k = 0$ for some $k \geq 1$, then $\sigma_n = \tau_n = 0$ for all $n \geq 1$ so that u_0 and u_1 must be classical of the same type.*

To prove Theorem 3.4, we need a lemma.

LEMMA 3.5. *Let $\{S_n(x)\}_{n=0}^\infty$ and $\{T_n(x)\}_{n=0}^\infty$ be two MOPSs satisfying three-term recurrence relations*

$$\begin{cases} S_{n+1}(x) = (x - \delta_n)S_n(x) - \epsilon_n S_{n-1}(x), \\ T_{n+1}(x) = (x - \tilde{\delta}_n)T_n(x) - \tilde{\epsilon}_n T_{n-1}(x), \end{cases} \quad n \geq 0. \quad (3.16)$$

If moreover,

$$T_n(x) = S_n(x) + d_n S_{n-1}(x) + e_n S_{n-2}(x) + f_n, \quad n \geq 0 \quad (3.17)$$

($S_{-1} = S_{-2} = 0$ and $d_0 = e_0 = e_1 = f_0 = f_1 = f_2 = 0$), then

$$d_n = \sum_{k=0}^{n-1} (\delta_k - \tilde{\delta}_k), \quad n \geq 1 \quad (3.18)$$

$$e_n = \sum_{k=1}^{n-1} \left\{ \epsilon_k - \tilde{\epsilon}_k + d_k (\delta_{k-1} - \tilde{\delta}_k) \right\}, \quad n \geq 2 \quad (3.19)$$

$$f_n = 0, \quad n \geq 0, \quad (3.20)$$

and

$$-d_n \epsilon_{n-1} - e_n \delta_{n-2} + e_n \tilde{\delta}_n + d_{n-1} \tilde{\epsilon}_n = 0, \quad n \geq 2 \quad (3.21)$$

$$-e_n \epsilon_{n-2} + e_{n-1} \tilde{\epsilon}_n = 0, \quad n \geq 3. \quad (3.22)$$

In particular,

(i) *If $e_k = 0$ for some $k \geq 2$, then $e_n = 0$, $n \geq 0$.*

(ii) *If $e_j = 0$ for some $j \geq 2$ and $d_k = 0$ for some $k \geq 1$, then $d_n = \epsilon_n = 0$, $n \geq 0$.*

Proof. Equations (3.18)–(3.22) follow immediately from $0 = T_{n+1} - (x - \tilde{\delta}_n)T_n + \tilde{\epsilon}_n T_{n-1}$, $n \geq 0$, (3.16), and (3.17). Assume $e_k = 0$ for some $k \geq 2$. Then (3.22) implies $e_n = 0$ for all $n \geq 0$, since $\epsilon_n, \tilde{\epsilon}_n \neq 0$, $n \geq 1$. Assume $e_j = 0$ for some $j \geq 2$ and $d_k = 0$ for some $k \geq 1$. Then (3.21) and (3.22) yield $d_n = e_n = 0$ for all $n \geq 0$. ■

Proof of Theorem 3.4. Assume u_0 is classical. Then $\{Q_n(x)\}_{n=0}^\infty = \{\frac{1}{n+1}P'_{n+1}(x)\}_{n=0}^\infty$ is also an MOPS so that the conclusion follows by Lemma 3.5. Assume u_1 is strongly classical. Then $\{R_n(x)\}_{n=0}^\infty = \{\frac{1}{n+1}T'_{n+1}(x)\}_{n=0}^\infty$ for some classical MOPS $\{T_n(x)\}_{n=0}^\infty$. Then (3.3) becomes

$$T_n(x) = P_n(x) + d_n P_{n-1}(x) + e_n P_{n-2}(x) + f_n, \quad n \geq 0, \quad (3.23)$$

where $d_n = \frac{-n}{n+1}\sigma_{n-1}$, $n \geq 2$, $e_2 = T_2(0) - P_2(0) + 2\sigma_1 P_1(0)$, $e_n = -\frac{n}{n-2}\tau_{n-2}$, $n \geq 3$, and $f_n = T_n(0) - P_n(0) + \frac{n}{n-1}\sigma_{n-1}P_{n-1}(0) + \frac{n}{n-2}\tau_{n-2}P_{n-2}(0)$, $n \geq 3$. Hence, the conclusion follows again by Lemma 3.5. ■

In [3], Branquinho and Marcellán found necessary and sufficient conditions for the PS $\{T_n(x)\}_{n=0}^\infty$ defined by the relation (3.17) with $f_n = 0$, $n \geq 0$, and $e_n \neq 0$, $n \geq 2$ to be an MOPS assuming that only $\{S_n(x)\}_{n=0}^\infty$ is an MOPS. Compare Theorem 3.4 with the analog for 2-term coherence analyzed in [20].

In the following, let g_n and h be the leading coefficients of $G_n(x)$ and $H(x)$, respectively.

PROPOSITION 3.6. *If u_0 is classical, then $G_1 u_1$ is also classical of the same type as u_0 . If moreover $\deg(G_1) = 0$, then $\sigma_n = \tau_n = 0$, $n \geq 1$. If moreover $\deg(G_1) = 1$, i.e., $G_1(x) = g_1(x - \xi)(g_1 \neq 0)$, then $Q_n(\xi) \neq 0$, $\sigma_n = (R_{n-1}(\xi)/R_n(\xi))\gamma_n$, $\tau_n = 0$, $n \geq 1$, and $\{Q_n(x)\}_{n=0}^\infty = \{R_n^*(\xi; x)\}_{n=0}^\infty$.*

Proof. Assume u_0 is a classical moment functional satisfying $(\varphi u_0)' = \psi u_0$ with $0 \leq \deg(\varphi) \leq 2$ and $\deg(\psi) = 1$. Then (cf. (2.3))

$$\varphi P_n'' + \psi P_n' = \lambda_n P_n, \quad n \geq 0,$$

where $\lambda_n = \frac{1}{2}n(n-1)\varphi''(x) + n\psi'(x)$ and $\lambda_n \neq 0$, $n \geq 1$. Hence $\psi(x) = \lambda_1 P_1(x)$ so that by (3.4) for $n = 1$

$$(\varphi u_0)' = \lambda_1 P_1 u_0 = \lambda_1 p_1 (G_1 u_1)'.$$

Therefore $G_1 u_1 = (\lambda_1 p_1)^{-1} \varphi u_0$ is also a classical moment functional of the same type as u_0 . If $\deg(G_1) = 0$, then $\sigma_1 = \tau_1 = 0$ (cf. (3.5)) so that $\sigma_n = \tau_n = 0$, $n \geq 1$ by Theorem 3.4. If $\deg(G_1) = 1$, then $\sigma_1 \neq 0$ and $\tau_1 = 0$ so that $\sigma_n \neq 0$, $\tau_n = 0$, $n \geq 1$, and $(x - \xi)u_1 = (\lambda_1 p_1 g_1)^{-1} \varphi u_0$. Hence, $(x - \xi)u_1$ is quasi-definite so that $Q_n(\xi) \neq 0$, $n \geq 1$, $\{Q_n(x)\}_{n=0}^\infty = \{R_n^*(\xi; x)\}_{n=0}^\infty$, and $\sigma_n = (R_{n-1}(\xi)/R_n(\xi))\gamma_n$, $n \geq 1$ (see (2.4)). ■

COROLLARY 3.7. *If u and u_1 are classical, then*

- (i) u_0, u_1 , and $G_1 u_1$ are classical of the same type;
- (ii) $u_0 \neq u_H$ if $\deg(G_1) = 1$ or 2 .

Proof. Assume that u_0 and u_1 are classical and $(\varphi u_0)' = \psi u_0$ ($0 \leq \deg(\varphi) \leq 2, \deg(\psi) = 1$). Then by Proposition 3.6, $G_1 u_1$ is also classical of the same type as u_0 . Since $G_1 u_1$ must be of the same type as u_1 , u_1 is of the same type as u_0 . Hence there exist polynomials $\gamma(x)$ and $\tilde{\gamma}(x)$ of degree 1 such that

$$(\varphi u_1)' = \gamma u_1 \quad \text{and} \quad (\varphi G_1 u_1)' = \tilde{\gamma} G_1 u_1.$$

Hence $\tilde{\gamma} G_1 u_1 = G_1' \varphi u_1 + G_1 (\varphi u_1)' = G_1' \varphi u_1 + G_1 \gamma u_1$ so that $G_1' \varphi = (\tilde{\gamma} - \gamma) G_1$. Therefore, $\deg(\varphi) \geq 1$ so that $u_0 \neq u_H$ if $\deg(G_1) = 1$ or 2 . ■

PROPOSITION 3.8. *If u_1 is classical, then $\rho_1(x)$, $\nu(x)$, and $H(x)$ must have a common factor $\pi(x)$ with $\max(\deg(\rho_1) - 2, \deg(\nu) - 1) \leq \deg(\pi) \leq 3$ and $u_1 \neq u_j^{(0,0)}$, the Legendre moment functional.*

Proof. Assume that u_1 is a classical moment functional satisfying $(\varphi_1 u_1)' = \psi_1 u_1$ with $0 \leq \deg(\varphi_1) \leq 2$ and $\deg(\psi_1) = 1$. Since $(\rho_1 u_1)' = \eta_1 u_1$, $\rho_1 = \varphi_1 \pi$ for some polynomial $\pi(x) (\neq 0)$ with $\deg(\pi) = \deg(\rho_1) - \deg(\varphi_1)$. Then $\nu u_1 = \rho_1 u_1' = \pi \varphi_1 u_1' = \pi(\psi_1 - \varphi_1') u_1$ so that $\nu = (\psi_1 - \varphi_1') \pi$ and $\psi_1(x) - \varphi_1'(x) \neq 0$. Hence, $\max(\deg(\rho_1) - 1, \deg(\nu) - 1) \leq \deg(\pi) \leq 3$ and $u_1 \neq u_j^{(0,0)}$ since $\psi_1(x) - \varphi_1'(x) \equiv 0$ in case $u_1 = u_j^{(0,0)}$. Finally assume $\deg(\pi) \geq 1$ but $\pi(x)$ does not divide $H(x)$. Then, $\pi(x)$ has a zero ξ such that $H(\xi) \neq 0$. Then by (3.10), $P_n(\xi) = 0$, $n \geq 1$, since $\rho_1(\xi) = \nu(\xi) = 0$, which is a contradiction. ■

We now examine the semiclassical character of u_0 and u_1 depending on $\deg(H)$: Let s_i be the class numbers of u_i for $i = 0, 1$.

Case 1. $\deg(H) = 0$, that is, $\sigma_2 = \tau_1 = \tau_2 = 0$. Then, there are two cases: $\sigma_1 = 0$ and $\sigma_1 \neq 0$.

Case 11. $\sigma_1 = \sigma_2 = \tau_1 = \tau_2 = 0$. Then $\deg(\rho_0), \deg(\rho_1) \leq 2$, and $\deg(\eta_0) = \deg(\eta_1) = 1$ so that u_0 and u_1 are classical of the same type and $\sigma_n = \tau_n = 0$ for all $n \geq 1$ by Theorem 3.4.

Case 12. $\sigma_2 = \tau_1 = \tau_2 = 0$ and $\sigma_1 \neq 0$. Then $\deg(\rho_1) = \deg(\rho_0) = 3$ and $\deg(\eta_1) = \deg(\eta_0) = 2$. Hence u_0 and u_1 are semiclassical of class ≤ 1 . If either u_0 is classical or u_1 is strongly classical, then by Theorem 3.4, $\sigma_n = \tau_n = 0$, $n \geq 1$. This contradicts $\sigma_1 \neq 0$. Hence $s_0 = 1$, $s_1 = 0$ or 1 , and u_1 cannot be strongly classical.

Case 2. $\deg(H) = 1$, that is, (i) $\tau_1 \neq 0$, $\tau_2 = 0$, and $\sigma_1\sigma_2 + \tau_1 = 0$ or (ii) $\sigma_1 = \tau_1 = \tau_2 = 0$ and $\sigma_2 \neq 0$. Then u_0 cannot be classical and u_1 cannot be strongly classical by Theorem 3.4.

Case 21. $\tau_1 \neq 0$, $\tau_2 = 0$, and $\sigma_1\sigma_2 + \tau_1 = 0$. Then $\deg(\rho_0) \leq 4$, $\deg(\rho_1) \leq 4$, $1 \leq \deg(\eta_0) \leq 4$, $1 \leq \deg(\eta_1) \leq 3$ so that $1 \leq s_0 \leq 3$ and $0 \leq s_1 \leq 2$.

Case 22. $\sigma_1 = \tau_1 = \tau_2 = 0$ and $\sigma_2 \neq 0$. Then $1 \leq s_0 \leq 2$ and $0 \leq s_1 \leq 1$.

Case 3. $\deg(H) = 2$, that is, (i) $\sigma_1 = \tau_1 = 0$ and $\tau_2 \neq 0$ or (ii) $\tau_1 \neq 0$, $\tau_2 = 0$, and $\sigma_1\sigma_2 + \tau_1 \neq 0$ or (iii) $\tau_1 = \tau_2 = 0$ and $\sigma_1\sigma_2 \neq 0$.

In cases (i) and (ii), u_0 cannot be classical and u_1 cannot be strongly classical by Theorem 3.4 and $1 \leq s_0 \leq 4$, $0 \leq s_1 \leq 2$.

In case (iii), there are three cases: $H(x) = h(x - \xi)^2$ or $H(x) = h(x - \xi_1)(x - \xi_2)$ ($\xi_1 \neq \xi_2$) and $\tau_n = 0$, $n \geq 1$ or $H(x) = h(x - \xi_1)(x - \xi_2)$ ($\xi_1 \neq \xi_2$) and $\tau_n \neq 0$ for some $n \geq 3$.

THEOREM 3.9. Assume $\tau_1 = \tau_2 = 0$ and $\sigma_1\sigma_2 \neq 0$ so that $\deg(H) = 2$.

(i) If $H(x) = h(x - \xi)^2$, then u_0 and G_1u_1 are classical of the same type, $\deg(\eta_1) = 2$, and $\sigma_n \neq 0$, $\tau_n = 0$, $n \geq 1$.

(ii) If $H(x) = h(x - \xi_1)(x - \xi_2)$ ($\xi_1 \neq \xi_2$), $\tau_n = 0$, $n \geq 1$, then u_1 is classical. Moreover, if u_1 is strongly classical, then $\sigma_n \neq 0$, $n \geq 1$.

(iii) If $\tau_n \neq 0$ for some $n \geq 3$, then $H(x) = h(x - \xi_1)(x - \xi_2)$ ($\xi_1 \neq \xi_2$) and $1 \leq s_0 \leq 3$, $0 \leq s_1 \leq 1$, and u_1 cannot be strongly classical.

Proof. Note that $\deg(G_1) = 1$ and $\deg(G_2) = 2$ when $\tau_1 = \tau_2 = 0$ and $\sigma_1\sigma_2 \neq 0$.

(i) The following proof is essentially the same as that of Theorem 1 in [24], where it is assumed that $\sigma_n \neq 0$ and $\tau_n = 0$, $n \geq 1$.

Assume $H(x) = h(x - \xi)^2$ ($h = g_1g_2$). Then

$$H(\xi) = G_1(\xi)G_2'(\xi) - g_1G_2(\xi) = 0 \quad \text{and}$$

$$H'(\xi) = 2g_2G_1(\xi) = 0$$

so that $G_1(\xi) = G_2(\xi) = 0$. Hence $G_1(\xi) = g_1(x - \xi)$ and $G_2(x) = G_1(x)\tilde{G}_2(x)$, $\deg(\tilde{G}_2) = 1$. Then

$$\rho_1(x) = G_1(2p_1P_2 - p_2P_1\tilde{G}_2) = (x - \xi)\tilde{\rho}_1(x), \quad 0 \leq \deg(\tilde{\rho}_1) \leq 2$$

$$\eta_1(x) = G_1(2p_1P_2' - p_2P_1'\tilde{G}_2) = (x - \xi)\tilde{\eta}_1(x), \quad 0 \leq \deg(\tilde{\eta}_1) \leq 1.$$

Using (3.13), we have

$$\tilde{\rho}_1 u_0 = p_1 p_2 g_2 G_1 u_1 = p_1 p_2 h(x - \xi) u_1 \quad (3.24)$$

so that by (3.11), $(\tilde{\rho}_1 u_0)' = p_2 g_2 (p_1 G_1 u_1)' = p_2 g_2 P_1 u_0$. Therefore, u_0 is classical and $G_1 u_1$ is also classical of the same type as u_0 by Proposition 3.6 satisfying

$$(\tilde{\rho}_1 G_1 u_1)' = \tilde{\eta}_1 G_1 u_1.$$

Hence $\deg(\tilde{\eta}_1) = 1$ and so $\deg(\eta_1) = 2$. Finally $\sigma_n \neq 0$ and $\tau_n = 0$, $n \geq 1$ by Theorem 3.4.

(ii) It is also proved by Meijer (see Theorem 2 in [24]) assuming $\sigma_n \neq 0$ and $\tau_n = 0$, $n \geq 1$. But, the inspection of the proof of Theorem 2 in [24] reveals that we only need $\sigma_1 \sigma_2 \neq 0$ and $\tau_n = 0$, $n \geq 1$. Then, by Theorem 3.4, $\sigma_n \neq 0$, $n \geq 1$, if u_1 is strongly classical.

(iii) Assume $\tau_n \neq 0$ for some $n \geq 3$. Then $H(x)$ cannot have a double zero by (i) so that $H(x) = h(x - \xi_1)(x - \xi_2)$ ($\xi_1 \neq \xi_2$) and the conclusion follows from Theorem 3.4. ■

In particular, we have:

COROLLARY 3.10. *If $\tau_1 = \tau_2 = 0$, $\sigma_1 \sigma_2 \neq 0$, and $H(x) = h(x - \xi)^2$, then $R_n(\xi) \neq 0$, $n \geq 0$ and $\{R_n^*(\xi; x)\}_{n=0}^\infty$ is the classical MOPS relative to $\tilde{\rho}_1 u_0$.*

Proof. It follows immediately from (3.24) since $\tilde{\rho}_1 u_0$ is a classical moment functional. ■

The relation (3.24) between u_0 and u_1 also holds in case Theorem 3.9(ii) (see Theorem 2 in [24]) for $\xi = \xi_1$ or ξ_2 . Hence we have in case (i) or (ii) in Theorem 3.9

$$p_1 p_2 h u_1 = (x - \xi)^{-1} \tilde{\rho}_1 u_0 + p_1 p_2 h r_0 \delta(x - \xi).$$

Case 4. $\deg(H) = 3$, that is, $\tau_1 = 0$, $\tau_2 \neq 0$, and $\sigma_1 \neq 0$. Then $1 \leq s_0 \leq 5$, $0 \leq s_1 \leq 2$, and u_1 cannot be strongly classical.

Case 5. $\deg(H) = 4$, that is, $\tau_1 \tau_2 \neq 0$.

THEOREM 3.11. *If G_1 divides G_2 , then u_0 and $G_1 u_1$ are classical of the same type, $\deg(\eta_1) = 3$, and $\tau_n \neq 0$, $n \geq 1$. To be precise, we have:*

(i) *If $H(x) = h(x - \xi)^4$, then G_1 divides G_2 and*

$$\begin{aligned} u_1 = & (p_1 p_2 g_2)^{-1} (x - \xi)^{-2} \tilde{\rho}_1 u_0 + r_0 \delta(x - \xi) \\ & + (R_1(0) + \xi) r_0 \delta'(x - \xi). \end{aligned} \quad (3.25)$$

(ii) If $H(x) = h(x - \xi_1)^2(x - \xi_2)^2$ ($\xi_1 \neq \xi_2$) and $G_1(\xi_1)G_1(\xi_2) = 0$, then G_1 divides G_2 and

$$u_1 = (p_1 p_2 g_2)^{-1} (x - \xi_1)^{-1} (x - \xi_2)^{-1} \tilde{\rho}_1 u_0 \\ + \frac{r_0}{\xi_1 - \xi_2} [(R_1(0) + \xi_1) \delta(x - \xi_2) - (R_1(0) + \xi_2) \delta(x - \xi_1)]. \quad (3.26)$$

Here, $\rho_1(x) = G_1(x) \tilde{\rho}_1(x)$.

Proof. Since $\tau_1, \tau_2 \neq 0$, $\deg(G_1) = 2$ and $\deg(G_2) = 3$. Assume that G_1 divides G_2 . Set $G_2 = G_1 \tilde{G}_2$, $\deg(\tilde{G}_2) = 1$. Then

$$\rho_1 = G_1(2p_1 P_2 - p_2 P_1 \tilde{G}_2) = G_1 \tilde{\rho}_1 \quad (0 \leq \deg(\tilde{\rho}_1) \leq 2)$$

$$\eta_1 = G_1(2p_1 P'_2 - p_2 P'_1 \tilde{G}_2) = G_1 \tilde{\eta}_1 \quad (0 \leq \deg(\tilde{\eta}_1) \leq 1).$$

As in the proof of Theorem 3.9(i), by (3.13), we obtain

$$\tilde{\rho}_1 u_0 = p_1 p_2 G_1 \tilde{G}'_2 u_1 = p_1 p_2 \frac{g_2}{g_1} G_1 u_1 \quad (3.27)$$

so that by (3.4), $(\tilde{\rho}_1 u_0)' = p_2 \tilde{G}'_2 (h_1 G_1 u_1)' = p_2 \tilde{G}'_2 P_1 u_0$. Hence u_0 and $G_1 u_1$ are classical of the same type by Proposition 3.6 and $(\tilde{\rho}_1 G_1 u_1)' = \tilde{\eta}_1 G_1 u_1$. Hence, $\deg(\tilde{\eta}_1) = 1$ and so $\deg(\eta_1) = 3$. Finally, $\tau_n \neq 0$, $n \geq 1$ by Theorem 3.4.

(i) $H(x) = h(x - \xi)^4$, that is, $H(\xi) = H'(\xi) = H''(\xi) = H'''(\xi) = 0$. Since $H'''(x) = 12g_2 G'_1(x)$, $G'_1(\xi) = 0$. Hence we have

$$H(\xi) = G_1(\xi) G'_2(\xi) = 0 \quad \text{and}$$

$$H''(\xi) = 6g_2 G_1(\xi) - 2g_1 G'_2(\xi) = 0$$

so that $G_1(\xi) = G'_2(\xi) = 0$. Then $H'(\xi) = -2g_1 G_2(\xi) = 0$ so that $G_2(\xi) = 0$. Hence $G_1 = g_1(x - \xi)^2$ divides G_2 . Finally, (3.25) comes from (3.27) and Lemma 2.4(iii).

(ii) Assume $H(x) = h(x - \xi_1)^2(x - \xi_2)^2$ ($\xi_1 \neq \xi_2$) and, for example, $G_1(\xi_1) = 0$. Then $G_1(x) = g_1(x - \xi_1)(x - \xi)$ and

$$H'''(x) = 2G'_1 G''_2 = 12h(2x - \xi_1 - \xi) = 12h(2x - \xi_1 - \xi_2)$$

so that $\xi = \xi_2$. Hence $G_1(\xi_1) = G_1(\xi_2) = 0$ and so $G_2(\xi_1) = G_2(\xi_2) = 0$ since $H(\xi_1) = H(\xi_2) = 0$. Therefore, G_1 divides G_2 . Finally, (3.26) comes from (3.27) and Lemma 2.4(ii). ■

Finally, in this section, let us consider the case when $\sigma_n = 0$, $n \geq 1$, so that the coherency (3.3) becomes

$$R_n = Q_n - \tau_{n-1}Q_{n-2}, \quad n \geq 0. \quad (3.28)$$

Note that if u_0 is symmetric, then u_1 must also be symmetric but if u_1 is symmetric, then u_0 may or may not be symmetric in general. When u_0 is symmetric and $\tau_n \neq 0$, $n \geq 1$, (3.28) is the symmetric coherence introduced by Iserles *et al.* [9]. In case of (3.28),

$$\deg(H) = \begin{cases} 4 & \text{if } \tau_1\tau_2 \neq 0 \\ 2 & \text{if } \tau_1 = 0, \tau_2 \neq 0 \text{ or } \tau_1 \neq 0, \tau_2 = 0 \\ 0 & \text{if } \tau_1 = \tau_2 = 0. \end{cases}$$

As before, if $\deg(H) = 0$, then u_0 and u_1 must be classical of the same type and $\tau_n = 0$, $n \geq 1$ and if either $\deg(H) = 2$ or $\deg(H) = 4$ and $\tau_n \neq 0$ for some $n \geq 3$, then u_0 cannot be classical and u_1 cannot be strongly classical. When $\deg(H) = 4$ and u_0 is symmetric, Meijer proved (see Theorem 3 and Theorem 4 in [24]):

THEOREM 3.12. *Assume $\sigma_n = 0$, $n \geq 1$, $\tau_1\tau_2\tau_3 \neq 0$, and u_0 is symmetric. Set*

$$G_1G'_3 - G'_1G_3 = xB(x),$$

where $B(x) = ax^4 + bx^2 + c$ ($a \neq 0$).

(i) *If $B(x) = a(x^2 - \xi^2)^2$, then u_0 and G_1u_1 are classical of the same type and $\tau_n \neq 0$, $n \geq 1$.*

(ii) *If $B(x) = a(x^2 - \xi_1^2)(x^2 - \xi_2^2)$ ($\xi_1 \neq \xi_2$), then u_1 is classical. Moreover, if u_1 is strongly classical, then $\tau_n \neq 0$, $n \geq 1$.*

Meijer assumed $\tau_n \neq 0$, $n \geq 1$ but it is easy to see that the same proof works assuming only $\tau_1\tau_2\tau_3 \neq 0$ and then $\tau_n \neq 0$, $n \geq 1$, by Theorem 3.4.

We have seen several cases of generalized coherency where either u_0 or u_1 is classical. In the next section, we give a complete characterization of companion moment functionals when either u_0 is classical or u_1 is strongly classical together with examples.

4. GENERATING GENERALIZED COHERENT PAIRS

Coherent (i.e., $\sigma_n \neq 0$ and $\tau_n = 0$, $n \geq 1$) and symmetrically coherent (i.e., $\sigma_n = 0$, $\tau_n \neq 0$, $n \geq 1$, and u_0 is symmetric) pairs of positive-definite moment functionals $\{u_0, u_1\}$ are classified in [24] (see also [20]). We now

consider the following problems:

- Characterize generalized coherent pairs $\{u_0, u_1\}$, where either u_0 or u_1 is classical.
- How do we generate generalized coherent pairs?

We will consider only 2-term and 3-term coherent pairs since if otherwise, i.e., $\sigma_n = \tau_n = 0$, $n \geq 1$, then $\{u_0, u_1\}$ is a generalized coherent pair if and only if u_0 is a classical moment functional satisfying $(\varphi u_0)' = \psi u_0$ ($0 \leq \deg(\varphi) \leq 2$, $\deg(\psi) = 1$) and $u_1 = \varphi u_0$.

As in Section 3, let $\{P_n(x)\}_{n=0}^\infty$ or $\{R_n(x)\}_{n=0}^\infty$ be the MOPSs relative to u_0 or u_1 , respectively, when u_0 or u_1 is quasi-definite and $Q_n(x) = \frac{1}{n+1}P'_{n+1}(x)$, $n \geq 0$.

THEOREM 4.1. *Assume that u_0 is a classical moment functional satisfying $(\varphi u_0)' = \psi u_0$ ($0 \leq \deg(\varphi) \leq 2$, $\deg(\psi) = 1$). Then u_1 (not assumed to be quasi-definite a priori) is a 2-term coherent companion of u_0 if and only if*

$$u_1 = (x - \xi)^{-1} \varphi u_0 + a \delta(x - \xi) \quad (4.1)$$

for some complex numbers ξ and a satisfying

$$aQ_n(\xi) + \langle u_0, \varphi \rangle Q_{n-1}^{(1)}(\xi) \neq 0, \quad n \geq 0.$$

In this case, $Q_n(x) = R_n^*(\xi; x)$, $n \geq 0$, $\sigma_n \neq 0$, $n \geq 1$, and

$$R_n(x) = Q_n(x) - \frac{R_{n-1}(\xi)}{R_n(\xi)} \frac{\langle u_1, R_n^2 \rangle}{\langle u_1, R_{n-1}^2 \rangle} Q_{n-1}(x), \quad n \geq 1. \quad (4.2)$$

Proof. Assume that u_1 is a 2-term companion of u_0 , that is,

$$R_n(x) = Q_n(x) - \sigma_n Q_{n-1}(x), \quad n \geq 1$$

and $\sigma_n \neq 0$ for some $n \geq 1$. Then $\sigma_n \neq 0$, $n \geq 1$, by Theorem 3.4. Moreover, by Proposition 2.3, $\{Q_n(x)\}_{n=0}^\infty = \{R_n^*(\xi; x)\}_{n=0}^\infty$ for some ξ with $R_n(\xi) \neq 0$, $n \geq 1$ and $(x - \xi)u_1 = \varphi u_0$ since $\{Q_n(x)\}_{n=0}^\infty$ is the classical MOPS relative to φu_0 . Hence

$$u_1 = (x - \xi)^{-1} \varphi u_0 + \langle u_0, 1 \rangle \delta(x - \xi)$$

so that we have (4.1) by Lemma 2.4(i).

Conversely assume (4.1). Then u_1 is quasi-definite by Lemma 2.5. Since $(x - \xi)u_1 = \varphi u_0$ is quasi-definite (in fact, classical), $Q_n(x) = R_n^*(\xi; x)$, $n \geq 0$, and (4.2) holds (cf. (2.4)). ■

Note that (4.1) implies $(x - \xi)u_1 = \varphi u_0$ is classical so that s_1 (the class number of u_1) is either 0 or 1. If $s_1 = 0$, i.e., u_1 is classical, then $\varphi(\xi) = 0$ so that $u_i \neq u_H$, $i = 0, 1$.

COROLLARY 4.2. *Let $\{u_0, u_1\}$ be a 2-term coherent pair of real moment functionals where u_0 is classical.*

(i) *If u_1 is positive-definite and $[a, b](-\infty \leq a < b \leq \infty)$ is the true interval of orthogonality of u_1 (see Definition 5.2 on p. 29 in [6]), then $Q_n(x)$, $n \geq 1$, has n real simple zeros which interlace with the zeros of $R_n(x)$ and $R_{n+1}(x)$. Moreover, if $\xi \leq a$ (resp. $\xi \geq b$) then φu_0 (resp. $-\varphi u_0$) is positive-definite.*

(ii) *If φu_0 is positive-definite, then $R_n(x)$, $n \geq 1$, has n real simple zeros which interlace with the zeros of $Q_n(x)$ and $Q_{n+1}(x)$.*

Proof. By Theorem 4.1, we have $(x - \xi)u_1 = \varphi u_0$ for some real number ξ with $R_n(\xi) \neq 0$, $n \geq 1$. Then (ii) and the first part of (i) come from Proposition 2.3. For the second part of (i), see Theorem 7.1 on p. 36 in [6].

■

Furthermore,

THEOREM 4.3. *Let u_1 be a strongly classical moment functional satisfying $(\varphi u_1)' = \psi u_1$ ($0 \leq \deg(\varphi) \leq 2$, $\deg(\psi) = 1$) and $\{T_n(x)\}_{n=0}^\infty$ the classical MOPS relative to w with $\frac{1}{n+1}T'_{n+1}(x) = R_n(x)$, $n \geq 0$. Then u_0 is a 2-term coherent companion of u_1 if and only if for some complex number ξ with $T_n(\xi) \neq 0$, $n \geq 1$*

$$u_0 = (x - \xi)w. \quad (4.3)$$

In this case, $P_n(x) = T_n^(\xi; x)$, $n \geq 0$, $\sigma_n \neq 0$, $n \geq 1$, and*

$$R_n(x) = Q_n(x) - \frac{n}{n+1} \frac{T_n(\xi)}{T_{n+1}(\xi)} \frac{\langle w, T_n^2 \rangle}{\langle w, T_{n+1}^2 \rangle} Q_{n-1}(x), \quad n \geq 1. \quad (4.4)$$

Proof. Assume that there are complex numbers σ_n , $n \geq 1$, not all zero such that

$$R_n(x) = Q_n(x) - \sigma_n Q_{n-1}(x), \quad n \geq 1.$$

Then $\sigma_n \neq 0$ ($n \geq 1$) by Theorem 3.4. Integrating both sides of the equation, we obtain

$$T_n(x) = P_n(x) - \frac{n}{n-1} \sigma_{n-1} P_{n-1}(x) - t_n, \quad n \geq 2 \quad (4.5)$$

and $T_1(x) = P_1(x) - \sigma_0 P_0(x)$ ($\sigma_0 \neq 0$). Then $t_n = 0$, $n \geq 2$, and $\sigma_0 \neq 0$ by Lemma 3.5 so that $\langle u_0, T_n \rangle = 0$, $n \geq 2$, and $\langle u_0, T_1 \rangle \neq 0$. Hence $u_0 = (x - \xi)w$ for some ξ by Lemma 2.6 and so $T_n(\xi) \neq 0$, $n \geq 1$, and $P_n(x) = T_n^*(\xi; x)$, $n \geq 0$. Then we have from (4.5) and (2.4)

$$\sigma_n = \frac{n}{n+1} \frac{T_n(\xi)}{T_{n+1}(\xi)} \frac{\langle w, T_n^2 \rangle}{\langle w, T_{n+1}^2 \rangle}, \quad n \geq 1$$

and (4.4) by differentiating (4.5). Conversely assume (4.3) and $T_n(\xi) \neq 0$, $n \geq 1$. Then u_0 is quasi-definite and $P_n(x) = T_n^*(\xi; x)$, $n \geq 0$ so that

$$T_n(x) = P_n(x) - \alpha_n P_{n-1}(x), \quad n \geq 0 \ (\alpha_n \neq 0, n \geq 1).$$

Hence, $R_n(x) = Q_n(x) - (\alpha_{n+1}/(n+1))Q_{n-1}(x)$, $n \geq 1$ so that $\{u_0, u_1\}$ is a 2-term coherent pair. ■

Note that (4.3) implies that s_0 (the class number of u_0) is either 0 or 1. If $s_0 = 0$, i.e., u_0 is classical, then $\varphi(\xi) = 0$ so that $u_i \neq u_H$, $i = 0, 1$.

COROLLARY 4.4. *Let $\{u_0, u_1\}$ be a 2-term coherent pair of real moment functionals where u_1 is strongly classical. If u_0 (resp. u_1) is positive-definite, then $T_n(x)$ (resp. $P_n(x)$), $n \geq 1$, has n real simple zeros which interlace with the zeros of $P_n(x)$ and $P_{n+1}(x)$ (resp. $T_n(x)$ and $T_{n+1}(x)$).*

Proof. In this case, we have (4.3) for some real number ξ with $T_n(\xi) \neq 0$, $n \geq 1$. Hence the conclusion follows from Proposition 2.3. ■

Moreover, if u_0 in Corollary 4.4 is positive-definite, then $Q_n(x)$, $n \geq 1$, also has n real simple zeros. In fact, $\{Q_n(x)\}_{n=0}^\infty$ is a Sturm sequence but not necessarily an OPS unless u_0 is classical. See Theorem 2.3 in [10].

In particular, Theorem 4.1 and Theorem 4.3 imply that for any classical moment functional u_0 (resp. strongly classical moment functional u_1), there are infinitely many companions u_1 (resp. u_0) with $\sigma_n \neq 0$ and $\tau_n = 0$, $n \geq 1$.

EXAMPLES. By Theorem 4.1 and Theorem 4.3, we can relax restrictions on parameters in coherent pairs found by Meijer [24]. For example, in case $u_1 = u_L^{(\alpha)}$ ($\alpha \neq 0, -1, -2, \dots$), $\{u_0, u_L^{(\alpha)}\}$ is coherent if and only if $u_0 = u_L^{(\alpha)} - \xi u_L^{(\alpha-1)}$ and $L_n^{(\alpha-1)}(\xi) \neq 0$, $n \geq 1$. For $\alpha = 0$, $u_L^{(0)}$ is not strongly classical but $\{u_L^{(0)} + a\delta(x), u_L^{(0)}\}$ is coherent for any complex number a satisfying

$$1 + a \sum_{k=0}^n \frac{(k!)^2}{\langle u_L^{(0)}, L_k^{(0)}(x)^2 \rangle} \neq 0, \quad n \geq 0, \quad (4.6)$$

where $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ are the monic Laguerre polynomials. In fact, the condition (4.6) is the necessary and sufficient condition for $u_L^{(0)} + a\delta(x)$ to be quasi-definite (see [15, 17]) and coherency follows from the identity (see (3.14) in [24])

$$L_n^{(\alpha)}(0) = (-1)^n n! \binom{n+\alpha}{n} \quad \text{and} \quad L_n^{(\alpha)}(x) = L_n^{(\alpha+1)}(x) + nL_{n-1}^{(\alpha+1)}(x)$$

(see (5.1.7) and (5.1.13) in [28]).

In particular, $u_L^{(\alpha)}$ ($\alpha \neq -1, -2, \dots$) is self-coherent (i.e., $\{u_L^{(\alpha)}, u_L^{(\alpha)}\}$ is coherent). In fact, by the characterization of classical OPSs by Marcellán *et al.* [16], any classical moment functional u is self-coherent (1-term if $u = u_H$, 2-term if $u = u_L^{(\alpha)}$, symmetric if $u = u_J^{(\alpha, \alpha)}$, 3-term if $u = u_J^{(\alpha, \beta)}, u_B^{(\alpha)}$).

Hermite moment functional u_H has no positive-definite (2-term) companion but $\{(x - \xi)u_H, u_H\}$ is coherent for any ξ with $H_n(\xi) \neq 0$, $n \geq 1$.

We now consider 3-term coherent pairs $\{u_0, u_1\}$, one of which is classical.

THEOREM 4.5. *Let u_0 be a classical moment functional satisfying $(\varphi u_0)' = \psi u_0$ ($0 \leq \deg(\varphi) \leq 2, \deg(\psi) = 1$). Assume $\langle u_0, \varphi \rangle = 1$. Then u_1 is a 3-term companion of u_0 if and only if either*

$$u_1 = (x - \xi_1)^{-1}(x - \xi_2)^{-1}\varphi u_0 + a\delta(x - \xi_1) + b\delta(x - \xi_2) \quad (4.7)$$

or

$$u_1 = (x - \xi_1)^{-2}\varphi u_0 + a\delta(x - \xi_1) + b\delta'(x - \xi_1) \quad (4.8)$$

for some complex numbers ξ_1, ξ_2, a , and b satisfying

(i) in case of (4.7)

$$\left\{ \begin{array}{l} a + b \neq 0, \\ a(\xi_1 - a\xi_1 - b\xi_2)^2 + b(\xi_2 - a\xi_1 - b\xi_2)^2 \neq 1, \\ \left| \begin{array}{cc} a(\xi_2 - \xi_1)Q_{n+1}(\xi_1) - Q_n^{(1)}(\xi_1) & a(\xi_2 - \xi_1)Q_n(\xi_1) - Q_{n-1}^{(1)}(\xi_1) \\ b(\xi_2 - \xi_1)Q_{n+1}(\xi_2) - Q_n^{(1)}(\xi_2) & b(\xi_2 - \xi_1)Q_n(\xi_2) - Q_{n-1}^{(1)}(\xi_2) \end{array} \right| \neq 0, \end{array} \right\} \quad n \geq 0;$$

(ii) in case of (4.9)

$$\left\{ \begin{array}{l} a \neq 0, a - b^2 \neq 0, \\ \left| \begin{array}{l} bQ_{n+1}(\xi_1) + Q_n^{(1)}(\xi_1) \\ b(bQ'_{n+1}(\xi_1) + Q_n^{(1)' }(\xi_1)) + aQ_{n+1}(\xi_1) \end{array} \right| \\ \left| \begin{array}{l} bQ_n(\xi_1) + Q_{n-1}^{(1)}(\xi_1) \\ b(bQ'_n(\xi_1) + Q_{n-1}^{(1)' }(\xi_1)) + aQ_n(\xi_1) \end{array} \right| \end{array} \right\} \neq 0, \quad n \geq 0. \quad (4.10)$$

Proof. Assume that there are complex numbers σ_n and τ_n , $n \geq 1$, such that not all τ_n are zero and

$$R_n(x) = Q_n(x) - \sigma_n Q_{n-1}(x) - \tau_n Q_{n-2}(x), \quad n \geq 2.$$

Then $\tau_n \neq 0$, $n \geq 1$ by Theorem 3.4 and $\langle \varphi u_0, R_n \rangle = 0$, $n \geq 3$, and $\langle \varphi u_0, R_2 \rangle \neq 0$. Hence, $\varphi u_0 = \pi_2 u_1$ by Lemma 2.6 for some polynomial $\pi_2(x)$ of degree 2, which we may assume to be monic. Then we have (4.7) by Lemma 2.4(ii) and (4.9) holds by Theorem 6 in [3] since $\varphi u_0 = \pi_2 u_1$ is quasi-definite if $\pi_2(x) = (x - \xi_1)(x - \xi_2)$ ($\xi_1 \neq \xi_2$). Similarly if $\pi_2(x) = (x - \xi_1)^2$, then (4.8) and (4.10) hold by Lemma 2.4(iii) and Corollary 8 in [3].

Conversely, assume that (4.7) and (4.9) hold. Then $(x - \xi_1)(x - \xi_2)u_1 = \varphi u_0$ and u_1 is quasi-definite. Write $R_n = Q_n + \sum_{j=0}^{n-1} c_{n,j} Q_j$, $n \geq 1$. Then

$$c_{n,j} \langle \varphi u_0, Q_j^2 \rangle = \langle \varphi u_0, R_n Q_j \rangle = \langle u_1, R_n (x - \xi_1)(x - \xi_2) Q_j \rangle, \quad 0 \leq j \leq n-1$$

so that $c_{n,j} = 0$, $0 \leq j < n-2$, and $c_{n,n-2} \neq 0$. Hence u_1 is a 3-term coherent companion of u_0 . In case of (4.8) and (4.10), the proof is the same. ■

Furthermore,

THEOREM 4.6. *Let u_1 be a strongly classical moment functional satisfying $(\varphi u_1)' = \psi u_1$ ($0 \leq \deg(\varphi) \leq 2$, $\deg(\psi) = 1$) and $\{T_n(x)\}_{n=0}^\infty$ the classical MOPS relative to w with $\frac{1}{n+1} T'_{n+1}(x) = R_n(x)$, $n \geq 0$. Then u_0 is a 3-term companion of u_1 if and only if either*

$$u_0 = (x - \xi_1)(x - \xi_2)w \quad (4.11)$$

for some complex numbers $\xi_1 \neq \xi_2$ satisfying

$$\left| \begin{array}{cc} T_n(\xi_1) & T_{n+1}(\xi_1) \\ T_n(\xi_2) & T_{n+1}(\xi_2) \end{array} \right| \neq 0, \quad n \geq 1 \quad (4.12)$$

or

$$u_0 = (x - \xi_1)^2 w \quad (4.13)$$

for some complex number ξ_1 satisfying

$$\begin{vmatrix} T_n(\xi_1) & T_{n+1}(\xi_1) \\ T'_n(\xi_1) & T'_{n+1}(\xi_1) \end{vmatrix} \neq 0, \quad n \geq 1. \quad (4.14)$$

Proof. Assume that there are complex numbers σ_n and τ_n , $n \geq 1$, such that not all τ_n are zero and

$$R_n(x) = Q_n(x) - \sigma_n Q_{n-1}(x) - \tau_{n-1} Q_{n-2}(x), \quad n \geq 2.$$

Then, $\tau_n \neq 0$, $n \geq 1$, by Theorem 3.4 and

$$T_n(x) = P_n(x) - \frac{n}{n-1} \sigma_{n-1} P_{n-1}(x) - \frac{n}{n-2} \tau_{n-2} P_{n-2}(x) - t_n, \quad n \geq 3,$$

$T_2(x) = P_2(x) - 2\sigma_1 P_1(x) - \tau_0 P_0(x)$ ($\tau_0 \neq 0$). Then, $t_n = 0$, $n \geq 3$ by Lemma 3.5 so that $\langle u_0, T_n \rangle = 0$, $n \geq 3$, and $\langle u_0, T_2 \rangle \neq 0$. Hence, $u_0 = \pi_2 w$ for some monic polynomial $\pi_2(x)$ of degree 2. If $\pi_2(x) = (x - \xi_1)(x - \xi_2)$ ($\xi_1 \neq \xi_2$) or $\pi_2(x) = (x - \xi_1)^2$, then (4.12) or (4.14) must hold by Theorem 2.1 in [1] since u_0 is quasi-definite.

Conversely assume that (4.11) and (4.12) hold. Then u_0 is quasi-definite. Write $T_n = P_n + \sum_{j=0}^{n-1} c_{n,j} P_j$, $n \geq 1$. Then

$$c_{n,j} \langle u_0, P_j^2 \rangle = \langle u_0, T_n P_j \rangle = \langle w, (x - \xi_1)(x - \xi_2) T_n P_j \rangle, \quad 0 \leq j \leq n-1,$$

so that $c_{n,j} = 0$, $0 \leq j < n-2$, and $c_{n,n-2} \neq 0$. Hence

$$T_n(x) = P_n(x) + c_{n,n-1} P_{n-1}(x) + c_{n,n-2} P_{n-2}(x), \quad n \geq 2$$

which implies, by differentiation, that u_0 is a 3-term companion of u_1 . In case of (4.13) and (4.14), the proof is the same. ■

Theorem 4.5 and Theorem 4.6 imply that for any classical moment functional u_0 (resp. strongly classical moment functional u_1), there are infinitely many 3-term companions u_1 (resp. u_0) with $\tau_n \neq 0$, $n \geq 1$.

EXAMPLES. Again by Theorem 4.5 and Theorem 4.6, we can relax restrictions on parameters in symmetric coherent pairs found by Meijer

[24]. For example, Meijer [24] showed that

$$\left\{ e^{-x^2} dx, \frac{1}{x^2 + \xi^2} e^{-x^2} dx \right\} \quad (\xi \neq 0)$$

is symmetrically coherent. More generally, let $x^2 + ax + b = (x - \xi) \cdot (x - \bar{\xi})$ be a real polynomial with $a^2 - 4b < 0$. Then

$$\left\{ e^{-x^2} dx, \frac{1}{x^2 + ax + b} e^{-x^2} dx \right\}$$

is a 3-term coherent pair (not symmetric unless $a = 0$). The moment functional u_1 with integral representation

$$\langle u_1, \pi(x) \rangle = \int_{-\infty}^{\infty} \frac{\pi(x)}{x^2 + ax + b} e^{-x^2} dx$$

can be expressed as in (4.7). Let

$$\langle u_H, \pi(x) \rangle = \int_{-\infty}^{\infty} \pi(x) e^{-x^2} dx.$$

Then

$$\begin{aligned} \langle (x^2 + ax + b)^{-1} u_H, \pi \rangle &= \left\langle u_H, \frac{\pi(x) - \phi(x)}{x^2 + ax + b} \right\rangle \\ &= \int_{-\infty}^{\infty} \frac{\pi(x) - \phi(x)}{x^2 + ax + b} e^{-x^2} dx, \end{aligned}$$

where $\phi(x) = \frac{i}{2\operatorname{Im} \xi} \{ \pi(\xi)(\bar{\xi} - x) + \pi(\bar{\xi})(x - \xi) \}$. Hence

$$\begin{aligned} &\langle (x^2 + ax + b)^{-1} u_H, \pi \rangle \\ &= \langle u_1, \pi \rangle - \frac{i}{2\operatorname{Im} \xi} \{ \langle u_1, \bar{\xi} - x \rangle \pi(\xi) + \langle u_1, x - \xi \rangle \pi(\bar{\xi}) \} \end{aligned}$$

so that

$$\begin{aligned} u_1 &= (x^2 + ax + b)^{-1} u_H \\ &\quad + \frac{i}{2\operatorname{Im} \xi} \{ \langle u_1, \bar{\xi} - x \rangle \delta(x - \xi) + \langle u_1, x - \xi \rangle \delta(x - \bar{\xi}) \}. \end{aligned}$$

Similarly, the Gegenbauer case handled in [24] can be extended also to include non-symmetric generalized coherent pairs. For example,

$$\{u_j^{(\alpha, \beta)}, u_j^{(\alpha, \beta)} + a\delta(x-1) + b\delta(x+1)\}$$

is a 3-term coherent pair (not symmetric unless $\alpha = \beta$ and $a = b$) as long as $u_1 = u_j^{(\alpha, \beta)} + a\delta(x-1) + b\delta(x+1)$ is quasi-definite. See Theorem 3.1 in [15] or Theorem 4 in [7] for the conditions for u_1 to be quasi-definite.

Generalized Jacobi polynomials $\{P_n^{(\alpha, \beta; a, b)}(x)\}_{n=0}^\infty$, orthogonal with respect to $u_j^{(\alpha, \beta)} + a\delta(x-1) + b\delta(x+1)$ were first introduced by Koornwinder [11] for $\alpha, \beta > -1$ and $a, b \geq 0$.

We now consider $\{T_n(x)\}_{n=0}^\infty$ the monic Chebychev polynomials of the first kind, which are orthogonal with respect to $u_j^{(-1/2, -1/2)}$. Then

$$\frac{1}{n+1}T'_{n+1}(x) = U_n(x), \quad n \geq 0$$

are the monic Chebychev polynomials of the second kind. For them the three-term recurrence relation is

$$U_{n+1}(x) = xU_n(x) - \frac{1}{4}U_{n-1}(x), \quad n \geq 1 \quad (U_0(x) = 1, U_1(x) = x).$$

Now, define a monic PS $\{R_n(x)\}_{n=0}^\infty$ by

$$R_n(x) = U_n(x) + aU_{n-1}(x) + bU_{n-2}(x), \quad n \geq 0 \quad (U_{-1} = U_{-2} = 0), \quad (4.15)$$

where a and b are complex numbers with $|a| + |b| \neq 0$. Then it is easy to prove (see [3]) that the $\{R_n(x)\}_{n=0}^\infty$ satisfy

$$R_{n+1}(x) = xR_n(x) - \frac{1}{4}R_{n-1}(x), \quad n \geq 2$$

$$R_2(x) = xR_1(x) - \left(\frac{1}{4} - b\right)R_0(x), \quad R_1(x) = x + a.$$

Hence, $\{R_n(x)\}_{n=0}^\infty$ is an MOPS (Bernstein polynomials; see p. 31 in [28]) (resp. symmetric MOPS) if and only if $b \neq \frac{1}{4}$ (resp. $a = 0$ and $b \neq \frac{1}{4}$). The relation (4.15) implies that $\{u_j^{(-1/2, -1/2)}, u_1\}$ is a 2-term coherent pair or a 3-term coherent pair if $b = 0$ and $b \neq 0, \frac{1}{4}$, respectively, where u_1 is the canonical moment functional of $\{R_n(x)\}_{n=0}^\infty$.

When $b = 0$ (so that $a \neq 0$), we have $(x + \frac{5}{4}a)u_1 = u_j^{(1/2, 1/2)}$ so that

$$u_1 = \left(x + \frac{5}{4}a\right)^{-1} u_j^{(\frac{1}{2}, \frac{1}{2})} + \delta\left(x + \frac{5}{4}a\right).$$

When $a = 0$ and $b \neq 0, \frac{1}{4}$, we have

$$\left\{x^2 + \frac{1}{b}\left(b - \frac{1}{4}\right)^2\right\}u_1 = (x - \xi_1)(x - \xi_2)u_1 = u_j^{(\frac{1}{2}, \frac{1}{2})}.$$

Thus (cf. Lemma 2.4(ii))

$$\begin{aligned} u_1 &= \left\{x^2 + \frac{1}{b}\left(b - \frac{1}{4}\right)^2\right\}^{-1} u_j^{(\frac{1}{2}, \frac{1}{2})} \\ &\quad + \frac{1}{\xi_2 - \xi_1} \{\xi_2 \delta(x - \xi_1) - \xi_1 \delta(x - \xi_2)\} \end{aligned} \quad (4.16)$$

and $\{u_j^{(-1/2, -1/2)}, u_1\}$ is symmetrically coherent. Some special cases of (4.16) are handled in [3] for $b = 1$ and [25] for $\frac{1}{8} < b < \frac{1}{4}$.

We now give two more ways to generate generalized coherent pairs $\{u_0, u_1\}$, where none of u_0 and u_1 need to be classical.

THEOREM 4.7. *Let u be a classical moment functional satisfying $(\varphi u)' = \psi u$ ($0 \leq \deg(\varphi) \leq 2, \deg(\psi) = 1$). If $\{u_0, \varphi u\}$ and $\{u, u_1\}$ are 2-term coherent pairs, then $\{u_0, u_1\}$ is a 3-term coherent pair.*

Proof. Let $\{S_n(x)\}_{n=0}^\infty$ and $\{T_n(x)\}_{n=0}^\infty$ be the MOPSs relative to φu and u , respectively. Then $\frac{1}{n+1}T'_{n+1}(x) = S_n(x)$, $n \geq 0$, and

$$S_n = \frac{1}{n+1}P'_{n+1} - \frac{a_n}{n}P'_n, \quad n \geq 1, \quad (4.17)$$

$$R_n = \frac{1}{n+1}T'_{n+1} - \frac{b_n}{n}T'_n, \quad n \geq 1. \quad (4.18)$$

Hence $a_n \neq 0, b_n \neq 0, n \geq 1$ by Lemma 3.5 and

$$R_n = S_n - b_n S_{n-1} = \frac{1}{n+1}P'_{n+1} - \frac{a_n + b_n}{n}P'_n + \frac{a_{n-1}b_n}{n-1}P'_{n-1}, \quad n \geq 2,$$

so that $\{u_0, u_1\}$ is a 3-term coherent pair. ■

For example (cf. Meijer [24]),

$$\left\{ (x - \xi_1)u_L^{(\alpha)}, \frac{u_L^{(\alpha+1)}}{x - \xi_2} + a\delta(x - \xi_2) \right\}$$

is a 3-term coherent pair for any complex numbers ξ_1 , ξ_2 , and $a \neq 0$ satisfying $L_n^{(\alpha)}(\xi_1) \neq 0$ and $aL_n^{(\alpha+1)}(\xi_2) + \langle u_L^{(\alpha)}, x \rangle L_{n-1}^{(\alpha+1,1)}(\xi_2) \neq 0$, $n \geq 1$ ($\alpha \neq -1, -2, \dots$), where $\{L_n^{(\alpha+1,1)}(x)\}_{n=0}^\infty$ are the monic numerator polynomials (or the associated polynomials of the first kind) for $\{L_n^{(\alpha+1)}(x)\}_{n=0}^\infty$.

THEOREM 4.8. *Let $\{u, u_1\}$ be a 2-term coherent pair satisfying (4.18). Then for any $\xi \in \mathbb{C}$ with $T_n(\xi) \neq 0$, $n \geq 1$, $\{(x - \xi)u, u_1\}$ is a 3-term coherent pair.*

Proof. For any $\xi \in \mathbb{C}$ with $T_n(\xi) \neq 0$, $n \geq 1$, $(x - \xi)u$ is quasi-definite and its corresponding MOPS is $\{P_n(x)\}_{n=0}^\infty = \{T_n^*(\xi; x)\}_{n=0}^\infty$. Then by (2.4)

$$T_n = P_n - \alpha_n P_{n-1}, \quad \alpha_n = \frac{T_{n-1}(\xi)}{T_n(\xi)} \frac{\langle u, T_n^2 \rangle}{\langle u, T_{n-1}^2 \rangle}, \quad n \geq 1.$$

Hence

$$R_n = \frac{1}{n+1} P'_{n+1} - \left(\frac{\alpha_{n+1}}{n+1} + \frac{b_n}{n} \right) P'_n + \frac{\alpha_n b_n}{n} P'_{n-1}, \quad n \geq 2.$$

■

Combining Theorem 4.1 and Theorem 4.3 with Theorem 4.8, we get:

COROLLARY 4.9. (i) *Let u_0 be a classical moment functional satisfying $(\varphi u_0)' = \psi u_0$ ($0 \leq \deg(\varphi) \leq 2$, $\deg(\psi) = 1$). Then*

$$\left\{ (x - \xi_1)u_0, (x - \xi_2)^{-1} \varphi u_0 + a\delta(x - \xi_2) \right\}$$

is a 3-term coherent pair if

$$P_n(\xi_1) \neq 0 \quad \text{and} \quad aQ_n(\xi_2) + \langle u_0, \varphi \rangle Q_{n-1}^{(1)}(\xi_2) \neq 0, \quad n \geq 0.$$

(ii) *Let u_1 be a strongly classical moment functional satisfying $(\varphi u_1)' = \psi u_1$ ($0 \leq \deg(\varphi) \leq 2$, $\deg(\psi) = 1$) and w a classical moment functional with $\varphi w = u_1$. Then*

$$\left\{ (x - \xi_1)(x - \xi_2)w, u_1 \right\}$$

is a 3-term coherent pair if

$$T_n(\xi_1) \neq 0 \quad \text{and} \quad S_n(\xi_2) \neq 0, \quad n \geq 1,$$

where $\{S_n(x)\}_{n=0}^\infty$ and $\{T_n(x)\}_{n=0}^\infty$ are the MOPSs relative to w and $(x - \xi_2)w$, respectively.

5. ZEROS AND GENERALIZED FOURIER SERIES COEFFICIENTS

Now assume u_0 and u_1 are two positive-definite moment functionals defined by

$$\langle u_i, \pi(x) \rangle = \int_a^b \pi(x) d\mu_i(x), \quad i = 0, 1,$$

where μ_0 and μ_1 are monotonic increasing functions with infinite set of points as support and finite moments of all orders. As before, let $\{P_n(x)\}_{n=0}^\infty$ and $\{R_n(x)\}_{n=0}^\infty$ be the MOPSs relative to u_0 and u_1 , respectively, satisfying the three-term recurrence relations (3.1) and (3.2).

Let $\{P_n^{(\lambda)}(x)\}_{n=0}^\infty$ be the monic Sobolev OPS relative to the Sobolev inner product

$$\phi_\lambda(f, g) := \langle f, g \rangle_0 + \lambda \langle f', g' \rangle_1, \quad (5.1)$$

where $\lambda \geq 0$ and

$$\langle f, g \rangle_0 := \langle u_0, fg \rangle = \int_a^b f(x)g(x) d\mu_0(x),$$

$$\langle f, g \rangle_1 := \langle u_1, fg \rangle = \int_a^b f(x)g(x) d\mu_1(x).$$

From now on, we assume that $\{u_0, u_1\}$ is a generalized coherent pair. Then, de Bruin and Meijer [4] proved the following:

THEOREM 5.1 (see [4, Theorem 4.1]). *For any fixed $n \geq 3$, we have the following:*

(i) *If $\tau_{n-2} > 0$, then for sufficiently large λ , $P_n^{(\lambda)}(x)$ has n real simple zeros, interlacing with the zeros of $P_{n-1}(x)$ and $R_{n-1}(x)$. At most two of the zeros of $P_n^{(\lambda)}(x)$ are outside (a, b) ;*

(ii) *If $\tau_{n-2} = 0$ and $\sigma_{n-1} \neq 0$, then for sufficiently large λ , $P_n^{(\lambda)}(x)$ has n real simple zeros, interlacing with the zeros of $P_n(x)$, $P_{n-1}(x)$, and $R_{n-1}(x)$. At most one of the zeros of $P_n^{(\lambda)}(x)$ is outside (a, b) .*

Hence, we are interested in generalized coherent pairs, where $\tau_n \geq 0$.

If $\{P_n(x)\}_{n=0}^\infty$ is a classical OPS, then $\{Q_n(x)\}_{n=0}^\infty$ with $Q_n = \frac{1}{n+1}P'_{n+1}$ is also a classical OPS satisfying

$$Q_{n+1}(x) = (x - \tilde{b}_n)Q_n(x) - \tilde{c}_n Q_{n-1}(x). \quad (5.2)$$

If $\{R_n(x)\}_{n=0}^\infty$ is a strongly classical OPS, then $\{T_n(x)\}_{n=0}^\infty$ with $R_n = \frac{1}{n+1}T'_{n+1}$ is also a classical OPS satisfying

$$T_{n+1}(x) = (x - \tilde{\delta}_n)T_n(x) - \tilde{\epsilon}_n T_{n-1}(x). \quad (5.3)$$

THEOREM 5.2. *Let $\{u_0, u_1\}$ be a generalized coherent pair.*

(i) *If u_0 is classical and $\tau_1 = \gamma_1 - \tilde{c}_1 + (\tilde{b}_2 - \beta_1)(\beta_0 - \tilde{b}_0) > 0$ (resp. $\tau_1 = 0$), then $\tau_n > 0$ (resp., $\tau_n = 0$) for all $n \geq 1$.*

(ii) *Assume that u_1 is strongly classical. If $\{T_n(x)\}_{n=0}^\infty$ is a positive-definite OPS and $\tilde{\tau}_0 := \tilde{\epsilon}_1 - c_1 + (b_2 - \tilde{\delta}_1)(\tilde{\delta}_0 - b_0) > 0$ (resp. $\tilde{\tau}_0 = 0$), then $\tau_n > 0$ (resp. $\tau_n = 0$) for all $n \geq 1$.*

Proof. (i) If u_0 is classical, then $\{Q_n(x)\}_{n=0}^\infty$ is also a classical MOPS satisfying (5.2). Then by (3.3) and Lemma 3.5 we have

$$\tau_n = \tau_{n-1} \frac{\gamma_{n+1}}{\tilde{c}_{n-1}}, \quad n \geq 2,$$

$$\tau_1 = \gamma_1 - \tilde{c}_1 + (\tilde{b}_2 - \beta_1)(\beta_0 - \tilde{b}_0).$$

For any $f \in W_2^1([-\infty, \infty), d\mu_0, d\mu_1] := \{f | f \in L_{\mu_0}^2, f' \in L_{\mu_1}^2\}$, we can expand f in a Fourier series for the monic Sobolev orthogonal polynomials $\{P_n^{(\lambda)}(x)\}_{n=0}^\infty$,

$$f \sim \sum_{n=0}^{\infty} \frac{f_n}{s_n} P_n^{(\lambda)}(x),$$

where $f_n = f_n(\lambda) = \phi_\lambda(f, P_n^{(\lambda)})$ and $s_n = s_n(\lambda) = \phi_\lambda(P_n^{(\lambda)}, P_n^{(\lambda)})$. We assume that $\{u_0, u_1\}$ is a generalized coherent pair. Then it is easy to see (see Eq. (5.3) in [4]) that

$$P_{n+1} - \tilde{\sigma}_n P_n - \tilde{\tau}_{n-1} P_{n-1} = P_{n+1}^{(\lambda)} - a_n P_n^{(\lambda)} - b_{n-1} P_{n-1}^{(\lambda)}, \quad n \geq 0, \quad (5.4)$$

where $\tilde{\sigma}_n = \frac{n+1}{n} \sigma_n$, $\tilde{\tau}_n = \frac{n+2}{n} \tau_n$, $n \geq 1$, $\tilde{\sigma}_0 = \tilde{\tau}_{-1} = \tilde{\tau}_0 = a_0 = b_{-1} = b_0 = P_{-1}(x) = P_{-1}^{(\lambda)}(x) = 0$, and

$$a_n s_n = \tilde{\sigma}_n \langle P_n, P_n^{(\lambda)} \rangle_0 + \tilde{\tau}_{n-1} \langle P_{n-1}, P_n^{(\lambda)} \rangle_0, \quad n \geq 1,$$

$$b_n s_n = \tilde{\tau}_n \langle P_n, P_n^{(\lambda)} \rangle_0, \quad n \geq 1.$$

Since

$$\langle P_n, P_n^{(\lambda)} \rangle_0 = \langle P_n, P_n \rangle_0 = p_n, \quad n \geq 0$$

and

$$\begin{aligned} \langle P_{n-1}, P_n^{(\lambda)} \rangle_0 &= \langle P_{n-1}, a_{n-1} P_{n-1}^{(\lambda)} - \tilde{\sigma}_{n-1} P_{n-1} \rangle_0 = (a_{n-1} - \tilde{\sigma}_{n-1}) p_{n-1}, \\ & \quad n \geq 1, \\ a_n s_n &= \tilde{\sigma}_n p_n + \tilde{\tau}_{n-1} (a_{n-1} - \tilde{\sigma}_{n-1}) p_{n-1}, \quad n \geq 1 \end{aligned} \quad (5.5)$$

and

$$b_n s_n = \tilde{\tau}_n p_n, \quad n \geq 1. \quad (5.6)$$

First, to evaluate $\{s_n\}$, we have by (5.4)

$$\begin{aligned} s_{n+1} &= \phi_\lambda(P_{n+1}^{(\lambda)}, P_{n+1}^{(\lambda)}) = \phi_\lambda(P_{n+1}^{(\lambda)} - a_n P_n^{(\lambda)} - b_{n-1} P_{n-1}^{(\lambda)}, P_{n+1}^{(\lambda)}) \\ &= \phi_\lambda(P_{n+1} - \tilde{\sigma}_n P_n - \tilde{\tau}_{n-1} P_{n-1}, P_{n+1}^{(\lambda)}), \quad n \geq 0 \end{aligned}$$

so that

$$\begin{aligned} s_{n+1} &= p_{n+1} + \tilde{\sigma}_n (\tilde{\sigma}_n - a_n) p_n \\ &\quad + \tilde{\tau}_{n-1} [\tilde{\tau}_{n-1} - b_{n-1} - a_n (a_{n-1} - \tilde{\sigma}_{n-1})] p_{n-1} + \lambda(n+1)^2 r_n, \\ & \quad n \geq 0 \end{aligned}$$

with $s_0 = p_0$.

Finally, to evaluate $\{f_n\}$, we have by (5.1)

$$f_n = \langle f, P_n^{(\lambda)} \rangle_0 + \lambda \langle f', P_n^{(\lambda)'} \rangle_1, \quad n \geq 0. \quad (5.8)$$

Since $\{P_n(x)\}_{n=0}^\infty$ is a positive-definite OPS, $\{Q_n(x)\}_{n=0}^\infty$ is also a positive-definite OPS so that $\tilde{c}_n > 0$ and $\gamma_n > 0$ for all $n \geq 1$. Hence we have the result.

(ii) Assume that $\{T_n(x)\}_{n=0}^\infty$ is a positive-definite classical MOPS. Then, by (3.3) and Lemma 3.5 we have

$$\begin{aligned} T_n(x) &= P_n(x) - \tilde{\sigma}_{n-1} P_{n-1}(x) - \tilde{\tau}_{n-1} P_{n-2}(x), \quad n \geq 0 \\ & \quad (P_{-1}(x) = P_{-2}(x) = 0), \end{aligned}$$

where $\tilde{\sigma}_{-1} = \tilde{\tau}_{-2} = \tilde{\tau}_{-1} = 0$, $\tilde{\sigma}_0 = \tilde{\delta}_0 - b_0$, $\tilde{\sigma}_n = \frac{n+1}{n}\sigma_n$, $\tilde{\tau}_n = \frac{n+2}{n}\tau_n$, $n \geq 1$ so that

$$\tilde{\tau}_n = \tilde{\tau}_{n-1} \frac{\tilde{\epsilon}_{n+2}}{c_n}, \quad n \geq 1,$$

$$\tilde{\tau}_0 = \tilde{\epsilon}_1 - c_1 + (b_2 - \tilde{\delta}_1)(\tilde{\delta}_0 - b_0).$$

Since $\{P_n(x)\}_{n=0}^\infty$ and $\{T_n(x)\}_{n=0}^\infty$ are positive-definite OPSs, $c_n > 0$ and $\tilde{\epsilon}_n > 0$ for all $n \geq 1$. Hence we have the result. ■

THEOREM 5.3. *Let u_0 , u_1 , and u be positive-definite moment functionals as in Theorem 4.7. Let $\{S_n(x)\}_{n=0}^\infty$ and $\{T_n(x)\}_{n=0}^\infty$ be the MOPs relative to φu and u , respectively, satisfying the three-term recurrence relations (3.16). If $\tau_1 = -a_1 b_2 > 0$, then $\tau_n > 0$ for all $n \geq 1$.*

Proof. By Theorem 4.7, we have

$$R_n = \frac{1}{n+1}P'_{n+1} - \frac{a_n + b_n}{n}P'_n + \frac{a_{n-1}b_n}{n-1}P'_{n-1}, \quad n \geq 2$$

so that $\tau_n = -a_n b_{n+1}$, $n \geq 1$. Since $\{u_0, \varphi u\}$ is a 2-term coherent pair satisfying (4.17), we have by Lemma 3.5

$$\tilde{a}_n = \tilde{a}_{n-1} \frac{\tilde{\epsilon}_{n+1}}{c_n}, \quad n \geq 1,$$

where $\tilde{a}_n = \frac{n+1}{n}a_n$, $n \geq 1$, and $\tilde{a}_0 = \tilde{\delta}_0 - b_0$. Since $\{u, u_1\}$ is also a 2-term coherent pair satisfying (4.18) and u is classical, we have by Lemma 3.5

$$b_n = b_{n-1} \frac{\gamma_n}{\epsilon_{n-1}}, \quad n \geq 2, \quad \text{and} \quad b_1 = \beta_0 = \delta_0.$$

Hence

$$\tau_n = \frac{n^2}{(n-1)(n+1)} \frac{\tilde{\epsilon}_{n+1}\gamma_{n+1}}{c_n\epsilon_n} \tau_{n-1}, \quad n \geq 2.$$

Since u is positive-definite, φu is also positive-definite so that c_n , γ_n , ϵ_n , and $\tilde{\epsilon}_n$ are all positive for $n \geq 1$. Hence the conclusion follows. ■

In [9], Iserles *et al.* found an efficient algorithm to evaluate the generalized Fourier coefficients when $\{u_0, u_1\}$ is a coherent pair. Here, we consider the same problem when $\{u_0, u_1\}$ is a generalized coherent pair.

By (5.4), we have

$$\begin{aligned}\langle f, P_{n+1}^{(\lambda)} \rangle_0 &= \langle f, P_{n+1} - \tilde{\sigma}_n P_n - \tilde{\tau}_{n-1} P_{n-1} \rangle_0 \\ &\quad + a_n \langle f, P_n^{(\lambda)} \rangle_0 + b_{n-1} \langle f, P_{n-1}^{(\lambda)} \rangle_0, \quad n \geq 0.\end{aligned}\quad (5.9)$$

Since $(n+1)R_n = P_{n+1}^{(\lambda)'} - a_n P_n^{(\lambda)'} - b_{n-1} P_{n-1}^{(\lambda)'}$, $n \geq 0$ by (3.3) and (5.4), we also have

$$\begin{aligned}\langle f', P_{n+1}^{(\lambda)'} \rangle_1 &= (n+1) \langle f', R_n \rangle_1 + a_n \langle f', P_n^{(\lambda)'} \rangle_1 + b_{n-1} \langle f', P_{n-1}^{(\lambda)'} \rangle_1, \\ &\quad n \geq 0.\end{aligned}\quad (5.10)$$

Hence by (5.8), (5.9), and (5.10), we have

$$\begin{aligned}f_{n+1} &= a_n f_n + b_{n-1} f_{n-1} + \langle f, P_{n+1} - \tilde{\sigma}_n P_n - \tilde{\tau}_{n-1} P_{n-1} \rangle_0 \\ &\quad + \lambda(n+1) \langle f', R_n \rangle_1, \quad n \geq 0\end{aligned}\quad (5.11)$$

with $f_0 = \langle f, 1 \rangle_0$. Hence, to evaluate f_n , it is enough to evaluate $\phi_\lambda(f, P_{n+1} - \tilde{\sigma}_n P_n - \tilde{\tau}_{n-1} P_{n-1})$, that is, $\langle f, P_{n+1} - \tilde{\sigma}_n P_n - \tilde{\tau}_{n-1} P_{n-1} \rangle_0$ and $(n+1) \langle f', R_n \rangle_1$, and then use the recursion (5.11).

Next we will describe the algorithm in order to compute $\{a_n\}$, $\{b_n\}$, and $\{s_n\}$.

STARTING DATA:

$$a_0 = 0 \quad s_1 = p_1 + \lambda r_0 \quad b_0 = 0$$

STEP 1 FROM (5.6) AND s_1 WE GET b_1

FROM (5.5) AND s_1 WE GET a_1

USE (5.7) TO GET s_2 FROM THE ABOVE

INFORMATION

$$a_1 \quad s_2 \quad b_1$$

STEP 2 FROM (5.6) AND s_2 WE GET b_2

FROM (5.5) AND s_2 WE GET a_2

USE (5.7) TO GET S_3 FROM THE ABOVE
INFORMATION

$$\begin{array}{ccc} a_2 & s_3 & b_2 \\ & \vdots & \\ & \vdots & \end{array}$$

STEP n FROM (5.6) AND s_n WE GET b_n

FROM (5.5) AND s_n WE GET a_n

USE (5.7) TO GET s_{n+1} FROM THE ABOVE
INFORMATION

On the other hand, from (5.11) for $n \geq 0$

$$f_{n+1} = a_n f_n + b_{n-1} f_{n-1} + d_{n+1} - \tilde{\sigma}_n d_n - \tilde{\tau}_{n-1} d_{n-1} + \lambda(n+1)e_n, \quad (5.12)$$

where $d_n = \langle f, P_n \rangle_0$ and $r_n = \langle f', R_n \rangle_1$. This means that we need initial conditions $f_0 = \langle f, 1 \rangle_0$, $f_1 = \langle f, P_1^{(\lambda)} \rangle = \langle f, P_1 \rangle = d_1 + \lambda e_0$. Of course, all the other elements which are involved in (5.12) are known in every step.

Note that we do not need to find the explicit expression of the Sobolev orthogonal polynomials $P_n^{(\lambda)}$ in the above computation.

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