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NOTE

An Extension of Carlson's Inequality

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An extension of Carlson's inequality is made by using the Euler–Maclaurin summation formula. The integral analogues of this inequality are also presented.

Key Words: Carlson's inequality; Euler–Maclaurin summation formula; integral analogues of Carlson's inequality.

1. INTRODUCTION

If $a_n \geq 0$ and $0 < \sum_{n=1}^{\infty} n^2 a_n^2 < \infty$, then Carlson's inequality (see Carlson [1], Beckenbach and Bellman [2], and Kuang [3]) is

$$\left(\sum_{n=1}^{\infty} a_n\right)^4 < \pi^2 \left(\sum_{n=1}^{\infty} a_n^2\right)\left(\sum_{n=1}^{\infty} n^2 a_n^2\right), \tag{1.1}$$

where π^2 is the best possible.

If $f(x) \geq 0$, then the integral analogue of Carlson's inequality is

$$\left(\int_0^{\infty} f(x)\, dx\right)^4 \leq \pi^2 \left(\int_0^{\infty} f^2(x)\, dx\right)\left(\int_0^{\infty} x^2 f^2(x)\, dx\right), \tag{1.2}$$

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where π^2 is the best possible, provided the integrals on the right-hand side of (1.2) exist.

Remark 1. It is a well-known fact that (1.1) holds for $a_n > 0$ and that not all a_n are zero. Mikhlin [4, p. 7] noted that $f(x) \geq 0$ and $f(x) \neq 0$ represent conditions for inequality in (1.2). However, these conditions are not sufficient since equality in (1.2) holds when $f(x) = (\alpha + \beta x^2)^{-1}$ for any positive numbers α and β.

In recent years, considerable attention has been given to generalizations of Carlson's inequalities (1.1) and (1.2) by several authors, including Mikhlin [4]. As pointed out by Mikhlin, inequality (1.1) may be strengthened as

$$\left(\sum_{n=1}^{\infty} a_n\right)^4 < \pi^2 \left(\sum_{n=1}^{\infty} a_n^2\right)\left(\sum_{n=1}^{\infty}\left(n - \frac{1}{2}\right)^2 a_n^2\right). \tag{1.3}$$

The main purpose of this paper is to study an improvement and extension of (1.1) and (1.2). More precisely, we prove the following:

THEOREM 1. *Let* $S_\alpha = \sum_{n=1}^{\infty} n^\alpha a_n^p$, $S_\beta = \sum_{n=1}^{\infty} n^\beta a_n^p$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \beta < p - 1 < \alpha$. *Suppose* $a_n \geq 0$, $n = 1, 2, \ldots,$ *and* $0 < S_\alpha, S_\beta < \infty$, *then*

$$\left(\sum_{n=1}^{\infty} a_n\right)^p < 2\left\{\frac{S_\alpha^{\lambda_\alpha}}{(\alpha - \beta) S_\beta^{\lambda_\beta}} B(\lambda_\beta, (-\lambda_\alpha)) - c(p, \alpha, \beta)\right\}^{p/q}$$
$$\times \left(\sum_{n=1}^{\infty} n^\alpha a_n^p\right)\left(\sum_{n=1}^{\infty} n^\beta a_n^p\right), \tag{1.4}$$

where

$$\lambda_\alpha = \frac{p - \alpha q}{p(\alpha - \beta)}, \qquad \lambda_\beta = \frac{p - \beta q}{p(\alpha - \beta)}, \tag{1.5}$$

and

$$c(p, \alpha, \beta) = \int_0^1 \frac{dx}{(S_\beta x^\alpha + S_\alpha x^\beta)^{q/p}} - \frac{1}{(S_\alpha + S_\beta)^{q/p}} > 0 \tag{1.6}$$

and $B(u, v)$ *is the Beta function*.

Remark 2. When $p = q = \alpha = 2$ and $\beta = 0$, inequality (1.4) reduces to

$$\left(\sum_{n=1}^{\infty} a_n\right)^4 < (\pi - 2c)^2 \left(\sum_{n=1}^{\infty} a_n^2\right)\left(\sum_{n=1}^{\infty} n^2 a_n^2\right), \tag{1.7}$$

where

$$c = G(S) = \arctan S - \frac{S}{1 + S^2} > 0, \qquad S = \left(\frac{S_0}{S_2}\right)^{1/2}, \tag{1.8}$$

and

$$S_0 = \sum_{n=1}^{\infty} a_n, \qquad S_2 = \sum_{n=1}^{\infty} n^2 a_n^2.$$

Hence, inequalities (1.4) and (1.7) are new improvement and extension of (1.1).

THEOREM 2. *Let $f(x) \geq 0$, $a > 0$, $0 < \beta < p - 1 < \alpha$, $\frac{1}{p} + \frac{1}{q} = 1$, and the integrals $S_\alpha = \int_a^\infty x^\alpha f^p(x)\,dx$ and $S_\beta = \int_a^\infty x^\beta f^p(x)\,dx$ exist. If $0 < S_\alpha$, $S_\beta < \infty$, then $f \in L[a, \infty)$ and*

$$\left(\int_a^\infty f(x)\,dx\right)^p \leq 2\left\{\frac{S_\alpha^{\lambda_\alpha}}{(\alpha - \beta)S_\beta^{\lambda_\beta}} B(\lambda_\beta, (-\lambda_\alpha)) - c(p, \alpha, \beta)\right\}^{p/q} \times \left(\int_a^\infty x^\alpha f^p(x)\,dx\right)\left(\int_a^\infty x^\beta f^p(x)\,dx\right), \tag{1.9}$$

with equality occurring only if $f(x) = (S_\beta x^\alpha + S_\alpha x^\beta)^{-q/p}$, where

$$c(p, \alpha, \beta) = \int_0^a (S_\beta x^\alpha + S_\alpha x^\beta)^{-q/p}\,dx > 0 \tag{1.10}$$

and λ_α, λ_β are defined by (1.5).

Remark 3. When $p = q = \alpha = 2$ and $\beta = 0$, then the inequality (1.9) reduces to

$$\left(\int_a^\infty f(x)\,dx\right)^4 < \left\{\pi - 2\arctan\left(\frac{S_0}{S_2}\right)^{1/2} a\right\}^2 \left(\int_a^\infty f^2(x)\,dx\right) \times \left(\int_a^\infty x^2 f^2(x)\,dx\right), \tag{1.11}$$

with equality occurring only if $f(x) = (S_2 + S_0 x^2)^{-1}$.

When $a \to 0$, Inequality (1.11) reduces to (1.2). This shows that Inequalities (1.9) and (1.11) are a new improvement and extension of (1.2).

2. PROOFS OF THEOREMS 1 AND 2

Proof of Theorem 1. Let g be defined as

$$g(x) = (S_\beta x^\alpha + S_\alpha x^\beta)^{-q/p}. \tag{2.1}$$

If $0 < \beta < p - 1 < \alpha$, $\frac{1}{p} + \frac{1}{q} = 1$, we prove that

$$\sum_{n=1}^{\infty} g(n) < \frac{S_\alpha^{\lambda_\alpha}}{(\alpha - \beta)S_\beta^{\lambda_\beta}} B(\lambda_\beta, (-\lambda_\alpha)) - c(p, \alpha, \beta), \tag{2.2}$$

where λ_α, λ_β, and $c(p, \alpha, \beta)$ are defined by (1.5) and (1.6), respectively.

In fact, by the Euler–Maclaurin summation formula, we have

$$\sum_{n=1}^{\infty} g(n) < \int_1^{\infty} g(x)\,dx + g(1) = \int_0^{\infty} g(x)\,dx - \left(\int_0^1 g(x)\,dx - g(1)\right).$$

Direct computation yields

$$\int_0^{\infty} g(x)\,dx = \int_0^{\infty} \frac{dx}{\left(S_\beta x^\alpha + S_\alpha x^\beta\right)^{q/p}}$$

$$= S_\beta^{-q/p} \int_0^{\infty} \frac{x^{-q\beta/p}\,dx}{\left(S_\alpha/S_\beta + x^{(\alpha-\beta)}\right)^{q/p}} = \frac{S_\alpha^{\lambda_\alpha}}{(\alpha-\beta)S_\beta^{\lambda_\beta}} B(\lambda_\beta, (-\lambda_\alpha)).$$

Since g is a strictly decreasing function on [0, 1], i.e.,

$$g(x) > g(1) \qquad (x < 1),$$

$c(p, \alpha, \beta) = \int_0^1 g(x)\,dx - g(1) > 0$. This yields (2.2).

Applying the Hölder inequality and (2.2), we get

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n (S_\beta n^\alpha + S_\alpha n^\beta)^{1/p} (S_\beta n^\alpha + S_\alpha n^\beta)^{-1/p}$$

$$\leq \left\{ \sum_{n=1}^{\infty} a_n^p (S_\beta n^\alpha + S_\alpha n^\beta) \right\}^{1/p} \left\{ \sum_{n=1} (S_\beta n^\alpha + S_\alpha n^\beta)^{-q/p} \right\}^{1/q}$$

$$= (2S_\alpha S_\beta)^{1/p} \left\{ \sum_{n=1}^{\infty} g(n) \right\}^{1/q}$$

$$< (2S_\alpha S_\beta)^{1/p} \left\{ \frac{S_\alpha^{\lambda_\alpha}}{(\alpha-\beta)S_\beta^{\lambda_\beta}} B(\lambda_\beta, (-\lambda_\alpha)) - c(p, \alpha, \beta) \right\}^{1/q}.$$

This proves (1.4). Theorem 1 is proved.

Proof of Theorem 2. Application of Hölder's inequality yields

$$\int_a^{\infty} f(x)dx = \int_a^{\infty} f(x)(S_\beta x^\alpha + S_\alpha x^\beta)^{1/p} (S_\beta x^\alpha + S_\alpha x^\beta)^{-1/p} dx$$

$$\leq \left\{ \int_a^{\infty} f^p(x)(S_\beta x^\alpha + S_\alpha x^\beta) dx \right\}^{1/p} \left\{ \int_a^{\infty} (S_\beta x^\alpha + S_\alpha x^\beta)^{-q/p} dx \right\}^{1/q}$$

$$= (2S_\alpha S_\beta)^{1/p} \left\{ \int_a^{\infty} (S_\beta x^\alpha + S_\alpha x^\beta)^{-q/p} dx \right\}^{1/q}. \tag{2.3}$$

If $g(x)=(S_\beta x^\alpha+S_\alpha x^\beta)^{-q/p}$, then direct computation yields

$$\int_0^\infty g(x)dx=\frac{S_\alpha^{\lambda_\alpha}}{(\alpha-\beta)S_\beta^{\lambda_\beta}}B(\lambda_\beta,(-\lambda_\alpha)). \tag{2.4}$$

Hence, from (2.3) and (2.4), we obtain

$$\begin{aligned}\left(\int_a^\infty f(x)dx\right)^p &\le 2S_\alpha S_\beta\left(\int_a^\infty g(x)dx\right)^{p/q}\\ &=2S_\alpha S_\beta\left\{\int_0^\infty g(x)dx-\int_0^a g(x)dx\right\}^{p/q}\\ &=2S_\alpha S_\beta\left\{\frac{S_\alpha^\lambda}{(\alpha-\beta)S_\beta^\lambda}B(\lambda_\beta,(-\lambda_\alpha))-\int_0^a g(x)dx\right\}^{p/q}.\end{aligned}$$

This is the desired result. The proof is complete.

Remark 4. Our proofs are simpler than those in [1, 2].

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