

# On $\alpha$ -times integrated C-cosine functions and abstract Cauchy problem I<sup>☆</sup>

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## Abstract

In this paper we first investigate some basic properties concerning nondegenerate  $\alpha$ -times integrated C-cosine functions on a Banach space  $X$ , and then characterize their generator  $A$  in terms of the unique existence of strong solutions of the following abstract Cauchy problem:  $ACP(f, x, y) \quad u''(t) = Au(t) + f(t)$  for  $t > 0$ ,  $u(0) = x$ ,  $u'(0) = y$ .

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## Introduction

Let  $A$  be a closed linear operator with domain  $D(A)$  and range  $R(A)$  in a Banach space  $(X, \|\cdot\|)$ , and let  $[D(A)]$  denote the Banach space  $D(A)$  with the graph norm  $\|x\|_A = \|x\| + \|Ax\|$  for  $x \in D(A)$ . The abstract Cauchy problem associated with  $A$  is the following initial value problem:

$$ACP(f, x, y) \quad \begin{cases} u''(t) = Au(t) + f(t) & \text{for } t > 0, \\ u(0) = x, \quad u'(0) = y, \end{cases}$$

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where  $x, y \in X$  and  $f$  is an  $X$ -valued function defined on a subset of  $\mathbb{R}$  containing  $(0, \infty)$ . A function  $u$  is called a solution of  $\text{ACP}(f, x, y)$ , if  $u \in C^2((0, \infty), X) \cap C^1([0, \infty), X) \cap C((0, \infty), [D(A)])$  and satisfies  $\text{ACP}(f, x, y)$ . It is well known that  $\text{ACP}$  is closely related to the theory of cosine functions (see [1–3, 11]).

Recently many authors have introduced exponentially bounded  $n$ -times integrated  $C$ -cosine functions which are given as follows. Let  $C: X \rightarrow X$  be a bounded linear operator. A strongly continuous family  $\{C(t) \mid t \geq 0\}$  of bounded linear operators on  $X$  is called:

- (i) an  $n$ -times integrated  $C$ -cosine function on  $X$  for some  $n \in \mathbb{N}$ , if  $CC(\cdot) = C(\cdot)C$ ,  $C(0) = 0$ , and for all  $s, t \geq 0, x \in X$ ,

$$\begin{aligned} 2C(t)C(s)x = & \frac{1}{(n-1)!} \left\{ (-1)^n \int_0^{|s-t|} (|s-t|-r)^{n-1} C(r)Cx \, dr \right. \\ & + \left( \int_0^{s-t} - \int_0^t - \int_0^s \right) (t+s-r)^{n-1} C(r)Cx \, dr \\ & \left. + \int_0^t (s-t+r)^{n-1} C(r)Cx \, dr + \int_0^s (t-s+r)^{n-1} C(r)Cx \, dr \right\}; \end{aligned}$$

- (ii) a (0-times integrated)  $C$ -cosine function on  $X$ , if  $CC(\cdot) = C(\cdot)C$ ,  $C(0) = C$ , and  $2C(t)C(s) = C(t+s)C + C(|t-s|)C$  for all  $s, t \geq 0$ .

In this case,  $\text{ACP}(j_{n-1}(\cdot)x, 0, 0)$  has a unique solution for every  $x \in C(D(A^{n+1}))$  (see [7–11, 14–16]), where  $j_{-1}(\cdot)$  is defined as the zero function on  $[0, \infty)$ . In particular, for the case  $n = 0$ , the function  $C^{-1}C(\cdot)x$  is the unique solution of  $\text{ACP}(0, x, 0)$  for every  $x \in C(D(A))$  (see [4, 6, 13]).

In this paper we attempt to investigate a more general class of operator families, namely  $\alpha$ -times integrated  $C$ -cosine functions, where  $\alpha > 0$  may not be a positive integer.

In Section 1, some basic properties of nondegenerate  $\alpha$ -times integrated  $C$ -cosine functions are discussed. However, as an example given in Kuo and Shaw [6], we can show that an  $\alpha$ -times integrated  $C$ -cosine function on a Banach space may not be exponentially bounded. If  $A$  is the generator of an  $\alpha$ -times integrated  $C$ -cosine function  $C(\cdot)$  on  $X$  with  $\alpha > 0$  and  $C$  is injective, then  $u(\cdot) = C^{-1}S(\cdot)x$  is the unique solution of  $\text{ACP}(j_{\alpha-1}(\cdot)x, 0, 0)$  in  $C^1([0, \infty), [D(A)])$  for every  $x \in C(D(A))$  (see Proposition 1.5). Here  $S(t)x = \int_0^t C(r)x \, dr$  and  $j_{\alpha-1}(t) = t^{\alpha-1}/\Gamma(\alpha)$  for  $t > 0$ .

In Section 2, we first show that if  $C$  is injective and  $A$  is a closed linear operator which commutes with  $C$ , then  $C^{-1}AC$  generates an  $\alpha$ -times integrated  $C$ -cosine function on  $X$  when the problem  $\text{ACP}(j_\alpha(\cdot)Cx, 0, 0)$  has a unique solution in  $C^2([0, \infty), X) \cap C([0, \infty), [D(A)])$  for each  $x \in X$  (see Theorem 2.2). In order we show that the converse is also true, and is equivalent to the problem  $\text{ACP}(j_\alpha(\cdot)Cx + j_\alpha * Cg(\cdot), 0, 0)$  has a unique solution in  $C^2([0, \infty), X) \cap C([0, \infty), [D(A)])$  for each  $x \in X$  and  $g \in L^1_{\text{loc}}([0, \infty), X)$ .

when  $C^{-1}AC = A$  (see Theorem 2.3). This can be also applied to deduce that  $C^{-1}AC$  generates a nondegenerate  $(\alpha + 1)$ -times (respectively  $\alpha$ -times) integrated  $C$ -cosine function on  $X$  when the problem  $ACP(j_{\alpha-1}(\cdot)Cx, 0, 0)$  has a unique solution in  $C([0, \infty), [D(A)])$  (respectively in  $C^1([0, \infty), [D(A)])$ ) for each  $x \in D(A)$  with  $(\lambda - A)x \in R(C)$  (see Theorems 2.4 and 2.5). Since  $C^{-1}AC = A$  when  $\rho(A) \neq \emptyset$  and  $A$  commutes with  $C$ , we can also deduce that a closed linear operator  $A$  with  $\rho(A) \neq \emptyset$  is the generator of an  $(\alpha + 1)$ -times (respectively  $\alpha$ -times) integrated  $C$ -cosine function on  $X$  if and only if  $A$  commutes with  $C$  and  $ACP(j_{\alpha-1}(\cdot)x, 0, 0)$  has a unique solution in  $C([0, \infty), [D(A)])$  for every  $x \in C(D(A))$  (respectively in  $C^1([0, \infty), [D(A)])$  for every  $x \in C(D(A))$ ) (see Corollaries 2.7 and 2.8). Finally we show that  $A$  generates a nondegenerate  $(\alpha + 1)$ -times (respectively  $\alpha$ -times) integrated  $C$ -cosine function on  $X$  is equivalent to  $ACP(j_{\alpha-1}(\cdot)Cx, 0, 0)$  has a unique solution  $u(\cdot, Cx)$  in  $C([0, \infty), [D(A)])$  which depends continuously on  $x$  (respectively in  $C^1([0, \infty), [D(A)])$  which depends continuously differentiable on  $x$ ) for each  $x \in D(A)$  when  $A = C^{-1}AC$  is densely defined (see Theorems 2.9 and 2.10). Further characterizations of generators in terms of the unique existence of strong and weak solutions of  $ACP(f, x, y)$  will be established in [5].

## 1. Some basic properties of $\alpha$ -times integrated $C$ -cosine functions

From now on we shall always denote  $B(X)$  to be the set of all bounded linear operators from a Banach space  $X$  into itself.

**Definition 1.1.** Let  $C \in B(X)$ , and  $\alpha$  be a positive number. A family  $\{C(t) \mid t \geq 0\}$  in  $B(X)$  is called an  $\alpha$ -times integrated  $C$ -cosine function on  $X$ , if:

$$C(\cdot)C = CC(\cdot); \quad (1.1)$$

$C(\cdot)$  is strongly continuous. That is,

$$C(\cdot)x : [0, \infty) \rightarrow X \quad \text{is continuous for each } x \in X; \quad (1.2)$$

for all  $x \in X$  and  $t, s \geq 0$ ,

$$\begin{aligned} 2C(t)C(s)x = \frac{1}{\Gamma(\alpha)} & \left\{ \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) (t+s-r)^{\alpha-1} C(r)Cx \, dr \right. \\ & + \int_{|t-s|}^t (s-t+r)^{\alpha-1} C(r)Cx \, dr + \int_{|t-s|}^s (t-s+r)^{\alpha-1} C(r)Cx \, dr \\ & \left. + \int_0^{|t-s|} (|t-s|+r)^{\alpha-1} C(r)Cx \, dr \right\}. \end{aligned} \quad (1.3)$$

Moreover, we say that  $C(\cdot)$  is nondegenerate, provided that if  $x \in X$  and  $C(t)x = 0$  for all  $t \geq 0$ , then  $x = 0$ .

When  $C = I$ , an  $\alpha$ -times integrated C-cosine function is also called an  $\alpha$ -times integrated cosine function (see [17]). In general, just like in the case of C-cosine function, an  $\alpha$ -times integrated C-cosine function may not be exponentially bounded. For example, let  $\{C(t) \mid t \geq 0\}$  be a family of bounded linear operators on  $X := L^2(\mathbb{R})$ , defined by

$$(C(t)f)(s) = \frac{1}{\Gamma(\alpha)} e^{-s^2} f(s) \int_0^t (t-r)^{\alpha-1} \left[ \frac{e^{rs} + e^{-rs}}{2} \right] dr.$$

It is clear that  $C(\cdot)$  is an  $\alpha$ -times integrated C-cosine function on  $X$  with  $Cf(s) = e^{-s^2} f(s)$  for all  $s \in \mathbb{R}$  and  $f \in X$ , and

$$\begin{aligned} \|C(t)\| &= \sup_{s \in \mathbb{R}} \left[ \frac{1}{\Gamma(\alpha)} e^{-s^2} \int_0^t (t-r)^{\alpha-1} \left( \frac{e^{rs} + e^{-rs}}{2} \right) dr \right] \\ &= \sup_{s \in \mathbb{R}} \left[ \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} e^{r^2/4} \left( \frac{e^{-(s-r/2)^2} + e^{-(s+r/2)^2}}{2} \right) dr \right] \\ &\geq \sup_{s \geq 0} \left[ \frac{1}{2\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} e^{r^2/4} e^{-(s-r/2)^2} dr \right] \\ &\geq \sup_{s \geq 0} \left[ \frac{1}{2\Gamma(\alpha)} e^{t^2/4^2} \int_{t/2}^t (t-r)^{\alpha-1} e^{-(s-r/2)^2} dr \right] \\ &\geq \sup_{s \geq \frac{3t}{8}} \left[ \frac{1}{2\Gamma(\alpha)} e^{t^2/4^2} \int_{t/2}^t (t-r)^{\alpha-1} e^{-(s-t/4)^2} dr \right] \\ &\geq \sup_{s \geq \frac{3t}{8}} \left[ \frac{1}{2\Gamma(\alpha)} e^{t^2/4^2} \frac{(\frac{t}{2})^\alpha}{\alpha} e^{-(s-t/4)^2} \right] = \frac{1}{2\Gamma(\alpha+1)} e^{t^2/4^2} \left( \frac{t}{2} \right)^\alpha e^{-(t/8)^2} \\ &\geq \frac{1}{2\Gamma(\alpha+1)} e^{3t^2/8^2} \quad \text{for all } t \geq 2. \end{aligned}$$

Under the assumption that  $C(\cdot)$  is nondegenerate, one can define its generator as below.

**Definition 1.2.** Let  $\alpha > 0$ , the (integral) generator  $A: D(A) \rightarrow X$  of a nondegenerate  $\alpha$ -times integrated C-cosine function  $C(\cdot)$  on  $X$  is defined by

$$\begin{aligned} x \in D(A) \quad \text{and} \quad Ax = y \quad \text{if and only if} \quad C(t)x - j_\alpha(t)Cx &= \int_0^t S(s)y ds \\ \text{for all } t \geq 0, \quad \text{where } S(s)y &= \int_0^s C(r)y dr. \end{aligned} \tag{1.4}$$

**Remark 1.3.** Let  $A$  be the generator of a nondegenerate  $\alpha$ -times integrated  $C$ -cosine function  $C(\cdot)$  on  $X$ . Then

$$C \text{ is injective;} \quad (1.5)$$

$$A \text{ is well defined, linear and closed.} \quad (1.6)$$

**Remark 1.4.** Let  $C(\cdot)$  be an  $\alpha$ -times integrated  $C$ -cosine function on  $X$ . Then

(i)  $S(\cdot)$  is an  $(\alpha + 1)$ -times integrated  $C$ -cosine function on  $X$ ;

(ii)  $C(0) = 0$  on  $X$ , if  $\alpha > 0$  and  $C(\cdot)$  is nondegenerate.

**Proposition 1.5.** Let  $A$  be the generator of a nondegenerate  $\alpha$ -times integrated  $C$ -cosine function  $C(\cdot)$  on  $X$  for some  $\alpha > 0$ . Then

$$C(t)x \in D(A) \quad \text{and} \quad AC(t)x = C(t)Ax \quad \text{for all } x \in D(A) \text{ and } t \geq 0; \quad (1.7)$$

$$\int_0^t S(r)x \, dr \in D(A) \quad \text{and} \quad A \int_0^t S(r)x \, dr = C(t)x - j_\alpha(t)Cx$$

for all  $t \geq 0$  and  $x \in X$ ; (1.8)

$$C^{-1}AC = A. \quad (1.9)$$

**Proof.** We shall first show that (1.7) holds. Indeed, if  $x \in D(A)$  is given, then from (1.4) we have  $C(s)C(t)x - j_\alpha(t)C(s)Cx = C(s) \int_0^t S(r)y \, dr$ , and so  $C(t)C(s)x - j_\alpha(t)CC(s)x = \int_0^t S(r)C(s)Ax \, dr$  for all  $t \geq 0$ . Hence  $C(s)x \in D(A)$  and  $AC(s)x = C(s)Ax$  for all  $s \geq 0$ . In order, we shall show that (1.8) holds. Clearly, it suffices to show that for all  $x \in X$ ,  $s > 0$  and  $t \geq 0$

$$\frac{d}{ds}C(s) \int_0^t S(r)x \, dr = S(s)[C(t)x - j_\alpha(t)Cx] + j_{\alpha-1}(s)C \int_0^t S(r)x \, dr,$$

which needs only to be shown that for all  $t, s \geq 0$

$$2C(s) \int_0^t S(r)x \, dr + 2j_\alpha(t) \int_0^s S(r)Cx \, dr = 2C(t) \int_0^s S(r)x \, dr + 2j_\alpha(s) \int_0^t S(r)Cx \, dr.$$

Indeed, let  $\tilde{S}(t)x = \int_0^t S(r)x \, dr$  for all  $x \in X$  and  $t \geq 0$ , then  $\tilde{S}(\cdot)$  is an  $(\alpha + 2)$ -times integrated  $C$ -cosine function on  $X$ , and for each  $x \in X$  and  $t \geq 0$ ,  $\tilde{S}(\cdot)\tilde{S}(t)x \in C^2([0, \infty), X)$ . We shall first show that for each  $x \in X$  and  $t, s \geq 0$

$$\begin{aligned} & 2C(t)\tilde{S}(s)x + 2j_\alpha(s)\tilde{S}(t)Cx \\ &= \frac{1}{\Gamma(\alpha)} \left[ \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) (t+s-r)^{\alpha-1} \tilde{S}(r)Cx \, dr \right] \end{aligned}$$

$$\begin{aligned}
& + \int_{|t-s|}^t (s-t+r)^{\alpha-1} \tilde{S}(r) Cx \, dr + \int_{|t-s|}^s (t-s+r)^{\alpha-1} \tilde{S}(r) Cx \, dr \\
& + \int_0^{|t-s|} (|t-s|+r)^{\alpha-1} \tilde{S}(r) Cx \, dr \Big]. \tag{1.10}
\end{aligned}$$

Indeed, for  $t \geq s$ , we have

$$\begin{aligned}
& 2C(t)\tilde{S}(s)x + 2j_\alpha(s)\tilde{S}(t)Cx \\
& = \frac{d^2}{dt^2} 2\tilde{S}(t)\tilde{S}(s)x + 2j_\alpha(s)\tilde{S}(t)Cx \\
& = \frac{d^2}{dt^2} \frac{1}{\Gamma(\alpha+2)} \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) (t+s-r)^{\alpha+1} \tilde{S}(r) Cx \, dr \\
& \quad + \frac{d^2}{dt^2} \frac{1}{\Gamma(\alpha+2)} \int_{t-s}^t (s-t+r)^{\alpha+1} \tilde{S}(r) Cx \, dr \\
& \quad + \frac{d^2}{dt^2} \frac{1}{\Gamma(\alpha+2)} \int_0^s (t-s+r)^{\alpha+1} \tilde{S}(r) Cx \, dr + 2j_\alpha(s)\tilde{S}(t)Cx \\
& = \frac{d}{dt} \left[ \frac{1}{\Gamma(\alpha+1)} \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) (t+s-r)^\alpha \tilde{S}(r) Cx \, dr - j_{\alpha+1}(s)\tilde{S}(t)Cx \right] \\
& \quad + \frac{d}{dt} \left[ j_{\alpha+1}(s)\tilde{S}(t)Cx - \frac{1}{\Gamma(\alpha+1)} \int_{t-s}^t (s-t+r)^\alpha \tilde{S}(r) Cx \, dr \right] \\
& \quad + \frac{1}{\Gamma(\alpha)} \int_0^s (t-s+r)^{\alpha-1} \tilde{S}(r) Cx \, dr + 2j_\alpha(s)\tilde{S}(t)Cx \\
& = \frac{1}{\Gamma(\alpha)} \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) (t+s-r)^{\alpha-1} \tilde{S}(r) Cx \, dr - 2j_\alpha(s)\tilde{S}(t)Cx \\
& \quad + \frac{1}{\Gamma(\alpha)} \int_{t-s}^t (s-t+r)^{\alpha-1} \tilde{S}(r) Cx \, dr \\
& \quad + \frac{1}{\Gamma(\alpha)} \int_0^s (t-s+r)^{\alpha-1} \tilde{S}(r) Cx \, dr + 2j_\alpha(s)\tilde{S}(t)Cx
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\Gamma(\alpha)} \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) (t+s-r)^{\alpha-1} \tilde{S}(r) Cx \, dr \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t-s}^t (s-t+r)^{\alpha-1} \tilde{S}(r) Cx \, dr + \frac{1}{\Gamma(\alpha)} \int_0^s (t-s+r)^{\alpha-1} \tilde{S}(r) Cx \, dr \\
&= \frac{1}{\Gamma(\alpha)} \left[ \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) (t+s-r)^{\alpha-1} \tilde{S}(r) Cx \, dr \right. \\
&\quad + \int_{|t-s|}^t (s-t+r)^{\alpha-1} \tilde{S}(r) Cx \, dr + \int_{|t-s|}^s (t-s+r)^{\alpha-1} \tilde{S}(r) Cx \, dr \\
&\quad \left. + \int_0^{|t-s|} (|t-s|+r)^{\alpha-1} \tilde{S}(r) Cx \, dr \right].
\end{aligned}$$

Similarly, we can show that (1.10) holds for  $s \geq t$ . Clearly, the right-hand side of (1.10) is symmetric in  $s, t$ . So we have  $2C(t)\tilde{S}(s)x + 2j_\alpha(s)\tilde{S}(t)Cx = 2C(s)\tilde{S}(t)x + 2j_\alpha(t)\tilde{S}(s)Cx$  for all  $t, s \geq 0$  and  $x \in X$ , which implies that  $C(s)\tilde{S}(t)x = C(t)\tilde{S}(s)x + j_\alpha(s)\tilde{S}(t)Cx - j_\alpha(t)\tilde{S}(s)Cx$  for all  $t, s \geq 0$  and  $x \in X$ .

Finally, we shall show that (1.9) holds. We first claim that  $A \subset C^{-1}AC$ . Indeed, if  $x \in D(A)$ , then by (1.8) and the definition of  $A$ , we have

$$C(t)Cx - j_\alpha(t)C^2x = C[C(t)x - j_\alpha(t)Cx] = C \int_0^t S(r)Ax \, dr = \int_0^t S(r)CAx \, dr$$

for all  $t \geq 0$ , or equivalently,  $Cx \in D(A)$  and  $ACx = CAx \in R(C)$ . That is,  $A \subset C^{-1}AC$ . Conversely, for  $x \in D(C^{-1}AC)$ , we have  $Cx \in D(A)$  and  $ACx \in R(C)$ . By the definition of generator and the commutativity of  $C$  and  $C(\cdot)$ , we also have

$$\begin{aligned}
C[C(t)x - j_\alpha(t)Cx] &= C(t)Cx - j_\alpha(t)C^2x = \int_0^t S(r)ACx \, dr \\
&= \int_0^t S(r)CC^{-1}ACx \, dr = C \int_0^t S(r)C^{-1}ACx \, dr.
\end{aligned}$$

Since  $C$  is injective, this implies that  $x \in D(A)$  and  $Ax = C^{-1}ACx$ . Consequently, we have  $A \supset C^{-1}AC$ .  $\square$

**Lemma 1.6.** *Let  $A$  be the generator of a nondegenerate  $\alpha$ -times integrated  $C$ -cosine function  $C(\cdot)$  on  $X$  and let  $t_0 > 0$  be fixed. Assume that  $u \in C([0, t_0], X)$  satisfies  $u(t) = A \int_0^t (t-s)u(s) \, ds$  for all  $0 \leq t < t_0$ . Then  $u \equiv 0$  on  $[0, t_0]$ .*

**Proof.** We know from (1.8) that  $\int_0^t \int_0^s C(r)x \, dr \, ds \in D(A)$  and

$$A \int_0^t \int_0^s C(r)x \, dr \, ds = C(t)x - j_\alpha(t)Cx,$$

so

$$\begin{aligned} \int_0^t C(t-s)u(s) \, ds &= \int_0^t C(t-s)A \int_0^s (s-r)u(r) \, dr \, ds \\ &= \int_0^t AC(t-s) \int_0^s (s-r)u(r) \, dr \, ds \\ &= A \int_0^t C(t-s) \int_0^s (s-r)u(r) \, dr \, ds \\ &= \int_0^t C(t-s)u(s) \, ds - C \int_0^t j_\alpha(t-s)u(s) \, ds \end{aligned}$$

for each  $0 \leq t < t_0$ .

Hence  $\int_0^t j_\alpha(t-s)u(s) \, ds = 0$  for each  $0 \leq t < t_0$ . Now, let  $n = [\alpha] + 1$ , then

$$0 = j_{n-\alpha-1} * (j_\alpha * u)(t) = j_n * u(t) = T^{n+1}u(t), \quad (1.11)$$

where

$$j_\beta * u(t) = \frac{1}{\Gamma(\beta+1)} \int_0^t (t-s)^\beta u(s) \, ds \quad \text{and} \quad T^{n+1}u(t) = \int_0^t (T^n u)(s) \, ds.$$

Differentiating the right-hand side of the last equality of (1.11)  $(n+1)$ -times, we have  $u(t) = 0$  for  $0 \leq t < t_0$ .  $\square$

As a direct consequence of Lemma 1.6 we can obtain the following result.

**Remark 1.7.** Any closed linear operator in  $X$  generates at most one  $\alpha$ -times integrated  $C$ -cosine function on  $X$ .

**Proposition 1.8.** Let  $A$  be the generator of a nondegenerate  $(\alpha+1)$ -times integrated  $C$ -cosine function  $C(\cdot)$  on  $X$ . Then for each  $x \in D(A)$ ,  $C(\cdot)x$  is the unique solution of  $ACP(j_{\alpha-1}(\cdot)Cx, 0, 0)$  in  $C([0, \infty), [D(A)])$ .

**Proposition 1.9.** Let  $A$  be the generator of a nondegenerate  $\alpha$ -times integrated  $C$ -cosine function  $C(\cdot)$  on  $X$  and  $C^1 = \{x \in X \mid C(\cdot)x \text{ is continuously differentiable on } (0, \infty)\}$ . Then

$$S(t)C^1 \subset D(A) \quad \text{for each } t > 0; \quad (1.12)$$

$$S(\cdot)x \text{ is the unique solution of } ACP(j_{\alpha-1}(\cdot)Cx, 0, 0) \quad \text{for each } x \in C^1; \quad (1.13)$$



$S(\cdot)x$  is the unique solution of  $\text{ACP}(j_{\alpha-1}(\cdot)Cx, 0, 0)$  in  $C^1([0, \infty), [D(A)])$  for each  $x \in D(A)$ . (1.14)

**Proof of (1.13).** Let  $x \in C^1$  be fixed. Then  $S(\cdot)x \in C^2((0, \infty), X) \cap C^1([0, \infty), X)$ . Differentiating the equation in (1.8), we have  $S(t)x \in D(A)$  and  $AS(t)x = \frac{d^2}{dt^2}S(t)x - j_{\alpha-1}(t)Cx$  for all  $t \geq 0$ , which implies that  $S(\cdot)x$  is a solution of  $\text{ACP}(j_{\alpha-1}(\cdot)Cx, 0, 0)$  in  $C^2((0, \infty), X) \cap C^1([0, \infty), X) \cap C((0, \infty), [D(A)])$ . The uniqueness of solutions of  $\text{ACP}(j_{\alpha-1}(\cdot)Cx, 0, 0)$  follows from Lemma 1.6.  $\square$

**Proof of (1.14).** Let  $x \in D(A)$  be fixed, then  $AS(\cdot)x = S(\cdot)Ax \in C^1([0, \infty), X)$ . From (1.7) and (1.8), we have  $\frac{d^2}{dt^2}S(t)x = \frac{d}{dt}C(t)x = j_{\alpha-1}(t)Cx + S(t)Ax$  for all  $t > 0$ . Consequently,  $S(\cdot)x$  is the unique solution of  $\text{ACP}(j_{\alpha-1}(\cdot)Cx, 0, 0)$  in  $C^2((0, \infty), X) \cap C^1([0, \infty), [D(A)])$ .  $\square$

**Proposition 1.10.** Let  $A$  be the generator of a nondegenerate  $\alpha$ -times integrated  $C$ -cosine function  $C(\cdot)$  on  $X$  and  $x \in X$ . Assume that  $C(t)x \in R(C)$  for all  $t \geq 0$  and  $C^{-1}C(\cdot)x$  is continuously differentiable on  $(0, \infty)$ . Then  $C^{-1}S(t)x \in D(A)$  for all  $t > 0$  and  $C^{-1}S(\cdot)x$  is the unique solution of  $\text{ACP}(j_{\alpha-1}(\cdot)x, 0, 0)$ .

**Proof.** By Lemma 1.6, we need only to show that  $C^{-1}S(\cdot)x$  is a solution of  $\text{ACP}(j_{\alpha-1}(\cdot)x, 0, 0)$ . Clearly,  $C(\cdot)x = CC^{-1}C(\cdot)x$  is continuously differentiable on  $(0, \infty)$ . It follows that  $S(\cdot)x \in C^2((0, \infty), X) \cap C^1([0, \infty), X) \cap C((0, \infty), [D(A)])$ . By Proposition 1.9, we have

$$C \frac{d^2}{dt^2} C^{-1} S(t)x = \frac{d^2}{dt^2} S(t)x = AS(t)x + j_{\alpha-1}(t)Cx = ACC^{-1}S(t)x + j_{\alpha-1}(t)Cx$$

for all  $t > 0$ . Hence  $C^{-1}S(t)x \in D(C^{-1}AC) = D(A)$  and

$$\frac{d^2}{dt^2} C^{-1} S(t)x = (C^{-1}AC)C^{-1}S(t)x + j_{\alpha-1}(t)x = AC^{-1}S(t)x + j_{\alpha-1}(t)x$$

for all  $t > 0$ , which implies that  $C^{-1}S(\cdot)x$  is a solution of  $\text{ACP}(j_{\alpha-1}(\cdot)x, 0, 0)$ .  $\square$

## 2. Abstract Cauchy problems

In this section we shall always assume that  $C \in B(X)$  is injective and  $A$  is a closed linear operator in  $X$  which commutes with  $C$ , (that is,  $CA \subset AC$ ) and then try to characterize the generator of an  $\alpha$ -times integrated  $C$ -cosine function in terms of existence of the unique solution of  $\text{ACP}(f, x, y)$ .

**Lemma 2.1.** For all  $\alpha > 0$  and  $s, t \geq 0$ , we have

$$\begin{aligned} & \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) (t+s-r)^{\alpha} r^{\alpha-1} dr + \int_{|t-s|}^t (s-t+r)^{\alpha} r^{\alpha-1} dr \\ & + \int_{|t-s|}^s (t-s+r)^{\alpha} r^{\alpha-1} dr + \int_0^{|t-s|} (|t-s|+r)^{\alpha} r^{\alpha-1} dr = 0. \end{aligned} \quad (2.1)$$

**Proof.** Since  $C(\cdot) = j_\alpha(\cdot)$  is an  $\alpha$ -times integrated cosine function on  $\mathbb{R}$ , we have

$$\begin{aligned} 2j_\alpha(t)j_\alpha(s) &= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\alpha+1)} \left[ \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) (t+s-r)^{\alpha-1} r^\alpha dr \right. \\ &\quad + \int_{|t-s|}^t (s-t+r)^{\alpha-1} r^\alpha dr + \int_{|t-s|}^s (t-s+r)^{\alpha-1} r^\alpha dr \\ &\quad \left. + \int_0^{|t-s|} (|t-s|+r)^{\alpha-1} r^\alpha dr \right] \end{aligned}$$

for all  $t, s \geq 0$ . Using (1.3) and integration by parts, one has, for  $t \geq s$

$$\begin{aligned} &2 \frac{t^\alpha}{\Gamma(\alpha+1)} \frac{s^\alpha}{\Gamma(\alpha+1)} \\ &= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\alpha+1)} \left[ \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) (t+s-r)^{\alpha-1} r^\alpha dr \right. \\ &\quad \left. + \int_{t-s}^t (s-t+r)^{\alpha-1} r^\alpha dr + \int_0^s (t-s+r)^{\alpha-1} r^\alpha dr \right] \\ &= \frac{1}{\Gamma(\alpha+1)} \frac{1}{\Gamma(\alpha+1)} \left[ 2t^\alpha s^\alpha + \alpha \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) (t+s-r)^\alpha r^{\alpha-1} dr \right. \\ &\quad \left. + t^\alpha s^\alpha - \alpha \int_{t-s}^t (s-t+r)^\alpha r^{\alpha-1} dr + t^\alpha s^\alpha - \alpha \int_0^s (t-s+r)^\alpha r^{\alpha-1} dr \right], \end{aligned}$$

and so

$$\begin{aligned} 2t^\alpha s^\alpha &= 2t^\alpha s^\alpha + \alpha \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) (t+s-r)^\alpha r^{\alpha-1} dr + t^\alpha s^\alpha \\ &\quad - \alpha \int_{t-s}^t (s-t+r)^\alpha r^{\alpha-1} dr + t^\alpha s^\alpha - \alpha \int_0^s (t-s+r)^\alpha r^{\alpha-1} dr. \end{aligned}$$

Hence

$$0 = \alpha \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) (t+s-r)^\alpha r^{\alpha-1} dr + 2t^\alpha s^\alpha - \alpha \int_{t-s}^t (s-t+r)^\alpha r^{\alpha-1} dr$$

$$-\alpha \int_0^s (t-s+r)^\alpha r^{\alpha-1} dr.$$

As in the proof of [7, Lemma 3.1], we have

$$-t^\alpha s^\alpha = \alpha \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) (t+s-r)^\alpha r^{\alpha-1} dr,$$

and so

$$\begin{aligned} 0 &= -\alpha \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) (t+s-r)^\alpha r^{\alpha-1} dr - \alpha \int_{t-s}^t (s-t+r)^\alpha r^{\alpha-1} dr \\ &\quad - \alpha \int_0^s (t-s+r)^\alpha r^{\alpha-1} dr. \end{aligned}$$

Hence

$$\begin{aligned} 0 &= \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) (t+s-r)^\alpha r^{\alpha-1} dr + \int_{|t-s|}^t (s-t+r)^\alpha r^{\alpha-1} dr \\ &\quad + \int_{|t-s|}^s (t-s+r)^\alpha r^{\alpha-1} dr + \int_0^{|t-s|} (|t-s|+r)^\alpha r^{\alpha-1} dr \end{aligned}$$

for  $t \geq s$ . Similarly, we can show that (2.1) holds when  $t \leq s$ .  $\square$

**Theorem 2.2.**  $C^{-1}AC$  generates a nondegenerate  $\alpha$ -times integrated C-cosine function on  $X$ , if  $ACP(j_\alpha(\cdot)Cx, 0, 0)$  has a unique solution in  $C^2([0, \infty), X) \cap C([0, \infty), [D(A)])$  for each  $x \in X$ .

**Proof.** Indeed, if the unique solution of  $ACP(j_\alpha(\cdot)Cx, 0, 0)$  is denoted by  $u(\cdot; Cx)$  for each  $x \in X$ . Then we set  $C(t)x = u''(t; Cx)$  and  $\tilde{S}(t)x = u(t; Cx)$  for each  $t \geq 0$  and  $x \in X$ . Clearly,  $C(\cdot)$  and  $\tilde{S}(\cdot)$  both are strongly continuous and nondegenerate. Combining the uniqueness of solutions for  $ACP(j_\alpha(\cdot)Cx, 0, 0)$  with the assumption  $CA \subset AC$ , we have  $u(\cdot; C^2x) = Cu(\cdot; Cx)$  for each  $x \in X$ , which implies that  $C(t): X \rightarrow X$  and  $\tilde{S}(t): X \rightarrow X$  both are linear and commute with  $C$  for each  $t \geq 0$ . We next set

$$\begin{aligned} v_s(t; x) &= \frac{1}{\Gamma(\alpha+2)} \left\{ \left[ \int_0^{t+s} - \int_0^t - \int_0^s \right] (t+s-r)^{\alpha+1} \tilde{S}(r)Cx dr \right. \\ &\quad \left. + \int_{|t-s|}^t (s-t+r)^{\alpha+1} \tilde{S}(r)Cx dr + \int_{|t-s|}^s (t-s+r)^{\alpha+1} \tilde{S}(r)Cx dr \right. \end{aligned}$$

$$+ \int_0^{|t-s|} (|t-s|+r)^{\alpha+1} \tilde{S}(r) Cx \, dr \Big\}$$

for all  $t, s \geq 0$  and  $x \in X$ . Then from Lemma 2.1, we have, for  $t \geq s$

$$\begin{aligned} Av_s(t; x) &= \frac{1}{\Gamma(\alpha+2)} \left[ \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) (t+s-r)^{\alpha+1} Au(r; C^2x) \, dr \right. \\ &\quad \left. + \int_{t-s}^t (s-t+r)^{\alpha+1} Au(r; C^2x) \, dr + \int_0^s (t-s+r)^{\alpha+1} Au(r; C^2x) \, dr \right] \\ &= \frac{1}{\Gamma(\alpha+2)} \left[ \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) (t+s-r)^{\alpha+1} \{u''(r; C^2x) - j_\alpha(r) C^2x\} \, dr \right. \\ &\quad \left. + \int_{t-s}^t (s-t+r)^{\alpha+1} \{u(r; C^2x) - j_\alpha(r) C^2x\} \, dr \right. \\ &\quad \left. + \int_0^s (t-s+r)^{\alpha+1} \{u''(r; C^2x) - j_\alpha(r) C^2x\} \, dr \right] \\ &= \frac{1}{\Gamma(\alpha+2)} \left[ \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) (t+s-r)^{\alpha+1} u''(r; C^2x) \, dr \right. \\ &\quad \left. + \int_{t-s}^t (s-t+r)^{\alpha+1} u''(r; C^2x) \, dr + \int_0^s (t-s+r)^{\alpha+1} u''(r; C^2x) \, dr \right]. \end{aligned}$$

Using integration by parts, we also have, for  $\alpha > 1$  and  $t \geq s$

$$\begin{aligned} &\frac{1}{\Gamma(\alpha+2)} \left[ \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) (t+s-r)^{\alpha+1} u''(r; C^2x) \, dr \right. \\ &\quad \left. + \int_{t-s}^t (s-t+r)^{\alpha+1} u''(r; C^2x) \, dr + \int_0^s (t-s+r)^{\alpha+1} u''(r; C^2x) \, dr \right] \\ &= \frac{1}{\Gamma(\alpha+1)} \left[ \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) (t+s-r)^\alpha u'(r; C^2x) \, dr \right. \\ &\quad \left. + \int_{t-s}^t (s-t+r)^\alpha u'(r; C^2x) \, dr + \int_0^s (t-s+r)^\alpha u'(r; C^2x) \, dr \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\Gamma(\alpha)} \left[ \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) (t+s-r)^{\alpha-1} u(r; \mathbb{C}^2 x) dr \right. \\
&\quad \left. + \int_{t-s}^t (s-t+r)^{\alpha-1} u(r; \mathbb{C}^2 x) dr + \int_0^s (t-s+r)^{\alpha-1} u(r; \mathbb{C}^2 x) dr \right] \\
&= \frac{1}{\Gamma(\alpha)} \left[ \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) (t+s-r)^{\alpha-1} u(r; \mathbb{C}^2 x) dr \right. \\
&\quad + \int_{|t-s|}^t (s-t+r)^{\alpha-1} u(r; \mathbb{C}^2 x) dr + \int_{|t-s|}^s (t-s+r)^{\alpha-1} u(r; \mathbb{C}^2 x) dr \\
&\quad \left. + \int_0^{|t-s|} (|t-s|+r)^{\alpha-1} u(r; \mathbb{C}^2 x) dr \right] - 2j_\alpha(s)u(t; \mathbb{C}^2 x) - 2j_\alpha(t)u(s; \mathbb{C}^2 x), \\
\frac{d}{dt} v_s(t) &= \frac{1}{\Gamma(\alpha+1)} \left[ \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) (t+s-r)^\alpha u(r; \mathbb{C}^2 x) dr \right. \\
&\quad \left. - \int_{t-s}^t (s-t+r)^\alpha u(r; \mathbb{C}^2 x) dr + \int_0^s (t-s+r)^\alpha u(r; \mathbb{C}^2 x) dr \right],
\end{aligned}$$

and

$$\begin{aligned}
\frac{d^2}{dt^2} v_s(t; x) &= \frac{1}{\Gamma(\alpha)} \left[ \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) (t+s-r)^{\alpha-1} u(r; \mathbb{C}^2 x) dr \right. \\
&\quad \left. + \int_{t-s}^t (s-t+r)^{\alpha-1} u(r; \mathbb{C}^2 x) dr + \int_0^s (t-s+r)^{\alpha-1} u(r; \mathbb{C}^2 x) dr \right] \\
&\quad - 2j_\alpha(s)u(r; \mathbb{C}^2 x) \\
&= \frac{1}{\Gamma(\alpha)} \left[ \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) (t+s-r)^{\alpha-1} u(r; \mathbb{C}^2 x) dr \right. \\
&\quad + \int_{|t-s|}^t (s-t+r)^{\alpha-1} u(r; \mathbb{C}^2 x) dr + \int_{|t-s|}^s (t-s+r)^{\alpha-1} u(r; \mathbb{C}^2 x) dr \\
&\quad \left. + \int_0^{|t-s|} (|t-s|+r)^{\alpha-1} u(r; \mathbb{C}^2 x) dr \right] - 2j_\alpha(s)u(t; \mathbb{C}^2 x).
\end{aligned}$$

Consequently, for  $\alpha > 1$  and  $t \geq s$

$$\frac{d^2}{dt^2} v_s(t; x) = A v_s(t; x) + 2j_\alpha(t) u(s; C^2 x). \quad (2.2)$$

Similarly, we can show that (2.2) holds when  $t \leq s$ . The uniqueness of solutions implies that  $v_s(\cdot; x) = u(\cdot; 2Cu(s; Cx))$  for each  $s \geq 0$ , and so

$$\begin{aligned} & \frac{1}{\Gamma(\alpha+2)} \left[ \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) (t+s-r)^{\alpha+1} \tilde{S}(r) Cx \, dr \right. \\ & \quad + \int_{|t-s|}^t (s-t+r)^{\alpha+1} \tilde{S}(r) Cx \, dr + \int_{|t-s|}^s (t-s+r)^{\alpha+1} \tilde{S}(r) Cx \, dr \\ & \quad \left. + \int_0^{|t-s|} (|t-s|+r)^{\alpha+1} \tilde{S}(r) Cx \, dr \right] = 2\tilde{S}(t)\tilde{S}(s)x \end{aligned} \quad (2.3)$$

for all  $t, s \geq 0$ . In order, if  $\alpha = 1$ , then using integration by parts, we have, for  $t \geq s$

$$\begin{aligned} A v_s(t; x) &= \left( \int_0^{t+s} - \int_0^t - \int_0^s \right) u'(r; C^2 x) \, dr + \int_{t-s}^t u'(r; C^2 x) \, dr + \int_0^s u'(r; C^2 x) \, dr \\ &= u(t+s; C^2 x) + u(|t-s|; C^2 x) - 2u(t; C^2 x) - 2u(s; C^2 x) \end{aligned}$$

and

$$\frac{d^2}{dt^2} v_s(t; x) = u(t+s; C^2 x) + u(|t-s|; C^2 x) - 2u(t; C^2 x).$$

It follows that

$$\frac{d^2}{dt^2} v_s(t; x) = A v_s(t; x) + 2u(s; C^2 x) \quad \text{for all } t \geq s. \quad (2.4)$$

Similarly, we can show that (2.4) holds when  $t \leq s$ . As in the proof of the case  $\alpha > 1$ , we can show that (2.3) holds when  $\alpha = 1$ . Consequently, (2.4) holds for all  $\alpha \geq 1$ . We now turn to the case  $0 < \alpha < 1$ . The uniqueness of solutions for  $\text{ACP}(j_\alpha(\cdot)Cx, 0, 0)$  implies that  $\int_0^\cdot u(r; Cx) \, dr$  is the unique solution of  $\text{ACP}(j_{\alpha+1}(\cdot)Cx, 0, 0)$  for each  $x \in X$ . Next let  $\tilde{S}(t)x = \int_0^t u(r; Cx) \, dr$  for all  $x \in X$  and  $t \geq 0$ . The previous arguments have shown that (2.3) holds when  $\tilde{S}(\cdot)$  and  $\alpha$  are replaced by  $\tilde{S}(\cdot)$  and  $\alpha + 1$ , respectively, which together with the fact  $\int_0^\cdot u(r; Cx) \, dr \in C^1([0, \infty), X)$  implies that (2.3) holds when  $0 < \alpha < 1$ . Now let  $C([0, \infty), [D(A)])$  be a Fréchet space with the quasi-norm  $|\cdot|$  defined by  $|v| = \sum_{k=1}^\infty \frac{|v|_k}{2^k(1+|v|_k)}$  for  $v \in C([0, \infty), [D(A)])$ , where  $|v|_k = \max_{t \in [0, k]} |v(t)|_A$  for  $k \in \mathbb{N}$ . We consider the linear map  $\eta: X \rightarrow C([0, \infty), [D(A)])$  defined by  $\eta(x) = u(\cdot; Cx)$  for  $x \in X$ , and shall show that  $\eta$  is a closed linear operator. Indeed, if  $x_n \rightarrow x$  in  $X$  and  $\eta(x_n) \rightarrow v(\cdot)$  in  $C([0, \infty), [D(A)])$ . Then for each  $k \in \mathbb{N}$ ,  $u(\cdot; Cx_n) = \tilde{S}(\cdot)x_n \rightarrow v(\cdot)$  and  $A\tilde{S}(\cdot)x_n = Au(\cdot; Cx_n) = u''(\cdot; Cx_n) - j_\alpha(\cdot)Cx_n \rightarrow Av(\cdot)$  uniformly on  $[0, k]$ , so that

$v(0) = 0$  and  $u''(\cdot; Cx_n) = A\tilde{S}(\cdot)x_n + j_\alpha(\cdot)Cx_n \rightarrow Av(\cdot) + j_\alpha(\cdot)Cx$  uniformly on  $[0, k]$ . Hence  $u'(\cdot; Cx_n) \rightarrow \int_0^t Av(r) dr + j_{\alpha+1}(\cdot)Cx$  uniformly on  $[0, k]$ , which implies that  $v$  is twice differentiable on  $[0, k]$ ,  $v'(t) = \int_0^t Av(r) dr + j_{\alpha+1}(t)Cx$  and  $v''(t) = Av(t) + j_\alpha(t)Cx$  for all  $0 \leq t \leq k$ . The uniqueness of solutions for  $ACP(j_\alpha(\cdot)Cx, 0, 0)$  implies that  $v(\cdot) = u(\cdot; Cx) = \tilde{S}(\cdot)x = \eta(x)$ . This shows that  $\eta$  is closed, and so is continuous. Hence  $\tilde{S}(t)$  belongs to  $B(X)$  for each  $t \geq 0$ . Consequently,  $\tilde{S}(\cdot)$  is a nondegenerate  $(\alpha + 2)$ -times integrated C-cosine function on  $X$ . Clearly, the continuity of  $\eta$  also implies that  $C(t) \in B(X)$  for each  $t \geq 0$ . We now prove that  $C(\cdot)$  is an  $\alpha$ -times integrated C-cosine function on  $X$ . Indeed, if  $x \in X$  is given, then for  $t \geq s \geq 0$ , we have

$$\begin{aligned} 2\tilde{S}(t)C(s)x &= \frac{d^2}{ds^2} 2\tilde{S}(t)\tilde{S}(s)x \\ &= \frac{d}{ds} \frac{1}{\Gamma(\alpha+1)} \left\{ \left[ \int_0^{t+s} - \int_0^t - \int_0^s \right] (t+s-r)^\alpha \tilde{S}(r)Cx dr \right. \\ &\quad \left. + \int_{t-s}^t (s-t+r)^\alpha \tilde{S}(r)Cx dr + \int_0^s (t-s+r)^\alpha \tilde{S}(r)Cx dr \right\} \\ &= \frac{1}{\Gamma(\alpha)} \left\{ \left[ \int_0^{t+s} - \int_0^t - \int_0^s \right] (t+s-r)^{\alpha-1} \tilde{S}(r)Cx dr \right. \\ &\quad \left. + \int_{t-s}^t (s-t+r)^{\alpha-1} \tilde{S}(r)Cx dr + \int_0^s (t-s+r)^{\alpha-1} \tilde{S}(r)Cx dr \right\}. \end{aligned}$$

Using integration by parts twice, we obtain

$$\begin{aligned} 2\tilde{S}(t)C(s)x &= \frac{1}{\Gamma(\alpha+2)} \left\{ \left[ \int_0^{t+s} - \int_0^t - \int_0^s \right] (t+s-r)^{\alpha+1} C(r)Cx dr \right. \\ &\quad \left. + \int_{t-s}^t (s-t+r)^{\alpha+1} C(r)Cx dr + \int_0^s (t-s+r)^{\alpha+1} C(r)Cx dr \right\} \end{aligned}$$

for all  $t \geq s \geq 0$ , which implies that

$$\begin{aligned} 2C(t)C(s)x &= 2\frac{d^2}{dt^2} \tilde{S}(t)C(s)x \\ &= \frac{1}{\Gamma(\alpha)} \left\{ \left[ \int_0^{t+s} - \int_0^t - \int_0^s \right] (t+s-r)^{\alpha-1} C(r)Cx dr \right. \\ &\quad \left. + \int_{t-s}^t (s-t+r)^{\alpha-1} C(r)Cx dr + \int_0^s (t-s+r)^{\alpha-1} C(r)Cx dr \right\} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\Gamma(\alpha)} \left\{ \left[ \int_0^{t+s} - \int_0^t - \int_0^s \right] (t+s-r)^{\alpha-1} C(r) Cx \, dr \right. \\
&\quad + \int_{|t-s|}^t (s-t+r)^{\alpha-1} C(r) Cx \, dr + \int_{|t-s|}^s (t-s+r)^{\alpha-1} C(r) Cx \, dr \\
&\quad \left. + \int_0^{|t-s|} (|t-s|+r)^{\alpha-1} C(r) Cx \, dr \right\} \quad (2.5)
\end{aligned}$$

for all  $t \geq s \geq 0$ . Similarly, we can show that (2.5) holds for all  $s \geq t \geq 0$ . Consequently,  $C(\cdot)$  is a nondegenerate  $\alpha$ -times integrated C-cosine function on  $X$ . Finally, we shall show that  $C^{-1}AC$  is its generator. Indeed, if  $B$  denotes the generator of  $C(\cdot)$ , then for each  $x \in D(B)$ , we have

$$A\tilde{S}(t)x = C(t)x - j_\alpha(t)Cx = \int_0^t \int_0^s C(r)Bx \, dr \, ds = \tilde{S}(t)Bx$$

for all  $t \geq 0$ . The closedness of  $A$  and the fact  $\tilde{S}(\cdot)Bx \in C^2([0, \infty), X) \cap C([0, \infty), [D(A)])$  imply that  $\int_0^t \int_0^s \tilde{S}(r)Bx \, dr \, ds \in D(A)$  and

$$\begin{aligned}
A \int_0^t \int_0^s \tilde{S}(r)Bx \, dr \, ds &= \int_0^t \int_0^s A\tilde{S}(r)Bx \, dr \, ds = \int_0^t \int_0^s [C(r)Bx - j_\alpha(r)CBx] \, dr \, ds \\
&= \tilde{S}(t)Bx - j_{\alpha+2}(t)CBx
\end{aligned}$$

for all  $t \geq 0$ . Since  $j_{\alpha+2}(t)Cx = \tilde{S}(t)x - \int_0^t \int_0^s \tilde{S}(r)Bx \, dr \, ds \in D(A)$  for  $t \geq 0$ , we have  $Cx \in D(A)$  and

$$\begin{aligned}
j_{\alpha+2}(t)ACx &= A\tilde{S}(t)x - A \int_0^t \int_0^s \tilde{S}(r)Bx \, dr \, ds = \tilde{S}(t)Bx - [\tilde{S}(t)Bx - j_{\alpha+2}(t)CBx] \\
&= j_{\alpha+2}(t)CBx
\end{aligned}$$

for all  $t \geq 0$ , which implies that  $x \in D(C^{-1}AC)$  and  $C^{-1}ACx = Bx$ . Hence  $B \subset C^{-1}AC$ . Conversely, if  $x \in D(C^{-1}AC)$ , then we set  $w(t) = \int_0^t \int_0^s \tilde{S}(r)C^{-1}ACx \, dr \, ds + j_{\alpha+2}(t)Cx$  for all  $t \geq 0$ . Since

$$C(\cdot)C^{-1}ACx = A\tilde{S}(\cdot)C^{-1}ACx + j_\alpha(\cdot)CC^{-1}ACx = A[\tilde{S}(\cdot)C^{-1}ACx + j_\alpha(\cdot)Cx],$$

we have

$$\begin{aligned}
w''(t) - j_\alpha(t)Cx &= \tilde{S}(t)C^{-1}ACx = \int_0^t \int_0^s A[\tilde{S}(r)C^{-1}ACx + j_\alpha(r)Cx] \, dr \, ds \\
&= Aw(t)
\end{aligned}$$



for all  $t \geq 0$ . The uniqueness of solutions for  $\text{ACP}(j_\alpha(\cdot)Cx, 0, 0)$  implies that  $w(\cdot) = u(\cdot; Cx) = \tilde{S}(\cdot)x$ , which implies that  $C(t)x = w''(t) = \tilde{S}(t)C^{-1}ACx + j_\alpha(t)Cx$  for all  $t \geq 0$ . This means that  $x \in D(B)$  and  $Bx = C^{-1}ACx$ . That is,  $C^{-1}AC \subset B$ . Consequently,  $B = C^{-1}AC$ .  $\square$

As an application of Theorem 2.2, we can deduce the following theorems and corollaries.

**Theorem 2.3.** *The following statements are equivalent:*

- (i) *A is the generator of a nondegenerate  $\alpha$ -times integrated C-cosine function  $C(\cdot)$  on space X;*
- (ii)  *$C^{-1}AC = A$  and the problem  $\text{ACP}(j_\alpha(\cdot)Cx + j_\alpha * Cg(\cdot), 0, 0)$  has a unique solution in  $C^2([0, \infty), X) \cap C([0, \infty), [D(A)])$  for each  $x \in X$  and  $g \in L^1_{\text{loc}}([0, \infty), X)$ ;*
- (iii)  *$C^{-1}AC = A$  and the problem  $\text{ACP}(j_\alpha(\cdot)Cx, 0, 0)$  has a unique solution in  $C^2([0, \infty), X) \cap C([0, \infty), [D(A)])$ .*

*In this case,  $\tilde{S}(\cdot)x + \tilde{S} * g(\cdot)$  is the unique solution of  $\text{ACP}(j_\alpha(\cdot)Cx + j_\alpha * Cg(\cdot), 0, 0)$ , where  $j_\alpha * Cg(t) = \int_0^t j_\alpha(t-s)Cg(s)ds$  and  $\tilde{S} * g(t) = \int_0^t \tilde{S}(t-s)g(s)ds$  for all  $t \geq 0$ .*

**Proof.** From Theorem 2.2 we need only to be shown that (i)  $\Rightarrow$  (ii) holds. Indeed, if  $x \in X$  and  $g \in L^1_{\text{loc}}([0, \infty), X)$  are given, then we set  $u(\cdot) = \hat{S}(\cdot)x + \hat{S} * g(\cdot)$ , so that  $u \in C^2([0, \infty), X) \cap C([0, \infty), [D(A)])$ ,  $u(0) = u'(0) = 0$ , and

$$\begin{aligned} Au(t) &= A\tilde{S}(t)x + A \int_0^t \tilde{S}(t-s)g(s)ds \\ &= C(t)x - j_\alpha(t)Cx + \int_0^t [C(t-s) - j_\alpha(t-s)C]g(s)ds \\ &= C(t)x + \int_0^t C(t-s)g(s)ds - [j_\alpha(t)Cx + j_\alpha * Cg(t)] \\ &= u''(t) - [j_\alpha(t)Cx + j_\alpha * Cg(t)] \end{aligned}$$

for all  $t \geq 0$ . That is,  $u$  is a solution of  $\text{ACP}(j_\alpha(\cdot)Cx + j_\alpha * Cg(\cdot), 0, 0)$  in  $C^2([0, \infty), X) \cap C([0, \infty), [D(A)])$ . The uniqueness of solutions for  $\text{ACP}(j_\alpha(\cdot)Cx + j_\alpha * Cg(\cdot), 0, 0)$  follows directly from the uniqueness of solutions for  $\text{ACP}(0, 0, 0)$ . Hence the proof of this theorem is complete.  $\square$

**Theorem 2.4.** *Assume that  $R(C) \subset R(\lambda - A)$  for some  $\lambda \in \mathbb{C}$  and  $\text{ACP}(j_{\alpha-1}(\cdot)x, 0, 0)$  has a unique solution in  $C([0, \infty), [D(A)])$  for each  $x \in D(A)$  with  $(\lambda - A)x \in R(C)$ . Then  $C^{-1}AC$  generates a nondegenerate  $(\alpha + 1)$ -times integrated C-cosine function on X.*

**Proof.** From Theorem 2.2, it suffices to show that the integral equation

$$v(\cdot; x) = A \int_0^t \int_0^s v(r; x) dr ds + j_{\alpha+1}(\cdot)Cx \quad (2.6)$$

has a (unique) solution  $v(\cdot; x)$  in  $C([0, \infty), X)$  for each  $x \in X$ . Indeed, if  $x \in X$  is given, then there exists a  $y_x \in D(A)$  such that  $(\lambda - A)y_x = Cx$ . By hypothesis,  $ACP(j_{\alpha-1}(\cdot)y_x, 0, 0)$  has a unique solution  $u(\cdot; y_x)$  in  $C([0, \infty), [D(A)])$ . It follows from the closedness of  $A$  and the continuity of  $Au(\cdot)$  that we have  $\int_0^t \int_0^s u(r; y_x) dr ds \in D(A)$  and  $A \int_0^t \int_0^s u(r; y_x) dr ds = \int_0^t \int_0^s Au(r; y_x) dr ds = u(t; y_x) - j_{\alpha+1}(t)y_x \in D(A)$  for all  $t \geq 0$ , which implies that

$$\begin{aligned} (\lambda - A)u(t; y_x) &= (\lambda - A) \left[ A \int_0^t \int_0^s u(r; y_x) dr ds + j_{\alpha+1}(t)y_x \right] \\ &= A \int_0^t \int_0^s (\lambda - A)u(r; y_x) dr ds + j_{\alpha+1}(t)Cx \end{aligned} \quad (2.7)$$

for all  $t \geq 0$ . That is,  $v(\cdot; x) = (\lambda - A)u(\cdot; y_x)$  is a solution of (2.6).  $\square$

**Theorem 2.5.** Assume that  $R(C) \subset R(\lambda - A)$  for some  $\lambda \in \mathbb{C}$  and  $ACP(j_{\alpha-1}(\cdot)x, 0, 0)$  has a unique solution in  $C^1([0, \infty), [D(A)])$  for each  $x \in D(A)$  with  $(\lambda - A)x \in R(C)$ . Then  $C^{-1}AC$  generates a nondegenerate  $\alpha$ -times integrated  $C$ -cosine function on  $X$ .

**Proof.** Let  $S(\cdot)x = v(\cdot; x) = (\lambda - A)u(\cdot; y_x)$  for each  $x \in X$ , where  $u(\cdot; y_x) \in C^2([0, \infty), X) \cap C^1([0, \infty), [D(A)])$  is the unique solution of  $ACP(j_{\alpha-1}(\cdot)y_x, 0, 0)$  given as in (2.7) when  $y_x \in D(A)$  is chosen so that  $(\lambda - A)y_x = Cx$ . Then  $S(\cdot)$  is a nondegenerate  $(\alpha + 1)$ -times integrated  $C$ -cosine function on  $X$ . Now if  $x \in X$  is given, then from (2.7),  $S(\cdot)x$  is continuously differentiable on  $[0, \infty)$  and  $\frac{d}{dt}S(t)x = (\lambda - A)u'(t; y_x) = A \int_0^t v(r; x) dr + j_{\alpha}(t)Cx$  for all  $t \geq 0$ . In this case, we set  $C(t)x = \frac{d}{dt}S(t)x$  for all  $t \geq 0$  and  $x \in X$ . Then  $C(t): X \rightarrow X$  for  $t \geq 0$ , are linear, and for each  $x \in X$ ,  $C(\cdot)x: [0, \infty) \rightarrow X$  is continuous. Next we shall show that the linear map  $\phi$  from  $X$  into the Fréchet space  $C([0, \infty), X)$  defined by  $\phi(x) = C(\cdot)x$  for all  $x \in X$ , is continuous. From the closed graph theorem it suffices to show that  $\phi$  is closed. Indeed, if  $x_n \rightarrow x$  and  $\phi(x_n) \rightarrow v$  in  $C([0, \infty), X)$ . Then  $A \int_0^t v(r; x_n) dr = C(\cdot)x_n - j_{\alpha}(\cdot)Cx_n = \phi(x_n) - j_{\alpha}(\cdot)Cx_n \rightarrow v(\cdot) - j_{\alpha}(\cdot)Cx$  uniformly on compact subsets of  $[0, \infty)$ . Clearly,  $v(\cdot; x_n) = S(\cdot)x_n \rightarrow S(\cdot)x = v(\cdot; x)$  uniformly on compact subsets of  $[0, \infty)$ , which implies that  $\int_0^t v(r; x_n) dr \rightarrow \int_0^t v(r; x) dr$  uniformly on compact subsets of  $[0, \infty)$ . From the closedness of  $A$ , we have  $A \int_0^t v(r; x) dr = v(t) - j_{\alpha}(t)Cx$  for all  $t \geq 0$ , or equivalently,  $v(\cdot) = A \int_0^t v(r; x) dr + j_{\alpha}(\cdot)Cx = \phi(x)$  on  $[0, \infty)$ . That is,  $\phi$  is closed. In particular,  $C(t) \in B(X)$  for each  $t \geq 0$ . An easy computation shows that  $C(\cdot)$  is a nondegenerate  $\alpha$ -times integrated  $C$ -cosine function on  $X$  with  $C^{-1}AC$  as its generator.  $\square$

**Lemma 2.6** (see [12]). *Let  $\lambda \in \mathbb{C}$ , and  $A$  be a closed linear operator in  $X$  such that  $CA \subset AC$ . Then*

- (i)  $C(D(A)) \subset C(D(C^{-1}AC)) \subset (\lambda - A)^{-1}C(X)$ ;
- (ii)  $C(D(A)) = (\lambda - A)^{-1}C(X)$  if and only if  $\lambda \in \rho(A)$ ;
- (iii)  $C^{-1}AC = A$  if  $\rho(A) \neq \emptyset$ .

**Corollary 2.7.** *Let  $A$  be a closed linear operator with nonempty resolvent set. Then  $A$  is the generator of a nondegenerate  $(\alpha + 1)$ -times integrated  $C$ -cosine function  $S(\cdot)$  on  $X$  if and only if  $ACP(j_{\alpha-1}(\cdot)Cx, 0, 0)$  has a unique solution  $u(t; Cx)$  in  $C([0, \infty), [D(A)])$  for every  $x \in D(A)$ . In this case,  $u(\cdot; Cx) = S(\cdot)x$ .*

**Proof.** Indeed, if  $S(\cdot)$  is a nondegenerate  $(\alpha + 1)$ -times integrated  $C$ -cosine function on  $X$  with generator  $A$ , then from the consequence of Proposition 1.5, we have  $AS(\cdot)x = S(\cdot)Ax \in C([0, \infty), X)$ ,  $\frac{d}{dt}S(t)x = j_\alpha(t)Cx + A \int_0^t S(s)x ds = j_\alpha(t)Cx + \int_0^t S(s)Ax ds$  for  $t \geq 0$ , and  $\frac{d^2}{dt^2}S(t)x = j_{\alpha-1}(t)Cx + S(t)Ax = j_{\alpha-1}(t)Cx + AS(t)x$  for  $t > 0$ . Consequently,  $S(\cdot)x$  is the unique solution of  $ACP(j_{\alpha-1}(\cdot)Cx, 0, 0)$  in  $C([0, \infty), [D(A)])$ . The converse is the direct consequences of Lemma 2.6 and Theorem 2.3.  $\square$

**Corollary 2.8.** *Let  $A$  be a closed linear operator with nonempty resolvent set. Then  $A$  is the generator of a nondegenerate  $\alpha$ -times integrated  $C$ -cosine function  $C(\cdot)$  on  $X$  if and only if  $ACP(j_{\alpha-1}(\cdot)Cx, 0, 0)$  has a unique solution  $u(\cdot; Cx)$  in  $C^1([0, \infty), [D(A)])$  for every  $x \in D(A)$ . In this case,  $u'(\cdot; Cx) = C(\cdot)x$ .*

**Theorem 2.9.** *Let  $A : D(A) \subset X \rightarrow X$  be a densely defined closed linear operator satisfying  $C^{-1}AC = A$ . Then the following are equivalent:*

- (i)  $A$  generates a nondegenerate  $(\alpha + 1)$ -times integrated  $C$ -cosine function  $S(\cdot)$  on  $X$ ;
- (ii) for each  $x \in D(A)$ ,  $ACP(j_{\alpha-1}(\cdot)Cx, 0, 0)$  has a unique solution  $u(\cdot; Cx)$  in  $C([0, \infty), [D(A)])$  which depends continuously on  $x$ . That is, if  $\{x_n\}_{n=1}^\infty$  is a Cauchy sequence in  $(D(A), \|\cdot\|)$ , then  $\{u(\cdot; Cx_n)\}_{n=1}^\infty$  converges uniformly on compact subsets of  $[0, \infty)$ .

**Proof.** (i)  $\Rightarrow$  (ii). Clearly,  $S(\cdot)x$  is the unique solution of  $ACP(j_{\alpha-1}(\cdot)Cx, 0, 0)$  in  $C([0, \infty), [D(A)])$  which depends continuously on  $x \in D(A)$ .

(ii)  $\Rightarrow$  (i). In view of Theorem 2.2, we need only to be shown that  $ACP(j_{\alpha+1}(\cdot)Cx, 0, 0)$  has a unique solution in  $C^2([0, \infty), X) \cap C([0, \infty), [D(A)])$  for each  $x \in X$ . Indeed, if  $x \in X$  is given, then from the denseness of  $D(A)$ , we have  $x_m \rightarrow x$  in  $X$  for some sequence  $\{x_m\}_{m=1}^\infty$  in  $D(A)$ . Now let  $u(\cdot; Cx_m)$  denote the unique solution of  $ACP(j_{\alpha-1}(\cdot)Cx_m, 0, 0)$  in  $C([0, \infty), [D(A)])$ . Then we set  $v_m(t) = \int_0^t \int_0^s u(r; Cx_m) dr ds$  for  $t \geq 0$  and  $m \in \mathbb{N}$ . By hypothesis, we have  $u(\cdot; Cx_m) \rightarrow u(\cdot)$  uniformly on compact subsets of  $[0, \infty)$  for some  $u \in C([0, \infty), X)$ , so that  $v_m(\cdot) \rightarrow \int_0^t \int_0^s u(r) dr ds$  uniformly on compact subsets of  $[0, \infty)$ . Next let  $v(t) = \int_0^t \int_0^s u(r) dr ds$ , then  $v \in C^2([0, \infty), X)$ . Since  $Au(\cdot; Cx_m) = u''(\cdot; Cx_m) - j_{\alpha-1}(\cdot)Cx_m$  on  $(0, \infty)$ , we have

$$\begin{aligned}
 Av_m(\cdot) &= A \int_0^{\cdot} \int_0^s u(r; Cx_m) dr ds = \int_0^{\cdot} \int_0^s Au(r; Cx_m) dr ds \\
 &= u(\cdot; Cx_m) - j_{\alpha+1}(\cdot)Cx_m
 \end{aligned} \tag{2.8}$$

on  $[0, \infty)$  for  $m \in \mathbb{N}$ . Clearly, the right-hand side of the last equality of (2.8) converges uniformly to  $u(\cdot) - j_{\alpha+1}(\cdot)Cx$  on compact subsets of  $[0, \infty)$ . It follows from the closedness of  $A$  that  $v(t) \in D(A)$  for  $t \geq 0$  and  $Av(\cdot) = u(\cdot) - j_{\alpha+1}(\cdot)Cx = v''(\cdot) - j_{\alpha+1}(\cdot)Cx$  belongs to  $C([0, \infty), X)$ , which implies that  $v(\cdot)$  is a solution of  $ACP(j_{\alpha+1}(\cdot)Cx, 0, 0)$  in  $C^2([0, \infty), X) \cap C([0, \infty), [D(A)])$ . Hence the proof is complete.  $\square$

**Theorem 2.10.** *Let  $A : D(A) \subset X \rightarrow X$  be a densely defined closed linear operator satisfying  $C^{-1}AC = A$ . Then the following are equivalent:*

- (i) *A generates a nondegenerate  $\alpha$ -times integrated C-cosine function  $C(\cdot)$  on  $X$ ;*
- (ii) *for each  $x \in D(A)$ ,  $ACP(j_{\alpha-1}(\cdot)Cx, 0, 0)$  has a unique solution  $u(\cdot; Cx)$  in  $C^1([0, \infty), [D(A)])$  which depends continuously differentiable on  $x$ . That is, if  $\{x_n\}_{n=1}^{\infty}$  is a Cauchy sequence in  $(D(A), \|\cdot\|)$ , then  $\{u(\cdot; Cx_n)\}_{n=1}^{\infty}$  and  $\{u'(\cdot; Cx_n)\}_{n=1}^{\infty}$  both converge uniformly on compact subsets of  $[0, \infty)$ .*

**Proof.** (i)  $\Rightarrow$  (ii). Indeed, if we set  $S(t)x = \int_0^t C(r)x dr$  for  $t \geq 0$  and  $x \in X$ . Then  $S(\cdot)x$  is the unique solution of  $ACP(j_{\alpha-1}(\cdot)Cx, 0, 0)$  in  $C^1([0, \infty), [D(A)])$  for each  $x \in D(A)$ . Now if  $\{x_n\}_{n=1}^{\infty}$  is a Cauchy sequence in  $(D(A), \|\cdot\|)$ , then we set  $u(\cdot; Cx_n) = S(\cdot)x_n$  for  $n \in \mathbb{N}$ , so that  $u(\cdot; Cx_n) = S(\cdot)x_n$  and  $u'(\cdot; Cx_n) = C(\cdot)x_n$  both converge uniformly on compact subsets of  $[0, \infty)$ .

(ii)  $\Rightarrow$  (i). For each  $x \in X$  and  $t \geq 0$ , we define  $S(t)x = \lim_{n \rightarrow \infty} u(t; Cx_n)$  and  $C(t)x = \lim_{n \rightarrow \infty} u'(t; Cx_n)$  whenever  $\{x_n\}_{n=1}^{\infty}$  is a sequence in  $D(A)$  which converges to  $x$  in  $X$ . Clearly,  $C(t)x = \frac{d}{dt}S(t)x$  for  $t \geq 0$  and  $x \in X$ . As in the proof of Theorem 2.9 we can show that  $S(\cdot)$  is a nondegenerate  $(\alpha + 1)$ -times integrated C-cosine function on  $X$  with generator  $A$ . An easy computation shows that  $C(\cdot)$  is a nondegenerate  $\alpha$ -times integrated C-cosine function on  $X$  with generator  $A$ .  $\square$

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