

A hybrid mean value formula involving Kloosterman sums and Hurwitz zeta-function [☆]

Ren Ganglian

Department of Mathematics, Northwest University, Xi'an, Shaanxi 710069, China

Received 22 November 2005

Available online 15 September 2006

Submitted by B.C. Berndt

Abstract

The main purpose of this paper is using the mean value theorem of Dirichlet L -function and the estimates for character sums to study the asymptotic properties of a hybrid mean value of Kloosterman sums with the weight of Hurwitz zeta-function and the Cochrane sums, and give an interesting mean value formula for it. © 2006 Elsevier Inc. All rights reserved.

Keywords: Cochrane sums; Hurwitz zeta-function; Character sums; Mean value; Asymptotic formula

1. Introduction

For a positive integer q and an arbitrary integer h , the Dedekind sum $S(h, q)$ is defined by

$$S(h, q) = \sum_{a=1}^q \left(\left(\frac{a}{q} \right) \right) \left(\left(\frac{ah}{q} \right) \right),$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer,} \\ 0, & \text{if } x \text{ is an integer.} \end{cases}$$

The sum $S(h, q)$ plays an important role in the transformation theory of the Dedekind η function. The various properties of $S(h, q)$ were investigated by many authors. For example, Apostol [6]

[☆] This work is supported by the N.S.F. (60472068) and the P.S.F. (2004A09) of PR China.
E-mail address: rgl70718@sina.com.

and Carlitz [3] obtained a reciprocity theorem of $S(h, q)$. Conrey et al. [5] studied the mean value distribution of $S(h, q)$, and proved an important asymptotic formula for $\sum_{h=1}^q |S(h, q)|^{2k}$, where $\sum_{h=1}^q$ denotes the summation over all $1 \leq h \leq q$ such that $(h, q) = 1$. Professor Zhang [7] improved the error term of [5] for $k = 1$.

In October 2000, Professor Todd Cochrane introduced a sum analogous to the Dedekind sum as follows:

$$C(h, q) = \sum_{a=1}^q \left(\left(\frac{\bar{a}}{q} \right) \right) \left(\left(\frac{ah}{q} \right) \right),$$

where $\bar{a}$ is defined by the equation $a\bar{a} \equiv 1 \pmod{q}$. About this sum, some scholars have studied the arithmetical properties and mean value distribution of it. For example, Professor Zhang found that there exist some relations between $C(h, q)$ and Kloosterman sums

$$K(m, n; q) = \sum_{b=1}^q e\left(\frac{mb + n\bar{b}}{q}\right),$$

where $e(y) = e^{2\pi i y}$, he [2] obtained the following asymptotic formula: If q is a square-full number,

$$\sum_{h=1}^q K(h, 1; q) C(h, q) = \frac{-1}{2\pi^2} q \phi(q) + O\left(q \exp\left(\frac{3 \ln q}{\ln \ln q}\right)\right) \quad (1)$$

where $\phi(q)$ is the Euler function.

In this paper, we shall study the asymptotic properties of the hybrid mean value

$$\sum_{a=1}^q \sum_{b=1}^q K(a, 1; q) \zeta\left(\frac{1}{2}, \frac{b}{q}\right) C(ab, q),$$

where $\zeta(s, a)$ is Hurwitz zeta-function.

This problem is interesting, because it can help us to find some relations between the Cochrane sums, Kloosterman sums and Hurwitz zeta-function. By using the estimates for character sums and the mean value theorems of Dirichlet L -functions, we shall prove the following main conclusion:

Theorem. For any positive integer q , we have the asymptotic formula

$$\sum_{a=1}^q \sum_{b=1}^q K(a, 1; q) \zeta\left(\frac{1}{2}, \frac{b}{q}\right) C(ab, q) = -\frac{q^{\frac{3}{2}} \phi(q)}{2\pi^2} \prod_{p \parallel q} \left(1 - \frac{1}{p(p-1)}\right) + O(q^{2+\varepsilon}),$$

where ε denotes any fixed positive number, $\prod_{p \parallel q}$ denotes the product over all prime divisors p of q such that $p \mid q$ and $p^2 \nmid q$.

2. Several lemmas

To complete the proof of Theorem, we need the following lemmas.

Lemma 1. Let integer $q \geq 3$ and $(a, q) = 1$. Then we have

$$C(a, q) = \frac{-1}{\pi^2 \phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1) = -1}} \bar{\chi}(a) \left(\sum_{n=1}^{\infty} \frac{G(\chi, n)}{n} \right)^2,$$

where χ denotes a Dirichlet character modulo q with $\chi(-1) = -1$, and $G(\chi, n) = \sum_{b=1}^q \chi(b) e(\frac{bn}{q})$ is the Gauss sums corresponding to χ .

Proof. See Lemma 1 of Ref. [2]. $\square$

Lemma 2. For any positive integer q , let χ be a non-primitive character modulo q , and q^* denotes the conductor of χ with $\chi = \chi_{q^*}^0 \chi_q^0$, then we have the following results:

(I) if $(n, q) > 1$, we have

$$G(\chi, n) = \begin{cases} \bar{\chi}^*\left(\frac{n}{(n, q)}\right) \chi^*\left(\frac{q}{q^*(n, q)}\right) \mu\left(\frac{q}{q^*(n, q)}\right) \phi(q) \phi^{-1}\left(\frac{q}{(n, q)}\right) \tau(\chi^*), & \text{if } q^* = \frac{q_1}{(n, q_1)}, \\ 0, & \text{otherwise,} \end{cases}$$

where χ_q^0 denotes the principal character modulo q , q_1 is the largest divisor of q that has the same factors with q^* , and $\mu(n)$ is the Möbius function;

(II) if $(n, q) = 1$, we have

$$G(\chi, n) = \bar{\chi}^*(n) \chi^*\left(\frac{q}{q^*}\right) \mu\left(\frac{q}{q^*}\right) \tau(\chi^*);$$

(III) if χ is a primitive character modulo q , we have

$$G(\chi, n) = \bar{\chi}(n) \tau(\chi).$$

Proof. See Lemma 3 of Ref. [8]. $\square$

Lemma 3. Let q and r be two integers with $q \geq 2$ and $(r, q) = 1$. Then we have the identities

$$\sum_{\chi \bmod q}^* \chi(r) = \sum_{d|(q, r-1)} \mu\left(\frac{q}{d}\right) \phi(d)$$

and

$$J(q) = \sum_{d|q} \mu(d) \phi\left(\frac{q}{d}\right),$$

where $\sum_{\chi \bmod q}^*$ denotes the summation over all primitive characters modulo q and $J(q)$ denotes the number of primitive characters modulo q .

Proof. See Lemma 3 of Ref. [2]. $\square$

Lemma 4. Let $q = uv$, where $(u, v) = 1$, u be a square-full integer or $u = 1$, v be a square-free integer. Then we have the asymptotic formula

$$\begin{aligned}
& \sum_{d|v} \sum_{d_1|\frac{v}{d}} \sum_{d_2|\frac{v}{d}} \sum_{g_1|\frac{v}{dd_1}} \sum_{g_2|\frac{v}{dd_2}} \frac{u^2 d^2 \mu(\frac{v}{dd_1 g_1}) \mu(\frac{v}{dd_2 g_2})}{g_1 g_2 d_1 d_2 \phi(\frac{q}{d_1}) \phi(\frac{q}{d_2})} \\
& \quad \times \sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}}^* \bar{\chi}(g_1 g_2 d_1 d_2) L^2(1, \bar{\chi}) L\left(\frac{1}{2}, \overline{\chi \chi_q^0}\right) \\
& = \frac{q}{2} \prod_{p|q} \left(1 - \frac{1}{p(p-1)}\right) + O(q^{\frac{1}{2}+\varepsilon}),
\end{aligned}$$

where χ_q^0 is the principal character modulo q .

Proof. For convenience, we define

$$\begin{aligned}
\sum_{\gamma, \delta}^{(q)} F(\gamma, \delta) &= \sum^{(q)} F(ud, d_1 d_2 g_1 g_2) \\
&= \sum_{d|v} \sum_{d_1|\frac{v}{d}} \sum_{d_2|\frac{v}{d}} \sum_{g_1|\frac{v}{dd_1}} \sum_{g_2|\frac{v}{dd_2}} \frac{\mu(\frac{v}{dd_1 g_1}) \mu(\frac{v}{dd_2 g_2})}{\phi(\frac{q}{d_1}) \phi(\frac{q}{d_2})} F(ud, d_1 d_2 g_1 g_2),
\end{aligned}$$

where $F(\gamma, \delta)$ denotes an arbitrary function with $\gamma = ud$, $\delta = d_1 d_2 g_1 g_2$. Then we can further define:

$$\begin{aligned}
\sum_{\gamma, \delta}^{B(q)} F(\gamma, \delta) &= \sum^{(q)} \gamma^2 \delta^{-1} F(\gamma, \delta), & \sum_{\gamma, \delta}^{C(q)} F(\gamma, \delta) &= \sum^{(q)} \gamma^2 F(\gamma, \delta), \\
\sum_{\gamma, \delta}^{B(q)'} F(\gamma, \delta) &= \sum^{(q)'} \gamma^2 \delta^{-1} F(\gamma, \delta) \quad \text{and} \quad \sum_{\gamma, \delta}^{C(q)'} F(\gamma, \delta) &= \sum^{(q)'} \gamma^2 F(\gamma, \delta),
\end{aligned}$$

where the definition of $\sum^{(q)'}$ is similar to that of $\sum^{(q)}$, but with each factor of the form $\mu(k)$ replaced by $|\mu(k)| = \mu^2(k)$.

We also put

$$A(y, \chi) = \sum_{M < n \leq y} \chi(n),$$

where χ is any character modulo ud ($= \gamma$) and M is any parameter with $M \geq ud$.

By Abel's identity we have

$$L(s, \bar{\chi}) = \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n^s} = \sum_{1 \leq n \leq M} \frac{\bar{\chi}(n)}{n^s} + s \int_M^{\infty} \frac{A(y, \bar{\chi})}{y^{s+1}} dy,$$

so we get

$$\begin{aligned}
 & \sum_{\substack{\chi \bmod \gamma \\ \chi(-1)=-1}}^* \bar{\chi}(\delta) L^2(1, \bar{\chi}) L\left(\frac{1}{2}, \overline{\chi \chi_q^0}\right) \\
 &= \sum_{\substack{\chi \bmod \gamma \\ \chi(-1)=-1}}^* \bar{\chi}(\delta) \left(\sum_{1 \leq l \leq M} \frac{\bar{\chi}(l)}{l} + \int_M^\infty \frac{A(x, \bar{\chi})}{x^2} dx \right) \left(\sum_{1 \leq m \leq M} \frac{\bar{\chi}(m)}{m} + \int_M^\infty \frac{A(y, \bar{\chi})}{y^2} dy \right) \\
 & \quad \times \left(\sum_{1 \leq n \leq M} \frac{\overline{\chi \chi_q^0}(n)}{n^{\frac{1}{2}}} + \frac{1}{2} \int_M^\infty \frac{A(z, \overline{\chi \chi_q^0})}{z^{\frac{3}{2}}} dz \right). \tag{2}
 \end{aligned}$$

Note that for $(a, q) = 1$, from Lemma 3 we know that

$$\begin{aligned}
 \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^* \chi(a) &= \frac{1}{2} \sum_{\chi \bmod q}^* (1 - \chi(-1)) \chi(a) = \frac{1}{2} \sum_{\chi \bmod q}^* \chi(a) - \frac{1}{2} \sum_{\chi \bmod q}^* \chi(-a) \\
 &= \frac{1}{2} \sum_{u|(q, a-1)} \mu\left(\frac{q}{u}\right) \phi(u) - \frac{1}{2} \sum_{u|(q, a+1)} \mu\left(\frac{q}{u}\right) \phi(u).
 \end{aligned}$$

So we have

$$\begin{aligned}
 & \sum_{\gamma, \delta}^{B(q)} \sum_{\substack{\chi \bmod \gamma \\ \chi(-1)=-1}}^* \bar{\chi}(\delta) \left(\sum_{1 \leq l \leq M} \frac{\bar{\chi}(l)}{l} \right) \left(\sum_{1 \leq m \leq M} \frac{\bar{\chi}(m)}{m} \right) \left(\sum_{1 \leq n \leq M} \frac{\bar{\chi} \chi_q^0(n)}{n^{\frac{1}{2}}} \right) \\
 &= \sum_{\gamma, \delta}^{B(q)} \sum_{1 \leq l \leq M} \sum_{1 \leq m \leq M} \sum_{1 \leq n \leq M} \frac{n^{\frac{1}{2}} \chi_q^0(n)}{lmn} \sum_{\substack{\chi \bmod \gamma \\ \chi(-1)=-1}}^* \bar{\chi}(\delta lmn) \\
 &= \frac{1}{2} \sum_{\gamma, \delta}^{B(q)} \sum_{1 \leq l \leq M} \sum_{1 \leq m \leq M} \sum_{1 \leq n \leq M} \frac{n^{\frac{1}{2}} \chi_q^0(n)}{lmn} \sum_{s|(\gamma, \delta lmn-1)} \mu\left(\frac{\gamma}{s}\right) \phi(s) \\
 & \quad - \frac{1}{2} \sum_{\gamma, \delta}^{B(q)} \sum_{1 \leq l \leq M} \sum_{1 \leq m \leq M} \sum_{1 \leq n \leq M} \frac{n^{\frac{1}{2}} \chi_q^0(n)}{lmn} \sum_{s|(\gamma, \delta lmn+1)} \mu\left(\frac{\gamma}{s}\right) \phi(s) \\
 &= \frac{1}{2} \sum_{\gamma, \delta}^{C(q)} \sum_{s|\gamma} \mu\left(\frac{\gamma}{s}\right) \phi(s) \sum_{1 \leq l \leq M} \sum_{\substack{1 \leq m \leq M \\ (lm, \gamma)=1}} \sum_{\substack{1 \leq n \leq M \\ (n, q)=1 \\ \delta lmn \equiv 1(s)}} \frac{n^{\frac{1}{2}}}{\delta lmn} \\
 & \quad - \frac{1}{2} \sum_{\gamma, \delta}^{C(q)} \sum_{s|\gamma} \mu\left(\frac{\gamma}{s}\right) \phi(s) \sum_{1 \leq l \leq M} \sum_{\substack{1 \leq m \leq M \\ (lm, \gamma)=1}} \sum_{\substack{1 \leq n \leq M \\ (n, q)=1 \\ \delta lmn \equiv -1(s)}} \frac{n^{\frac{1}{2}}}{\delta lmn}.
 \end{aligned}$$

The main term in the above formula comes from the term corresponding to $\delta lmn = g_1 g_2 d_1 d_2 lmn \equiv 1$ with $g_1 = g_2 = d_1 = d_2 = l = m = n = 1$, and all the other terms are the error terms. In fact, from the above formula we have

$$\begin{aligned}
 & \sum_{\gamma, \delta}^{B(q)} \sum_{\substack{\chi \bmod \gamma \\ \chi(-1)=-1}}^* \bar{\chi}(\delta) \left(\sum_{1 \leq l \leq M} \frac{\bar{\chi}(l)}{l} \right) \left(\sum_{1 \leq m \leq M} \frac{\bar{\chi}(m)}{m} \right) \left(\sum_{1 \leq n \leq M} \frac{\bar{\chi} \chi_q^0(n)}{n^{\frac{1}{2}}} \right) \\
 &= \frac{1}{2} \sum_{d|v} \frac{\gamma^2}{\phi^2(q)} \sum_{s|\gamma} \mu\left(\frac{\gamma}{s}\right) \phi(s) \\
 &\quad + O\left(M^{\frac{1}{2}} \sum_{\gamma, \delta}^{C(q)'} \sum_{s|\gamma} \left| \mu\left(\frac{\gamma}{s}\right) \right| \phi(s) \sum_{1 \leq h \leq M^3 \delta/s} (hs+1)^{\varepsilon-1}\right) \\
 &\quad + O\left(M^{\frac{1}{2}} \sum_{\gamma, \delta}^{C(q)'} \sum_{s|\gamma} \left| \mu\left(\frac{\gamma}{s}\right) \right| \phi(s) \sum_{\substack{1 \leq h \leq M^3 \delta/s \\ hs > 1}} (hs-1)^{\varepsilon-1}\right) \\
 &= \frac{1}{2} \sum_{d|v} \frac{u^2 d^2}{\phi^2(q)} \sum_{s|ud} \mu\left(\frac{ud}{s}\right) \phi(s) + O(M^{\frac{1}{2}} (Mq)^{\varepsilon}) \\
 &= \frac{u^2}{2\phi^2(q)} \sum_{d|v} d^2 J(ud) + O(M^{\frac{1}{2}} (Mq)^{\varepsilon}) \\
 &= \frac{u^2 J(u)}{2\phi^2(q)} \sum_{d|v} d^2 J(d) + O(M^{\frac{1}{2}} (Mq)^{\varepsilon}) \\
 &= \frac{u\phi^2(u)}{2\phi^2(q)} \prod_{p|v} \left(p(p-1)^2 \left(1 - \frac{1}{p(p-1)} \right) \right) + O(M^{\frac{1}{2}} (Mq)^{\varepsilon}) \\
 &= \frac{q}{2} \prod_{p|v} \left(1 - \frac{1}{p(p-1)} \right) + O(M^{\frac{1}{2}} (Mq)^{\varepsilon}), \tag{3}
 \end{aligned}$$

where we have used the fact that v is a square-free number, u is a square-full number, and the identity $J(u) = \phi^2(u)/u$, if u is a square-full number.

In the following we shall estimate the other error terms. For any non-principal Dirichlet character $\chi_1 \bmod q$, and any $z > M$, from the periodic property one has

$$\sum_{M \leq n \leq z} \chi_1(n) = \sum_{M \leq n \leq M + \{(z-M)/q\}q} \chi_1(n),$$

where $\{x\}$ denotes the fractional part of x , so we have

$$\begin{aligned}
& \sum_{\gamma, \delta}^{B(q)} \sum_{\substack{\chi \bmod \gamma \\ \chi(-1)=-1}}^* \bar{\chi}(\delta) \sum_{1 \leq l \leq M} \sum_{1 \leq m \leq M} \frac{\bar{\chi}(ml)}{ml} \int_M^\infty \frac{A(z, \bar{\chi} \chi_q^0)}{z^{\frac{3}{2}}} dz \\
&= \sum_{\gamma, \delta}^{B(q)} \int_M^\infty \left(\sum_{1 \leq l \leq M} \sum_{1 \leq m \leq M} \sum_{M \leq n \leq z} \frac{\chi_q^0(n)}{ml} \sum_{\substack{\chi \bmod \gamma \\ \chi(-1)=-1}}^* \bar{\chi}(\delta lmn) \right) z^{-\frac{3}{2}} dz \\
&\ll \sum_{\gamma, \delta}^{B(q)'} \int_M^\infty \left(\sum_{1 \leq l \leq M} \sum_{1 \leq m \leq M} \sum_{1 \leq n \leq q} \frac{1}{lm} \sum_{\substack{s|\gamma \\ \delta lmn \equiv 1(s)}} \phi(s) \right) z^{-\frac{3}{2}} dz \\
&\ll \sum_{\gamma, \delta}^{B(q)'} \int_M^\infty \left(\sum_{s|\gamma} \phi(s) \frac{q}{s} \ln^2 M \right) z^{-\frac{3}{2}} dz \\
&\ll q^{1+\varepsilon} M^{-\frac{1}{2}} \ln^2 M.
\end{aligned} \tag{4}$$

With the same method as for (4), we can also get the estimates of other error terms:

$$\begin{aligned}
& \sum_{\gamma, \delta}^{B(q)} \sum_{\substack{\chi \bmod \gamma \\ \chi(-1)=-1}}^* \bar{\chi}(\delta) \sum_{1 \leq l \leq M} \frac{\bar{\chi}(l)}{l} \int_M^\infty \frac{A(y, \bar{\chi})}{y^2} dy \int_M^\infty \frac{A(z, \bar{\chi} \chi_q^0)}{z^{\frac{3}{2}}} dz \\
&\ll q^{2+\varepsilon} M^{-\frac{3}{2}} \ln M,
\end{aligned} \tag{5}$$

$$\begin{aligned}
& \sum_{\gamma, \delta}^{B(q)} \sum_{\substack{\chi \bmod \gamma \\ \chi(-1)=-1}}^* \bar{\chi}(\delta) \sum_{1 \leq l \leq M} \sum_{1 \leq n \leq M} \frac{\bar{\chi}(nl) \chi_q^0(n)}{n^{\frac{1}{2}} l} \int_M^\infty \frac{A(y, \bar{\chi})}{y^2} dy \\
&\ll q^{1+\varepsilon} M^{-\frac{1}{2}} \ln^2 M,
\end{aligned} \tag{6}$$

$$\begin{aligned}
& \sum_{\gamma, \delta}^{B(q)} \sum_{\substack{\chi \bmod \gamma \\ \chi(-1)=-1}}^* \bar{\chi}(\delta) \int_M^\infty \frac{A(x, \bar{\chi})}{x^2} dx \int_M^\infty \frac{A(y, \bar{\chi})}{y^2} dy \int_M^\infty \frac{A(z, \bar{\chi} \chi_q^0)}{z^{\frac{3}{2}}} dz \\
&\ll q^{3+\varepsilon} M^{-\frac{5}{2}},
\end{aligned} \tag{7}$$

$$\begin{aligned}
& \sum_{\gamma, \delta}^{B(q)} \sum_{\substack{\chi \bmod \gamma \\ \chi(-1)=-1}}^* \bar{\chi}(\delta) \sum_{1 \leq m \leq M} \frac{\bar{\chi}(m)}{m} \int_M^\infty \frac{A(x, \bar{\chi})}{x^2} dx \int_M^\infty \frac{A(z, \bar{\chi} \chi_q^0)}{z^{\frac{3}{2}}} dz \\
&\ll q^{2+\varepsilon} M^{-\frac{3}{2}} \ln M,
\end{aligned} \tag{8}$$

$$\begin{aligned}
& \sum_{\gamma, \delta}^{B(q)} \sum_{\substack{\chi \bmod \gamma \\ \chi(-1)=-1}}^* \bar{\chi}(\delta) \sum_{1 \leq m \leq M} \sum_{1 \leq n \leq M} \frac{\bar{\chi}(nm) \chi_q^0(n)}{n^{\frac{1}{2}} m} \int_M^\infty \frac{A(x, \bar{\chi})}{x^2} dx \\
&\ll q^{1+\varepsilon} M^{-\frac{1}{2}} \ln^2 M,
\end{aligned} \tag{9}$$

$$\sum_{\gamma, \delta}^{B(q)} \sum_{\substack{\chi \bmod \gamma \\ \chi(-1)=-1}}^* \bar{\chi}(\delta) \sum_{1 \leq n \leq M} \frac{\bar{\chi} \chi_q^0(n)}{n^{\frac{1}{2}}} \int_M^{\infty} \frac{A(x, \bar{\chi})}{x^2} dx \int_M^{\infty} \frac{A(y, \bar{\chi})}{y^2} dy \\ \ll q^{2+\varepsilon} M^{-\frac{3}{2}} \ln M. \quad (10)$$

Taking $M = q$ and combining (2)–(10), we obtain

$$\sum_{\gamma, \delta}^{B(q)} \sum_{\substack{\chi \bmod \gamma \\ \chi(-1)=-1}}^* \bar{\chi}(\delta) L^2(1, \bar{\chi}) L\left(\frac{1}{2}, \overline{\chi \chi_q^0}\right) = \frac{q}{2} \prod_{p|q} \left(1 - \frac{1}{p(p-1)}\right) + O(q^{\frac{1}{2}+\varepsilon}).$$

This completes the proof of Lemma 4. $\square$

3. Proof of the theorem

In this section, we complete the proof of Theorem. For any complex number $s = \sigma + it$ with $1/2 \leq \sigma < 1$, from [1] we know that

$$L(s, \bar{\chi}) = \frac{1}{q^s} \sum_{a=1}^q \bar{\chi}(a) \zeta\left(s, \frac{a}{q}\right)$$

and

$$\sum_{a=1}^q \bar{\chi}(a) K(a, 1; q) = \sum_{a=1}^q \sum_{c=1}^{q'} \bar{\chi}(a) e\left(\frac{ac + \bar{c}}{q}\right) \\ = \sum_{c=1}^{q'} \bar{\chi}(c) e\left(\frac{\bar{c}}{q}\right) \sum_{a=1}^q \bar{\chi}(a) e\left(\frac{a}{q}\right) = \tau^2(\bar{\chi}).$$

Applying these identities and Lemma 1, we have

$$\sum_{a=1}^q \sum_{b=1}^{q'} K(a, 1; q) \zeta\left(\frac{1}{2}, \frac{b}{q}\right) C(ab, q) \\ = \frac{-1}{\pi^2 \phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \left(\sum_{a=1}^q \bar{\chi}(a) K(a, 1; q) \right) \\ \times \left(\sum_{b=1}^q \bar{\chi}(b) \zeta\left(\frac{1}{2}, \frac{b}{q}\right) \right) \times \left(\sum_{n=1}^{\infty} \frac{G(\chi, n)}{n} \right)^2 \\ = -\frac{q^{\frac{1}{2}}}{\pi^2 \phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \tau^2(\bar{\chi}) L\left(\frac{1}{2}, \bar{\chi}\right) \left(\sum_{n=1}^{\infty} \frac{G(\chi, n)}{n} \right)^2.$$

Let $q = uv$ with $(u, v) = 1$, u be a square-full integer or $u = 1$, v be a square-free integer, for any non-primitive character χ modulo q , there is one and only one $m \mid q$ with a unique primitive character χ^* modulo m such that $\chi = \chi^* \chi_q^0$, where χ_q^0 is the principal character modulo q , from the properties of Gauss sums (see Ref. [4]) we have $\tau(\chi^*) \tau(\bar{\chi}^*) = -m$, if $\chi(-1) = -1$.

It is clear that $\bar{\chi}^*(q/m)\mu(q/m) \neq 0$ if and only if $m = ud$, where $d \mid v$. For any non-primitive character χ modulo q with $\chi = \chi^* \chi_q^0$, from Lemma 2 we can obtain the following identities:

$$\tau(\chi) = \chi^* \left(\frac{q}{m} \right) \mu \left(\frac{q}{m} \right) \tau(\chi^*) \neq 0$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{G(\chi^* \chi_q^0, n)}{n} &= \sum_{\substack{n=1 \\ (n, ud)=1}}^{\infty} \frac{\bar{\chi}^* \left(\frac{n}{(n, uv)} \right) \chi^* \left(\frac{uv}{ud(n, uv)} \right) \mu \left(\frac{uv}{ud(n, uv)} \right) \phi(q) \tau(\chi^*)}{n \phi \left(\frac{uv}{(n, uv)} \right)} \\ &= \sum_{d_1 \mid \frac{v}{d}} \sum_{\substack{n=1 \\ (\frac{n}{d_1}, \frac{v}{dd_1})=1}}^{\infty} \frac{\bar{\chi}^* \left(\frac{n}{d_1} \right) \chi^* \left(\frac{v}{dd_1} \right) \mu \left(\frac{v}{dd_1} \right) \phi(q) \tau(\chi^*)}{n \phi \left(\frac{uv}{d_1} \right)} \\ &= \sum_{d_1 \mid \frac{v}{d}} \sum_{m=1}^{\infty} \sum_{\substack{g \mid m \\ g \mid \frac{v}{dd_1}}}^{\infty} \frac{\mu(g) \bar{\chi}^*(m) \chi^* \left(\frac{v}{dd_1} \right) \mu \left(\frac{v}{dd_1} \right) \phi(q) \tau(\chi^*)}{md_1 \phi \left(\frac{uv}{d_1} \right)} \\ &= \sum_{d_1 \mid \frac{v}{d}} \sum_{g \mid \frac{v}{dd_1}} \frac{\mu(g) \bar{\chi}^*(g) \chi^* \left(\frac{v}{dd_1} \right) \mu \left(\frac{v}{dd_1} \right) \phi(q) \tau(\chi^*) L(1, \bar{\chi}^*)}{gd_1 \phi \left(\frac{uv}{d_1} \right)}. \end{aligned}$$

Therefore, by Lemmas 2 and 4 we have

$$\begin{aligned} &\sum_{a=1}^q \sum_{b=1}^{q'} K(a, 1; q) \zeta \left(\frac{1}{2}, \frac{b}{q} \right) C(ab, q) \\ &= -\frac{q^{\frac{1}{2}}}{\pi^2 \phi(q)} \sum_{d \mid v} \sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}}^* \bar{\chi}^2 \left(\frac{v}{d} \right) \tau^2(\bar{\chi}) L \left(\frac{1}{2}, \bar{\chi} \chi_q^0 \right) \\ &\quad \times \left(\sum_{d_1 \mid \frac{v}{d}} \sum_{g \mid \frac{v}{dd_1}} \frac{\mu(g) \bar{\chi}(g) \chi \left(\frac{v}{dd_1} \right) \mu \left(\frac{v}{dd_1} \right) \phi(q) \tau(\chi) L(1, \bar{\chi})}{gd_1 \phi \left(\frac{uv}{d_1} \right)} \right)^2 \\ &= -\frac{q^{\frac{1}{2}} \phi(q)}{\pi^2} \sum_{d \mid v} \sum_{d_1 \mid \frac{v}{d}} \sum_{d_2 \mid \frac{v}{d}} \sum_{g_1 \mid \frac{v}{dd_1}} \sum_{g_2 \mid \frac{v}{dd_2}} \frac{u^2 d^2 \mu \left(\frac{v}{dd_1 g_1} \right) \mu \left(\frac{v}{dd_2 g_2} \right)}{g_1 g_2 d_1 d_2 \phi \left(\frac{q}{d_1} \right) \phi \left(\frac{q}{d_2} \right)} \\ &\quad \times \sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}}^* \bar{\chi}(g_1 g_2 d_1 d_2) L^2(1, \bar{\chi}) L \left(\frac{1}{2}, \bar{\chi} \chi_q^0 \right) \\ &= -\frac{1}{2\pi^2} q^{\frac{3}{2}} \phi(q) \prod_{p \mid q} \left(1 - \frac{1}{p(p-1)} \right) + O(q^{2+\varepsilon}). \end{aligned}$$

This completes the proof of Theorem.

Acknowledgment

The author expresses his gratitude to the referee for his very helpful comments and valuable suggestions, which greatly improved this paper.

References

- [1] Tom M. Apostol, Introduction to Analytic Number Theory, Springer-Verlag, New York, 1976.
- [2] Zhang Wenpeng, On a Cochrane sum and its hybrid mean value formula, *J. Math. Anal. Appl.* 267 (2002) 89–96.
- [3] L. Carlitz, The reciprocity theorem of Dedekind sums, *Pacific J. Math.* 3 (1953) 513–522.
- [4] L.K. Hua, Introduction to Number Theory, Science Press, Beijing, 1979.
- [5] J.B. Conrey, E. Fransen, R. Klein, C. Scott, Mean value of Dedekind sums, *J. Number Theory* 56 (1996) 214–226.
- [6] Tom M. Apostol, Modular Function and Dirichlet Series in Number Theory, Springer-Verlag, New York, 1976.
- [7] Zhang Wenpeng, On the mean value of Dedekind sums, *J. Théor. Nombres Bordeaux* 8 (1996) 429–442.
- [8] C. Pan, C. Pan, Goldbach Conjecture, Science Press, Beijing, 1981.