

Fourth order wave equations with nonlinear strain and source terms[☆]

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Abstract

In this paper we study the initial boundary value problem for fourth order wave equations with nonlinear strain and source terms. First we introduce a family of potential wells and prove the invariance of some sets and vacuum isolating of solutions. Then we obtain a threshold result of global existence and nonexistence. Finally we discuss the global existence of solutions for the problem with critical initial condition $I(u_0) \geq 0$, $E(0) = d$. So the Esquivel-Avila's results are generalized and improved.

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1. Introduction

In the study of a weakly nonlinear analysis of elasto-plastic-microstructure models for longitudinal motion of an elasto-plastic bar there arises the model equation

$$u_{tt} + u_{xxxx} = a(u_x^2)_x + f(x) \quad (1.1)$$

where $a < 0$ is a constant (see [1]).

In [2] Chen and Yang studied the initial boundary value problem for more general equations

$$u_{tt} + u_{xxxx} = \sigma(u_x)_x + f(x, t), \quad x \in (0, 1), \quad t > 0, \quad (1.2)$$

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under the assumption $\sigma'(s) \geq C$ (bounded below) they obtained the global existence of generalized solutions and classical solutions and gave the sufficient conditions for the finite time blow up of solutions. In [3] Zhang and Chen studied the initial boundary value problem of Eq. (1.2) again with initial boundary conditions

$$\begin{aligned} u(0, t) = u(1, t) = 0, \quad u_x(0, t) = u_x(1, t) = 0, \\ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x). \end{aligned}$$

By using potential well method they proved the global existence of weak solutions, generalized solutions and classical solutions under some assumptions which are satisfied by Eq. (1.1).

In [4] Yang first studied the initial boundary value problem of multidimensional form of Eq. (1.2) with linear damping term

$$u_{tt} + \Delta^2 u + \lambda u_t = \sum_{i=1}^n \frac{\partial}{\partial x_i} \sigma_i(u_{x_i}).$$

By introducing a potential well without depth he obtained the global existence and asymptotic behavior of weak solutions and generalized solutions. In addition, the finite time blow up of solutions was proved for the case $E(0) < 0$.

Recently in [5] J.A. Esquivel-Avila studied the initial boundary value problem for fourth order wave equation with nonlinear strain and source terms

$$\begin{aligned} u_{tt} + \Delta^2 u - \alpha \Delta u \pm \beta \sum_{i=1}^n \frac{\partial}{\partial x_i} (|u_{x_i}|^{m-2} u_{x_i}) &= \mu |u|^{r-2} u, \quad x \in \Omega, \quad t > 0, \\ u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \\ u = \frac{\partial u}{\partial n} &= 0, \quad \text{or} \quad u = \Delta u = 0, \quad x \in \partial \Omega, \quad t \geq 0, \end{aligned} \quad (1.3)$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain, α , β and μ are positive constants,

$$2 < r \leq \frac{2(n-2)}{n-4} \quad \text{if } n \geq 5, \quad 2 < m \leq \frac{2(n-1)}{n-2} \quad \text{if } n \geq 3.$$

For problem (1.3) he defined

$$\begin{aligned} W &= \{u \mid I(u) > 0, J(u) < d\} \cup \{0\}, \\ V &= \{u \mid I(u) < 0, J(u) < d\}, \\ E(t) &= \frac{1}{2} \|u_t\|^2 + J(u), \end{aligned}$$

where

$$\begin{aligned} J(u) &= \frac{1}{2} \|\Delta u\|^2 + \frac{\alpha}{2} \|\nabla u\|^2 \mp \frac{\beta}{m} \|\nabla u\|_m^m - \frac{\mu}{r} \|u\|_r^r, \\ I(u) &= \|\Delta u\|^2 + \alpha \|\nabla u\|^2 \mp \beta \|\nabla u\|_m^m - \mu \|u\|_r^r, \\ d &= \inf J(u) \end{aligned}$$

subject to $I(u) = 0, u \neq 0$.

In this paper we denote $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{W^{k,p}(\Omega)}$ by $\|\cdot\|_p$ and $\|\cdot\|_{k,p}$, $\|\cdot\| = \|\cdot\|_2$ and $(u, v) = \int_{\Omega} uv \, dx$.

The main results obtained in [5] are as follows:

- (i) Assume that $E(0) < d$. Then when $u_0 \in W$ problem (1.3) admits a global weak solution, and when $u_0 \in V$ the solution of problem (1.3) blows up in finite time.
- (ii) Assume that $E(0) = d$, $I(u_0) > 0$ or $u_0 = 0$. Then problem (1.3) admits a global weak solution.

Although there have been a lot of works using potential well method, much of them are that on second order nonlinear evolution equations and there are only a few works on high order nonlinear evolution equations [3–31]. In particular, on the fourth order wave equations with nonlinear strain terms there are only works mentioned above [3–5].

In this paper we consider the initial boundary value problem for more general fourth order wave equations with nonlinear strain and source terms

$$u_{tt} + \Delta^2 u - \alpha \Delta u + \sum_{i=1}^n \frac{\partial}{\partial x_i} \sigma_i(u_{x_i}) = f(u), \quad x \in \Omega, \quad t > 0, \quad (1.4)$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \quad (1.5)$$

$$u = \frac{\partial u}{\partial n} = 0, \quad \text{or} \quad u = \Delta u = 0, \quad x \in \partial\Omega, \quad t \geq 0, \quad (1.6)$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain, $\alpha \geq 0$ is a constant. $f(s)$ and $\sigma_i(s)$ ($1 \leq i \leq n$) satisfy the following conditions

- (H₁):
- (i) $f(s) \in C^1$ and $f(0) = f'(0) = 0$;
 - (ii) $f(s)$ is monotone for $-\infty < s < \infty$ and is convex for $s > 0$, concave for $s < 0$;
 - (iii) $|f(s)| \leq a_1 |s|^{q_1}$ and $(p_1 + 1)F(s) \leq sf(s)$ for some $a_1 > 0$, $1 < p_1, q_1 < \infty$ if $n \leq 4$;
 $1 < p_1, q_1 < \frac{n+4}{n-4}$ if $n \geq 5$,

$$F(s) = \int_0^s f(\tau) d\tau;$$

and

- (H₂):
- (i) $\sigma_i(s) \in C^1$ and $\sigma_i(0) = \sigma'_i(0) = 0$, $1 \leq i \leq n$;
 - (ii) $\sigma_i(s)$ are monotone for $-\infty < s < \infty$ and are convex for $s > 0$, concave for $s < 0$,
 $1 \leq i \leq n$;
 - (iii) $|\sigma_i(s)| \leq a_2 |s|^{q_2}$ and $(p_2 + 1)G_i(s) \leq s\sigma_i(s)$ for some $a_2 > 0$, $1 < p_2, q_2 < \infty$ if
 $n = 1, 2$; $1 < p_2, q_2 < \frac{n+2}{n-2}$ if $n \geq 3$,

$$G_i(s) = \int_0^s \sigma_i(\tau) d\tau, \quad 1 \leq i \leq n.$$

Clearly problem (1.3) with $+\beta$ is a special case of problem (1.4)–(1.6).

In this paper we generalize some results of [5] from problem (1.3) to problem (1.4)–(1.6), from one potential well W and V to a family of potential wells W_δ and sets V_δ . The main results are as follows:

- (i) For the more general problem (1.4)–(1.6) we again obtain the threshold result of global existence and nonexistence of solutions by using new method.
- (ii) For the more general problem (1.4)–(1.6) with critical initial condition $I(u_0) \geq 0$, $E(0) = d$ we again obtain the global existence of solutions by using new method.
- (iii) For problem (1.4)–(1.6) we obtain the vacuum isolating of solutions which was not obtained in [5].
- (iv) For problem (1.4)–(1.6) we obtain some new invariant sets and global existence theorems of solutions which were not obtained in [5].
- (v) For the special case problem (1.3) of problem (1.4)–(1.6) we generalize exponents r and m from $r \leq \frac{2(n-2)}{n-4}$ to $r < \frac{2n}{n-4}$ for $n \geq 5$; from $m \leq \frac{2(n-1)}{n-2}$ to $m < \frac{2n}{n-2}$ for $n \geq 3$.

In Eq. (1.4) taking $f(s) = 0$ or $\sigma_i(s) = 0$ ($1 \leq i \leq n$) respectively we obtain equations

$$u_{tt} + \Delta^2 u - \alpha \Delta u + \sum_{i=1}^n \frac{\partial}{\partial x_i} \sigma_i(u_{x_i}) = 0, \quad x \in \Omega, \quad t > 0, \quad (1.7)$$

and

$$u_{tt} + \Delta^2 u - \alpha \Delta u = f(u), \quad x \in \Omega, \quad t > 0. \quad (1.8)$$

Therefore all results obtained in this paper are applicable for problem (1.7), (1.5), (1.6) and problem (1.8), (1.5), (1.6).

Lemma 1.1. [10,32] For any $u \in H^2(\Omega) \cap H_0^1(\Omega)$, $\|\Delta u\|$ is equivalent to $\|u\|_{2,2}$.

Let

$$H = \left\{ u \in H^2(\Omega) \cap H_0^1(\Omega) \mid \frac{\partial u}{\partial n} = 0 \text{ or } \Delta u = 0 \text{ on } \partial\Omega \right\}$$

and

$$\|u\|_H^2 = \|\Delta u\|^2 + \alpha \|\nabla u\|^2, \quad \alpha \geq 0.$$

Then we have

Corollary 1.2. For any $u \in H$, $\|u\|_H$ is equivalent to $\|u\|_{2,2}$.

Corollary 1.3. [32] Let q_1 and q_2 be defined in (H_1) and (H_2) . Then

- (i) $H \hookrightarrow L^{q_1+1}(\Omega)$ compactly and $\|u\|_{q_1+1} \leq C_1 \|u\|_H$,
- (ii) $H \hookrightarrow W^{1,q_2+1}(\Omega)$ compactly and $\|u\|_{1,q_2+1} \leq C_2 \|u\|_H$,

where C_1 and C_2 are constants independent on u .

2. Introducing of the family of potential wells

In this section for problem (1.4)–(1.6) we shall introduce a family of potential wells W_δ and corresponding family of outside sets V_δ of W_δ and give a series of properties of them.

First for problem (1.4)–(1.6) we define

$$J(u) = \frac{1}{2} \|\Delta u\|^2 + \frac{\alpha}{2} \|\nabla u\|^2 - \int_{\Omega} F(u) \, dx - \sum_{i=1}^n \int_{\Omega} G_i(u_{x_i}) \, dx,$$

$$I(u) = \|\Delta u\|^2 + \alpha \|\nabla u\|^2 - \int_{\Omega} u f(u) \, dx - \sum_{i=1}^n \int_{\Omega} u_{x_i} \sigma_i(u_{x_i}) \, dx,$$

the potential well

$$W = \{u \in H \mid I(u) > 0, J(u) < d\} \cup \{0\}$$

and

$$V = \{u \in H \mid I(u) < 0, J(u) < d\},$$

where the depth of W

$$d = \inf J(u)$$

subject to the conditions $u \in H$, $I(u) = 0$ and $\|u\|_H \neq 0$.

Lemma 2.1. [7] *Let (H_1) and (H_2) hold. Then*

- (i) $F(s) \geq A_1 |s|^{p_1+1}$ for $|s| \geq 1$ and some $A_1 > 0$; $G_i(s) \geq A_2 |s|^{p_2+1}$ for $|s| \geq 1$ and some $A_2 > 0$, $1 \leq i \leq n$;
- (ii) $s(sf'(s) - f(s)) \geq 0$; $s(s\sigma'_i(s) - \sigma_i(s)) \geq 0$, $1 \leq i \leq n$

where the equalities hold only for $s = 0$.

Corollary 2.2. *Let (H_1) and (H_2) hold. Then*

$$\begin{aligned} sf(s) &\geq (p_1 + 1)A_1 |s|^{p_1+1} \quad \text{for } |s| \geq 1, \\ s\sigma_i(s) &\geq (p_2 + 1)A_2 |s|^{p_2+1} \quad \text{for } |s| \geq 1, 1 \leq i \leq n. \end{aligned}$$

Lemma 2.3. *Let (H_1) and (H_2) hold. Then for any $u \in H$, $\|u\|_H \neq 0$, we have*

$$\lim_{\lambda \rightarrow 0} J(\lambda u) = 0, \quad \lim_{\lambda \rightarrow +\infty} J(\lambda u) = -\infty.$$

Proof. (i) From (H_1) and (H_2) we have

$$\begin{aligned} 0 \leq F(s) &\leq \frac{1}{p_1 + 1} sf(s) \leq \frac{a_1}{p_1 + 1} |s|^{q_1+1}, \\ 0 \leq G_i(s) &\leq \frac{1}{p_2 + 1} s\sigma_i(s) \leq \frac{a_2}{p_2 + 1} |s|^{q_2+1}, \quad 1 \leq i \leq n. \end{aligned}$$

Hence $\lim_{\lambda \rightarrow 0} J(\lambda u) = 0$ follows from

$$\begin{aligned} 0 \leq \int_{\Omega} F(\lambda u) \, dx &\leq \frac{a_1 \lambda^{q_1+1}}{p_1 + 1} \int_{\Omega} |u|^{q_1+1} \, dx = \frac{a_1 \lambda^{q_1+1}}{p_1 + 1} \|u\|_{q_1+1}^{q_1+1}, \\ 0 \leq \int_{\Omega} G_i(\lambda u_{x_i}) \, dx &\leq \frac{a_2 \lambda^{q_2+1}}{p_2 + 1} \int_{\Omega} |u_{x_i}|^{q_2+1} \, dx = \frac{a_2 \lambda^{q_2+1}}{p_2 + 1} \|u_{x_i}\|_{q_2+1}^{q_2+1}. \end{aligned}$$

(ii) From Lemma 2.1 we have

$$\int_{\Omega} F(\lambda u) \, dx \geq A_1 \lambda^{p_1+1} \int_{\Omega_{\lambda}} |u|^{p_1+1} \, dx,$$

where

$$\Omega_{\lambda} = \left\{ x \mid x \in \Omega, |u| \geq \frac{1}{\lambda} \right\}.$$

Note that

$$\lim_{\lambda \rightarrow +\infty} \int_{\Omega_{\lambda}} |u|^{p_1+1} \, dx = \|u\|_{p_1+1}^{p_1+1}$$

we get

$$\int_{\Omega} F(\lambda u) \, dx \geq B_1 \lambda^{p_1+1} \quad \text{for some } B_1 > 0 \text{ as } \lambda \rightarrow +\infty. \quad (2.1)$$

By the same way we can obtain

$$\int_{\Omega} G_i(\lambda u_{x_i}) \, dx \geq B_2 \lambda^{p_2+1} \quad \text{for some } B_2 > 0 \text{ as } \lambda \rightarrow +\infty, \quad (2.2)$$

$\lim_{\lambda \rightarrow \infty} J(\lambda u) = -\infty$ follows from (2.1) and (2.2). \square

Lemma 2.4. *Let (H_1) and (H_2) hold,*

$$\varphi(\lambda) = \frac{1}{\lambda} \int_{\Omega} u f(\lambda u) \, dx, \quad \psi_i(\lambda) = \frac{1}{\lambda} \int_{\Omega} u_{x_i} \sigma_i(\lambda u_{x_i}) \, dx, \quad 1 \leq i \leq n.$$

Then for any $u \in H$, $\|u\|_H \neq 0$, we have

- (i) $\varphi(\lambda)$ and $\psi_i(\lambda)$ ($1 \leq i \leq n$) are increasing on $0 < \lambda < \infty$;
- (ii) $\lim_{\lambda \rightarrow 0} \varphi(\lambda) = \lim_{\lambda \rightarrow 0} \psi_i(\lambda) = 0$, $\lim_{\lambda \rightarrow +\infty} \varphi(\lambda) = \lim_{\lambda \rightarrow +\infty} \psi_i(\lambda) = +\infty$, $1 \leq i \leq n$.

Proof. Clearly it is enough to prove this lemma for $\varphi(\lambda)$.

(i) From Lemma 2.1 we have

$$\begin{aligned} \varphi'(\lambda) &= \frac{1}{\lambda^2} \int_{\Omega} (\lambda u^2 f'(\lambda u) - u f(\lambda u)) \, dx \\ &= \frac{1}{\lambda^3} \int_{\Omega} \lambda u (\lambda u f'(\lambda u) - f(\lambda u)) \, dx > 0. \end{aligned}$$

Hence $\varphi(\lambda)$ is strictly increasing on $0 < \lambda < \infty$.

(ii) From

$$0 \leq \varphi(\lambda) = \frac{1}{\lambda^2} \int_{\Omega} \lambda u f(\lambda u) \, dx \leq \frac{a_1}{\lambda^2} \int_{\Omega} |\lambda u|^{q_1+1} \, dx = a_1 \lambda^{q_1-1} \|u\|_{q_1+1}^{q_1+1}$$

we get $\lim_{\lambda \rightarrow 0} \varphi(\lambda) = 0$. On the other hand, from Lemma 2.2 we have

$$\begin{aligned}\varphi(\lambda) &= \frac{1}{\lambda^2} \int_{\Omega} \lambda u f(\lambda u) \, dx \geq \frac{(p_1 + 1)A_1}{\lambda^2} \int_{\Omega_\lambda} |\lambda u|^{p_1+1} \, dx \\ &= (p_1 + 1)A_1 \lambda^{p_1-1} \int_{\Omega_\lambda} |u|^{p_1+1} \, dx,\end{aligned}$$

where Ω_λ is the same as that in the proof of Lemma 2.3. Again by

$$\lim_{\lambda \rightarrow +\infty} \int_{\Omega_\lambda} |u|^{p_1+1} \, dx = \|u\|_{p_1+1}^{p_1+1}$$

we obtain

$$\lim_{\lambda \rightarrow +\infty} \varphi(\lambda) = +\infty. \quad \square$$

Lemma 2.5. *Let (H_1) and (H_2) hold. Then for any $u \in H$, $\|u\|_H \neq 0$, we have*

(i) *on the interval $0 < \lambda < \infty$ there exists a unique $\lambda^* = \lambda^*(u)$ such that*

$$\left. \frac{d}{d\lambda} J(\lambda u) \right|_{\lambda=\lambda^*} = 0;$$

- (ii) *$J(\lambda u)$ is increasing on $0 \leq \lambda \leq \lambda^*$, decreasing on $\lambda^* \leq \lambda < \infty$ and takes the maximum at $\lambda = \lambda^*$;*
 (iii) *$I(\lambda u) > 0$ for $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ for $\lambda^* < \lambda < \infty$ and $I(\lambda^* u) = 0$.*

Proof. (i) First we have

$$\begin{aligned}\frac{d}{d\lambda} J(\lambda u) &= \lambda \|\Delta u\|^2 + \alpha \lambda \|\nabla u\|^2 - \int_{\Omega} u f(\lambda u) \, dx - \sum_{i=1}^n \int_{\Omega} u_{x_i} \sigma_i(\lambda u_{x_i}) \, dx \\ &= \lambda (\|u\|_H^2 - \Phi(\lambda)),\end{aligned}$$

where

$$\Phi(\lambda) = \varphi(\lambda) + \sum_{i=1}^n \psi_i(\lambda), \quad (2.3)$$

$\varphi(\lambda)$ and $\psi_i(\lambda)$ are defined in Lemma 2.4. Lemma 2.4 shows that $\Phi(\lambda)$ is increasing on $0 \leq \lambda < \infty$, $\lim_{\lambda \rightarrow 0} \Phi(\lambda) = 0$ and $\lim_{\lambda \rightarrow +\infty} \Phi(\lambda) = +\infty$. Hence there exists a unique $\lambda^* = \lambda^*(u)$ such that $\Phi(\lambda^*) = \|u\|_H^2$ and $\frac{d}{d\lambda} J(\lambda u)|_{\lambda=\lambda^*} = 0$.

(ii) From (i) and the increasing of $\Phi(\lambda)$ it follows that

$$\begin{aligned}\frac{d}{d\lambda} J(\lambda u) &> 0 \quad \text{for } 0 < \lambda < \lambda^*, \\ \frac{d}{d\lambda} J(\lambda u) &< 0 \quad \text{for } \lambda^* < \lambda < \infty.\end{aligned}$$

Hence $J(\lambda u)$ is increasing on $0 \leq \lambda \leq \lambda^*$, decreasing on $\lambda^* \leq \lambda < \infty$.

(iii) The conclusion follows from the proof of part (ii) and

$$\begin{aligned} I(\lambda u) &= \lambda^2 \|\Delta u\|^2 + \alpha \lambda^2 \|\nabla u\|^2 - \int_{\Omega} \lambda u f(\lambda u) \, dx - \sum_{i=1}^n \int_{\Omega} \lambda u_{x_i} \sigma_i(\lambda u_{x_i}) \, dx \\ &= \lambda \frac{d}{d\lambda} J(\lambda u). \quad \square \end{aligned}$$

Now for $\delta > 0$ we further define

$$\begin{aligned} I_{\delta}(u) &= \delta(\|\Delta u\|^2 + \alpha \|\nabla u\|^2) - \int_{\Omega} u f(u) \, dx - \sum_{i=1}^n \int_{\Omega} u_{x_i} \sigma_i(u_{x_i}) \, dx, \\ d(\delta) &= \inf J(u) \end{aligned}$$

subject to the conditions $u \in H$, $I_{\delta}(u) = 0$ and $\|u\|_H \neq 0$.

Lemma 2.6. *Let (H_1) and (H_2) hold. Assume that $0 < \|u\|_H < r(\delta)$. Then $I_{\delta}(u) > 0$. In particular, if $0 < \|u\|_H < r(1)$, then $I(u) > 0$, where $r(\delta)$ is the unique real root of equation $h(r) = \delta$,*

$$\begin{aligned} h(r) &= a_1 C_1^{q_1+1} r^{q_1-1} + a_2 C_2^{q_2+1} r^{q_2-1}, \\ C_1 &= \sup_{u \in H} \frac{\|u\|_{q_1+1}}{\|u\|_H}, \quad C_2 = \sup_{u \in H} \frac{\left(\sum_{i=1}^n \|u_{x_i}\|_{q_2+1}^{q_2+1}\right)^{\frac{1}{q_2+1}}}{\|u\|_H}. \end{aligned}$$

Proof. If $0 < \|u\|_H < r(\delta)$, then we have

$$\begin{aligned} &\int_{\Omega} u f(u) \, dx + \sum_{i=1}^n \int_{\Omega} u_{x_i} \sigma_i(u_{x_i}) \, dx \\ &\leq a_1 \|u\|_{q_1+1}^{q_1+1} + a_2 \sum_{i=1}^n \|u_{x_i}\|_{q_2+1}^{q_2+1} \leq a_1 C_1^{q_1+1} \|u\|_H^{q_1+1} + a_2 C_2^{q_2+1} \|u\|_H^{q_2+1} \\ &= h(\|u\|_H) \|u\|_H^2 < \delta \|u\|_H^2 \end{aligned}$$

and $I_{\delta}(u) > 0$. \square

Lemma 2.7. *Let (H_1) and (H_2) hold. Assume that $I_{\delta}(u) < 0$. Then $\|u\|_H > r(\delta)$. In particular, if $I(u) < 0$, then $\|u\|_H > r(1)$.*

Proof. $I_{\delta}(u) < 0$ implies $\|u\|_H \neq 0$ and from

$$\begin{aligned} \delta \|u\|_H^2 &= \delta(\|\Delta u\|^2 + \alpha \|\nabla u\|^2) < \int_{\Omega} u f(u) \, dx + \sum_{i=1}^n \int_{\Omega} u_{x_i} \sigma_i(u_{x_i}) \, dx \\ &\leq h(\|u\|_H) \|u\|_H^2 \end{aligned}$$

we get $h(\|u\|_H) > \delta$ and $\|u\|_H > r(\delta)$. \square

Lemma 2.8. *Let (H_1) and (H_2) hold. Assume that $I_{\delta}(u) = 0$. Then $\|u\|_H \geq r(\delta)$ or $\|u\|_H = 0$. In particular, if $I(u) = 0$, then $\|u\|_H \geq r(1)$ or $\|u\|_H = 0$.*

Proof. If $\|u\|_H = 0$, then $I_\delta(u) = 0$. If $I_\delta(u) = 0$, $\|u\|_H \neq 0$, then from

$$\begin{aligned} \delta \|u\|_H^2 &= \delta (\|\Delta u\|^2 + \alpha \|\nabla u\|^2) = \int_{\Omega} u f(u) \, dx + \sum_{i=1}^n \int_{\Omega} u_{x_i} \sigma_i(u_{x_i}) \, dx \\ &\leq h(\|u\|_H) \|u\|_H^2 \end{aligned}$$

we get $h(\|u\|_H) \geq \delta$ and $\|u\|_H \geq r(\delta)$. \square

Lemma 2.9. Let (H_1) and (H_2) hold. Then

- (i) $d(\delta) \geq a(\delta)r^2(\delta)$ for $a(\delta) = \frac{1}{2} - \frac{\delta}{p+1}$, $0 < \delta < \frac{p+1}{2}$, $p = \min\{p_1, p_2\}$;
- (ii) $\lim_{\delta \rightarrow 0} d(\delta) = 0$, and there exists $\delta_0 \geq \frac{p+1}{2}$ such that $d(\delta_0) = 0$ and $d(\delta) > 0$ for $0 < \delta < \delta_0$;
- (iii) $d(\delta)$ is strictly increasing on $0 < \delta \leq 1$, strictly decreasing on $1 \leq \delta < \delta_0$ and takes the maximum $d(1) = d$ at $\delta = 1$.

Proof. (i) If $I_\delta(u) = 0$ and $\|u\|_H \neq 0$, then by Lemma 2.8 we have $\|u\|_H \geq r(\delta)$. Hence by (H_1) and (H_2) we have

$$\begin{aligned} J(u) &= \frac{1}{2} \|u\|_H^2 - \int_{\Omega} F(u) \, dx - \sum_{i=1}^n \int_{\Omega} G_i(u_{x_i}) \, dx \\ &\geq \frac{1}{2} \|u\|_H^2 - \frac{1}{p_1+1} \int_{\Omega} u f(u) \, dx - \frac{1}{p_2+1} \sum_{i=1}^n \int_{\Omega} u_{x_i} \sigma_i(u_{x_i}) \, dx \\ &\geq \left(\frac{1}{2} - \frac{\delta}{p+1} \right) \|u\|_H^2 + \frac{1}{p+1} I_\delta(u) \\ &= a(\delta) \|u\|_H^2 \geq a(\delta) r^2(\delta). \end{aligned}$$

Hence (i) holds.

(ii) For any $u \in H$, $\|u\|_H \neq 0$, we define $\lambda = \lambda(\delta)$ by

$$\delta (\|\Delta(\lambda u)\|^2 + \alpha \|\nabla(\lambda u)\|^2) = \int_{\Omega} \lambda u f(\lambda u) \, dx + \sum_{i=1}^n \int_{\Omega} \lambda u_{x_i} \sigma_i(\lambda u_{x_i}) \, dx, \quad (2.4)$$

i.e.

$$\delta \|u\|_H^2 = \Phi(\lambda), \quad (2.5)$$

where $\Phi(\lambda)$ is defined by (2.3). From the properties of $\Phi(\lambda)$ given in the proof of Lemma 2.5 it follows that for any $\delta > 0$ there exists a unique $\lambda = \lambda(\delta) = \Phi^{-1}(\delta \|u\|_H^2)$ satisfying (2.4) which implies $I_\delta(\lambda u) = 0$. Clearly we have

$$\lim_{\delta \rightarrow 0} \lambda(\delta) = 0, \quad \lim_{\delta \rightarrow +\infty} \lambda(\delta) = +\infty.$$

So by Lemma 2.3 we get

$$\begin{aligned} \lim_{\delta \rightarrow 0} J(\lambda(\delta)u) &= \lim_{\lambda \rightarrow 0} J(\lambda u) = 0, \\ \lim_{\delta \rightarrow +\infty} J(\lambda(\delta)u) &= \lim_{\lambda \rightarrow +\infty} J(\lambda u) = -\infty. \end{aligned}$$

Hence we have $\lim_{\delta \rightarrow 0} d(\delta) = 0$ and $\lim_{\delta \rightarrow +\infty} d(\delta) = -\infty$ which and (i) of this lemma show that there exists $\delta_0 \geq \frac{p+1}{2}$ such that $d(\delta_0) = 0$ and $d(\delta) > 0$ for $0 < \delta < \delta_0$.

(iii) We prove that $d(\delta') < d(\delta'')$ for any $0 < \delta' < \delta'' < 1$ or $1 < \delta'' < \delta' < \delta_0$. Clearly it is enough to prove that for any $0 < \delta' < \delta'' < 1$ or $1 < \delta'' < \delta' < \delta_0$ and any $u \in H$, $I_{\delta''}(u) = 0$ and $\|u\|_H \neq 0$ there exist $v \in H$ satisfying $I_{\delta'}(v) = 0$ and $\|v\|_H \neq 0$, and a constant $\varepsilon(\delta', \delta'') > 0$ such that $J(v) < J(u) - \varepsilon(\delta', \delta'')$. In fact, for above u we also define $\lambda(\delta)$ by (2.4), then $I_\delta(\lambda(\delta)u) = 0$, $\lambda(\delta'') = 1$ and (2.5) holds. Let $g(\lambda) = J(\lambda u)$. Then

$$\begin{aligned} \frac{d}{d\lambda} g(\lambda) &= \frac{1}{\lambda} \left(\|\lambda u\|_H^2 - \int_{\Omega} \lambda u f(\lambda u) dx - \sum_{i=1}^n \int_{\Omega} \lambda u_{x_i} \sigma_i(\lambda u_{x_i}) dx \right) \\ &= \frac{1}{\lambda} ((1 - \delta) \|\lambda u\|_H^2 + I_\delta(\lambda u)) = (1 - \delta) \lambda \|u\|_H^2. \end{aligned}$$

Take $v = \lambda(\delta')u$, then $I_{\delta'}(v) = 0$ and $\|v\|_H \neq 0$.

If $0 < \delta' < \delta'' < 1$, then $\lambda(\delta') < \lambda(\delta'') = 1$,

$$J(u) - J(v) = g(1) - g(\lambda(\delta')) > (1 - \delta'')r^2(\delta'')\lambda(\delta')(1 - \lambda(\delta')) \equiv \varepsilon(\delta', \delta'').$$

If $1 < \delta'' < \delta' < \delta_0$, then $\lambda(\delta') > \lambda(\delta'') = 1$,

$$J(u) - J(v) = g(1) - g(\lambda(\delta')) > (\delta'' - 1)r^2(\delta'')\lambda(\delta'')(\lambda(\delta') - 1) \equiv \varepsilon(\delta', \delta''). \quad \square$$

Lemma 2.10. Let (H_1) and (H_2) hold, $0 < \delta < \frac{p+1}{2}$. Assume that $I_\delta(u) > 0$ and $J(u) \leq d(\delta)$. Then

$$0 < \|u\|_H^2 < \frac{d(\delta)}{a(\delta)}.$$

In particular, if $I(u) > 0$ and $J(u) \leq d$, then

$$0 < \|u\|_H^2 < \frac{2(p+1)}{p-1}d.$$

Proof. This lemma follows from

$$a(\delta)\|u\|_H^2 + \frac{1}{p+1}I_\delta(u) \leq J(u) \leq d(\delta). \quad \square \quad (2.6)$$

Again by (2.6) we can obtain the following lemma.

Lemma 2.11. Let (H_1) and (H_2) hold, $0 < \delta < \frac{p+1}{2}$. Assume that $J(u) \leq d(\delta)$ and

$$\|u\|_H^2 > \frac{d(\delta)}{a(\delta)},$$

then $I_\delta(u) < 0$. In particular, if $J(u) \leq d$ and

$$\|u\|_H^2 > \frac{2(p+1)}{p-1}d,$$

then $I(u) < 0$.

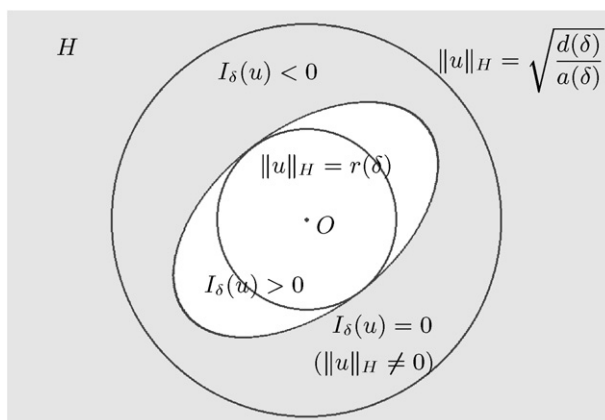


Fig. 1.

Lemma 2.12. Let (H_1) and (H_2) hold, $0 < \delta < \frac{p+1}{2}$. Assume that $I_\delta(u) = 0$, $\|u\|_H \neq 0$ and $J(u) \leq d(\delta)$. Then u belongs to the set of extremals where $J(u) = d(\delta)$ and

$$r^2(\delta) \leq \|u\|_H^2 \leq \frac{d(\delta)}{a(\delta)}.$$

In particular, if $I(u) = 0$, $\|u\|_H \neq 0$ and $J(u) \leq d$, then u belongs to the set of extremals where $J(u) = d$ and

$$r^2(1) \leq \|u\|_H^2 \leq \frac{2(p+1)}{p-1}d.$$

Proof. From $I_\delta(u) = 0$, $\|u\|_H \neq 0$ and the definition of $d(\delta)$ we get $J(u) \geq d(\delta)$, hence we have $J(u) = d(\delta)$. The remainder of the proof follows from Lemma 2.8 and (2.6). \square

Remark 2.13. The results of Lemmas 2.6–2.8 and Lemmas 2.10–2.12 show that the space H is divided into two parts $I_\delta(u) > 0$ and $I_\delta(u) < 0$ by the surface $I_\delta(u) = 0$ ($\|u\|_H \neq 0$). The inside part of $I_\delta(u) = 0$ is $I_\delta(u) > 0$ and the outside part of $I_\delta(u) = 0$ is $I_\delta(u) < 0$. Sphere $\|u\|_H = r(\delta)$ lies inside of $I_\delta(u) > 0$ and sphere $\|u\|_H^2 = \frac{d(\delta)}{a(\delta)}$ lies inside of $I_\delta(u) < 0$ (see Fig. 1).

Further for $0 < \delta < \delta_0$ we define

$$W_\delta = \{u \in H \mid I_\delta(u) > 0, J(u) < d(\delta)\} \cup \{0\},$$

$$V_\delta = \{u \in H \mid I_\delta(u) < 0, J(u) < d(\delta)\},$$

$$B_\delta = \{u \in H \mid \|u\|_H < r(\delta)\},$$

$$\bar{B}_\delta = B_\delta \cup \partial B_\delta = \{u \in H \mid \|u\|_H \leq r(\delta)\},$$

$$B_\delta^c = \{u \in H \mid \|u\|_H > r(\delta)\}.$$

Lemma 2.14. Let (H_1) and (H_2) hold, $0 < \delta < \frac{p+1}{2}$. Then

$$B_{r_1(\delta)} \subset W_\delta \subset B_{r_2(\delta)}, \quad V_\delta \subset B_\delta^c,$$

where

$$B_{r_1(\delta)} = \{u \in H \mid \|u\|_H^2 < \min\{r^2(\delta), 2d(\delta)\}\},$$

$$B_{r_2(\delta)} = \left\{u \in H \mid \|u\|_H^2 < \frac{d(\delta)}{a(\delta)}\right\}.$$

Proof. First from Lemma 2.6 it follows that $\|u\|_H < r(\delta)$ gives $I_\delta(u) > 0$ or $\|u\|_H = 0$. On the other hand, from $J(u) \leq 12\|u\|_H^2$ and $\|u\|_H^2 < 2d(\delta)$ we get $J(u) < d(\delta)$. Hence we obtain $B_{r_1(\delta)} \subset W_\delta$. The remainder of this proof follows from Lemmas 2.10 and 2.7. \square

From the definitions of W_δ , V_δ and Lemma 2.9 we can obtain the following

Lemma 2.15. *Let (H_1) and (H_2) hold. Then*

- (i) *If $0 < \delta' < \delta'' \leq 1$, then $W_{\delta'} \subset W_{\delta''}$.*
- (ii) *If $1 \leq \delta'' < \delta' < \delta_0$, then $V_{\delta'} \subset V_{\delta''}$.*

Lemma 2.16. *Let (H_1) and (H_2) hold. Assume that $0 < J(u) < d$ for some $u \in H$, (δ_1, δ_2) is the maximal interval such that $d(\delta) > J(u)$ for $\delta \in (\delta_1, \delta_2)$. Then the sign of $I_\delta(u)$ is unchangeable for $\delta_1 < \delta < \delta_2$.*

Proof. First $J(u) > 0$ implies $\|u\|_H \neq 0$. If the sign of $I_\delta(u)$ is changed, then there exists $\bar{\delta} \in (\delta_1, \delta_2)$ such that $I_{\bar{\delta}}(u) = 0$. Hence we have $J(u) \geq d(\bar{\delta})$ which contradicts $J(u) < d(\bar{\delta})$. \square

3. Invariant sets and vacuum isolating of solutions

In this section we study the invariance of some sets under the flow of (1.4)–(1.6) and vacuum isolating behavior of solutions for problem (1.4)–(1.6). First for problem (1.4)–(1.6) we define the energy

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{\alpha}{2}\|\nabla u\|^2 - \int_{\Omega} F(u) \, dx - \sum_{i=1}^n \int_{\Omega} G_i(u_{x_i}) \, dx$$

$$\equiv \frac{1}{2}\|u_t\|^2 + J(u).$$

Definition 3.1. $u = u(x, t)$ is called a weak solution of problem (1.4)–(1.6) on $\Omega \times [0, t]$ if $u \in L^\infty(0, T; H)$, $u_t \in L^\infty(0, T; L^2(\Omega))$ and satisfies

$$(i) \quad (u_t, v) + \int_0^t a(u, v) \, d\tau = \int_0^t b(u, v) \, d\tau + \int_0^t c(u, v) \, d\tau + (u_1, v),$$

$$\forall v \in H, \, t \in (0, T), \quad (3.1)$$

where

$$a(u, v) = (\Delta u, \Delta v) + \alpha(\nabla u, \nabla v), \quad b(u, v) = (f(u), v),$$

$$c(u, v) = \sum_{i=1}^n (\sigma_i(u_{x_i}), v_{x_i}).$$

- (ii) $u(x, 0) = u_0(x)$ in H .
 (iii) $E(t) \leq E(0), \quad \forall t \in [0, T]$. (3.2)

Theorem 3.1. Let (H_1) and (H_2) hold, $u_0(x) \in H$, $u_1(x) \in L^2(\Omega)$. Assume that $0 < e < d$, (δ_1, δ_2) is the maximal interval such that $d(\delta) > e$ for $\delta \in (\delta_1, \delta_2)$. Then

- (i) All solutions of problem (1.4)–(1.6) with $E(0) = e$ belong to W_δ for $\delta_1 < \delta < \delta_2$, provided $I(u_0) > 0$ or $\|u_0\|_H = 0$.
 (ii) All solutions of problem (1.4)–(1.6) with $E(0) = e$ belong to V_δ for $\delta_1 < \delta < \delta_2$, provided $I(u_0) < 0$.

Proof. (i) Let $u(t)$ be any solution of problem (1.4)–(1.6) with $E(0) = e$ and $I(u_0) > 0$ or $\|u_0\|_H = 0$, T be the existence time of $u(t)$. If $\|u_0\|_H = 0$, then $u_0(x) \in W_\delta$ for $0 < \delta < \delta_0$. If $I(u_0) > 0$, then from the proof of Lemma 2.16 and

$$\frac{1}{2}\|u_1\|^2 + J(u_0) = E(0) < d(\delta), \quad \delta_1 < \delta < \delta_2, \quad (3.3)$$

it follows that $I_\delta(u_0) > 0$ and $J(u_0) < d(\delta)$, i.e. $u_0(x) \in W_\delta$ for $\delta_1 < \delta < \delta_2$. Next we prove $u(t) \in W_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 < t < T$. If it is false, we must have $t_0 \in (0, T)$ such that $u(t_0) \in \partial W_\delta$ for some $\delta \in (\delta_1, \delta_2)$, i.e. $I_\delta(u(t_0)) = 0$, $\|u(t_0)\|_H \neq 0$ or $J(u(t_0)) = d(\delta)$. From (3.2) we have

$$\frac{1}{2}\|u_t\|^2 + J(u) \leq E(0) < d(\delta), \quad \delta_1 < \delta < \delta_2, \quad 0 < t < T, \quad (3.4)$$

which implies $J(u(t_0)) = d(\delta)$ is impossible. On the other hand, if $I_\delta(u(t_0)) = 0$ and $\|u(t_0)\|_H \neq 0$, then by the definition of $d(\delta)$ we have $J(u(t_0)) \geq d(\delta)$ which contradicts (3.4).

(ii) Let $u(t)$ be any solution of problem (1.4)–(1.6) with $E(0) = e$ and $I(u_0) < 0$, T be the existence time of $u(t)$. First from (3.3) and the proof of Lemma 2.16 we can obtain $I_\delta(u_0) < 0$ and $J(u_0) < d(\delta)$, i.e. $u_0(x) \in V_\delta$ for $\delta_1 < \delta < \delta_2$. Next we prove $u(t) \in V_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 < t < T$. If it is false, let $t_0 \in (0, T)$ be the first time such that $u(t) \in V_\delta$ for $0 \leq t < t_0$ and $u(t_0) \in \partial V_\delta$, i.e. $I_\delta(u(t_0)) = 0$ or $J(u(t_0)) = d(\delta)$ for some $\delta \in (\delta_1, \delta_2)$. Again (3.4) shows that $J(u(t_0)) = d(\delta)$ is impossible. If $I_\delta(u(t_0)) = 0$, then $I_\delta(u(t)) < 0$ for $0 < t < t_0$ and Lemma 2.7 yield $\|u(t)\|_H > r(\delta)$ and $\|u(t_0)\|_H \geq r(\delta)$. Hence by the definition of $d(\delta)$ we get $J(u(t_0)) \geq d(\delta)$ which contradicts (3.4). \square

From Theorem 3.1 and Lemma 2.9 we can obtain the following

Theorem 3.2. If in Theorem 3.1 the assumption $E(0) = e$ is replaced by $0 < E(0) \leq e$, then the conclusion of Theorem 3.1 also holds.

Theorem 3.3. Let (H_1) and (H_2) hold, $u_i(x)$ ($i = 0, 1$), e and δ_i ($i = 1, 2$) be the same as those in Theorem 3.1. Then for any $\delta \in (\delta_1, \delta_2)$ both sets W_δ and V_δ are invariant, thereby both sets

$$W_{\delta_1\delta_2} = \bigcup_{\delta_1 < \delta < \delta_2} W_\delta \quad \text{and} \quad V_{\delta_1\delta_2} = \bigcup_{\delta_1 < \delta < \delta_2} V_\delta$$

are invariant respectively under the flow of (1.4)–(1.6), provided $0 < E(0) \leq e$.

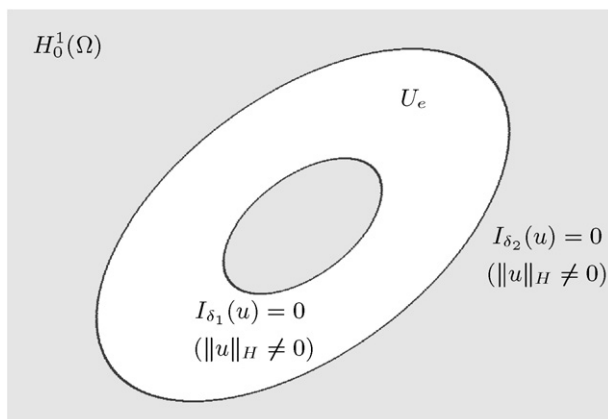


Fig. 2.

From (3.4) and the definition of $d(\delta)$ we see that if $0 < E(0) \leq e < d$, then $I_\delta(u) = 0$, $\delta_1 < \delta < \delta_2$, and $\|u\|_H \neq 0$ is impossible. So for the set of all solutions of problem (1.4)–(1.6) with $0 < E(0) \leq e$ there is a vacuum region

$$U_e = \{u \in H \mid \|u\|_H \neq 0, I_\delta(u) = 0, \delta_1 < \delta < \delta_2\}$$

such that there is no any solution of problem (1.4)–(1.6) (see Fig. 2).

The vacuum region U_e become bigger and bigger with decreasing of e . As the limit case we obtain

$$U_0 = \{u \in H \mid \|u\|_H \neq 0, I_\delta(u) = 0, 0 < \delta < \delta_0\}.$$

In fact, we have the following

Theorem 3.4. Let (H_1) and (H_2) hold, $u_0(x) \in H$, $u_1(x) \in L^2(\Omega)$. Then all nontrivial solutions of problem (1.4)–(1.6) with $E(0) = 0$ belong to

$$B_{r_0}^c = \{u \in H \mid \|u\|_H \geq r_0\},$$

where r_0 is the unique real root of equation $h_1(r) = \frac{1}{2}$,

$$h_1(r) = \frac{a_1 C_1^{q_1+1}}{p_1 + 1} r^{q_1-1} + \frac{a_2 C_2^{q_2+1}}{p_2 + 1} r^{q_2-1}.$$

Proof. Let $u(t)$ be any solution of problem (1.4)–(1.6) with $E(0) = 0$, T be the existence time of $u(t)$. First from

$$E(t) = \frac{1}{2} \|u_t\|^2 + J(u) \leq E(0) = 0$$

it follows that $J(u) \leq 0$ for $0 \leq t < T$. Hence from

$$\frac{1}{2} \|u\|_H^2 = \frac{1}{2} \|\Delta u\|^2 + \frac{\alpha}{2} \|\nabla u\|^2 \leq \int_{\Omega} F(u) dx + \sum_{i=1}^n \int_{\Omega} G_i(u_{x_i}) dx$$

$$\begin{aligned}
&\leq \frac{1}{p_1+1} \int_{\Omega} u f(u) \, dx + \frac{1}{p_2+1} \sum_{i=1}^n \int_{\Omega} u_{x_i} \sigma_i(u_{x_i}) \, dx \\
&\leq \frac{a_1}{p_1+1} \|u\|_{q_1+1}^{q_1+1} + \frac{a_2}{p_2+1} \sum_{i=1}^n \|u_{x_i}\|_{q_2+1}^{q_2+1} \\
&\leq \frac{a_1}{p_1+1} C_1^{q_1+1} \|u\|_H^{q_1+1} + \frac{a_2}{p_2+1} C_2^{q_2+1} \|u\|_H^{q_2+1} = h_1(\|u\|_H) \|u\|_H^2
\end{aligned}$$

it follows that either $h(\|u\|_H) \geq \frac{1}{2}$, $\|u\|_H \geq r_0$ or $\|u\|_H = 0$. If $\|u_0\|_H = 0$, then $\|u(t)\|_H \equiv 0$ for $0 < t < T$. Otherwise there exists $t_0 \in (0, T)$ such that $0 < \|u(t_0)\|_H < r_0$ which is impossible. By a similar argument we can prove that if $\|u_0\|_H \geq r_0$, then $\|u(t)\|_H \geq r_0$ for $0 < t < T$. \square

4. Global existence and nonexistence of solutions

In this section we prove the global existence and nonexistence of solutions and obtain a threshold result of global existence and nonexistence of solutions for problem (1.4)–(1.6).

Theorem 4.1. *Let (H_1) and (H_2) hold, $u_0(x) \in H$, $u_1(x) \in L^2(\Omega)$. Assume that $E(0) < d$ and $I(u_0) > 0$ or $\|u_0\|_H = 0$. Then problem (1.4)–(1.6) admits a global weak solution $u(t) \in L^\infty(0, \infty; H)$ with $u_t(t) \in L^\infty(0, \infty; L^2(\Omega))$ and $u(t) \in W$ for $0 \leq t < \infty$.*

Proof. Let $\{w_j(x)\}$ be a system of base functions in H . Construct the approximate solutions of problem (1.4)–(1.6)

$$u_m(x, t) = \sum_{j=1}^m g_{jm}(t) w_j(x), \quad m = 1, 2, \dots,$$

satisfying

$$(u_{mtt}, w_s) + a(u_m, w_s) = b(u_m, w_s) + c(u_m, w_s), \quad s = 1, 2, \dots, m, \quad (4.1)$$

$$u_m(x, 0) = \sum_{j=1}^m a_{jm} w_j(x) \rightarrow u_0(x) \quad \text{in } H, \quad (4.2)$$

$$u_{mt}(x, 0) = \sum_{j=1}^m b_{jm} w_j(x) \rightarrow u_1(x) \quad \text{in } L^2(\Omega). \quad (4.3)$$

Multiplying (4.1) by $g'_{sm}(t)$ and summing for s we get

$$\frac{d}{dt} E_m(t) = 0$$

and

$$E_m(t) = \frac{1}{2} \|u_{mt}\|^2 + J(u_m) = E_m(0), \quad 0 \leq t < \infty. \quad (4.4)$$

First $E(0) < d$ and $I(u_0) > 0$ or $\|u_0\|_H = 0$ yield $u_0(x) \in W$. Then from (4.2) and (4.3) it follows that $E_m(0) < d$ and $u_m(0) \in W$ for sufficiently large m . Next from (4.4) and the argument

in the proof of Theorem 3.1 we can obtain $u_{mt}(t) \in W$ for sufficiently large m and $0 \leq t < \infty$. Hence from (4.4) and

$$J(u_m) \geq \frac{p-1}{2(p+1)} \|u_m\|_H^2 + \frac{1}{p+1} I(u_m) \geq \frac{p-1}{2(p+1)} \|u_m\|_H^2$$

we get

$$\frac{1}{2} \|u_{mt}\|^2 + \frac{p-1}{2(p+1)} \|u_m\|_H^2 < d, \quad 0 \leq t < \infty, \quad (4.5)$$

for sufficiently large m . (4.5) gives

$$\|u_m\|_H^2 < \frac{2(p+1)}{p-1} d, \quad 0 \leq t < \infty, \quad (4.6)$$

$$\|u_m\|_{q_1+1}^2 \leq C_1^2 \|u_m\|_H^2 < C_1^2 \frac{2(p+1)}{p-1} d, \quad 0 \leq t < \infty, \quad (4.7)$$

$$\|u_m\|_{1,q_2+1}^2 \leq C_2^2 \|u_m\|_H^2 < C_2^2 \frac{2(p+1)}{p-1} d, \quad 0 \leq t < \infty, \quad (4.8)$$

and

$$\|u_{mt}\|^2 < 2d, \quad 0 \leq t < \infty. \quad (4.9)$$

Furthermore we have

$$\begin{aligned} \|f(u_m)\|_{r_1}^{r_1} &\leq \int_{\Omega} (a_1 |u_m|^{q_1})^{r_1} dx = a_1^{r_1} \|u_m\|_{q_1+1}^{q_1+1} \leq a_1^{r_1} C_1^{q_1+1} \left(\frac{2(p+1)}{p-1} d \right)^{\frac{q_1+1}{2}}, \\ r_1 &= \frac{q_1+1}{q_1}, \quad 0 \leq t < \infty, \end{aligned} \quad (4.10)$$

$$\begin{aligned} \|\sigma_i(u_{mx_i})\|_{r_2}^{r_2} &\leq \int_{\Omega} (a_2 |u_{mx_i}|^{q_2})^{r_2} dx = a_2^{r_2} \|u_{mx_i}\|_{q_2+1}^{q_2+1} \leq a_2^{r_2} C_2^{q_2+1} \left(\frac{2(p+1)}{p-1} d \right)^{\frac{q_2+1}{2}}, \\ r_2 &= \frac{q_2+1}{q_2}, \quad 1 \leq i \leq n, \quad 0 \leq t < \infty. \end{aligned} \quad (4.11)$$

From (4.6)–(4.11) it follows that there exist u, χ, ξ_i ($1 \leq i \leq n$) and a subsequence $\{u_v\}$ of $\{u_m\}$ such that as $v \rightarrow \infty$,

$$\begin{aligned} u_v &\rightarrow u \text{ in } L^\infty(0, \infty; H) \text{ weakly star and a.e. in } Q = \Omega \times [0, \infty); \\ u_v &\rightarrow u \text{ in } L^{q_1+1}(\Omega) \text{ strongly for each fixed } t > 0; \\ u_{vx_i} &\rightarrow u_{x_i} \text{ in } L^\infty(0, \infty; L^{q_2+1}(\Omega)) \text{ weakly star and a.e. in } Q, \quad 1 \leq i \leq n; \\ u_{vx_i} &\rightarrow u_{x_i} \text{ in } L^{q_2+1}(\Omega) \text{ strongly for each fixed } t > 0; \\ u_{vt} &\rightarrow u_t \text{ in } L^\infty(0, \infty; L^2(\Omega)) \text{ weakly star}; \\ f(u_v) &\rightarrow \chi \text{ in } L^\infty(0, \infty; L^{r_1}(\Omega)) \text{ weakly star}; \\ \sigma_i(u_{vx_i}) &\rightarrow \xi_i \text{ in } L^\infty(0, \infty; L^{r_2}(\Omega)) \text{ weakly star}. \end{aligned}$$

By Lemma 1.3 in [10] we have $\chi = f(u)$ and $\xi_i = \sigma_i(u_{x_i})$.

Integrating (4.1) with respect to t we get

$$\begin{aligned} (u_{mt}, w_s) + \int_0^t a(u_m, w_s) \, d\tau \\ = \int_0^t b(u_m, w_s) \, d\tau + \int_0^t c(u_m, w_s) \, d\tau + (u_{mt}(0), w_s). \end{aligned} \quad (4.12)$$

In (4.12) for fixed s letting $m = v \rightarrow \infty$ we obtain

$$(u_t, w_s) + \int_0^t a(u, w_s) \, d\tau = \int_0^t b(u, w_s) \, d\tau + \int_0^t c(u, w_s) \, d\tau + (u_1, w_s), \quad \forall s,$$

and (3.1). On the other hand, (4.2) gives $u(x, 0) = u_0(x)$ in H .

Next we prove that u satisfies (3.2). First from above it follows that for each fixed $t > 0$ as $v \rightarrow \infty$,

$$\begin{aligned} & \left| \int_{\Omega} F(u_v) \, dx - \int_{\Omega} F(u) \, dx \right| \\ &= \left| \int_{\Omega} f(u + \theta_v u_v)(u_v - u) \, dx \right| \leq \|f(u + \theta_v u_v)\|_{r_1} \|u_v - u\|_{q_1+1} \\ &\leq C \|u_v - u\|_{q_1+1} \rightarrow 0, \\ & \left| \int_{\Omega} G_i(u_{vx_i}) \, dx - \int_{\Omega} G_i(u_{x_i}) \, dx \right| \\ &= \left| \int_{\Omega} \sigma_i(u_{x_i} + \theta_{v_i} u_{vx_i})(u_{vx_i} - u_{x_i}) \, dx \right| \leq \|\sigma_i(u_{x_i} + \theta_{v_i} u_{vx_i})\|_{r_2} \|u_{vx_i} - u_{x_i}\|_{q_2+1} \\ &\leq C \|u_{vx_i} - u_{x_i}\|_{q_2+1} \rightarrow 0, \end{aligned}$$

where $0 < Q_v = \theta_v(x) < 1$, $0 < \theta_{v_i} = \theta_{v_i}(x) < 1$, C is a constant independent of v . Hence we have

$$\begin{aligned} \lim_{v \rightarrow \infty} \int_{\Omega} F(u_v) \, dx &= \int_{\Omega} F(u) \, dx, \\ \lim_{v \rightarrow \infty} \sum_{i=1}^n \int_{\Omega} G_i(u_{vx_i}) \, dx &= \sum_{i=1}^n \int_{\Omega} G_i(u_{x_i}) \, dx. \end{aligned}$$

On the other hand, (4.2) and (4.3) give $E_v(0) \rightarrow E(0)$ as $v \rightarrow \infty$. Therefore from (4.4) we obtain

$$\begin{aligned} & \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|u\|_H^2 \\ &\leq \liminf_{v \rightarrow \infty} \frac{1}{2} \|u_{vt}\|^2 + \liminf_{v \rightarrow \infty} \frac{1}{2} \|u_v\|_H^2 \leq \liminf_{v \rightarrow \infty} \left(\frac{1}{2} \|u_{vt}\|^2 + \frac{1}{2} \|u_v\|_H^2 \right) \end{aligned}$$

$$\begin{aligned}
&= \liminf_{v \rightarrow \infty} \left(E_v(0) + \int_{\Omega} F(u_v) dx + \sum_{i=1}^n \int_{\Omega} G_i(u_{v x_i}) dx \right) \\
&= \lim_{v \rightarrow \infty} \left(E_v(0) + \int_{\Omega} F(u_v) dx + \sum_{i=1}^n \int_{\Omega} G_i(u_{v x_i}) dx \right) \\
&= E(0) + \int_{\Omega} F(u) dx + \sum_{i=1}^n \int_{\Omega} G_i(u_{x_i}) dx
\end{aligned}$$

which gives (3.2). Therefore $u(x, t)$ is a global weak solution of problem (1.4)–(1.6). Finally from Theorem 3.1 we get $u(t) \in W$ for $0 \leq t < \infty$. \square

Note that from

$$J(u_0) \geq \frac{p-1}{2(p+1)} \|u_0\|_H^2 + \frac{1}{p+1} I(u_0)$$

and $I(u_0) > 0$ we can get $J(u_0) > 0$ and $E(0) > 0$, hence from Theorems 4.1, 3.1 and Lemma 2.16, we can obtain the following

Theorem 4.2. *If in Theorem 4.1 the assumption $I(u_0) > 0$ is replaced by $I_{\delta_2}(u_0) > 0$, then problem (1.4)–(1.6) admits a global weak solution $u(t) \in L^\infty(0, \infty; H)$ with $u_t(t) \in L^\infty(0, \infty; L^2(\Omega))$ and $u(t) \in W_\delta$ for $\delta_1 < \delta < \delta_2$, $0 \leq t < \infty$, where (δ_1, δ_2) is the maximal interval such that $d(\delta) > E(0)$ for $\delta \in (\delta_1, \delta_2)$.*

From Theorem 4.2, Lemmas 2.6 and 2.10 we can obtain the following

Theorem 4.3. *If in Theorem 4.1 the assumption $I(u_0) > 0$ or $\|u_0\|_H = 0$ is replaced by $\|u_0\|_H < r(\delta_2)$, where δ_1 and δ_2 are the same as those in Theorem 4.2, then problem (1.4)–(1.6) admits a global weak solution $u(t) \in L^\infty(0, \infty; H)$ with $u_t(t) \in L^\infty(0, \infty; L^2(\Omega))$ and $\|u\|_H^2 < \frac{d(\delta)}{a(\delta)}$ for $\delta_1 < \delta < \min\{\delta_2, \frac{p+1}{2}\}$, $0 \leq t < \infty$; $\|u_t\|^2 < 2d(\delta)$ for $\delta_1 < \delta < \delta_2$, $0 \leq t < \infty$. In particular, we have $\|u\|_H^2 \leq \frac{d(\delta_1)}{a(\delta_1)}$, $\|u_t\|^2 \leq 2d(\delta_1)$ for $0 \leq t < \infty$.*

Theorem 4.4. *Let (H_1) and (H_2) hold, $u_0(x) \in H$, $u_1(x) \in L^2(\Omega)$. Assume that any one of the following conditions holds: (i) $E(0) < 0$; (ii) $E(0) = 0$ and $\|u_0\|_H \neq 0$; or (iii) $0 < E(0) < d$ and $I(u_0) < 0$. Then the existence time of any solution of problem (1.4)–(1.6) is finite.*

Proof. Let u be any solution of problem (1.4)–(1.6) satisfying above conditions, T be the existence time of u . If $E(0) < 0$, then from (3.2) we have

$$\frac{1}{2} \|u_t\|^2 + \frac{p-1}{2(p+1)} \|u\|_H^2 + \frac{1}{p+1} I(u) \leq \frac{1}{2} \|u_t\|^2 + J(u) \leq E(0) < 0 \quad (4.13)$$

and $I(u) < 0$ for $0 \leq t < T$; if $E(0) = 0$ and $\|u_0\|_H \neq 0$, then from Theorem 3.4 and (4.13) it follows that $\|u\|_H \geq r_0$ and $I(u) < 0$ for $0 \leq t < T$; if $0 < E(0) < d$ and $I(u_0) < 0$, then by Theorem 3.1 we get $I(u) < 0$ for $0 \leq t < T$. Therefore for all cases we always have $I(u) < 0$ for $0 \leq t < T$. Next we prove $T < +\infty$. If it is false, then $T = +\infty$. Let

$$\phi(t) = \|u(t)\|^2.$$

Then $\phi'(t) = 2(u_t, u)$, $\phi''(t) = 2\|u_t\|^2 + 2(u_{tt}, u)$. Hence we have

$$\phi''(t) = 2\|u_t\|^2 - 2I(u) > 0, \quad t > 0. \quad (4.14)$$

From

$$\int_{\Omega} u f(u) \, dx + \sum_{i=1}^n \int_{\Omega} u_{x_i} \sigma_i(u_{x_i}) \, dx \geq (p+1) \left(\int_{\Omega} F(u) \, dx + \sum_{i=1}^n \int_{\Omega} G_i(u_{x_i}) \, dx \right)$$

and (3.2) it follows that

$$\phi''(t) > (p+3)\|u_t\|^2 + (p-1)\|u\|_H^2 - 2(p+1)E(0). \quad (4.15)$$

(i) If $E(0) \leq 0$, then (4.15) gives

$$\phi''(t) > (p+3)\|u_t\|^2. \quad (4.16)$$

(ii) If $0 < E(0) < d$, then first from (4.15) we get

$$\phi''(t) > (p+3)\|u_t\|^2 + (p-1)\lambda\phi(t) - 2(p+1)E(0)$$

for some $\lambda > 0$.

Furthermore from Theorem 3.1 we have $u(t) \in V_{\delta}$ for $1 < \delta < \delta_2$ and $t \geq 0$, where (δ_1, δ_2) is the maximal interval such that $d(\delta) > E(0)$ for $\delta \in (\delta_1, \delta_2)$. Hence $I_{\delta}(u) < 0$ and $\|u\|_H > r(\delta)$ for $1 < \delta < \delta_2$ and $t \geq 0$. Therefore we get $I_{\delta_2}(u) \leq 0$ and $\|u\|_H \geq r(\delta_2)$ for $t \geq 0$. From this and (4.14) we obtain

$$\begin{aligned} \phi''(t) &\geq -2I(u) = 2(\delta_2 - 1)\|u\|_H^2 - 2I_{\delta_2}(u) \geq 2(\delta_2 - 1)\|u\|_H^2 \\ &\geq 2(\delta_2 - 1)r^2(\delta_2), \quad t \geq 0, \\ \phi'(t) &\geq 2(\delta_2 - 1)r^2(\delta_2)t + \phi'(0), \quad t \geq 0. \end{aligned}$$

Hence there exists $t_0 \geq 0$ such that $\phi'(t_0) > 0$ and

$$\phi(t) \geq \phi'(t_0)(t - t_0) + \phi(t_0) \geq \phi'(t_0)(t - t_0), \quad t \geq t_0.$$

So for sufficiently large t we have $(p-1)\lambda\phi(t) > 2(p+1)E(0)$ and (4.16) holds. As done in [7], from (4.16) we can obtain that there exists T_1 such that $0 < T_1 < \infty$ and

$$\lim_{t \rightarrow T_1} \|u(t)\|^2 = +\infty$$

which contradicts $T = +\infty$. \square

Remark 4.5. In [5] the proof of the global nonexistence of solutions for problem (1.3) mainly depends on some estimates for the depth d of the potential well W as follows

$$d < \frac{r-2}{2r}a(u) < \frac{r-2}{2r}b(u)$$

and

$$d < \frac{m-2}{2m}a(u) < \frac{m-2}{2m}c(u)$$

for any $u \in H$ and $I(u) < 0$, in particular for $u \in V$, where

$$a(u) = \|\Delta u\|^2 + \alpha \|\nabla u\|^2, \quad b(u) = \mu \|u\|_r^r, \quad c(u) = \beta \|\nabla u\|_m^m.$$

However these estimates only hold for the special problem (1.3) and do not hold for the general problem (1.4)–(1.6). Therefore the method used in [5] cannot be generalized to problem (1.4)–(1.6).

Note that from

$$\frac{1}{2}\|u_1\|^2 + \frac{p-1}{2(p+1)}\|u_0\|_H^2 + \frac{1}{p+1}I(u_0) \leq \frac{1}{2}\|u_1\|^2 + J(u_0) = E(0) \quad (4.17)$$

it follows that if $0 < E(0) < d$, then the definition of d shows that $I(u_0) = 0$, $\|u_0\|_H \neq 0$ is impossible. Otherwise we have $J(u_0) \geq d$ which contradicts (4.17). If $E(0) = 0$, then $I(u_0) > 0$ or $I(u_0) = 0$, $\|u_0\|_H \neq 0$ is impossible; if $E(0) < 0$, then $I(u_0) \geq 0$ is impossible. Thus all possible cases already have been considered in Theorems 4.1, 3.4 and 4.4, and a threshold result has been obtained as follows

Corollary 4.6. *Let (H_1) and (H_2) hold, $u_0(x) \in H$, $u_1(x) \in L^2(\Omega)$. Assume that $E(0) < d$. Then problem (1.4)–(1.6) admits a global weak solution, provided $I(u_0) \geq 0$; and problem (1.4)–(1.6) does not admit any global solution, provided $I(u_0) < 0$.*

Remark 4.7. In this paper the uniqueness, local existence and finite time blow up of solutions for problem (1.4)–(1.6) are not discussed and applied. However if besides (H_1) and (H_2) we further assume that $f(s)$ and $\sigma_i(s)$ ($1 \leq i \leq n$) are local Lipschitz continuous, and in (H_1) and (H_2) the assumptions $1 < q_1 < \frac{n+4}{n-4}$ if $n \geq 5$; $1 < q_2 < \frac{n+2}{n-2}$ if $n \geq 3$ are replaced by $1 < q_1 \leq \frac{n}{n-4}$ if $n \geq 5$; $1 < q_2 \leq \frac{n}{n-2}$ if $n \geq 3$, respectively, then by the normal method we can prove that the conclusion of Theorem 2.1 in [5] also holds for problem (1.4)–(1.6), and by which the uniqueness, local existence and finite time blow up of solutions for problem (1.4)–(1.6) can be obtained.

5. Problem (1.4)–(1.6) with critical initial conditions $I(u_0) \geq 0$, $E(0) = d$

In this section we shall prove the global existence of weak solutions for problem (1.4)–(1.6) with critical initial condition $I(u_0) \geq 0$, $E(0) = d$.

Theorem 5.1. *Let (H_1) and (H_2) hold, $u_0(x) \in H$, $u_1(x) \in L^2(\Omega)$. Assume that $E(0) = d$ and $I(u_0) \geq 0$. Then problem (1.4)–(1.6) admits a global weak solution $u(t) \in L^\infty(0, \infty; H)$ with $u_t(t) \in L^\infty(0, \infty; L^2(\Omega))$ and $u(t) \in \tilde{W} = W \cup \partial W$ for $0 \leq t < \infty$.*

Proof. We shall prove this theorem by considering the following two cases:

(i) $\|u_0\|_H \neq 0$. Taking a sequence $\{\lambda_m\}$ such that $0 < \lambda_m < 1$ ($m = 1, 2, \dots$) and $\lambda_m \rightarrow 1$ as $m \rightarrow \infty$ and letting $u_{0m} = \lambda_m u_0(x)$, consider the initial conditions

$$u(x, 0) = u_{0m}(x), \quad u_t(x, 0) = u_1(x) \quad (5.1)$$

and corresponding problem (1.4), (5.1), (1.6). First from $I(u_0) \geq 0$ and Lemma 2.5 it follows that $\lambda^* = \lambda^*(u_0) \geq 1$. Hence $I(u_{0m}) = I(\lambda_m u_0) > 0$ and

$$J(u_{0m}) \geq \frac{p-1}{2(p+1)}\|u_{0m}\|_H^2 + \frac{1}{p+1}I(u_{0m}) > 0.$$

From this and $J(u_{0m}) = J(\lambda_m u_0) < J(u_0)$ we can obtain

$$0 < E_m(0) = \frac{1}{2} \|u_1\|^2 + J(u_{0m}) < \frac{1}{2} \|u_1\|^2 + J(u_0) = E(0) = d.$$

Hence from Theorem 4.1 it follows that for each m problem (1.4), (5.1), (1.6) admits a global weak solution $u_m(t) \in L^\infty(0, \infty; H)$ with $u_{mt}(t) \in L^\infty(0, \infty; L^2(\Omega))$ and $u_m(t) \in W$ for $0 \leq t < \infty$ satisfying

$$(u_{mt}, v) + \int_0^t a(u_m, v) d\tau = \int_0^t b(u_m, v) d\tau + \int_0^t c(u_m, v) d\tau + (u_1, v),$$

$$\forall v \in H \text{ and } \forall t > 0, \quad (5.2)$$

and

$$\frac{1}{2} \|u_{mt}\|^2 + J(u_m) \leq E_m(0) < d, \quad 0 \leq t < \infty. \quad (5.3)$$

From (5.3) and

$$J(u_m) \geq \frac{p-1}{2(p+1)} \|u_m\|_H^2 + \frac{1}{p+1} I(u_m) \geq \frac{p-1}{2(p+1)} \|u_m\|_H^2$$

we can obtain

$$\|u_m\|_H^2 \leq \frac{2(p+1)}{p-1} d, \quad 0 \leq t < \infty, \quad (5.4)$$

$$\|u_m\|_{q_1+1}^2 \leq C_1^2 \|u_m\|_H^2 \leq C_1^2 \frac{2(p+1)}{p-1} d, \quad 0 \leq t < \infty, \quad (5.5)$$

$$\|u_m\|_{1,q_2+1}^2 \leq C_2^2 \|u_m\|_H^2 \leq C_2^2 \frac{2(p+1)}{p-1} d, \quad 0 \leq t < \infty, \quad (5.6)$$

$$\|u_{mt}\|^2 \leq 2d, \quad 0 \leq t < \infty, \quad (5.7)$$

$$\|f(u_m)\|_{r_1}^{r_1} \leq a_1^{r_1} \|u_m\|_{q_1+1}^{q_1+1} \leq a_1^{r_1} C_1^{q_1+1} \left(\frac{2(p+1)}{p-1} d \right)^{\frac{q_1+1}{2}},$$

$$r_1 = \frac{q_1+1}{q_1}, \quad 0 \leq t < \infty, \quad (5.8)$$

$$\|\sigma_i(u_{mx_i})\|_{r_2}^{r_2} \leq a_2^{r_2} \|u_{mx_i}\|_{q_2+1}^{q_2+1} \leq a_2^{r_2} C_2^{q_2+1} \left(\frac{2(p+1)}{p-1} d \right)^{\frac{q_2+1}{2}},$$

$$1 \leq i \leq n, \quad r_2 = \frac{q_2+1}{q_2}, \quad 0 \leq t < \infty. \quad (5.9)$$

From (5.4)–(5.9) it follows that there exist $u \in \bar{W}$ and a subsequence $\{u_v\}$ of $\{u_m\}$ such that as $v \rightarrow \infty$,

$$u_v \rightarrow u \text{ in } L^\infty(0, \infty; H) \text{ weakly star and a.e. in } Q = \Omega \times [0, \infty),$$

$$u_v \rightarrow u \text{ in } L^{q_1+1}(\Omega) \text{ strongly for each fixed } t > 0;$$

$$u_{vx_i} \rightarrow u_{x_i} \text{ in } L^\infty(0, \infty; L^2(\Omega)) \text{ weakly star,}$$

$$u_{vx_i} \rightarrow u_{x_i} \text{ in } L^{q_2+1}(\Omega) \text{ strongly for each fixed } t > 0;$$

$$f(u_v) \rightarrow f(u) \text{ in } L^\infty(0, \infty; L^{r_1}(\Omega)) \text{ weakly star,}$$

$$\sigma_i(u_{vx_i}) \rightarrow \sigma_i(u_{x_i}) \text{ in } L^\infty(0, \infty; L^{r_2}(\Omega)) \text{ weakly star, } 1 \leq i \leq n.$$

Letting $m = v \rightarrow \infty$ in (5.2) we get

$$(u_t, v) + \int_0^t a(u, v) d\tau = \int_0^t b(u, v) d\tau + \int_0^t c(u, v) d\tau + (u_1, v), \quad \forall v \in H \quad \text{and} \quad \forall t > 0.$$

On the other hand, clearly we have $u(x, 0) = u_0(x)$ in H . Finally from the proof of Theorem 4.1 we can get $E(t) \leq E(0)$. Therefore $u(x, t)$ is a global weak solution of problem (1.4)–(1.6).

(ii) $\|u_0\|_H = 0$. Since $\|u_0\|_H = 0$ implies $u_0 = 0$ and $J(u_0) = 0$ we have $\frac{1}{2}\|u_1\|^2 = E(0) = d$. Again take $\{\lambda_m\}$ as above and let $u_{1m}(x) = \lambda_m u_1(x)$. Consider the initial conditions

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_{1m}(x) \quad (5.10)$$

and problem (1.4), (5.10), (1.6). From $\|u_0\|_H = 0$,

$$0 < E_m(0) = \frac{1}{2}\|u_{1m}\|^2 + J(u_0) = \frac{1}{2}\|u_{1m}\|^2 < d$$

and Theorem 4.1 it follows that for each m problem (1.4), (5.10), (1.6) admits a global weak solution $u_m \in L^\infty(0, \infty; H)$ with $u_{mt} \in L^\infty(0, \infty; L^2(\Omega))$ and $u_m \in W$ for $0 \leq t < \infty$ satisfying (5.2) and (5.3). The remainder of this proof is the same as that in the proof of part (i) of this theorem. \square

Except the case for the particular forms of source and strain terms studied in [5], the global existence of solutions for problem (1.4)–(1.6) with critical initial condition $E(0) = d$ and $I(u_0) < 0$ and the asymptotic behavior of solutions for problem (1.4)–(1.6) are still open up to now.

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