Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 333 (2007) 567–576
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Positive solutions to the equations $AX = C$ and $XB = D$ for Hilbert space operators

Alegra Dajić, J.J. Koliha *

Department of Mathematics and Statistics, The University of Melbourne, Melbourne VIC 3010, Australia

Received 24 August 2006
Available online 13 December 2006
Submitted by R. Curto

Abstract

The paper studies the equation $AX = C$ for bounded linear operators between Hilbert spaces, gives conditions for the existence of hermitian solutions and positive solutions, and obtains the formula for the general form of these solutions. Then the common hermitian and positive solutions to the equations $AX = C$ and $XB = D$ are studied and new representations of the general solutions are given. Many results for matrices are recovered as special cases, and the results of Phadke and Thakare [S.V. Phadke, N.K. Thakare, Generalized inverses and operator equations, Linear Algebra Appl. 23 (1979) 191–199] are corrected.

Keywords: Hilbert space; Operator equation; Positive solution; Common positive solutions

1. Introduction

Positive solutions to the matrix equations of the title were studied by many authors, notably by Khatri and Mitra [5] in 1976, who gave necessary and sufficient conditions for the existence of positive solutions, and found the general solution based on generalized matrix inverses and the matrix rank. Phadke and Thakare [8] attempted to describe the hermitian and positive solutions for Hilbert space operators in 1979, but several of their results are incorrect or have incorrect proofs. One of the aims of the present paper is to find the correct version of these results.

* Corresponding author. Fax: +61 03 8344 4599.
E-mail addresses: a.dajic@ms.unimelb.edu.au (A. Dajić), j.koliha@ms.unimelb.edu.au (J.J. Koliha).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.11.016

2. Preliminaries

H, K and L denote complex Hilbert spaces, and $\mathcal{B}(H,K)$ the set of all bounded linear operators between H and K. We write $R(A)$ and $N(A)$ for the range and nullspace of $A \in \mathcal{B}(H,K)$. An operator $A \in \mathcal{B}(H,K)$ is *regular* if there is an operator $A^- \in \mathcal{B}(K,H)$ such that $AA^-A = A$; A^- is called an *inner inverse* of A. It is well known that A is regular if and only if A has a closed range. For the future use we record this well known result.

Lemma 2.1. *Let $A \in \mathcal{B}(H,K)$ be a closed range operator. Given a pair of topological complements M, N of $R(A)$, $N(A)$, respectively, there exists a unique inner inverse $A^- \in \mathcal{B}(K,H)$ of A with $R(AA^-) = R(A)$, $N(AA^-) = M$, $R(I - A^-A) = N(A^-A) = N(A)$ and $N(I - A^-A) = R(A^-A) = N$.*

An operator $A \in \mathcal{B}(H)$ is *hermitian* (or *self-adjoint*) if $A^* = A$, and *positive* if $\langle Ax, x\rangle \geqslant 0$ for all $x \in H$; we write $A \geqslant 0$ if A is positive. The set $\mathcal{B}(H)^+$ of the positive operators is a subset of the hermitian operators. The Moore–Penrose inverse of $A \in \mathcal{B}(H,K)$ is defined as the operator $A^\dagger \in \mathcal{B}(K,H)$ satisfying the Penrose equations (see [7]),

$$AA^\dagger A = A, \qquad A^\dagger AA^\dagger = A^\dagger, \qquad \left(A^\dagger A\right)^* = A^\dagger A, \qquad \left(AA^\dagger\right)^* = AA^\dagger. \tag{2.1}$$

An operator $A \in \mathcal{B}(H,K)$ has the (unique) Moore–Penrose inverse if and only if A has closed range, or equivalently if and only if it is regular. If a regular operator A is positive, then $A^\dagger \geqslant 0$.

In the construction of the general hermitian and positive common solution to $AX = C$ and $XB = D$ our main tools will be two lemmas for 2×2 operator matrices. One deals with the regularity and an inner inverse of such an operator matrix, the other gives conditions for its positivity. First some terminology.

Let $A \in \mathcal{B}(H)$, $B \in \mathcal{B}(K,H)$, $C \in \mathcal{B}(H,K)$ and $D \in \mathcal{B}(K)$ and let M be the operator matrix acting on $H \oplus K$ defined by

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{2.2}$$

If A has closed range, we define a *generalized Schur complement* $s(M)$ of M by

$$s(M) = D - CA^-B.$$

In this case the generalized Schur complement depends on the choice of the inner inverse A^- of A.

For block matrices the following result reduces to [3, Corollary 6.3.5].

Lemma 2.2. *Let $A \in \mathcal{B}(H)$, $B \in \mathcal{B}(K,H)$, $C \in \mathcal{B}(H,K)$ and $D \in \mathcal{B}(K)$ be such that A has closed range, $AA^-B = B$, $CA^-A = C$, and let M be the operator matrix acting on $H \oplus K$ defined by* (2.2). *Then M is regular if and only if the generalized Schur complement $F := s(M) = D - CA^-B$ is regular. An inner inverse of M is given by*

$$M^- = \begin{bmatrix} A^- + A^-BF^-CA^- & -A^-BF^- \\ -F^-CA^- & F^- \end{bmatrix}. \tag{2.3}$$

Proof. Suppose that A is regular and set

$$S = \begin{bmatrix} I & 0 \\ -CA^- & I \end{bmatrix}, \qquad T = \begin{bmatrix} I & -A^-B \\ 0 & I \end{bmatrix}; \tag{2.4}$$

then $SMT = \operatorname{diag}(A, F)$. Since S, T are invertible, M is regular if and only if $\operatorname{diag}(A, F)$ is regular, which occurs if and only if F is regular. Then (2.3) is calculated from the equation $M^- = T \operatorname{diag}(A^-, F^-)S$. □

The following result on positivity of operator matrices was derived by Albert [1] for block matrices and by Cvetković-Ilić et al. [4] for C^*-algebras for the special case of the Moore–Penrose inverse.

Lemma 2.3. *Let $A \in \mathcal{B}(H)$ have closed range, $B \in \mathcal{B}(K, H)$, $C \in \mathcal{B}(H, K)$ and $D \in \mathcal{B}(K)$ and let M be an operator matrix acting on $H \oplus K$ defined by* (2.2). *Then M is positive if and only if*

(i) $A = A^*$, $D = D^*$, $C = B^*$,
(ii) $A \geqslant 0$,
(iii) $AA^-B = B$,
(iv) $F := s(M) = D - CA^-B \geqslant 0$,

where A^- is an arbitrary inner inverse of A.

Proof. Suppose that M is positive on $H \oplus K$. Then there exists $N \in \mathcal{B}(H \oplus K)$ such that $M = N^*N$. We can write $N = [N_1\ N_2]$, where $N_1 \in \mathcal{B}(H, H \oplus K)$ and $N_2 \in \mathcal{B}(K, H \oplus K)$ are restrictions of N. Hence

$$M = \begin{bmatrix} N_1^* \\ N_2^* \end{bmatrix} [N_1\ N_2] = \begin{bmatrix} N_1^*N_1 & N_1^*N_2 \\ N_2^*N_1 & N_2^*N_2 \end{bmatrix},$$

where $A = N_1^*N_1$ and $D = N_2^*N_2$ are hermitian, and $C = N_2^*N_1 = (N_1^*N_2)^* = B^*$. Further, since $A = N_1^*N_1$ is a closed range operator, then so is N_1. Hence $AA^\dagger B = N_1^*N_1(N_1^*N_1)^\dagger \times N_1^*N_2 = N_1^*N_2 = B$ as $N_1(N_1^*N_1)^\dagger = (N_1^*)^\dagger$. If A^- is an arbitrary inner inverse of A, then $AA^-AA^\dagger = AA^\dagger$, and $AA^-B = B$.

We observe that for an operator matrix T defined as in (2.4), we have

$$T^*MT = \begin{bmatrix} A & 0 \\ -B^*(A^-)^*A + C & D - CA^-B \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} \tag{2.5}$$

as $B^*(A^-)^*A = (AA^-B)^* = B^* = C$. Since T^*MT is positive, then so are A and F.

Conversely assume that conditions (i)–(iv) hold. With T as above, we have $M = (T^{-1})^* \operatorname{diag}(A, F)T^{-1}$. Since both A and F are positive, so is M. □

Remark 2.4. Let M be the operator matrix (2.2), and let D be regular. Then we may apply the preceding two lemmas to the matrix N given by

$$N = \begin{bmatrix} D & C \\ B & A \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} M \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = ZMZ^{-1}.$$

Observing that $Z^{-1}N^-Z$ is an inner inverse for M, we get alternative results for M, in which the generalized Schur complement $D - CA^-B$ is replaced by $A - BD^-C$.

3. Hermitian solutions to $AX = C$

Theorem 3.1. *Let $A, C \in \mathcal{B}(H, K)$ and let A be a closed range operator. Then the equation $AX = C$ has a hermitian solution $X \in \mathcal{B}(H)$ if and only if $AA^-C = C$ and $AC^* \in \mathcal{B}(H)$ is hermitian. The general hermitian solution is of the form*

$$X = A^-C + \bigl(I - A^-A\bigr)\bigl(A^-C\bigr)^* + \bigl(I - A^-A\bigr)S\bigl(I - A^-A\bigr)^*, \quad \text{where } S \in \mathcal{B}(H) \text{ is hermitian.} \tag{3.1}$$

Proof. By A^- we denote an inner inverse of A. If $AX = C$ is solvable, which happens if and only if $AA^-C = C$, the general solution is of the form $X = A^-C + (I - A^-A)U$, where $U \in \mathcal{B}(H)$ is arbitrary.

Assume that $AA^-C = C$ and AC^* is hermitian. Then the operator $X_0 = A^-C + (I - A^-A)(A^-C)^*$ is a hermitian solution to $AX = C$ as $X_0 = A^-C + (A^-C)^* - A^-AC^*(A^-)^*$.

Suppose that $AX = C$ has a hermitian solution X. Then $AA^-C = C$ (the solvability condition), and $AC^* = A(AX)^* = AXA^*$ is hermitian. Since $X_0 = A^-C + (I - A^-A)(A^-C)^*$ is a hermitian solution, the operator $Z = X - X_0$ is a hermitian solution to $AZ = 0$. We have $Z = (I - A^-A)W$ for some $W \in \mathcal{B}(H)$, and $Z = (I - A^-A)Z$ as $I - A^-A$ is idempotent. Hence $Z = Z(I - A^-A)^* = (I - A^-A)Z(I - A^-A)^*$. Conversely, a direct verification shows that every operator of the form (3.1) is a hermitian solution to $AX = C$. □

Remark 3.2. For finite complex matrices we recover Khatri and Mitra's result [5, Theorem 2.1]. Phadke and Thakare [8, Theorem 2.1] state the theorem for bounded linear operators on a Hilbert space to itself, however they omit to include the condition $AA^-C = C$. Further, they do not prove the general form of the solution, but merely show that operators of the form (3.1) are hermitian solutions.

4. Common hermitian solutions to $AX = C$ and $XB = D$

We turn to the existence of common hermitian solutions for the equations $AX = C$ and $XB = D$ for operators between Hilbert spaces H, K, L. First we need a result on the regularity of an operator matrix $\begin{bmatrix} U \\ V \end{bmatrix}$. It can be deduced from a result of Patrício and Puystjens [6, Theorem 4] originally formulated for matrices with entries in an associative ring. A simple modification shows that it applies equally well to Hilbert space operators.

Lemma 4.1. *Let $U \in \mathcal{B}(H, K)$ and $V \in \mathcal{B}(H, L)$ be closed range operators. Then the regularity of any one of the following operators implies the regularity of the remaining three operators:*

$$M := V\bigl(I - U^-U\bigr), \qquad N := U\bigl(I - V^-V\bigr), \qquad \begin{bmatrix} U \\ V \end{bmatrix}, \qquad \begin{bmatrix} V \\ U \end{bmatrix}.$$

In this case an inner inverse of $\begin{bmatrix} U \\ V \end{bmatrix}$ is given by

$$\begin{bmatrix} U \\ V \end{bmatrix}^- = \bigl[\, U^- - \bigl(I - U^-U\bigr)M^-VU^- \quad \bigl(I - U^-U\bigr)M^- \,\bigr]. \tag{4.1}$$

For brevity, in the following theorem we use a generalized Schur complement $s(T)$ of the matrix

$$T = \begin{bmatrix} A & C \\ B^* & D^* \end{bmatrix}.$$

Theorem 4.2. *Let $A, C \in \mathcal{B}(H, K)$, $B, D \in \mathcal{B}(L, H)$, and let the operators A and B have closed range. Let $M = B^*(I - A^-A)$ have closed range, and let A^-, B^- and M^- be inner inverses of A, B and M, respectively. Then the equations*

$$AX = C, \qquad XB = D \tag{4.2}$$

have a common hermitian solution $X \in \mathcal{B}(H)$ if and only if

$$AA^-C = C, \quad DB^-B = D, \quad AD = CB; \quad AC^*,\ B^*D \text{ are hermitian.} \tag{4.3}$$

The general hermitian solution is given by

$$\begin{aligned} X = {} & A^-C + (I - A^-A)M^-s(T) \\ & + (I - A^-A)(I - M^-M)\big[A^-C + (I - A^-A)M^-s(T)\big]^* \\ & + (I - A^-A)(I - M^-M)U(I - M^-M)^*(I - A^-A)^*, \\ & \text{where } U \in \mathcal{B}(H) \text{ is hermitian.} \end{aligned} \tag{4.4}$$

Proof. It can be verified directly that the equations $AX = C$ and $XB = D$ have a common hermitian solution X if and only the operator matrix equation

$$\begin{bmatrix} A \\ B^* \end{bmatrix} X = \begin{bmatrix} C \\ D^* \end{bmatrix} \tag{4.5}$$

has a hermitian solution $X \in \mathcal{B}(H)$.

Assume first that a hermitian solution X common to both equations exists. The compatibility of the two equations imply $AA^-C = C$ and $DB^-B = D$. Further, $AD = A(XB) = (AX)B = CB$. The fact that AC^* and B^*D are hermitian follows from Theorem 3.1. This proves (4.3).

Conversely, assume that Eqs. (4.3) are satisfied. We prove that the hypotheses of Theorem 3.1 are satisfied for the matrix equation (4.5). Let us write

$$\boldsymbol{A} = \begin{bmatrix} A \\ B^* \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} C \\ D^* \end{bmatrix}, \quad \boldsymbol{Q} = \begin{bmatrix} AC^* & AD \\ B^*C^* & B^*D \end{bmatrix} = \boldsymbol{Q}^* \quad \text{as } AD = CB.$$

By Lemma 4.1, A is regular. Conditions $AA^-C = C$, $DB^-B = D$ and $AD = CB$ ensure that a common (not necessarily hermitian) solution $X = A^-C + DB^- - A^-ADB^-$ exists for $AX = C$ and $XB = D$. Hence the equation $\boldsymbol{A}X = \boldsymbol{C}$ is consistent, that is, $\boldsymbol{A}^-\boldsymbol{A}\boldsymbol{C} = \boldsymbol{C}$. Further, $\boldsymbol{A}\boldsymbol{C}^*$ is hermitian, as

$$\boldsymbol{A}\boldsymbol{C}^* = \begin{bmatrix} A \\ B^* \end{bmatrix} \begin{bmatrix} C \\ D^* \end{bmatrix}^* = \begin{bmatrix} A \\ B^* \end{bmatrix} \begin{bmatrix} C^* & D \end{bmatrix} = \begin{bmatrix} AC^* & AD \\ B^*C^* & B^*D \end{bmatrix} = \boldsymbol{Q}.$$

By Theorem 3.1, $\boldsymbol{A}X = \boldsymbol{C}$ has a hermitian solution X, which is then a common hermitian solution to $AX = C$ and $XB = D$. The general hermitian solution X to the equation $\boldsymbol{A}X = \boldsymbol{C}$ is given by

$$\begin{aligned} X = {} & \boldsymbol{A}^-\boldsymbol{C} + (I - \boldsymbol{A}^-\boldsymbol{A})(\boldsymbol{A}^-\boldsymbol{C})^* + (I - \boldsymbol{A}^-\boldsymbol{A})S(I - \boldsymbol{A}^-\boldsymbol{A})^*, \\ & \text{where } S \in \mathcal{B}(H) \text{ is hermitian.} \end{aligned}$$

For brevity write $P = I - A^- A$. Using Lemma 4.1, we have

$$\begin{aligned} I - \boldsymbol{A}^- \boldsymbol{A} &= I - \begin{bmatrix} A^- - PM^- B^* A^- & PM^- \end{bmatrix} \begin{bmatrix} A \\ B^* \end{bmatrix} \\ &= I - A^- A + PM^- B^* A^- A - PM^- B^* \\ &= P - PM^- B^* \big(I - A^- A\big) = P\big(I - M^- M\big), \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{A}^- \boldsymbol{C} &= \begin{bmatrix} A^- - PM^- B^* A^- & PM^- \end{bmatrix} \begin{bmatrix} C \\ D^* \end{bmatrix} \\ &= A^- C - PM^- B^* A^- C + PM^- D^* \\ &= A^- C + PM^- \big(D^* - B^* A^- C\big) \\ &= A^- C + PM^- s(T). \end{aligned}$$

Substituting this into the general formula for X above, we obtain (4.4). $\square$

We can obtain an alternative expression for the solution X in the preceding theorem if we interchange the order of the equations $AX = C$ and $B^* X = D^*$. By Lemma 4.1, $M = I - A^- A$ is regular if and only if $N = A(I - BB^-)^*$ is regular. This ensures that the alternative formula is obtained from (4.4) using the same hypotheses and the interchange $A \leftrightarrow B^*$ and $C \leftrightarrow D^*$.

Remark 4.3. For matrices we recover Khatri and Mitra's necessary and sufficient conditions for the existence of common hermitian solutions [5, Theorem 2.3]. The general form of solutions in [5] is given in terms of the block matrices $\boldsymbol{A}$, $\boldsymbol{C}$ and $\boldsymbol{Q}$ only. The explicit form of the general solution given in our equation (4.4) is new.

In [8], Phadke and Thakare state a version of the preceding theorem as Theorem 2.3(I). They assume only the regularity of A and B, and a condition equivalent to the self-adjointness of AC^* and $B^* D$. Simple counterexamples show that these conditions are not sufficient for the existence of a common hermitian solution.

5. Positive solutions to $AX = C$

Theorem 5.2 is one of the main results of this paper. For finite matrices we recover [5, Theorem 2.2]. In the proof of this theorem we will need the following auxiliary result.

Lemma 5.1. *Let N, R be closed subspaces of a Hilbert space H and let $N \cap R = \{0\}$. If $N \oplus R$ is closed, then there exists a closed complementary subspace M of N containing R.*

Proof. Let $L = (N \oplus R)^\perp$ be the orthogonal complement of the closed space $N \oplus R$. Then $H = (N \oplus R) \oplus L$, $R \oplus L$ is closed, and $M = R \oplus L$ is the desired complementary subspace of N. $\square$

Theorem 5.2. *Let $A, C \in \mathcal{B}(H, K)$ and let A and CA^* be closed range operators. Then $AX = C$ has a positive solution $X \in \mathcal{B}(H)$ if and only if*

$$CA^* \geqslant 0 \quad \text{and} \quad R(C) \subset R\big(CA^*\big). \tag{5.1}$$

The general positive solution is given by

$$X = C^*\big(CA^*\big)^{-}C + \big(I - A^{-}A\big)S\big(I - A^{-}A\big)^*, \quad S \in \mathcal{B}(H)^+, \tag{5.2}$$

where $(CA^*)^{-}$ *and* A^{-} *are arbitrary inner inverses of* CA^* *and* A, *respectively*; $X_0 = C^*(CA^*)^{-}C$ *is a particular positive solution to* $AX = C$, *independent of the choice of the inner inverse* $(CA^*)^{-}$.

Proof. Suppose first that X is a positive solution to $AX = C$. Then $CA^* = AXA^*$ is positive. We show that $N(CA^*) \subset N(C^*)$. Let $CA^*x = AXA^*x = 0$. By Reid's inequality for the positive operator X,

$$\|C^*x\|^2 = \|X\big(A^*x\big)\|^2 \leqslant \|X\|\langle XA^*x, A^*x\rangle = \|X\|\langle AXA^*x, x\rangle = 0.$$

From $N(CA^*) \subset N(C^*)$ it follows that $\overline{R(C)} \subset R(CA^*)$; in fact the two ranges are equal and closed.

Conversely, let (5.1) hold, and let $(CA^*)^\dagger$ be the Moore–Penrose inverse of the positive operator CA^*. Then $(CA^*)^\dagger$ is positive, and $(CA^*)(CA^*)^\dagger$ is the orthogonal projection of H onto $R(CA^*)$. Since $R(C) \subset R(CA^*)$, we have $(CA^*)(CA^*)^\dagger C = C$. Set $X_0 = C^*(CA^*)^\dagger C$. Then $X_0 \geqslant 0$ and

$$AX_0 = \big(AC^*\big)\big(CA^*\big)^\dagger C = \big(CA^*\big)\big(CA^*\big)^\dagger C = C.$$

The rest of the proof is concerned with the general form of the solution.

We show that $C^*(CA^*)^{-}C = X_0$ for any inner inverse $(CA^*)^{-}$ of CA^*. By [2, p. 50], $(CA^*)^{-} = (CA^*)^\dagger + W - (CA^*)^\dagger(CA^*)W(CA^*)(CA^*)^\dagger$ for some $W \in \mathcal{B}(K)$. Then

$$\begin{aligned} &C^*\big[W - \big(CA^*\big)^\dagger\big(CA^*\big)W\big(CA^*\big)\big(CA^*\big)^\dagger\big]C \\ &\quad = C^*WC - \big[\big(CA^*\big)\big(CA^*\big)^\dagger C\big]^* W\big(CA^*\big)\big(CA^*\big)^\dagger C \\ &\quad = C^*WC - C^*WC = 0. \end{aligned}$$

Thus $X_0 = C^*(CA^*)^{-}C$, and X_0 is a particular positive solution independent of the inner inverse $(CA^*)^{-}$ of CA^*.

Let X be a positive solution to $AX = C$. Then $X - X_0$ is a hermitian solution to $AZ = 0$, and by Theorem 3.1 with the right-hand side zero,

$$X - X_0 = (I - TA)U(I - TA)^*$$

with $T \in \mathcal{B}(K, H)$ an inner inverse of A and with $U \in \mathcal{B}(H)$ hermitian. Choose an inner inverse T of A such that $TAX_0 = X_0$. (The existence of such T is proved below.) Then

$$\begin{aligned} (I - TA)X(I - TA)^* &= (I - TA)X_0(I - TA)^* + (I - TA)U(I - TA)^* \\ &= (I - TA)U(I - TA)^*, \end{aligned}$$

and $S = (I - TA)U(I - TA)^*$ is positive. Let A^{-} be an arbitrary inner inverse of A. Then $(I - A^{-}A)(I - TA) = I - TA$, and

$$X = X_0 + \big(I - A^{-}A\big)S\big(I - A^{-}A\big)^*, \quad S \geqslant 0.$$

Conversely, if $X = X_0 + (I - A^{-}A)S(I - A^{-}A)^*$ with S positive, then X is a positive solution to $AX = C$.

To complete the proof we give a construction of an inner inverse T of A satisfying $TAX_0 = X_0$: since such an inner inverse T must satisfy $R(I - TA) = N(A)$, we first find a topological complement M of $N(A)$ containing $R(X_0)$.

We show that $R(X_0)$ is closed, $R(X_0) = R(C^*)$, and $N(A) \cap R(X_0) = \{0\}$. The closure of $R(X_0)$ follows from the fact that X_0 is regular with an inner inverse $C^\dagger CA^*(C^*)^\dagger$ (the verification is left to the reader). The inclusion $R(X_0) \subset R(C^*)$ is clear. The reverse inclusion follows from $AX_0 = C$ which implies $N(X_0) \subset N(C)$, which in turn implies $R(C^*) \subset R(X_0)$ ($R(C^*)$ is closed since $R(C)$ is). Finally, $N(A) \cap R(C^*) = \{0\}$ since $AC^*u = CA^*u = 0$ implies $C^*u = 0$ as shown previously.

To apply Lemma 5.1 we need to show that $N(A) \oplus R(X_0)$ is closed. This will follow when we show that

$$N(A) \oplus R(X_0) = A^{-1}\big(R(C)\big)$$

since $R(C)$ is closed and A continuous. The inclusion $A(N(A) \oplus R(X_0)) \subset R(C)$ gives $N(A) \oplus R(X_0) \subset A^{-1}(R(C))$. Conversely, let $x \in A^{-1}(R(C))$. Then $Ax = Cu = AX_0u = A(w + X_0u)$ for some $u \in H$ and $w \in N(A)$. Then $x - w - X_0u \in N(A)$, and $x \in N(A) \oplus R(X_0)$. Hence $A^{-1}(R(C)) \subset N(A) \oplus R(X_0)$ and $N(A) \oplus R(X_0)$ is closed. Therefore by Lemma 5.1 there exists a topological complement M of $N(A)$ in H containing $R(X_0)$.

Let N be an arbitrary topological complement of $R(A)$ in K. According to Lemma 2.1, there exists an inner inverse T of A such that $R(AT) = R(A)$, $N(AT) = N$, $R(TA) = M$ and $N(TA) = N(A)$. Then $TAX_0 = X_0$ as TA projects onto $M \supset R(X_0)$. □

Remark 5.3. The preceding proof takes as its starting point the proof given by Khatri and Mitra in [5, Theorem 2.2] for finite complex matrices. The details of our arguments for Hilbert spaces involve existence proofs which are not required in the finite dimensional case.

Remark 5.4. Phadke and Thakare [8, Theorem 2.2] sketch a proof of the sufficiency of conditions (5.1). In [8, Corollary 2.2] they give a formula for the general positive solution of the equation $AX = C$, but prove merely that every operator of the form (5.2) is a hermitian solution of $AX = C$. As we have seen, the converse is by far the most involved part of the proof.

6. Common positive solutions to $AX = C$ and $XB = D$

We address the existence of a common positive solution to operator equations (4.2). The formulation utilizing block operator matrices and their inner inverses owes its conciseness and elegance to Theorem 2.3 obtained by Khatri and Mitra [5] for finite matrices, but lacks explicit expressions in terms of the original operators. We rectify this in Theorem 6.3 obtaining results which are new even for finite matrices.

Theorem 6.1. *Let $A, C \in \mathcal{B}(H,K)$, $B, D \in \mathcal{B}(L,H)$, let*

$$\boldsymbol{A} = \begin{bmatrix} A \\ B^* \end{bmatrix}, \qquad \boldsymbol{C} = \begin{bmatrix} C \\ D^* \end{bmatrix}, \qquad \boldsymbol{Q} = \begin{bmatrix} CA^* & CB \\ (AD)^* & D^*B \end{bmatrix}, \tag{6.1}$$

and let $\boldsymbol{A}$ and $\boldsymbol{Q}$ be regular. Equations (4.2) *have a common positive solution $X \in \mathcal{B}(H)$ if and only if $\boldsymbol{Q}$ is positive and $R(\boldsymbol{C}) \subset R(\boldsymbol{Q})$. The general common positive solution is given by*

$$X = \boldsymbol{C}^*\boldsymbol{Q}^-\boldsymbol{C} + \big(I - \boldsymbol{A}^-\boldsymbol{A}\big)T\big(I - \boldsymbol{A}^-\boldsymbol{A}\big)^*, \quad T \in \mathcal{B}(H)^+, \tag{6.2}$$

where $X_0 = \boldsymbol{C}^*\boldsymbol{Q}^-\boldsymbol{C}$ is a particular common positive solution.

Proof. We apply Theorem 5.2 to $\boldsymbol{A}$, $\boldsymbol{C}$ and X, observing that Eqs. (4.2) have a common positive solution if and only if $\boldsymbol{A}X = \boldsymbol{C}$ has a positive solution. The key observation is that $\boldsymbol{C}A^* = \boldsymbol{Q}$. □

Remark 6.2. Theorem 2.3(II) of Phadke and Thakare [8] assumes merely that $\left[\begin{smallmatrix} CA^* & 0 \\ 0 & D^*B \end{smallmatrix}\right]$ is positive definite, which is not sufficient, and the formula given in their theorem for the general solution is incorrect.

In the next theorem we describe conditions for the existence of common positive solutions of (4.2) expressed in terms of the original operators. These conditions will be an explicit transcription of the block matrix requirements that $\boldsymbol{A}$ and $\boldsymbol{Q}$ are regular, $\boldsymbol{Q}$ is positive and $R(\boldsymbol{C})$ is contained in $R(\boldsymbol{Q})$. To do this we use Lemmas 4.1, 2.2, and 2.3, and present an explicit form of the general common solution. In the proof of the theorem, $\boldsymbol{A}$, $\boldsymbol{C}$ and $\boldsymbol{Q}$ are the operator matrices defined in (6.1). We also define two Schur complements

$$G := s\left(\begin{bmatrix} CA^* & C \\ (CB)^* & D^* \end{bmatrix}\right) = D^* - (CB)^*\bigl(CA^*\bigr)^- C,$$

$$F := s\left(\begin{bmatrix} CA^* & CB \\ (CB)^* & D^*B \end{bmatrix}\right) = D^*B - (CB)^*\bigl(CA^*\bigr)^- CB = GB,$$

which will be used in the statement of the theorem.

Theorem 6.3. *Let $A, C \in \mathcal{B}(H, K)$, $B, D \in \mathcal{B}(L, H)$ be operators satisfying the following conditions*:

$$A, B, M = B^*\bigl(I - A^-A\bigr),\ CA^*,\ F \text{ are regular};\ R(CB) \subset R\bigl(CA^*\bigr). \tag{6.3}$$

Then Eqs. (4.2) *have a common positive solution if and only if*

(i) $CA^* = AC^*$, $D^*B = B^*D$, $CB = AD$,
(ii) CA^* *and* F *are positive,*
(iii) $R(C) \subset R(CA^*)$ *and* $R(D^*) \subset R(D^*B)$.

The general common positive solution is given by

$$\begin{aligned} X = {} & C^*\bigl(CA^*\bigr)^- C + G^*F^-G \\ & + \bigl(I - A^-A\bigr)\bigl(I - M^-M\bigr)T\bigl(I - M^-M\bigr)^*\bigl(I - A^-A\bigr)^*, \end{aligned} \tag{6.4}$$

where $T \in \mathcal{B}(H)^+$.

Proof. According to Lemmas 4.1 and 2.2, (6.3) ensures the regularity of $\boldsymbol{A}$ and $\boldsymbol{Q}$ required by Theorem 6.1.

Let us assume that condition (i)–(iii) are satisfied. In view of Lemma 2.3, (i) and (ii) are equivalent to the positivity of $\boldsymbol{Q}$. We show that when (i) and (ii) hold, condition (iii) is equivalent to the inclusion $R(\boldsymbol{C}) \subset R(\boldsymbol{Q})$: Suppose that (iii) holds. For $x \in H$, $t \in K$ and $s \in L$,

$$\boldsymbol{C}x = \begin{bmatrix} C \\ D^* \end{bmatrix} x = \begin{bmatrix} Cx \\ D^*x \end{bmatrix}, \qquad \boldsymbol{Q}\begin{bmatrix} t \\ s \end{bmatrix} = \begin{bmatrix} CA^* & CB \\ (AD)^* & D^*B \end{bmatrix}\begin{bmatrix} t \\ s \end{bmatrix} = \begin{bmatrix} CA^*t + CBs \\ (AD)^*t + D^*Bs \end{bmatrix},$$

from which it follows that

$$\begin{aligned}R(\boldsymbol{C}) &= R(C) \oplus R(D^*) \subset R\big(CA^*\big) \oplus R\big(D^*B\big)\\ &\subset \big[R\big(CA^*\big) + R(CB)\big] \oplus \big[R\big((AD)^*\big) + R\big(D^*B\big)\big] \subset R(\boldsymbol{Q}).\end{aligned}$$

Conversely, if $R(\boldsymbol{C}) \subset R(\boldsymbol{Q})$, Eqs. (4.2) have a common positive solution by Theorem 6.1; then $R(C) \subset R(CA^*)$ and $R(D^*) \subset R(D^*B)$; thus condition (iii) is also necessary.

From Theorem 6.1 we know that $X_0 = \boldsymbol{C}^*\boldsymbol{Q}^-\boldsymbol{C}$ is a particular common positive solution. According to Lemma 2.2, $\boldsymbol{Q}^-$ is of the form

$$\boldsymbol{Q}^- = \begin{bmatrix} (CA^*)^- + (CA^*)^-CBF^-(CB)^*(CA^*)^- & -(CA^*)^-CBF^- \\ -F^-(CB)^*(CA^*)^- & F^- \end{bmatrix}.$$

After a calculation we obtain

$$X_0 = C^*\big(CA^*\big)^-C + G^*F^-G.$$

The expression for $I - \boldsymbol{A}^-\boldsymbol{A}$ is derived as in the proof of Theorem 4.2. Writing $M = B^*(I - A^-A)$, we obtain

$$I - \boldsymbol{A}^-\boldsymbol{A} = \big(I - A^-A\big)\big(I - M^-M\big).$$

Substituting this into (6.2) we obtain (6.4). □

We can again obtain an alternative expression for the solution X in the preceding theorem if we interchange the order of the equations $AX = C$ and $B^*X = D^*$, that is, if we apply the interchange $A \leftrightarrow B^*$ and $C \leftrightarrow D^*$ to the hypotheses and to (6.4).

Acknowledgments

The authors are indebted to Professors Pedro Patrício and Vladimir Rakočević for their advice and helpful suggestions, and to Raymond Lubansky for contributing some useful ideas.

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