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Limit relations for the complex zeros of Laguerre and q-Laguerre polynomials ☆

Mark V. DeFazio [a], Dharma P. Gupta [b], Martin E. Muldoon [b,*]

[a] Algoma University College, Sault Ste. Marie, Ontario P6A 2G4, Canada
[b] Department of Mathematics and Statistics, York University, Toronto, Ontario M3J 1P3, Canada

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Abstract

For each m $(= 1, \dots, n)$ the nth Laguerre polynomial $L_n^{(\alpha)}(x)$ has an m-fold zero at the origin when $\alpha = -m$. As the real variable $\alpha \to -m$, it has m simple complex zeros which approach 0 in a symmetric way. This symmetry leads to a finite value for the limit of the sum of the reciprocals of these zeros. There is a similar property for the zeros of the q-Laguerre polynomials and of the Jacobi polynomials and similar results hold for sums of other negative integer powers.

Keywords: Laguerre polynomials; q-Laguerre polynomials; Zeros; Jacobi polynomials

1. The Laguerre polynomials

The Laguerre polynomials are given by the explicit formula

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}, \tag{1}$$

valid for all $x, \alpha \in \mathbf{C}$, showing them to be polynomials in both x and α.

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* Corresponding author.
E-mail address: muldoon@yorku.ca (M.E. Muldoon).

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We also use the representation [6, (5.3.3)],

$$L_n^{(\alpha)}(x) = \binom{n+\alpha}{n} {}_1F_1(-n, \alpha+1; x) = \binom{n+\alpha}{n}\left[1 + \sum_{k=1}^{n} \frac{\binom{n}{k}(-x)^k}{(\alpha+1)_k}\right], \tag{2}$$

with the usual notation for the confluent hypergeometric function. This is valid for $\alpha \neq -1, \ldots, -n$, and with a limiting definition for $\alpha = -1, \ldots, -n$.

The Laguerre polynomials satisfy the three term recurrence relation

$$xL_n^{(\alpha)}(x) = -(n+1)L_{n+1}^{(\alpha)}(x) + (\alpha+2n+1)L_n^{(\alpha)}(x) - (\alpha+n)L_{n-1}^{(\alpha)}(x), \tag{3}$$

with initial conditions $L_{-1}^{(\alpha)}(x) = 0$ and $L_0^{(\alpha)}(x) = 1$ for all complex α and x. When $\alpha > -1$, this recurrence relation is positive definite and the Laguerre polynomials are orthogonal with respect to the weight function $x^\alpha e^{-x}$ on $[0, +\infty)$. From this it follows that the zeros of $L_n^{(\alpha)}(x)$ are positive, real, simple, increasing functions of α and they interlace with the zeros of $L_{n+1}^{(\alpha)}(x)$ [6]. When $\alpha \leqslant -1$ we no longer have orthogonality with respect to a positive weight function and the zeros can be non-real and non-simple. In the case $\alpha \leqslant -1$ the behaviour of the zeros of $L_n^{(\alpha)}(x)$ has been studied in detail in [1].

When $\alpha = -m$ $(1 \leqslant m \leqslant n)$ the explicit formula (1) yields [6, (5.2.1)],

$$L_n^{(-m)}(x) = (-x)^m \frac{(n-m)!}{n!} L_{n-m}^{(m)}(x), \tag{4}$$

so $L_n^{(-m)}(x)$ has a zero of order m at $x = 0$, the remaining $n - m$ zeros being those of $L_{n-m}^{(m)}(x)$. The zero of order m arises from the coalescence of m simple zeros as α approaches $-m$. This follows by the method used in [6, §6.72] to prove some corresponding results for the Jacobi polynomials. (See [1] for more details of the Laguerre case.)

To see that m zeros approach the origin as $\alpha \to -m$, we let $\epsilon = \alpha + m$ and normalize so that the coefficient of x^m is 1; that is, we consider the following multiple of $L_n^{(\alpha)}(x)$:

$$q_n(x, \epsilon) = \sum_{k=0}^{n} c_k(\epsilon) x^k, \tag{5}$$

where

$$c_k(\epsilon) = (-1)^{k-m} \frac{(n-m)!m!}{(n-k)!k!} \frac{(\epsilon - m + k + 1)_{n-k}}{(\epsilon+1)_{n-m}}. \tag{6}$$

For $0 \leqslant k \leqslant m-1$, $c_k(\epsilon)$ is a polynomial in ϵ which vanishes for $\epsilon = 0$; for $m - 1 \leqslant k \leqslant n$, it is a rational function of ϵ which remains bounded as $\epsilon \to 0$; also $c_m(\epsilon) = 1$. Thus we may write

$$q_n(x, \epsilon) = x^m + \epsilon c_0'(0) + r(x, \epsilon),$$

where $r(x, \epsilon)$ remains to be examined more closely. If we choose an x_0 so that $|x_0| = 2|\epsilon c_0'(0)|^{1/m}$, we find that

$$r(x_0, \epsilon) = O\bigl(\epsilon^{1+1/m}\bigr), \quad \epsilon \to 0.$$

so comparing $r(x_0, \epsilon)$ and $x^m + \epsilon c_0'(0)$, we find, by Rouché's theorem, that $q_n(x, \epsilon)$ has exactly m zeros in the disk $|x| < |x_0|$. This shows that the m-fold zero of the right-hand side of (4) arises from the confluence of m zeros of $L_n^{(\alpha)}(x)$, as $\alpha \to -m$.

For $n \geqslant 2$, although the reciprocal of each zero becomes infinite, as $\alpha \to -m$, the sum of their reciprocals approaches a finite negative value. We state this as follows:

Theorem 1. *Let $x_1(\alpha), \dots, x_m(\alpha)$ be the m $(2 \leqslant m \leqslant n)$ zeros of $L_n^{(\alpha)}(x)$ in a neighbourhood of $x = 0$ for $\alpha \sim -m$. Then*

$$\lim_{\alpha \to -m} \sum_{k=1}^{m} \frac{1}{x_k(\alpha)} = \frac{m(m-2n-1)}{m^2-1}. \tag{7}$$

Proof. Let α be close to $-m$. Let us number the zeros of $L_n^{(\alpha)}(x)$ so that $x_1, \dots, x_m$ are near zero and $x_{m+1}, \dots, x_n$ are near the positive real zeros of $L_{n-m}^{(m)}(x)$. The explicit formula (2) yields

$$\sum_{k=1}^{n} \frac{1}{x_k(\alpha)} = \frac{n}{\alpha+1}, \quad \alpha \neq -1, \dots, -n. \tag{8}$$

This is readily seen by considering the factorization

$$\binom{n+\alpha}{n}^{-1} L_n^{(\alpha)}(x) = \left(1 - \frac{x}{x_1(\alpha)}\right) \cdots \left(1 - \frac{x}{x_n(\alpha)}\right) = 1 - x \sum_{k=1}^{n} \frac{1}{x_k(\alpha)} + \cdots, \tag{9}$$

and comparing with the expansion (2):

$$\binom{n+\alpha}{n}^{-1} L_n^{(\alpha)}(x) = 1 - \frac{n}{\alpha+1} x + \cdots. \tag{10}$$

Letting $\alpha \to -m$ gives

$$\begin{aligned} \frac{n}{1-m} &= \lim_{\alpha \to -m} \sum_{k=1}^{n} \frac{1}{x_k(\alpha)} \\ &= \lim_{\alpha \to -m} \sum_{k=1}^{m} \frac{1}{x_k(\alpha)} + \lim_{\alpha \to -m} \sum_{k=m+1}^{n} \frac{1}{x_k(\alpha)}, \end{aligned}$$

where the zeros in the first sum are the ones in the neighbourhood of 0. But the zeros in the second sum on the right approach those of $L_{n-m}^{(m)}(x)$ and hence, using (8), we get

$$\frac{n}{1-m} = \lim_{\alpha \to -m} \sum_{k=1}^{m} \frac{1}{x_k(\alpha)} + \frac{n-m}{1+m}, \tag{11}$$

which gives (7). □

2. *q*-Laguerre polynomials

For $0 < q < 1$, the q-Laguerre polynomials may be defined by

$$L_n^{(\alpha)}(x, q) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} \sum_{k=0}^{n} \frac{(q^{-n}; q)_k q^{k(k+1)/2 + k(n+\alpha)}}{(q^{\alpha+1}; q)_k (q; q)_k} (1-q)^k x^k, \tag{12}$$

where we use the standard notation

$$(a; q)_n = \begin{cases} 1, & n = 0, \\ (1-a)(1-aq) \cdots (1-aq^{n-1}), & n = 1, 2, \dots. \end{cases}$$

The definition (12) is that given by Moak [5] and in [2, p. 210]. The definition given in [4, p. 108] has x replaced by $(1-q)^{-1}x$.

Note that $L_n^{(\alpha)}(x,q) \to L_n^{(\alpha)}(x)$ as $q \to 1^-$ so if we use $x_k(\alpha;q)$ for the zeros of $L_n^{(\alpha)}(x,q)$ then $x_k(\alpha;q) \to x_k(\alpha)$, as $q \to 1^-$.

It is an easy matter to show that

$$L_n^{(-m)}(x,q) = (-x)^m(1-q)^m \frac{(q;q)_{n-m}}{(q;q)_n} L_{n-m}^{(m)}(x,q), \quad m = 1,2,\dots,n. \tag{13}$$

We may proceed as in Section 1, to get

$$\sum_{k=1}^{n} \frac{1}{x_k(\alpha;q)} = \frac{q^{\alpha+1}(1-q^n)}{1-q^{\alpha+1}}, \quad \alpha \neq -1,\dots,-n, \tag{14}$$

which reduces to (8) as $q \to 1^-$, and

$$\lim_{\alpha \to -m} \sum_{k=1}^{m} \frac{1}{x_k(\alpha;q)} = \frac{q^{n-1}(1-q^{-m})(1+q-q^{m-n}-q^{-n})}{(1-q^{m-1})(1-q^{-m-1})}, \quad m = 2,\dots,n, \tag{15}$$

which reduces to (7) as $q \to 1^-$.

3. Sums of powers of the zeros

Here we consider power sums

$$S_r = \sum_{k=1}^{n} x_k^{-r}, \quad r = 1,2,\dots,$$

of the zeros of $L_n^{(\alpha)}(x)$. If we write

$$\binom{n+\alpha}{n}^{-1} L_n^{(\alpha)}(x) = \prod_{k=1}^{n}\left(1 - \frac{x}{x_k(\alpha)}\right) = 1 + \sum_{k=1}^{n} a_k x^k,$$

then the sums S_j are related to the coefficients a_j by [3, §3]:

$$S_1 = -a_1, \qquad S_r = -ra_r - \sum_{i=1}^{r-1} a_i S_{r-i}, \quad r = 2,3,\dots.$$

In particular $S_2 = a_1^2 - 2a_2$, giving, in the Laguerre case,

$$S_2 = \frac{n^2 + n(\alpha+1)}{(\alpha+1)^2(\alpha+2)}.$$

We may argue as in the proof of Theorem 1 that

$$\begin{aligned}\frac{n^2}{(1-m)^2} - \frac{n(n-1)}{(m-1)(m-2)} &= \lim_{\alpha \to -m} \sum_{k=1}^{n} \frac{1}{x_k(\alpha)^2} \\ &= \lim_{\alpha \to -m} \sum_{k=1}^{m} \frac{1}{x_k(\alpha)^2} + \lim_{\alpha \to -m} \sum_{k=m+1}^{n} \frac{1}{x_k(\alpha)^2}.\end{aligned}$$

As in Section 1, the zeros in the second sum on the right approach those of $L_{n-m}^{(m)}(x)$ and hence, with the notation of Theorem 1,

$$\frac{n^2}{(1-m)^2}-\frac{n(n-1)}{(m-1)(m-2)}=\lim_{\alpha\to -m}\sum_{k=1}^{m}\frac{1}{x_k(\alpha)^2}+\frac{(n-m)^2}{(1+m)^2}-\frac{(n-m)(n-m-1)}{(m+1)(m+2)}. \tag{16}$$

This leads to

$$\lim_{\alpha\to -m}\sum_{k=1}^{m}\frac{1}{x_k(\alpha)^2}=-\frac{(n-m)(n+1)}{(m+1)^2(m+2)}-\frac{n(n-m+1)}{(m-1)^2(m-2)},\quad 3\leqslant m\leqslant n. \tag{17}$$

It makes sense that the above sum is meaningless for $m=2$ since we do not expect the required cancellation in that case; $x_1(\alpha)^2$ and $x_2(\alpha)^2$ are positive numbers which both approach 0 as $\alpha\to -2^+$.

We may apply the same method to the q-Laguerre polynomials to get

$$\lim_{\alpha\to -m}\sum_{k=1}^{m}\frac{1}{x_k^2(\alpha;q)}=\frac{(1-q^n)^2}{(1-q^{m-1})^2}-\frac{2q(1-q^n)(1-q^{n-1})}{(1+q)(1-q^{m-1})(1-q^{m-2})}-\frac{q^{2m+2}(1-q^{n-m})^2}{(1-q^{m+1})^2}+\frac{2q^{2m+4}(1-q^{n-m})(1-q^{n-m-1})}{(1+q)(1-q^{m+1})(1-q^{m+2})}, \tag{18}$$

which reduces to (17) as $q\to 1^-$.

4. Jacobi polynomials

The reason for the described behaviour of the zeros of Laguerre polynomials lies in their hypergeometric character as exhibited in (2). In fact our results may be extended to a general class of ${}_pF_q$ with a numerator parameter $-n$ and a denominator parameter $\alpha+1$, even in some cases where the extra parameters depend on α. We confine attention to the case of Jacobi polynomials.

As in [6, (4.21.2)], we may define the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, for general complex numbers α and β by

$$P_n^{(\alpha,\beta)}(x)=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(n+\alpha+\beta+1)_k(\alpha+k+1)_{n-k}\left(\frac{x-1}{2}\right)^k \tag{19}$$

or by

$$P_n^{(\alpha,\beta)}(x)=\binom{n+\alpha}{n}{}_2F_1\left(-n,\,n+\alpha+\beta+1;\alpha+1;\frac{1-x}{2}\right), \tag{20}$$

with a limiting definition for $\alpha=-1,\dots,-n$, from which it follows [6, (4.22.2)] that, for $1\leqslant m\leqslant n$,

$$\binom{n}{m}P_n^{(-m,\beta)}(x)=\binom{n+\beta}{m}\left(\frac{x-1}{2}\right)^m P_{n-m}^{(m,\beta)}(x). \tag{21}$$

From the explicit formula (20) or from the differential equation for the Jacobi polynomials, we have, for $\alpha\neq -1,\dots,-n$,

$$\sum_{k=1}^{n}\frac{1}{1-x_k(\alpha,\beta)}=\frac{n(n+\alpha+\beta+1)}{2(\alpha+1)}. \tag{22}$$

From this we can get

Theorem 2. *Let $x_1(\alpha,\beta),\dots,x_k(\alpha,\beta)$ be the m $(2 \leqslant m \leqslant n)$ zeros of $P_n^{(\alpha,\beta)}$ in a neighbourhood of $x=1$ for $\alpha \sim -m$. Then*

$$\lim_{\alpha\to -m}\sum_{k=1}^{m}\frac{1}{1-x_k(\alpha,\beta)}=\frac{n(n-m+\beta+1)}{2(1-m)}-\frac{(n-m)(n+\beta+1)}{2(m+1)}. \tag{23}$$

Limits of sums of higher powers and q-extensions are obtainable as before.

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