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Limiting subdifferentials of indefinite integrals ☆

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Abstract

We compute the limiting subdifferential $\partial F(\bar{x})$ of the indefinite integral of the form $F(x)=\int_a^x f(t)\,dt$ where f is an essentially bounded measurable function, or a function continuous on an interval containing $\bar{x}\in\mathbb{R}$ (except for, possibly, $\bar{x}$), or a step-function which has a countable number of steps around $\bar{x}$. The related problem of computing the Aumann integral of the limiting subdifferential mapping $\partial f(\cdot)$, where f is a Lipschitz real function defined on an open set $U\subset\mathbb{R}^n$, is also investigated.

Keywords: Indefinite integral; Fréchet subdifferential; Limiting subdifferential; Aumann integral

1. Introduction

The Clarke subdifferential is one of the most significant concepts in nonsmooth analysis; see [4]. Its nonconvex counterpart, introduced in [7] and called the limiting subdifferential (or the Mordukhovich subdifferential), plays a central role in variational analysis and set-valued analysis (see [7–11]). The problem of computing or estimating the Clarke subdifferential of the integral functional

$$G(x)=\int_{\Omega} g(x,t)\,d\mu(t), \tag{1.1}$$

where g is a real function defined on $U\times\Omega$, U is an open subset of a Banach space and (Ω,μ) is a positive measure space, has been discussed in [4, Section 2.7]. As noted by Professor B.S. Mordukhovich, it would be interesting to obtain some formulae for the limiting subdifferential of $G(\cdot)$. In this general setting, the problem has not been solved so far.

In the first part of this paper, we will compute the limiting subdifferential $\partial F(\bar{x})$ of the indefinite integral

$$F(x)=\int_a^x f(t)\,dt, \tag{1.2}$$

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where f is an essentially bounded measurable function, or a function continuous on an interval containing $\bar{x} \in \mathbb{R}$ (except for, possibly, $\bar{x}$), or a step-function which has a countable number of steps around $\bar{x}$. Letting μ be the Lebesgue measure on $[a,b]$, $\Omega = [a,b]$, $g(x,t) = f(t)$ for $t \in [a,x]$ and $g(x,t) = 0$ for $t \in (x,b]$, we see that

$$F(x) = \int_a^x f(t)\,dt = \int_\Omega g(x,t)\,d\mu(t) =: G(x).$$

So, the function in (1.2) is a special case of that one given by (1.1). The limiting subdifferential $\partial F(\bar{x})$ is the Painlevé–Kuratowski limit of a family of the Fréchet subdifferentials $\hat{\partial} F(x)$, $x \in [a,b]$. We also present a formula for the Fréchet subdifferential of F.

The second part of the paper gives a representation formula for the Aumann integral (a set-valued integral) of the limiting subdifferential mapping $\partial f(\cdot)$, where $f : U \to \mathbb{R}$ is a Lipschitz function on an open set $U \subset \mathbb{R}^n$. Similar formulae for the case of the Clarke subdifferential mapping $\partial_C f(\cdot)$ were given in our preceding paper [6]. From the obtained results one can derive a formula of the Newton–Leibnitz type.

The rest of the paper is divided into 3 sections. The next section contains some definitions and results which are needed in the sequel. Section 3 computes the Fréchet subdifferential and the limiting subdifferential of the function F defined by (1.2). Formulae for the Aumann integral of the limiting subdifferential of a Lipschitz function are obtained in Section 4.

2. Preliminaries

Most of our notations are standard. Some special symbols are introduced when they are needed. Unless otherwise stated, all the spaces considered are Euclidean. In $\mathbb{R}^n$, we always select the *Euclidean norm* $\|x\| := (x_1^2 + \cdots + x_n^2)^{1/2}$ and the *Lebesgue measure* m. For a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, the expression

$$\operatorname*{Lim\,sup}_{u \to x} F(u) := \left\{x^* \in \mathbb{R}^n \;\middle|\; \exists u_k \to x,\ u_k^* \to x^*,\ u_k^* \in F(u_k) \text{ for } k = 1, 2, \ldots\right\}$$

denotes the *sequential Painlevé–Kuratowski upper/outer limit*. If $f : \mathbb{R}^n \to \overline{\mathbb{R}} := [-\infty, +\infty]$ is an extended-real-valued function, then $\limsup f(x)$ and $\liminf f(x)$ denote the upper and lower limits in the classical (scalar) sense. Recall that f is lower semicontinuous (l.s.c.) at a point x with $|f(x)| < \infty$ if

$$f(x) \leqslant \liminf_{u \to x} f(u).$$

We say that f is l.s.c. around x when it is l.s.c. at any point in some neighborhood of x.

Definition 2.1. (See [8].) Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ with $|f(x)| < \infty$ be l.s.c. around x. The *Fréchet subdifferential* of f at x is the set $\hat{\partial} f(x)$ defined by

$$\hat{\partial} f(x) := \left\{x^* \in \mathbb{R}^n \;\middle|\; \liminf_{u \to x} \frac{f(u) - f(x) - \langle x^*, u - x\rangle}{\|u - x\|} \geqslant 0\right\}.$$

We put $\hat{\partial} f(x) := \emptyset$ if $|f(x)| = \infty$. The *limiting subdifferential* of f at x is defined by

$$\partial f(x) := \operatorname*{Lim\,sup}_{u \xrightarrow{f} x} \hat{\partial} f(u),$$

where "Lim sup" stands for the sequential Painlevé–Kuratowski upper/outer limit, and $u \xrightarrow{f} x$ means $u \to x$ with $f(u) \to f(x)$.

The set $\hat{\partial} f(x)$ is convex while, in general, the limiting subdifferential $\partial f(x)$ may be nonconvex (see, e.g., Example 3.2). The notion of Fréchet subdifferential is an extension of the notion of Fréchet derivative.

Theorem 2.1. *(See [8, p. 90].) Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ with $|f(\bar{x})| < \infty$ be Fréchet differentiable at $\bar{x}$. Then $\hat{\partial} f(\bar{x}) = \{f'(\bar{x})\}$.*

Definition 2.2. (See [4].) Suppose that $f:\mathbb{R}^n\to\overline{\mathbb{R}}$ is Lipschitz near x. The *Clarke directional derivative* of f at x in direction $v\in\mathbb{R}^n$ is defined by

$$f^0(x;v):=\limsup_{x'\to x,\ t\to 0^+}\frac{f(x'+tv)-f(x')}{t}.$$

The *Clarke subdifferential* of f at x is the set

$$\partial_C f(x):=\big\{\xi\in\mathbb{R}^n\ \big|\ \langle\xi,v\rangle\leqslant f^0(x;v)\text{ for all }v\text{ in }\mathbb{R}^n\big\}.$$

The *directional derivative* of f at x in direction $v\in\mathbb{R}^n$, denoted by $f'(x;v)$, is defined by

$$f'(x;v):=\lim_{t\to 0^+}\frac{f(x+tv)-f(x)}{t},$$

if the limit on the right-hand side exists. One says that f is *Clarke regular* at x if, for every $v\in\mathbb{R}^n$, the directional derivative $f'(x;v)$ exists and $f'(x;v)=f^0(x;v)$.

The next theorem establishes a relationship between the Clarke subdifferential $\partial_C f(\bar{x})$ and the limiting subdifferential $\partial f(\bar{x})$. It was obtained in finite dimensions by B.S. Mordukhovich [7] and then in Asplund spaces by B.S. Mordukhovich and Y. Shao [10].

Theorem 2.2. *(See [10].) Let $f:\mathbb{R}^n\to\overline{\mathbb{R}}$ be Lipschitz near $\bar{x}$. Then*

$$\partial_C f(\bar{x})=\overline{\mathrm{co}}\,\partial f(\bar{x}),$$

where "$\overline{\mathrm{co}}$" stands for "closed convex hull."

The following result is due to J.M. Borwein and S.P. Fitzpatrick.

Theorem 2.3. *(See [14].) Let g be locally Lipschitz on an interval I and $x\in I$. If $\partial_C g(y)=[\alpha(y),\beta(y)]$ for $y\in I$, then*

$$\partial g(x)=\Big[\liminf_{y\to x}\alpha(y),\limsup_{y\to x^+}\beta(y)\Big]\cup\Big[\liminf_{y\to x^-}\alpha(y),\limsup_{y\to x}\beta(y)\Big].$$

We now recall the concept of integral of set-valued mappings due to R.J. Aumann [2]. Let $(\Omega,\mathcal{A},\mu)$ be a complete σ-finite measure space, and $G:\Omega\rightrightarrows\mathbb{R}^n$ be a set-valued mapping from Ω into closed nonempty subsets of $\mathbb{R}^n$. We denote by $\mathcal{G}$ the set of all integrable selections [1, p. 326] of G, that is

$$\mathcal{G}=\big\{g\in L^1\big(\Omega;\mathbb{R}^n,\mu\big)\ \big|\ g(x)\in G(x)\ \mu\text{-a.e. on }\Omega\big\}.$$

Definition 2.3. (See [1, p. 327].) The integral of G on Ω is the set of integrals of the integrable selections of G

$$\int_\Omega G\,d\mu:=\bigg\{\int_\Omega g\,d\mu\ \bigg|\ g\in\mathcal{G}\bigg\},$$

where $\int_\Omega g\,d\mu=(\int_\Omega g_1\,d\mu,\dots,\int_\Omega g_n\,d\mu)$ whenever $g=(g_1,\dots,g_n)$.

Recall that a set-valued mapping G is said to be *integrably bounded* if there exists a nonnegative function $k(\cdot)\in L^1(\Omega;\mathbb{R},\mu)$ such that $G(x)\subset k(x)B_{\mathbb{R}^n}$ a.e. on Ω, where

$$B_{\mathbb{R}^n}:=\big\{y\in\mathbb{R}^n\ \big|\ \|y\|\leqslant 1\big\}.$$

Theorem 2.4. *(See [1, p. 310].) Let $G:\Omega\rightrightarrows\mathbb{R}^n$, where Ω is a measurable subset of $\mathbb{R}^n$, be a closed set-valued mapping with nonempty images. Then G is measurable.*

The following property of the integral of set-valued mappings is used in the sequel.

Theorem 2.5. *(See [1, p. 330].) Let $G : \Omega \rightrightarrows \mathbb{R}^n$ be a measurable set-valued mapping with nonempty closed images, where Ω is a measurable subset of $\mathbb{R}^n$. If G is integrably bounded, then*

$$\int_\Omega G(x)\,m(dx) = \int_\Omega \overline{\mathrm{co}}\, G(x)\,m(dx).$$

3. Limiting subdifferential of an indefinite integral

In this section we present some formulae for computing the Fréchet subdifferential and the limiting subdifferential of some special classes of indefinite integrals.

Recall (see [12]) that a measurable function $f : [a,b] \to \overline{\mathbb{R}}$ is said to be essentially bounded on $[a,b]$ if there exists a real number M such that $|f(x)| \leqslant M$ a.e. for $x \in [a,b]$ (with respect to the Lebesgue measure). Let $L^\infty[a,b]$ denote the class of all essentially bounded measurable real functions on $[a,b]$. Let

$$\begin{aligned}
f^+(x) &:= \inf\bigl\{M \bigm| \exists\varepsilon > 0 \text{ such that } f(x') \leqslant M \text{ a.e. for } x' \in [x-\varepsilon, x+\varepsilon]\bigr\},\\
f^+_+(x) &:= \inf\bigl\{M \bigm| \exists\varepsilon > 0 \text{ such that } f(x') \leqslant M \text{ a.e. for } x' \in [x, x+\varepsilon]\bigr\},\\
f^-(x) &:= \sup\bigl\{M \bigm| \exists\varepsilon > 0 \text{ such that } f(x') \geqslant M \text{ a.e. for } x' \in [x-\varepsilon, x+\varepsilon]\bigr\},\\
f^-_-(x) &:= \sup\bigl\{M \bigm| \exists\varepsilon > 0 \text{ such that } f(x') \geqslant M \text{ a.e. for } x' \in [x-\varepsilon, x]\bigr\}.
\end{aligned}$$

It is clear that $f^-(x) \leqslant f^-_-(x) \leqslant f^+(x)$ and $f^-(x) \leqslant f^+_+(x) \leqslant f^+(x)$. Therefore,

$$\bigl[f^-(x), f^+_+(x)\bigr] \cup \bigl[f^-_-(x), f^+(x)\bigr] \subset \bigl[f^-(x), f^+(x)\bigr].$$

Theorem 3.1. *Let $f \in L^\infty[a,b]$ and $F(x) := \int_a^x f(t)\,dt$ for all $x \in [a,b]$. Then for each x in (a,b), one has*

$$\partial F(x) = \bigl[f^-(x), f^+_+(x)\bigr] \cup \bigl[f^-_-(x), f^+(x)\bigr]. \tag{3.1}$$

Proof. It is known (see [4, p. 34] or [5, p. 96]) that F is a Lipschitz function and

$$\partial_C F(y) = \bigl[f^-(y), f^+(y)\bigr] \tag{3.2}$$

for all $y \in [a,b]$. By Theorem 2.3 we obtain

$$\partial F(x) = \Bigl[\liminf_{y\to x} f^-(y), \limsup_{y\to x^+} f^+(y)\Bigr] \cup \Bigl[\liminf_{y\to x^-} f^-(y), \limsup_{y\to x} f^+(y)\Bigr]. \tag{3.3}$$

By (3.2), (3.3) and Theorem 2.2, one has

$$f^-(x) = \liminf_{y\to x} f^-(y) \quad \text{and} \quad f^+(x) = \limsup_{y\to x} f^+(y). \tag{3.4}$$

Next we will show that

$$f^+_+(x) = \limsup_{y\to x^+} f^+(y). \tag{3.5}$$

Indeed, for any $\delta > 0$ there are $M \in \mathbb{R}$ and $\varepsilon > 0$ such that

$$f(x') \leqslant M \quad \text{a.e. for } x' \in [x, x+\varepsilon]$$

and

$$f^+_+(x) + \delta \geqslant M.$$

Hence

$$f(x') \leqslant f^+_+(x) + \delta \quad \text{a.e. for } x' \in [x, x+\varepsilon]. \tag{3.6}$$

For any $y \in (x, x+\varepsilon)$, there exists $\varepsilon' > 0$ such that

$$[y-\varepsilon', y+\varepsilon'] \subset [x, x+\varepsilon].$$

By (3.6) we have

$$f^+(y) \leqslant f_+^+(x) + \delta \quad \text{for all } y \in (x, x+\varepsilon).$$

Hence

$$\limsup_{y \to x^+} f^+(y) \leqslant f_+^+(x) + \delta.$$

Since $\delta > 0$ is arbitrary, one has

$$\limsup_{y \to x^+} f^+(y) \leqslant f_+^+(x). \tag{3.7}$$

If (3.5) does not hold, then by (3.7) there is $\delta_0 > 0$ such that

$$\limsup_{y \to x^+} f^+(y) < f_+^+(x) - \delta_0.$$

Hence there exists $\varepsilon > 0$ such that

$$f^+(y) < f_+^+(x) - \delta_0 \quad \text{for all } y \in (x, x+\varepsilon). \tag{3.8}$$

Now we will prove that

$$f(x') \leqslant f_+^+(x) - \delta_0 \quad \text{a.e. for } x' \in (x, x+\varepsilon). \tag{3.9}$$

From (3.8) it follows that for each $y \in (x, x+\varepsilon)$ there exist $\varepsilon_y > 0$ and $M_y < f_+^+(x) - \delta_0$ such that

$$V_y := (y - \varepsilon_y, y + \varepsilon_y) \subset (x, x+\varepsilon)$$

and

$$f(x') \leqslant M_y \quad \text{a.e. for } x' \in V_y.$$

Hence

$$f(x') \leqslant f_+^+(x) - \delta_0 \quad \text{a.e. for } x' \in V_y. \tag{3.10}$$

Since $\{V_y \mid y \in (x, x+\varepsilon)\}$ is an open cover of $(x, x+\varepsilon)$, there is a countable subcover $\{V_{y_j} \mid j = 1, 2, \ldots\}$ of $\{V_y \mid y \in (x, x+\varepsilon)\}$. So (3.10) implies (3.9), and thus

$$f_+^+(x) \leqslant f_+^+(x) - \delta_0.$$

This is impossible. Therefore, (3.5) holds. Similarly, we have

$$f_-^-(x) = \liminf_{y \to x^-} f^-(y). \tag{3.11}$$

From (3.3)–(3.5) and (3.11) it follows that

$$\partial F(x) = \left[f^-(x), f_+^+(x)\right] \cup \left[f_-^-(x), f^+(x)\right].$$

This completes the proof. □

We now give several examples to illustrate the computation of the limiting subdifferential by using the formula (3.1). They also show that if $f \in L^\infty[a,b]$ and $F(x) := \int_a^x f(t)\,dt$ for all $x \in [a,b]$, then $\partial F(x)$ may be either a single closed interval or a disjoint union of two closed intervals.

Example 3.1. Let E be a measurable set in [0, 1] with the property that the intersection of any nonempty open interval in [0, 1] with both E and its complement has positive Lebesgue measure. Such sets do exist (see [13, p. 307]). Let χ_E be the characteristic function of E, i.e., $\chi_E(t) = 1$ if $t \in E$ and $\chi_E(t) = 0$ otherwise. Define $F(x) = \int_0^x \chi_E(t)\,dt$ for all $x \in [0,1]$ and put $f(t) := \chi_E(t)$. It is easy to see that $f \in L^\infty[0,1]$ and

$$f^+(x) = f_+^+(x) = 1 \quad \text{and} \quad f^-(x) = f_-^-(x) = 0 \quad \text{for all } x \in (0,1).$$

Thus, by Theorem 3.1 we have $\partial F(x) = [0,1]$ for all $x \in (0,1)$.

Remark 3.1. Since $\partial F(x) = [0,1]$ for all $x \in (0,1)$, by Theorem 2.2 one has $\partial_C F(x) = [0,1]$ for all $x \in (0,1)$. The last formula is due to R.T. Rockafellar (see [3, p. 191]).

Example 3.2. Let E be defined as in Example 3.1. Take any $x_0 \in E \cap (0,1)$, and define the function $f : [0,1] \to \mathbb{R}$ by

$$f(t) = \begin{cases} 1 & \text{if } t \in [x_0, 1] \cap E, \\ 0 & \text{if } t \in [x_0, 1] \backslash E, \\ 2 & \text{if } t \in [0, x_0) \cap E, \\ 3 & \text{if } t \in [0, x_0) \backslash E. \end{cases}$$

Let $F(x) := \int_a^x f(t)\,dt$ for all $x \in [0,1]$. It is clear that $f \in L^\infty[0,1]$ and

$$f^+(x_0) = 3, \qquad f_+^+(x_0) = 1, \qquad f^-(x_0) = 0, \qquad f_-^-(x_0) = 2.$$

Hence, by Theorem 3.1,

$$\partial F(x_0) = [0,1] \cup [2,3].$$

Corollary 3.1. *In addition to the assumptions of Theorem* 3.1*, suppose that* $\hat{\partial} F(x) \neq \emptyset$. *Then one has*

$$\partial F(x) = \left[f^-(x), f^+(x)\right],$$

and thus $\partial F(x) = \partial_C F(x)$.

Proof. We have

$$x^* \in \hat{\partial} F(x) \quad \Leftrightarrow \quad \begin{cases} \liminf\limits_{u \to x^+} \dfrac{\int_x^u f(t)\,dt}{u - x} \geqslant x^*, \\ \limsup\limits_{u \to x^-} \dfrac{\int_x^u f(t)\,dt}{u - x} \leqslant x^*. \end{cases} \tag{3.12}$$

For $\delta > 0$ there exist $\varepsilon > 0$ and $M \in \mathbb{R}$ such that

$$M < f_+^+(x) + \delta \quad \text{and} \quad f(x') \leqslant M \quad \text{a.e. for } x' \in [x, x + \varepsilon].$$

The latter implies that

$$f(x') \leqslant f_+^+(x) + \delta \quad \text{a.e. for } x' \in [x, x + \varepsilon].$$

Hence

$$\liminf_{u \to x^+} \frac{\int_x^u f(t)\,dt}{u - x} \leqslant f_+^+(x) + \delta.$$

Since $\delta > 0$ is arbitrary, we have

$$\liminf_{u \to x^+} \frac{\int_x^u f(t)\,dt}{u - x} \leqslant f_+^+(x). \tag{3.13}$$

Similarly, one has

$$f_-^-(x) \leqslant \limsup_{u \to x^-} \frac{\int_x^u f(t)\,dt}{u - x}. \tag{3.14}$$

Since $\hat{\partial} F(x) \neq \emptyset$, from (3.12)–(3.14) we deduce that $f_-^-(x) \leqslant f_+^+(x)$. Therefore, by Theorem 3.1 we obtain $\partial F(x) = [f^-(x), f^+(x)]$, which completes the proof. $\square$

Remark 3.2. From Corollary 3.1 it follows that if $\partial F(x)$ is nonconvex, then $\hat{\partial} F(x) = \emptyset$.

Example 3.3. Let $\{r_k\}_{k\in\mathbb{N}}$ be the set of rational numbers in $(a,b)\subset\mathbb{R}$, $a<b$. For each $k\in\mathbb{N}$, choose $\delta_k>0$ as small as $(r_k-\delta_k, r_k+\delta_k)\subset(a,b)$ and $\delta_k<2^{-(k+3)}(b-a)$. Put $A:=\bigcup_{k=0}^{\infty}(r_k-\delta_k, r_k+\delta_k)$ and $P:=[a,b]\backslash A$. Since A is an open set in $\mathbb{R}$, one has $A=\bigcup_{m=0}^{\infty}(a_m,b_m)$, where $\{(a_m,b_m)\}_{m\in\mathbb{N}}$ is a sequence of disjoint open intervals. Define a function $f:[a,b]\to\mathbb{R}$ by

$$f(x)=\begin{cases}0 & \text{if } x\in P,\\ (x-a_m)^2(x-b_m)^2\sin\frac{1}{(b_m-a_m)(x-a_m)(x-b_m)} & \text{if } x\in(a_m,b_m).\end{cases}$$

It is known (see [6]) that f is Lipschitz on $[a,b]$, and for every $x\in A$,

$$f'(x)=2(x-a_m)(x-b_m)(2x-a_m-b_m)\sin\frac{1}{(b_m-a_m)(x-a_m)(x-b_m)}$$
$$-\frac{2x-a_m-b_m}{b_m-a_m}\cos\frac{1}{(b_m-a_m)(x-a_m)(x-b_m)}.$$

Besides, $f'(x)=0$ for all $x\in P$. For each $x\in[a,b]$, let $F(x):=\int_a^x f'(t)\,dt$. Since f is Lipschitz on $[a,b]$, we have $f(x)=f(a)+F(x)$ for each $x\in[a,b]$. Hence

$$\partial_C F(x)=\partial_C f(x),\qquad \partial F(x)=\partial f(x)$$

and

$$\hat{\partial}F(x)=\hat{\partial}f(x)\neq\emptyset$$

for every $x\in[a,b]$. By Corollary 3.1, $\partial F(x)=\partial_C F(x)$ for all $x\in(a,b)$. On the other hand, by [6, Example 4.1] we have

$$\partial_C f(x)=\begin{cases}[-1,1] & \text{if } x\in P,\\ \{f'(x)\} & \text{if } x\in A.\end{cases}$$

Therefore,

$$\partial f(x)=\begin{cases}[-1,1] & \text{if } x\in P,\\ \{f'(x)\} & \text{if } x\in A.\end{cases}$$

Proposition 3.1. *Let $f\in L^1[a,b]$, where $L^1[a,b]$ is the space of Lebesgue integrable real functions on the interval $[a,b]$, $-\infty<a<\bar{x}<b<+\infty$, and $F(x):=\int_a^x f(t)\,dt$ for all $x\in[a,b]$. Suppose that f is continuous on some neighborhood of $\bar{x}$ except, possibly, at $\bar{x}$; and $\lim_{x\to\bar{x}^-}f(t):=\alpha\in\overline{\mathbb{R}}$, $\lim_{x\to\bar{x}^+}f(t):=\beta\in\overline{\mathbb{R}}$. Then we have the following assertions*:

(i) *If $\alpha=-\infty$ and $\beta=+\infty$, then $\hat{\partial}F(\bar{x})=\partial F(\bar{x})=\mathbb{R}$.*
(ii) *If $\alpha=-\infty$ and $\beta\in\mathbb{R}$, then $\hat{\partial}F(\bar{x})=\partial F(\bar{x})=(-\infty,\beta]$.*
(iii) *If $\alpha\in\mathbb{R}$ and $\beta=+\infty$, then $\hat{\partial}F(\bar{x})=\partial F(\bar{x})=[\alpha,+\infty)$.*
(iv) *If $\alpha=+\infty$ and $\beta=-\infty$, then $\hat{\partial}F(\bar{x})=\partial F(\bar{x})=\emptyset$.*
(v) *If $\alpha=+\infty$ and $\beta\in\mathbb{R}$, then $\hat{\partial}F(\bar{x})=\emptyset$ and $\partial F(\bar{x})=\{\beta\}$.*
(vi) *If $\alpha\in\mathbb{R}$ and $\beta=-\infty$, then $\hat{\partial}F(\bar{x})=\emptyset$ and $\partial F(\bar{x})=\{\alpha\}$.*
(vii) *If $\alpha,\beta\in\mathbb{R}$, then*

$$\hat{\partial}F(\bar{x})=\begin{cases}[\alpha,\beta] & \text{for } \alpha\leqslant\beta,\\ \emptyset & \text{for } \beta<\alpha\end{cases}\quad\text{and}\quad \partial F(\bar{x})=\begin{cases}[\alpha,\beta] & \text{for } \alpha\leqslant\beta,\\ \{\alpha,\beta\} & \text{for } \beta<\alpha.\end{cases}$$

Proof. Assume that f is continuous on $U:=(\bar{x}-\delta,\bar{x})\cup(\bar{x},\bar{x}+\delta)\subset[a,b]$ for some $\delta>0$. We have

$$x^*\in\hat{\partial}f(\bar{x})\quad\Leftrightarrow\quad\begin{cases}\liminf\limits_{x\to\bar{x}^+}\dfrac{\int_{\bar{x}}^x f(t)\,dt}{x-\bar{x}}\geqslant x^*,\\ \limsup\limits_{x\to\bar{x}^-}\dfrac{\int_{\bar{x}}^x f(t)\,dt}{x-\bar{x}}\leqslant x^*.\end{cases}\tag{3.15}$$

If $\beta \in \mathbb{R}$, then for any $x \in (\bar{x}, \bar{x}+\delta)$, one has

$$\frac{\int_{\bar{x}}^{x} f(t)\,dt}{x-\bar{x}} = f(\theta_x) \quad \text{for some } \theta_x \in (\bar{x}, x).$$

Therefore,

$$\liminf_{x\to\bar{x}^+} \frac{\int_{\bar{x}}^{x} f(t)\,dt}{x-\bar{x}} = \liminf_{x\to\bar{x}^+} f(\theta_x) = \beta. \tag{3.16}$$

If $\beta = +\infty$ or $\beta = -\infty$, then it is easy to see that

$$\liminf_{x\to\bar{x}^+} \frac{\int_{\bar{x}}^{x} f(t)\,dt}{x-\bar{x}} = \beta. \tag{3.17}$$

Similarly,

$$\limsup_{x\to\bar{x}^-} \frac{\int_{\bar{x}}^{x} f(t)\,dt}{x-\bar{x}} = \alpha. \tag{3.18}$$

From (3.15)–(3.18) it follows that

$$\hat{\partial} F(\bar{x}) = \begin{cases} [\alpha, \beta] & \text{if } \alpha, \beta \in \mathbb{R} \text{ and } \alpha \leqslant \beta, \\ \emptyset & \text{if } \alpha = +\infty \text{ or } \beta = -\infty \text{ or } \alpha > \beta, \\ \mathbb{R} & \text{if } \alpha = -\infty \text{ and } \beta = +\infty, \\ (-\infty, \beta] & \text{if } \alpha = -\infty \text{ and } \beta \in \mathbb{R}, \\ [\alpha, +\infty) & \text{if } \alpha \in \mathbb{R} \text{ and } \beta = +\infty. \end{cases}$$

Since f is continuous on U, $F(x)$ is Fréchet differentiable and $F'(x) = f(x)$ for all $x \in U$. By Theorem 2.1, $\hat{\partial} F(x) = \{f(x)\}$ for all $x \in U$. Hence

$$\operatorname*{Lim\,sup}_{x \xrightarrow{x\neq\bar{x}} \bar{x}} \hat{\partial} F(x) = \begin{cases} \{\alpha, \beta\} & \text{if } \alpha, \beta \in \mathbb{R}, \\ \{\alpha\} & \text{if } \alpha \in \mathbb{R} \text{ and } |\beta| = +\infty, \\ \{\beta\} & \text{if } |\alpha| = +\infty \text{ and } \beta \in \mathbb{R}, \\ \emptyset & \text{if } |\alpha| = |\beta| = +\infty. \end{cases}$$

From what has already been said we obtain the assertions (i)–(vii). □

Proposition 3.2. *Suppose $\{t_n\}$, $\{\tau_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ are sequences of real numbers such that $a = t_0 < t_1 < \cdots < t_n < t_{n+1} < \cdots < \bar{x} < \cdots < \tau_{n+1} < \tau_n < \cdots < \tau_1 < \tau_0 = b$, $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \tau_n = \bar{x}$, both $\sum_{n=1}^{\infty} \alpha_n(t_n - t_{n-1})$ and $\sum_{n=1}^{\infty} \beta_n(\tau_{n-1} - \tau_n)$ converge absolutely. Consider the indefinite integral $F : [a,b] \to \mathbb{R}$ given by $F(x) := \int_a^x f(t)\,dt$ for all $x \in [a,b]$, where*

$$f(t) = \begin{cases} \alpha_i & \text{if } t_{i-1} \leqslant t < t_i,\ i = 1, 2, \ldots, \\ \alpha_0 & \text{if } t = \bar{x}, \\ \beta_j & \text{if } \tau_j \leqslant t < \tau_{j-1},\ j = 1, 2, \ldots. \end{cases}$$

Putting

$$\alpha := -\liminf_{k\to\infty} \frac{\sum_{i=k+1}^{\infty} \alpha_i(t_i - t_{i-1})}{t_k - \bar{x}}, \qquad \beta := \liminf_{k\to\infty} \frac{\sum_{i=k+1}^{\infty} \beta_i(\tau_{i-1} - \tau_i)}{\tau_k - \bar{x}},$$

and

$$\Omega := \Big\{\lim_{i_k\to\infty} \alpha_{i_k}\Big\} \cup \Big\{\lim_{j_k\to\infty} \beta_{j_k}\Big\} \cup \operatorname*{Lim\,sup}_{i \xrightarrow{N_1} \infty} [\alpha_i, \alpha_{i+1}] \cup \operatorname*{Lim\,sup}_{j \xrightarrow{N_2} \infty} [\beta_{j+1}, \beta_j],$$

where $N_1 := \{i \in N \colon \alpha_i \leqslant \alpha_{i+1}\}$, $N_2 := \{j \in N \colon \beta_{j+1} \leqslant \beta_j\}$, $\{\lim_{i_k\to\infty} \alpha_{i_k}\}$ and $\{\lim_{j_k\to\infty} \beta_{j_k}\}$ are the set of all the cluster-points of sequences $\{\alpha_k\}$ and $\{\beta_k\}$, respectively, we have the following assertions:

(i) *If* $\alpha = -\infty$ *and* $\beta = +\infty$, *then* $\hat{\partial} F(\bar{x}) = \partial F(\bar{x}) = \mathbb{R}$.
(ii) *If* $\alpha = -\infty$ *and* $\beta \in \mathbb{R}$, *then* $\hat{\partial} F(\bar{x}) = (-\infty, \beta]$ *and* $\partial F(\bar{x}) = \Omega \cup (-\infty, \beta]$.
(iii) *If* $\alpha \in \mathbb{R}$ *and* $\beta = +\infty$, *then* $\hat{\partial} F(\bar{x}) = [\alpha, +\infty)$ *and* $\partial F(\bar{x}) = \Omega \cup [\alpha, +\infty)$.
(iv) *If* $\alpha = +\infty$ *or* $\beta = -\infty$, *then* $\hat{\partial} F(\bar{x}) = \emptyset$ *and* $\partial F(\bar{x}) = \Omega$.
(v) *If* $\alpha, \beta \in \mathbb{R}$, *then*

$$\hat{\partial} F(\bar{x}) = \begin{cases} [\alpha, \beta] & \text{for } \alpha \leqslant \beta, \\ \emptyset & \text{for } \beta < \alpha \end{cases} \quad \text{and} \quad \partial F(\bar{x}) = \begin{cases} [\alpha, \beta] & \text{for } \alpha \leqslant \beta, \\ \{\alpha, \beta\} & \text{for } \beta < \alpha. \end{cases}$$

Proof. Clearly, f is a measurable function and

$$\int_a^b \left| f(t) \right| dt = \sum_{n=1}^{\infty} \left| \alpha_n (t_n - t_{n-1}) \right| + \sum_{n=1}^{\infty} \left| \beta_n (\tau_{n-1} - \tau_n) \right|.$$

Since both the series $\sum_{n=1}^{\infty} \alpha_n (t_n - t_{n-1})$ and $\sum_{n=1}^{\infty} \beta_n (\tau_{n-1} - \tau_n)$ converge absolutely, $|f|$ is integrable on $[a, b]$ and so is f. By some easy computations, one has

$$F(x) = \begin{cases} \sum_{i=1}^{k} \alpha_i (t_i - t_{i-1}) + \alpha_{k+1}(x - t_k) & \text{if } x \in [t_k, t_{k+1}), \\ \sum_{i=1}^{\infty} \alpha_i (t_i - t_{i-1}) & \text{if } x = \bar{x}, \\ F(\bar{x}) + \sum_{i=k+1}^{\infty} \beta_i (\tau_{i-1} - \tau_i) + \beta_k (x - \tau_k) & \text{if } x \in (\tau_k, \tau_{k-1}]. \end{cases}$$

For $x \in (a, b) \backslash \{\bar{x}\}$, by Proposition 3.1, we get

$$\hat{\partial} F(x) = \begin{cases} \{\alpha_i\} & \text{if } x \in (t_{i-1}, t_i), \\ \{\beta_j\} & \text{if } x \in (\tau_j, \tau_{j-1}), \\ [\alpha_i, \alpha_{i+1}] & \text{if } x = t_i \text{ and } i \in N_1, \\ [\beta_{j+1}, \beta_j] & \text{if } x = \tau_j \text{ and } j \in N_2, \\ \emptyset & \text{otherwise,} \end{cases} \tag{3.19}$$

where $i, j \in \mathbb{N}$. It is clear that $F(x)$ is continuous at $\bar{x}$. Therefore,

$$\partial F(\bar{x}) = \operatorname*{Lim\,sup}_{x \to \bar{x}} \hat{\partial} F(x) = \hat{\partial} F(\bar{x}) \cup \operatorname*{Lim\,sup}_{x \xrightarrow{x \neq \bar{x}} \bar{x}} \hat{\partial} F(x).$$

It follows from (3.19) that $\operatorname{Lim\,sup}_{x \xrightarrow{x \neq \bar{x}} \bar{x}} \hat{\partial} F(x) = \Omega$. By Definition 2.1, we have

$$x^* \in \hat{\partial} f(\bar{x}) \quad \Leftrightarrow \quad \begin{cases} \liminf\limits_{x \to \bar{x}^+} \dfrac{F(x) - F(\bar{x})}{x - \bar{x}} \geqslant x^*, \\ \limsup\limits_{x \to \bar{x}^-} \dfrac{F(x) - F(\bar{x})}{x - \bar{x}} \leqslant x^*. \end{cases}$$

For each $k \in \mathbb{N}$ there exists $i \in \{k, k+1\}$ such that

$$\frac{F(\tau_i) - F(\bar{x})}{\tau_i - \bar{x}} = \min_{x \in [\tau_{k+1}, \tau_k]} \frac{F(x) - F(\bar{x})}{x - \bar{x}}.$$

Hence,

$$\liminf_{x \to \bar{x}^+} \frac{F(x) - F(\bar{x})}{x - \bar{x}} = \liminf_{k \to \infty} \frac{F(\tau_k) - F(\bar{x})}{\tau_k - \bar{x}} = \beta.$$

Similarly, $\limsup_{x \to \bar{x}^-} \frac{F(x) - F(\bar{x})}{x - \bar{x}} = \alpha$. Therefore,

$$\hat{\partial} F(\bar{x}) = \begin{cases} [\alpha, \beta] & \text{if } \alpha, \beta \in \mathbb{R} \text{ and } \alpha \leqslant \beta, \\ \emptyset & \text{if } \alpha = +\infty \text{ or } \beta = -\infty \text{ or } \alpha > \beta, \\ \mathbb{R} & \text{if } \alpha = -\infty \text{ and } \beta = +\infty, \\ (-\infty, \beta] & \text{if } \alpha = -\infty \text{ and } \beta \in \mathbb{R}, \\ [\alpha, +\infty) & \text{if } \alpha \in \mathbb{R} \text{ and } \beta = +\infty. \end{cases} \tag{3.20}$$

The desired assertions (i)–(v) follow from (3.19) and (3.20). □

4. The Aumann integral of the limiting subdifferential mapping

Our main result of this section reads as follows.

Theorem 4.1. *Let $f : U \to \mathbb{R}$ be a Lipschitz function defined on an open subset U of $\mathbb{R}^n$ and Ω be a measurable subset of U with $m(\Omega) < \infty$. Then we have*

$$\int_\Omega \partial f(x)\, m(dx) = \left\{ x^* \in \mathbb{R}^n \;\middle|\; \langle x^*, v\rangle \leqslant \int_\Omega f^0(x; v)\, m(dx) \text{ for all } v \in \mathbb{R}^n \right\}, \tag{4.1}$$

where $\partial f(x)$ and $f^0(x; v)$ denote, respectively, the limiting subdifferential of f at x and the Clarke generalized directional derivative of f at x in direction v.

Proof. Suppose that f is Lipschitz of rank ℓ on an open subset U of $\mathbb{R}^n$. Then the set-valued mapping $\partial f(\cdot) : U \rightrightarrows \mathbb{R}^n$ is closed, and thus $\partial f(x)$ is closed for all $x \in U$; see [3, p. 199]. By Theorem 2.2, one has $\overline{\text{co}}\, \partial f(x) = \partial_C f(x) \neq \emptyset$ (thus $\partial f(x) \neq \emptyset$) for all $x \in U$. Hence, by Theorem 2.4 the set-valued mapping $\partial f(\cdot)$ is measurable. Moreover, $m(\Omega) < \infty$, and $\|x^*\| \leqslant \ell$ for all $x^* \in \partial f(x)$ and for all $x \in U$; see [8, p. 86]. Hence, $\partial f(\cdot)$ is integrably bounded on Ω, and by Theorem 2.5 we get

$$\int_\Omega \partial f(x)\, m(dx) = \int_\Omega \overline{\text{co}}\, \partial f(x)\, m(dx).$$

On the other hand, $\overline{\text{co}}\, \partial f(x) = \partial_C f(x)$ for all $x \in \Omega$. Therefore,

$$\int_\Omega \partial f(x)\, m(dx) = \int_\Omega \partial_C f(x)\, m(dx). \tag{4.2}$$

The following assertions hold (see [4]):

(i) For each $v \in \mathbb{R}^n$, the mapping $f^0(\cdot; v)$ is measurable.
(ii) For all $v_1, v_2 \in \mathbb{R}^n$ and x in Ω, one has

$$\left| f^0(x; v_1) - f^0(x; v_1) \right| \leqslant k(x)\|v_1 - v_2\|,$$

where $k(x) := \ell$ for all $x \in \Omega$.
(iii) For each $x \in \Omega$, the function $f^0(x; \cdot)$ is convex (thus Clarke regular) on $\mathbb{R}^n$.

Since $m(\Omega) < \infty$, $k(\cdot) \in L^1(\Omega; \mathbb{R})$. Hence,

$$\partial_C F(v) = \int_\Omega \partial_C f^0(x; \cdot)(v)\, m(dx) \quad \text{for all } v \in \mathbb{R}^n,$$

where $F(v) := \int_\Omega f^0(x; v)\, m(dx)$; see [4, pp. 75–76]. Since F and $f^0(x; \cdot)$ are convex functions satisfying $F(0) = 0$ and $f^0(x; 0) = 0$, one has $\partial_C f^0(x; 0) = \partial_C f(x)$ and

$$\partial_C F(0) = \left\{ x^* \in \mathbb{R}^n \;\middle|\; \langle x^*, v\rangle \leqslant \int_\Omega f^0(x; v)\, m(dx) \text{ for all } v \in \mathbb{R}^n \right\}.$$

Therefore,

$$\int_\Omega \partial_C f(x)\, m(dx) = \left\{ x^* \in \mathbb{R}^n \;\middle|\; \langle x^*, v\rangle \leqslant \int_\Omega f^0(x; v)\, m(dx) \text{ for all } v \in \mathbb{R}^n \right\}. \tag{4.3}$$

Combining (4.2) and (4.3) we have (4.1). □

Recall [4, p. 30] that a mapping $f : X \to Y$, where X and Y are Banach spaces, is said to be (Hadamard) *strictly differentiable* at $x_0 \in X$ if there exists a continuous linear operator $D_s f(x_0) : X \to Y$ such that

$$\lim_{x \to x_0,\, t \to 0^+} \frac{f(x + tv) - f(x)}{t} = D_s f(x_0)v$$

and the convergence is uniform for v in compact sets. Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz near x. It is known (see [4, p. 33]) that $\partial_C f(x)$ reduces to a singleton if and only if f is strictly differentiable at x, and $\partial_C f(x) = \{D_s f(x)\}$. If f is Fréchet differentiable and Clarke regular at x, then $\partial_C f(x) = \{f'(x)\}$.

Corollary 4.1. *If f is a real Lipschitz function defined on an interval $[a, b] \subset \mathbb{R}$, then*

$$f(b) - f(a) \in \int_a^b \partial f(x)\, dx \tag{4.4}$$

and the equality

$$\int_a^b \partial f(x)\, dx = \big\{f(b) - f(a)\big\} \tag{4.5}$$

is valid if and only if f is strictly differentiable almost everywhere on $[a, b]$.

Proof. Since f is a Lipschitz function defined on an interval $[a, b] \subset \mathbb{R}$, f is Fréchet differentiable a.e. on $[a, b]$ and $f(b) - f(a) = \int_a^b f'(x)\, dx$. Put

$$D = \big\{x \in [a, b] \,\big|\, f \text{ is differentiable at } x\big\}.$$

We have

$$m\big([a, b] \backslash D\big) = 0 \quad \text{and} \quad \big\langle f'(x), v\big\rangle \leqslant f^0(x; v) \quad \text{for all } x \in D \text{ and } v \in \mathbb{R}, \tag{4.6}$$

where m stands for the Lebesgue measure on $\mathbb{R}$. Hence

$$\left\langle \int_a^b f'(x)\, dx, v \right\rangle = \int_a^b \big\langle f'(x), v\big\rangle dx \leqslant \int_a^b f^0(x; v)\, dx \quad \text{for all } v \in \mathbb{R}.$$

By Theorem 4.1, one has $f(b) - f(a) \in \int_a^b \partial f(x)\, dx$. If f is strictly differentiable a.e. on $[a, b]$, then $\partial f(x)$ reduces to a singleton a.e. for $x \in [a, b]$. Hence (4.5) holds. Now we suppose that the equality (4.5) is valid. For each $v \in \mathbb{R}$, it follows from (4.1), (4.5) and (4.6) that $\langle f'(x), v\rangle = f^0(x; v)$ a.e. for $x \in [a, b]$. Put $\Omega_1 := \{x \in D \cap [a, b] \mid f'(x) = f^0(x, 1)\}$ and $\Omega_{-1} := \{x \in D \cap [a, b] \mid -f'(x) = f^0(x, -1)\}$ and $\Omega = \Omega_1 \cap \Omega_{-1}$. One has $m([a, b] \backslash \Omega) = 0$ and $\langle f'(x), v\rangle = f^0(x; v)$ for all $x \in \Omega$ and $v \in \mathbb{R}$. This means that f is both Fréchet differentiable and Clarke regular at any $x \in \Omega$. Hence f is strictly differentiable at any $x \in \Omega$. This finishes the proof. □

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