



Properties of the probability density function of the non-central chi-squared distribution

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ARTICLE INFO

Article history:

Received 9 February 2008

Available online 28 May 2008

Submitted by V. Pozdnyakov

Dedicated to Prof. Matti Vuorinen on the occasion of his 60th birthday

Keywords:

Non-central chi-squared distribution

Modified Bessel function

Hoppe's formula

Turán-type inequality

Monotone form of l'Hospital's rule

ABSTRACT

In this paper we consider the probability density function (pdf) of a non-central χ^2 distribution with arbitrary number of degrees of freedom. For this function we prove that can be represented as a finite sum and we deduce a partial derivative formula. Moreover, we show that the pdf is log-concave when the degrees of freedom is greater or equal than 2. At the end of this paper we present some Turán-type inequalities for this function and an elegant application of the monotone form of l'Hospital's rule in probability theory is given.

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1. Partial derivative and finite sum formula for the pdf of the non-central χ^2 distribution with arbitrary degree of freedom

Let X_1, X_2, \dots, X_n be random variables that are normally distributed with unit variance and non-zero mean μ_i , where $i = 1, 2, \dots, n$. It is known that $X_1^2 + X_2^2 + \dots + X_n^2$ has the non-central χ^2 distribution with $n = 1, 2, 3, \dots$ degrees of freedom and non-centrality parameter $\lambda = \mu_1^2 + \mu_2^2 + \dots + \mu_n^2$. The probability density function $f_{n,\lambda} : [0, \infty) \rightarrow [0, \infty)$ of the non-central χ^2 distribution [15] is defined as

$$f_{n,\lambda}(x) = 2^{-n/2} e^{-(x+\lambda)/2} \sum_{k \geq 0} \frac{x^{n/2+k-1} (\lambda/4)^k}{\Gamma(n/2+k)k!}. \quad (1.1)$$

When $\mu_1 = \mu_2 = \dots = \mu_n = 0$, i.e. $\lambda = 0$, the above distribution reduces to the classical χ^2 distribution. The pdf $f_{n,0} : [0, \infty) \rightarrow [0, \infty)$ of this distribution is given by

$$f_{n,0}(x) = \frac{x^{n/2-1} e^{-x/2}}{2^{n/2} \Gamma(n/2)}. \quad (1.2)$$

The non-central chi-squared distribution is very close to the normal distribution and consequently frequently occurs in finance, estimation theory, decision theory and time series analysis [22]. Motivated by these applications, recently H. Ket-

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tani [17] pointed out that it is worth to find “the exact value of the distribution instead of appealing to various approximation algorithms known in the literature to approximate the infinite sum representation.” In this spirit H. Kettani—when the degree of freedom n is odd—reduced the infinite sum in (1.1) to a finite sum and he expressed the pdf $f_{n,\lambda}$ in terms of the $((n-1)/2)$ th partial derivative of the hyperbolic cosine function. Moreover he asked that what is the counterpart of the above mentioned results when the degree of freedom is even. Our main motivation to write this paper is this question, which we answer in this section: we present an alternative proof of Kettani’s results and we deduce the corresponding formulas when the degree of freedom is even. We note that our proofs are based on some simple properties of modified Bessel functions of the first kind and on Hoppe’s formula for the m th derivative of a composite function. Finally, in Section 2 we present some Turán-type inequalities for the pdf of χ^2 distribution and we show an elegant application of the monotone form of l’Hospital’s rule in probability theory.

Our first main result reads as follows.

Theorem 1.3. *The pdf $f_{n,\lambda}$ of the non-central χ^2 distribution is given by*

$$f_{n,\lambda}(x) = \begin{cases} \left(\frac{2x}{\lambda}\right)^m \frac{e^{-(x+\lambda)/2}}{\sqrt{2\pi x}} \cdot \frac{\partial^m \cosh(\sqrt{\lambda x})}{\partial x^m} & \text{if } n = 2m + 1, \\ \left(\frac{2x}{\lambda}\right)^m \frac{e^{-(x+\lambda)/2}}{2} \cdot \frac{\partial^m I_0(\sqrt{\lambda x})}{\partial x^m} & \text{if } n = 2m + 2, \end{cases}$$

where $m \in \mathbb{N}$ and I_0 is the modified Bessel function of the first kind of zero order.

Before we prove the above result let us recall some basic facts about modified Bessel functions of the first kind. The differential equation [23, p. 77]

$$x^2 y''(x) + xy'(x) - (x^2 + \mu^2)y(x) = 0, \quad (1.4)$$

which differs from Bessel’s equation only in the coefficient of y , is of frequent occurrence in problems of mathematical physics. The particular solution of (1.4) is called the modified Bessel function of the first kind of order μ , and is defined by the formula [23, p. 77] for all $x \in \mathbb{R}$,

$$I_\mu(x) = \sum_{k \geq 0} \frac{1}{k! \Gamma(\mu + k + 1)} \left(\frac{x}{2}\right)^{2k + \mu}. \quad (1.5)$$

Note that this special function is very useful in probability theory, more precisely the infinite divisibility of the Student t -distribution is strongly related to the behaviour of the modified Bessel functions of the first and third kind. We refer to the papers of E. Grosswald, M.E.H. Ismail, D.H. Kelker, K.S. Miller [11–14]. As we will see in this section the modified Bessel function of the first kind is also useful in the study of the non-central χ^2 distribution. For this let us consider the function $\gamma_\mu : [0, \infty) \rightarrow [1, \infty)$, defined by

$$\gamma_\mu(x) = 2^\mu \Gamma(\mu + 1) x^{-\mu/2} I_\mu(\sqrt{x}),$$

which is called sometimes as the normalized modified Bessel function of the first kind of order μ . Then in view of (1.5) γ_μ has the following MacLaurin series expansion:

$$\gamma_\mu(x) = \sum_{k \geq 0} \frac{\Gamma(\mu + 1)}{4^k \Gamma(\mu + k + 1) k!} x^k,$$

where $\mu \neq -1, -2, \dots$. Using this representation of the function γ_μ it is easy to see that for all $x \in \mathbb{R}$ and $\mu \neq -1, -2, \dots$ we have the following derivative formula:

$$4(\mu + 1) \gamma'_\mu(x) = \gamma_{\mu+1}(x), \quad (1.6)$$

which will be used in the sequel. We are now in position to prove Theorem 1.3.

Proof of Theorem 1.3. We note that the odd case was proved by H. Kettani [17]. However, we give here a completely different proof. It is known that the pdf of the non-central χ^2 distribution can be represented with the modified Bessel function of the first kind, i.e. using (1.1) and (1.5) we have

$$2f_{n,\lambda}(x) = e^{-(x+\lambda)/2} \left(\frac{x}{\lambda}\right)^{n/4-1/2} I_{n/2-1}(\sqrt{\lambda x}). \quad (1.7)$$

Taking into account the definition of γ_μ we have that

$$f_{n,\lambda}(x) = \left[\frac{x^{n/2-1} e^{-x/2}}{2^{n/2} \Gamma(n/2)} \right] e^{-\lambda/2} \gamma_{n/2-1}(\lambda x) = f_{n,0}(x) e^{-\lambda/2} \gamma_{n/2-1}(\lambda x). \quad (1.8)$$

On the other hand the repeated application of (1.6) yields

$$\lambda^m \gamma_{\mu+m}(\lambda x) = 4^m (\mu+1)(\mu+2) \cdots (\mu+m) \frac{\partial^m \gamma_{\mu}(\lambda x)}{\partial x^m} \quad (1.9)$$

for all $\lambda, x \geq 0$, $\mu \neq -1, -2, \dots$ and $m = 0, 1, 2, \dots$

Now assume that $n = 2m + 1$. Then applying (1.8) and (1.9) for $\mu = -1/2$ we have

$$\begin{aligned} f_{2m+1, \lambda}(x) &= \frac{e^{-(x+\lambda)/2} x^{m-1/2}}{2^{m+1/2} \Gamma(m+1/2)} \gamma_{-1/2+m}(\lambda x) \\ &= \frac{e^{-(x+\lambda)/2} x^{m-1/2}}{2^{m+1/2} \Gamma(m+1/2)} \cdot \frac{4^m (2m-1)!!}{\lambda^m 2^m} \cdot \frac{\partial^m \gamma_{-1/2}(\lambda x)}{\partial x^m} \\ &= \frac{e^{-(x+\lambda)/2} x^m}{\sqrt{2\pi x} \lambda^m} \cdot \frac{2^{2m} m! (2m-1)!!}{(2m)!} \cdot \frac{\partial^m \cosh(\sqrt{\lambda x})}{\partial x^m} \\ &= \left(\frac{2x}{\lambda}\right)^m \frac{e^{-(x+\lambda)/2}}{\sqrt{2\pi x}} \cdot \frac{\partial^m \cosh(\sqrt{\lambda x})}{\partial x^m}, \end{aligned}$$

where we have used that

$$I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cosh(x)$$

and

$$\Gamma\left(m + \frac{1}{2}\right) = \frac{(2m)! \sqrt{\pi}}{2^{2m} m!}, \quad \frac{2^{2m} m! (2m-1)!!}{(2m)!} = 2^m.$$

Suppose that $n = 2m + 2$. Then using again (1.8) and (1.9) for $\mu = 0$ we have

$$\begin{aligned} f_{2m+2, \lambda}(x) &= \frac{e^{-(x+\lambda)/2} x^m}{2^{m+1} \Gamma(m+1)} \gamma_m(\lambda x) \\ &= \frac{e^{-(x+\lambda)/2} x^m}{2^{m+1} \Gamma(m+1)} \cdot \frac{4^m m!}{\lambda^m} \cdot \frac{\partial^m \gamma_0(\lambda x)}{\partial x^m} \\ &= \left(\frac{2x}{\lambda}\right)^m \frac{e^{-(x+\lambda)/2}}{2} \cdot \frac{\partial^m I_0(\sqrt{\lambda x})}{\partial x^m} \end{aligned}$$

and hence the required result follows. \square

Now we would like to show that the pdf of the non-central χ^2 distribution can be represented actually as a finite sum involving modified Bessel functions of the first kind. Before we state the finite sum formula we need some auxiliary results.

Lemma 1.10. *The following assertions are true:*

(a) *If $p \in (0, 1)$ and $m = 1, 2, 3, \dots$, then*

$$\sum_{k=1}^m C_{2m-k-1}^{m-1} \left[\frac{1}{p^k} + \frac{1}{(1-p)^k} \right] = \frac{1}{p^m (1-p)^m}.$$

(b) *If $k = 1, 2, 3, \dots$ and $|x| < 1$, then*

$$(\sqrt{1+x} - 1)^k = \sum_{m \geq k} \frac{(-1)^{m-k}}{2^{2m-k}} \cdot \frac{k C_{2m-k}^m}{2m-k} \cdot x^m.$$

(c) *If $k = 0, 1, 2, \dots, m$ and m is a natural number, then*

$$\sum_{j=0}^k (-1)^{k-j} \frac{j(j-2) \cdots (j-2m+2)}{(k-j)! j!} = (-1)^{m-k} \frac{(2m-k-1)!}{(2m-2k)!! (k-1)!}. \quad (1.11)$$

Proof. (a) This result is a classical one related to the famous Banach's matches problem—see for example [9]—and we include here its proof only for the sake of completeness. Consider two matchboxes labeled with L and R , each containing m matches. When we want to use a match we choose a box and pick a match from it. If the probability of choosing the box with label L is p and the probability of choosing the other one is $(1-p)$ we emphasize the probability of the events L_k and R_k , where $k = 1, 2, \dots, m$, which are defined as follows:

1. L_k : when we observe that we have just picked the last match from the box R the box L contains exactly k matches.
2. R_k : when we observe that we have just picked the last match from the box L the box R contains exactly k matches.

It is clear that $L_i \cap L_j = \emptyset$ and $R_i \cap R_j = \emptyset$ for all $i \neq j$, moreover, $L_i \cap R_j = \emptyset$ for all i and j . Observe that

$$\bigcup_{k=1}^m (R_k \cup L_k)$$

is a sure event, thus we have

$$\sum_{k=1}^m [P(L_k) + P(R_k)] = 1.$$

But clearly

$$P(L_k) = (1-p)^{m-k} p^m C_{2m-k-1}^{m-1}$$

and

$$P(R_k) = p^{m-k} (1-p)^m C_{2m-k-1}^m,$$

because the $(2m-k)$ th pick is fixed and before this we need to choose $(m-1)$ times the R box and $(m-k)$ times the L box. Thus the stated relation represents the fact that the events $(L_k)_{1 \leq k \leq m}$ and $(R_k)_{1 \leq k \leq m}$ constitute the set of all outcomes.

(b) To prove the asserted equality first we compute the following sums:

$$S_1 = \sum_{k \geq 1} \frac{(-1)^k}{k} (\sqrt{1+x} - 1)^k \left[\frac{1}{p^k} + \frac{1}{(2-p)^k} \right]$$

and

$$S_2 = \sum_{k \geq 1} \left[\sum_{m \geq k} \frac{(-1)^m}{2^{2m-k}} \cdot \frac{C_{2m-k}^m}{2m-k} \cdot x^m \right] \left[\frac{1}{p^k} + \frac{1}{(2-p)^k} \right].$$

Now observe that the function $\varphi_1 : (-1, 1) \rightarrow \mathbb{R}$, defined by $\varphi_1(x) = (\sqrt{1+x} - 1)^k$, has a Taylor series expansion of the form

$$\sum_{m \geq k} c_{m,k} \cdot x^m.$$

Thus, if the sums S_1 and S_2 are equal, then the proof of (b) is complete because rearranging to the powers of x both sums, we obtain

$$\sum_{k=1}^m c_{m,k} \left[\frac{1}{p^k} + \frac{1}{(2-p)^k} \right] = \sum_{k=1}^m (-1)^m \frac{1}{2^{2m-k}} \frac{1}{m} C_{2m-k-1}^{m-1} \left[\frac{1}{p^k} + \frac{1}{(2-p)^k} \right]. \quad (1.12)$$

If we multiply both sides of (1.12) by p^m and then we let p tends to zero, we obtain $c_{m,m}$, hence we can reduce the number of terms to $m-1$ in both sums. After this we multiply by p^{m-1} both sides of

$$\sum_{k=1}^{m-1} c_{m,k} \left[\frac{1}{p^k} + \frac{1}{(2-p)^k} \right] = \sum_{k=1}^{m-1} (-1)^m \frac{1}{2^{2m-k}} \frac{1}{m} C_{2m-k-1}^{m-1} \left[\frac{1}{p^k} + \frac{1}{(2-p)^k} \right]$$

and let p tends to zero. From this we obtain $c_{m,m-1}$, etc. In fact this argument shows that the system of functions $\{f_k(p)\}_{1 \leq k \leq m}$, where $f_k : (0, 1) \rightarrow \mathbb{R}$ are defined by

$$f_k(p) = \frac{1}{p^k} + \frac{1}{(2-p)^k},$$

is a linear independent system, so we can identify the coefficients in (1.12).

By the other hand using the MacLaurin series expansion of the function $\varphi_2 : (-1, 1) \rightarrow \mathbb{R}$, defined by $\varphi_2(x) = \ln(1+x)$, we have

$$\begin{aligned} S_1 &= \sum_{k \geq 1} \frac{(-1)^k}{k} \left[\left(\frac{\sqrt{1+x}-1}{p} \right)^k + \left(\frac{\sqrt{1+x}-1}{2-p} \right)^k \right] = -\ln \left(1 + \frac{\sqrt{1+x}-1}{p} \right) - \ln \left(1 + \frac{\sqrt{1+x}-1}{2-p} \right) \\ &= -\ln \left[1 + \frac{x}{p(2-p)} \right]. \end{aligned}$$

Moreover,

$$\begin{aligned} S_2 &= \sum_{m \geq 1} \frac{(-1)^m x^m}{2^{2m}} \frac{1}{m} \sum_{m \geq k} C_{2m-k-1}^{m-1} \left[\left(\frac{2}{p} \right)^k + \left(\frac{2}{2-p} \right)^k \right] = \sum_{m \geq 1} \frac{(-1)^m x^m}{2^{2m}} \frac{1}{m} \frac{2^{2m}}{p^m (2-p)^m} \\ &= \sum_{m \geq 1} \frac{(-1)^m}{m} \left[\frac{x}{p(2-p)} \right]^m = -\ln \left[1 + \frac{x}{p(2-p)} \right], \end{aligned}$$

where we used (a) for $p/2$ instead of p and we changed the order of summation in S_2 .

(c) For convenience let us denote the left-hand side of (1.11) with β_k . Then using the part (b) of this lemma for the function φ_1 we have

$$\varphi_1^{(m)}(0) = \frac{(-1)^{m-k}}{2^{2m-k}} \frac{k \cdot (2m-k-1)!}{(m-k)!}.$$

By the other hand using the well-known Newton binomial formula we have that

$$[(\sqrt{1+x}-1)^k]^{(m)} = \sum_{j=0}^k \left[(-1)^{k-j} \frac{j(j-2) \cdots (j-2m+2)}{(k-j)!j!} \right] \cdot \frac{k!}{2^m} (x+1)^{j/2-m},$$

that is $k!\beta_k = 2^m \varphi_1^{(m)}(0)$. Thus to show (1.11) it remains to observe that

$$\beta_k = \frac{(-1/2)^{m-k} (2m-k-1)!}{(k-1)!(m-k)!} = (-1)^{m-k} \frac{(2m-k-1)!}{(2m-2k)!!(k-1)!}, \quad (1.13)$$

and with this the proof is complete. \square

Now we are ready to prove the second main result of this section, namely the finite sum formula for the pdf of χ^2 distribution for arbitrary degrees of freedom.

Theorem 1.14. *The pdf $f_{n,\lambda}$ of the non-central χ^2 distribution is given by*

$$f_{n,\lambda}(x) = \begin{cases} \frac{e^{-(x+\lambda)/2}}{\sqrt{2\pi x}} \left(\frac{x}{\lambda} \right)^{m/2} \sum_{i=0}^{m-1} \alpha_i(x) \cosh_{m-i}(\sqrt{\lambda x}) & \text{if } n = 2m + 1, \\ \frac{e^{-(x+\lambda)/2}}{2} \left(\frac{x}{\lambda} \right)^{m/2} \sum_{i=0}^{m-1} \alpha_i(x) \frac{\partial^{m-i} I_0(\sqrt{\lambda x})}{\partial x^{m-i}} & \text{if } n = 2m + 2, \end{cases}$$

where

$$\alpha_i(x) = (-1)^i (\lambda x)^{-i/2} \frac{(m+i-1)!}{(m-i-1)!(2i)!},$$

m is a natural number and

$$\cosh_{m-i}(\sqrt{\lambda x}) = \begin{cases} \cosh \sqrt{\lambda x} & \text{if } m-i \text{ is even,} \\ \sinh \sqrt{\lambda x} & \text{if } m-i \text{ is odd,} \end{cases}$$

is the alternate hyperbolic cosine/sine function.

Proof. It is worth mentioning here that the above finite sum formula—when the degrees of freedom is odd—is due to H. Kettani [17]. However, we give here a completely different proof for this part. For this let us recall the Hoppe formula [16] for the m th derivative of a composite function, namely

$$\frac{d^m}{dx^m} f(g(x)) = \sum_{k=0}^m \frac{f^{(k)}(g(x))}{k!} \sum_{j=0}^k (-1)^{k-j} C_k^j \cdot [g(x)]^{k-j} \cdot \frac{d^m}{dx^m} [g(x)]^j. \quad (1.15)$$

Now let us suppose that $n = 2m + 1$. If we consider the functions $f(x) = \cosh x$ and $g(x) = \sqrt{\lambda x}$, then clearly $f^{(k)}(x) = \cosh_k x$ and

$$2^m \partial^m [g(x)]^j / \partial x^m = \lambda^{j/2} j(j-2) \cdots (j-2m+2) \cdot x^{j/2-m}.$$

By using Theorem 1.3 and Hoppe's formula (1.15) we obtain that

$$f_{2m+1,\lambda}(x) = \frac{e^{-(x+\lambda)/2}}{\sqrt{2\pi x}} \lambda^{-m} \sum_{k=0}^m (\lambda x)^{k/2} [\cosh_k(\sqrt{\lambda x})] \cdot \beta_k, \quad (1.16)$$

thus changing in (1.16) and (1.13) k with $m - i$ we obtain the formula

$$f_{2m+1,\lambda}(x) = \frac{e^{-(x+\lambda)/2}}{\sqrt{2\pi x}} \left(\frac{x}{\lambda}\right)^{m/2} \sum_{i=0}^m \alpha_i(x) \cosh_{m-i}(\sqrt{\lambda x}).$$

Now rewriting $\alpha_i(x)$ as

$$(-1)^i (\lambda x)^{-i/2} \frac{(m+i)!(m-i)}{(m-i)!(m+i)(2i)!!}$$

we conclude that the last term in the above sum is zero, and hence the asserted result follows.

Finally, suppose that $n = 2m + 2$. Then applying Hoppe's formula to the functions $f(x) = I_0(x)$ and $g(x) = \sqrt{\lambda x}$, analogously with the first part of this theorem, the required result follows from the second part of Theorem 1.3 and from relation (1.11). With this the proof is complete. \square

2. Applications of the monotone form of l'Hospital rule in probability theory and inequalities involving the pdf of the non-central χ^2 distribution

In probability theory usually the cumulative distribution functions (cdf-s) does not have closed-form, and thus it is difficult to study their properties directly. In statistics, economics and industrial engineering frequently appears some problems which are related to the study of log-concavity (log-convexity) of some univariate distributions. An interesting unified exposition of related results on the log-concavity and log-convexity of many distributions—including applications—were communicated by M. Bagnoli and T. Bergstrom [5]. Some of their main results are included in the following theorem and in what follows we would like to present an alternative proof of these results by using the monotone form of l'Hospital rule.

Theorem 2.1. Let $f : (a, b) \rightarrow [0, \infty)$ be a continuously differentiable pdf. Moreover, let us consider the cdf $F : (a, b) \rightarrow [0, 1]$, the survival function $\bar{F} : (a, b) \rightarrow [0, 1]$, the left-hand integral of the cdf $G : (a, b) \rightarrow [0, \infty)$, the right-hand integral of the reliability function $H : (a, b) \rightarrow [0, \infty)$ and the failure rate $r : (a, b) \rightarrow [0, \infty)$, defined by

$$F(x) = \int_a^x f(t) dt, \quad \bar{F}(x) = \int_x^b f(t) dt, \quad G(x) = \int_a^x F(t) dt, \quad H(x) = \int_x^b \bar{F}(t) dt, \quad r(x) = \frac{f(x)}{\bar{F}(x)}.$$

Then the following implications are true:

- (a) f is log-concave $\Rightarrow F$ is log-concave $\Rightarrow G$ is log-concave.
- (b) $f(a) = 0$ and f is log-convex $\Rightarrow F$ is log-convex $\Rightarrow G$ is log-convex.
- (c) f is log-concave $\Rightarrow \bar{F}$ is log-concave $\Rightarrow H$ is log-concave.
- (d) $f(b) = 0$ and f is log-convex $\Rightarrow \bar{F}$ is log-convex $\Rightarrow H$ is log-convex.
- (e) f is log-concave (log-convex) $\Rightarrow r$ is increasing (decreasing).

Before we present an alternative proof of Theorem 2.1 let us recall the following simple and elegant result, which is known in literature as the monotone form of l'Hospital rule and was discovered by G.D. Anderson, M.K. Vamanamurthy and M. Vuorinen [2].

Lemma 2.2. For $a, b \in \mathbb{R}$ let $f_1, f_2 : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$, and differentiable on (a, b) . Further let $f_2'(x) \neq 0$ for all $x \in (a, b)$. If f_1'/f_2' is increasing (decreasing) on (a, b) , then so are

$$x \mapsto \frac{f_1(x) - f_1(a)}{f_2(x) - f_2(a)} \quad \text{and} \quad x \mapsto \frac{f_1(x) - f_1(b)}{f_2(x) - f_2(b)}.$$

It is worth mentioning here that various versions of this monotone form of l'Hospital rule was used since 1982 in different areas of mathematics, for example in differential geometry—see the papers of J. Cheeger, M. Gromov and M. Taylor [8, 10]—quasiconformal analysis—see the works of G.D. Anderson, M.K. Vamanamurthy and M. Vuorinen [2,3]—statistics and probability—see the papers of I. Pinelis [18–20]—analytic inequalities—see the papers of G.D. Anderson, M.K. Vamanamurthy and M. Vuorinen [4] and I. Pinelis [21], etc.

Proof of Theorem 2.1. (a) & (b). Since f is log-concave (log-convex), it follows that $x \mapsto f'(x)/f(x)$ is decreasing (increasing). Consequently, from Lemma 2.2 we conclude that the function

$$x \mapsto \frac{f(x) - f(a)}{F(x) - F(a)} = \frac{f(x) - f(a)}{F(x)}$$

is decreasing (increasing) on (a, b) . Because the distribution function F is increasing, this implies that $x \mapsto f(x)/F(x) = F'(x)/F(x)$ is decreasing (increasing) too on (a, b) , i.e. the cdf F is log-concave (log-convex). Applying again Lemma 2.2 we deduce that the function $x \mapsto F(x)/G(x) = G'(x)/G(x)$ is decreasing (increasing), hence the function G is log-concave (log-convex) too.

(c) & (d) & (e). Because f is log-concave (log-convex), clearly the function $x \mapsto f'(x)/f(x)$ is decreasing (increasing). From Lemma 2.2 we get that the function

$$x \mapsto -\frac{f(x) - f(b)}{\bar{F}(x) - \bar{F}(b)} = -\frac{f(x) - f(b)}{\bar{F}(x)}$$

is decreasing (increasing) on (a, b) . Since the survival function \bar{F} is decreasing, this implies that the function $x \mapsto -r(x) = -f(x)/\bar{F}(x) = \bar{F}'(x)/\bar{F}(x)$ is decreasing (increasing) too on (a, b) , i.e. the reliability function \bar{F} is log-concave (log-convex). Applying again Lemma 2.2 we deduce that the function $x \mapsto -\bar{F}(x)/H(x) = H'(x)/H(x)$ is decreasing (increasing), hence the function H is log-concave (log-convex) too. \square

H. Kettani [17] recently deduced some recurrence relations for the pdf $f_{n,\lambda}$ for n odd. We note that actually using the recurrence formula for the modified Bessel function can be proved easily a recurrence formula for $f_{n,\lambda}$ which is valid for all $n \geq 1$. The following results complete the results of H. Kettani [17], M. Bagnoli and T. Bergstrom [5].

Theorem 2.3. *The pdf of the non-central χ^2 distribution $f_{n,\lambda}$ —where $n = 1, 2, \dots$ and $\lambda, x \geq 0$ —has the following properties:*

- (a) *The function $x \mapsto f_{n,\lambda}(x)$ is log-concave on $[0, \infty)$ provided $n \geq 2$.*
- (b) *The function $\lambda \mapsto f_{n,\lambda}(x)$ is log-concave on $[0, \infty)$.*
- (c) *The recurrence relation*

$$xf_{n,\lambda}(x) - \lambda f_{n+4,\lambda}(x) = nf_{n+2,\lambda}(x)$$

holds true.

- (d) *The following Turán-type inequalities holds true:*

$$f_{n,\lambda}(x)f_{n+4,\lambda}(x) < [f_{n+2,\lambda}(x)]^2, \quad (2.4)$$

$$[\Gamma_{n+2}(x)]^2 \leq \Gamma_n(x)\Gamma_{n+4}(x), \quad (2.5)$$

$$\Gamma_{n_1+2}(x)\Gamma_{n_2}(x) \geq \Gamma_{n_2+2}(x)\Gamma_{n_1}(x), \quad (2.6)$$

where $\Gamma_n(x) = f_{n,\lambda}(x)/f_{n,0}(x)$ and $n_1 \geq n_2 > 0$.

Proof. (a) & (b). It is known [5] that the pdf $f_{n,0}$ of the χ^2 distribution is log-concave when $n \geq 2$. On the other hand [7, Theorem 2.2] the function γ_μ is log-concave if $\mu > -1$. Thus the function $x \mapsto \gamma_{n/2-1}(\lambda x)$ is log-concave too for all $n > 0$. Hence in view of (1.8) the function $f_{n,\lambda}$ is log-concave as a product of two log-concave functions. Moreover, it is easy to verify that the function $\lambda \mapsto \gamma_{n/2-1}(\lambda x)$ is log-concave too for all $n > 0$, and consequently $\lambda \mapsto f_{n,\lambda}$ becomes also log-concave for all $n = 1, 2, 3, \dots$

(c) Recall that for the modified Bessel function of the first kind I_μ the following recurrence relation [23, p. 79] hold

$$xI_{\mu-1}(x) - xI_{\mu+1}(x) = 2\mu I_\mu(x).$$

Changing in this relation μ with $n/2$ and x with $\sqrt{\lambda x}$ in view of (1.7) the asserted result follows.

(d) Amos' inequality [1, p. 243] states that $I_{\mu-1}(x)I_{\mu+1}(x) - I_\mu^2(x) < 0$ holds for all $x, \mu > 0$. Changing again μ with $n/2$ and x with $\sqrt{\lambda x}$, we easily get that

$$I_{n/2}^2(\sqrt{\lambda x}) - I_{n/2-1}(\sqrt{\lambda x})I_{n/2+1}(\sqrt{\lambda x}) > 0.$$

Application of (1.7) yields the required inequality (2.4).

Recently, the second author proved [6, Theorem 1] the following Turán-type inequality for the function γ_μ , namely that

$$[\gamma_{\mu+1}(x)]^2 \leq \gamma_\mu(x)\gamma_{\mu+2}(x)$$

holds for all $\mu > -1$ and $x \geq 0$. Changing μ with $n/2 - 1$ and x with λx we obtain the following

$$[\gamma_{n/2}(\lambda x)]^2 \leq \gamma_{n/2-1}(\lambda x)\gamma_{n/2+1}(\lambda x). \quad (2.7)$$

But from (1.8) it is clear that

$$\gamma_{n/2-1}(\lambda x) = e^{\lambda/2} f_{n,\lambda}(x)/f_{n,0}(x),$$

and consequently from (2.7) the asserted inequality (2.5) follows.

Let us consider $n_1 = n + 2$ and $n_2 = n$, then clearly (2.6) provides a generalization of (2.5). In order to generalize the Turán-type inequality $[\gamma_{\mu+1}(x)]^2 \leq \gamma_\mu(x)\gamma_{\mu+2}(x)$, recently the second author proved [6, Theorem 1] that the function $\mu \mapsto \gamma_{\mu+1}(x)/\gamma_\mu(x)$ is increasing on $(-1, \infty)$ for all $x \geq 0$ fixed. Thus, if $\mu_1 \geq \mu_2 > -1$, then for all $\lambda, x \geq 0$ we have

$$\gamma_{\mu_1+1}(\lambda x)\gamma_{\mu_2}(\lambda x) \geq \gamma_{\mu_2+1}(\lambda x)\gamma_{\mu_1}(\lambda x).$$

Now letting $\mu_1 = n_1/2 - 1$ and $\mu_2 = n_2/2 - 1$ we easily get

$$\gamma_{n_1/2}(\lambda x)\gamma_{n_2/2-1}(\lambda x) \geq \gamma_{n_2/2}(\lambda x)\gamma_{n_1/2-1}(\lambda x). \quad (2.8)$$

Finally, using (1.8) and (2.8) it follows that

$$\left[\frac{f_{n_1+2,\lambda}(x)}{f_{n_1+2,0}(x)} \right] \cdot \left[\frac{f_{n_2,\lambda}(x)}{f_{n_2,0}(x)} \right] \geq \left[\frac{f_{n_2+2,\lambda}(x)}{f_{n_2+2,0}(x)} \right] \cdot \left[\frac{f_{n_1,\lambda}(x)}{f_{n_1,0}(x)} \right],$$

and with this the proof is complete. \square

An immediate application of Theorems 2.1 and 2.3 is the following result.

Corollary 2.9. *Let $f_{n,\lambda}$ be the pdf of the non-central chi-squared distribution. Then for all $\lambda, x \geq 0$ and $n \geq 2$, the cdf $F_{n,\lambda}$ and the reliability function $\bar{F}_{n,\lambda}$, defined by*

$$F_{n,\lambda}(x) = \int_0^x f_{n,\lambda}(t) dt \quad \text{and} \quad \bar{F}_{n,\lambda}(x) = \int_x^\infty f_{n,\lambda}(t) dt,$$

are log-concave. Moreover, the failure rate $r_{n,\lambda}$, defined by

$$r_{n,\lambda}(x) = f_{n,\lambda}(x)/\bar{F}_{n,\lambda}(x),$$

is increasing.

Acknowledgments

The first author's research was partially supported by the Romanian Ministry of Education and Research, CNCSIS Grant type A 1465/2008. The second author's research was supported in part by the Institute of Mathematics, University of Debrecen, Hungary and in part by the Romanian Ministry of Education and Research, CNCSIS Grant type A 1472/2007. Both of the authors are grateful to Houssain Kettani for a copy of paper [17].

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