J. Math. Anal. Appl. 359 (2009) 240–251

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Lower semicontinuity property of multiparameter optimal stopping value and its application to multiparameter prophet inequalities

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ARTICLE INFO

Article history:
Received 12 December 2008
Available online 17 March 2009
Submitted by M. Ledoux

Keywords:
Stopping point
Multiparameter stochastic process
Convergence in distribution
Lower semicontinuity
Prophet inequality

ABSTRACT

This paper is concerned with the optimal stopping problem for discrete time multiparameter stochastic processes with the index set $\mathbf{N}^d$. The optimal stopping value of a discrete time multiparameter integrable stochastic process whose negative part is uniformly integrable, is lower semicontinuous for the topology of convergence in distribution. The multiparameter version of prophet inequality for the one-parameter optimal stopping problem is formulated and the lower semicontinuity property of the optimal stopping value is applied to the multiparameter prophet inequality.

1. Introduction

Let d be a fixed positive integer and $\mathbf{N}$ be the set of all nonnegative integers. In this paper we consider stochastic processes indexed by set $\mathbf{N}^d$, which is equipped with the following partial order: for $z = (z_1, z_2, \dots, z_d)$, $w = (w_1, w_2, \dots, w_d) \in \mathbf{N}^d$, $z \leqslant w$ if and only if $z_i \leqslant w_i$ for all i. Let e_i be the element for which the ith coordinate is 1 and all other coordinates are 0. We set $|z| = \sum_{i=1}^d z_i$ for $z = (z_1, z_2, \dots, z_d)$, $I = \{z \in \mathbf{N}^d\colon\ z \leqslant t\}$ for fixed $t \in \mathbf{N}^d$, and $I(z) = \{w \in I\colon\ w \leqslant z\}$, $p(z) = \{w \in I\colon\ w \leqslant z,\ |z - w| = 1\}$, $d(z) = \{w \in I\colon\ w \geqslant z,\ |w - z| = 1\}$ for $z \in I$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $\mathbf{X} = \{X(z),\ z \in I\}$ be a real valued integrable stochastic process and $\{\mathcal{F}_z^{\mathbf{X}},\ z \in I\}$ be the natural filtration of $\mathbf{X}$ which satisfies the following conditions: $\mathcal{F}_0^{\mathbf{X}}$ contains all P-null sets of $\mathcal{F}$, and if $z \leqslant w$, then $\mathcal{F}_z^{\mathbf{X}} \subseteq \mathcal{F}_w^{\mathbf{X}}$.

An $\{\mathcal{F}_z^{\mathbf{X}}\}$-stopping point is a random variable T taking values in I such that $\{T = z\} \in \mathcal{F}_z^{\mathbf{X}}$ for all $z \in I$. A tactic is a family $(\{\Gamma(k),\ 0 \leqslant k \leqslant |t|\}, \tau)$ which satisfies the following conditions: $\Gamma(0) = o$ P-a.e., $\Gamma(k)$ is an $\{\mathcal{F}_z^{\mathbf{X}}\}$-stopping point for all $k \leqslant |t|$, $\Gamma(k+1) \in d(\Gamma(k))$ P-a.e. for all $k \leqslant |t| - 1$, $\Gamma(k+1)$ is $\mathcal{F}_{\Gamma(k)}^{\mathbf{X}}$-measurable for all $k \leqslant |t| - 1$, and τ is an $\{\mathcal{F}_{\Gamma(k)}^{\mathbf{X}},\ 0 \leqslant k \leqslant |t|\}$-stopping time, where $\mathcal{F}_{\Gamma(k)}^{\mathbf{X}} = \{A \in \mathcal{F}\colon\ A \cap \{\Gamma(k) = z\} \in \mathcal{F}_z^{\mathbf{X}}$ for all $z \in I\}$. We call a stopping point T is accessible if there exists a tactic $(\{\Gamma(k)\}, \tau)$ such that $T = \Gamma(\tau)$ P-a.e., and denote the set of all accessible stopping points by $A(\mathbf{X})$. Then the multiparameter optimal stopping problem is to find a stopping point $T^* \in A(\mathbf{X})$ (a tactic $(\{\Gamma^*(k)\}, \tau^*)$) such that

$$V[\mathbf{X}] := E\big[X\big(T^*\big)\big] = \sup_{T \in A(\mathbf{X})} E\big[X(T)\big]\Big(E\big[X\big(\Gamma^*\big(\tau^*\big)\big)\big] = \sup_{(\{\Gamma(k)\}, \tau)} E\big[X\big(\Gamma(\tau)\big)\big]\Big).$$

The discrete time multiparameter optimal stopping problems have been studied by many authors, for example, Cairoli and Dalang [2], Krengel and Sucheston [13], Lawler and Vanderbei [14], Mandelbaum [15], Mandelbaum and Vanderbei [16],

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.03.035

and Mazziotto [17]. Furthermore the general multiparameter stochastic processes and the set-indexed stochastic processes have been studied by Edgar and Sucheston [4] and Ivanoff and Merzbach [11]. The notion of accessibility of a stopping point plays an important role in the general theory of multiparameter optimal stopping problems, hence the notions of tactic and optional increasing path have been introduced. In order to construct an optimal stopping point and to characterize an optimal value in the case where the index set is finite, the method of backwards induction is used. In this paper, we should remark the following: as stated above, we consider the multiparameter optimal stopping problems for the stochastic process indexed by the finite index set I in the class of all accessible stopping points. And also the optimal stopping point construction by using the method of backwards induction belongs to the class of accessible stopping points (see Cairoli and Dalang [2, pp. 79–82]). Therefore other conditions, for example, the conditional independence property, are not assumed excepting some conditions on the integrability.

Now in this paper we shall compare the expected reward of a player with complete foresight $E[\max_{z\in I} X(z)]$ and the expected reward of a player using stopping points $\sup_{T\in A(\mathbf{X})} E[X(T)]$. This relation is called a prophet inequality, which has been studied by many authors, for example, Hill [6,7], Hill and Kertz [8–10], Krengel and Sucheston [12] in the case of one-parameter optimal stopping problems, and Krengel and Sucheston [13], Tanaka [18,19] in the case of multiparameter optimal stopping problems. Especially, Hill and Kertz [10] contains very nice introduction to prophet theory for one-parameter optimal stopping problems.

By the way, it is well known that we give the topology of convergence in distribution on a family of random variables. The general theory of convergence in distribution is due to Billingsley [1]. Let $\mathbf{X}^n = \{X^n(z),\ z \in I\}$ and $\mathbf{X} = \{X(z),\ z \in I\}$, which are defined on $(\Omega^n, \mathcal{F}^n, P^n)$ and $(\Omega, \mathcal{F}, P)$ respectively, be real valued integrable stochastic processes. The corresponding probability distributions to these processes are defined on $(R^I, B(R^I))$, where $B(R^I)$ is the Borel σ-field of R^I. A sequence of stochastic processes $\{\mathbf{X}^n\}$ converges in distribution to the stochastic process $\mathbf{X}$ (we write $\mathbf{X}^n \overset{D}{\Longrightarrow} \mathbf{X}$) if, for all bounded continuous functions on R^I,

$$\lim_{n\to\infty} E^n\big[f\big(\mathbf{X}^n\big)\big] = E\big[f(\mathbf{X})\big],$$

where E^n and E denote the expectation with respect to P^n and P respectively.

Elton [5] discussed lower semicontinuity and continuity properties of optimal stopping value of a one-parameter discrete time stochastic process mainly in the case where the index set is finite (see Chow, Robbins and Siegmund [3] for the theory of a one-parameter optimal stopping problem).

We shall give the extension of Elton's results in the case of one-parameter stochastic processes to in the case of multiparameter stochastic processes in the first place, and show the existence of the best constant of a prophet inequality for a multiparameter optimal stopping problem, to which lower semicontinuity property is applied to the second.

This paper is organized as follows. In Section 2 we discuss the partition of a state space corresponding to a tactic. In Section 3 we prove the lower semicontinuity of optimal stopping value with respect to the multiparameter stochastic processes stated in this section. In Section 4 we give two examples which show the importance of the assumptions in the theorem of the previous section. In Section 5 we discuss a prophet inequality for a multiparameter optimal stopping problem, to which lower semicontinuity property is applied.

2. A partition corresponding to a tactic

Let $\Pi_{w,z} : R^{I(w)} \to R^{I(z)}$ be a projection for $z \leqslant w$ and we set $\overrightarrow{X(z)} := \{X(w):\ w \leqslant z\}$ for a stochastic process $\{X(z),\ z \in I\}$ and $z \in I$. We consider $(\Omega, \mathcal{F}, P)$, $\mathbf{X} = \{X(w),\ w \in I\}$ and $\{\mathcal{F}_z^{\mathbf{X}},\ z \in I\}$ introduced in Section 1.

Throughout this paper, for all $z \in I$, we say that a certain property discussed on $R^{I(z)}$, Q, holds with $P^{\overrightarrow{X(z)}}$ if $P(\overrightarrow{X(z)}$ satisfies the property $Q) = 1$, and denote the complement of a set S by S^C.

Lemma 2.1. *Let $(\{\Gamma(k),\ 0 \leqslant k \leqslant |t|\}, \tau)$ be a tactic with respect to $\{\mathcal{F}_z^{\mathbf{X}}\}$. Then there exist families of Borel sets $\{B_o^{e_i},\ i = 1,2,\dots,d\}$, $\{B_{e_i}^z,\ z \in d(e_i)\}$ $(i = 1,2,\dots,d)$, $\dots$, $\{B_z^w,\ w \in d(z)\}$ $(z \in I - \{t\})$, and $\{A_o^0\}, \{A_{e_i}^1,\ i = 1,2,\dots,d\}, \dots, \{A_z^k,\ |z| = k\}$ $(1 \leqslant k \leqslant |t| - 1)$, $\{A_t^{|t|}\}$ satisfying the following conditions*:

(1) *$B_o^{e_i} \in \mathcal{B}(R)$ and $\{B_o^{e_i},\ i = 1,2,\dots,d\}$ is a partition of R with $P^{X(o)}$.*

(2) (2.1) *For i and $z \in d(e_i)$, $B_{e_i}^z \in \mathcal{B}(R^{I(e_i)})$, and $\{B_{e_i}^z,\ z \in d(e_i)\}$ is a mutually distinct family of subsets of $R^{I(e_i)}$ and $\bigcup_{z\in d(e_i)} B_{e_i}^z = \Pi_{e_i,o}^{-1}(B_o^{e_i})$ with $P^{\overrightarrow{X(e_i)}}$.*

(2.2) *For $i \neq k$, $z \in d(e_i)$ and $w \in d(e_k)$, $\Pi_{t,e_i}^{-1}(B_{e_i}^z) \cap \Pi_{t,e_k}^{-1}(B_{e_k}^w) = \emptyset$ with $P^{\overrightarrow{X(t)}}$.*

(2.3) *$\bigcup_{i=1}^d \bigcup_{z\in d(e_i)} \Pi_{t,e_i}^{-1}(B_{e_i}^z) = R^I$ with $P^{\overrightarrow{X(t)}}$.*

(3) (3.1) *For z and $w \in d(z)$, $B_z^w \in \mathcal{B}(R^{I(z)})$, and $\{B_z^w,\ w \in d(z)\}$ is a mutually distinct family of subsets of $R^{I(z)}$ and $\bigcup_{w\in d(z)} B_z^w = \bigcup_{s\in p(z)} \Pi_{z,s}^{-1}(B_s^z)$ with $P^{\overrightarrow{X(z)}}$.*

(3.2) *For $z \neq w$ $(|z| = |w|)$, $r \in d(z)$ and $s \in d(w)$, $\Pi_{t,z}^{-1}(B_z^r) \cap \Pi_{t,w}^{-1}(B_w^s) = \emptyset$ with $P^{\overrightarrow{X(t)}}$.*

(3.3) *For k, $\bigcup_{z\in I,\ |z|=k} \bigcup_{s\in p(z)} \Pi_{t,s}^{-1}(B_s^z) = R^I$ with $P^{\overrightarrow{X(t)}}$.*

(4) $A_o^0 \in \mathcal{B}(R)$.

(5) (5.1) *For* i, $A_{e_i}^1 \in \mathcal{B}(R^{I(e_i)})$ *and* $A_{e_i}^1 \subseteq \bigcup_{z\in d(e_i)} B_{e_i}^z$ *with* $P^{\overrightarrow{X(e_i)}}$.

(5.2) *For* $i \neq k$, $\Pi_{t,e_i}^{-1}(A_{e_i}^1) \cap \Pi_{t,e_k}^{-1}(A_{e_k}^1) = \emptyset$ *with* $P^{\overrightarrow{X(t)}}$.

(5.3) $\bigcup_{i=1}^d \Pi_{t,e_i}^{-1}(A_{e_i}^1) \cap \Pi_{t,o}^{-1}(A_o^0) = \emptyset$ *with* $P^{\overrightarrow{X(t)}}$.

(6) (6.1) *For* k *and* z $(|z| = k)$, $A_z^k \in \mathcal{B}(R^{I(z)})$ *and* $A_z^k \subseteq \bigcup_{r\in d(z)} B_z^r$ *with* $P^{\overrightarrow{X(z)}}$.

(6.2) *For* k, z *and* w $(|z| = |w| = k)$, $\Pi_{t,z}^{-1}(A_z^k) \cap \Pi_{t,w}^{-1}(A_w^k) = \emptyset$ *with* $P^{\overrightarrow{X(t)}}$.

(6.3) *For* k, $\bigcup_{\ell=0}^{k-1} \bigcup_{r\in I,\,|r|=\ell} \Pi_{t,r}^{-1}(A_r^\ell) \cap \bigcup_{z\in I,\,|z|=k} \Pi_{t,z}^{-1}(A_z^k) = \emptyset$ *and* $A_z^k \subseteq \bigcap_{\ell=0}^{k-1} \bigcap_{r\in I,\,|r|=\ell,\,r\leqslant z} \Pi_{z,r}^{-1}((A_r^\ell)^C)$ *with* $P^{\overrightarrow{X(z)}}$.

(7) (7.1) $A_t^{|t|} \in \mathcal{B}(R^I)$.

(7.2) $A_t^{|t|} \cap \bigcup_{\ell=0}^{|t|-1} \bigcup_{r\in I,\,|r|=\ell} \Pi_{t,r}^{-1}(A_r^\ell) = \emptyset$ *and* $A_t^{|t|} \subseteq \bigcap_{\ell=0}^{|t|-1} \bigcap_{r\in I,\,|r|=\ell} \Pi_{t,r}^{-1}((A_r^\ell)^C)$ *with* $P^{\overrightarrow{X(t)}}$.

(7.3) $A_t^{|t|} \cup \bigcup_{\ell=0}^{|t|-1} \bigcup_{r\in I,\,|r|=\ell} \Pi_{t,r}^{-1}(A_r^\ell) = R^I$ *with* $P^{\overrightarrow{X(t)}}$.

(8) (8.1) *For* i, $\{\Gamma(1) = e_i\} = \{X(o) \in B_o^{e_i}\}$ *P-a.e.*

(8.2) *For* k, z $(|z| = k)$ *and* $w \in d(z)$, $\{\Gamma(k+1) = w, \Gamma(k) = z\} = \{\overrightarrow{X(z)} \in B_z^w\}$ *P-a.e.*

(8.3) $\{\tau = 0\} = \{X(o) \in A_o^0\}$ *P-a.e.*

(8.4) *For* k *and* z $(|z| = k)$, $\{\tau = k, \Gamma(k) = z\} = \{\overrightarrow{X(z)} \in A_z^k\} = \{\overrightarrow{X(t)} \in \Pi_{t,z}^{-1}(A_z^k)\}$ *P-a.e.*

Conversely, we can construct a tactic by using the families of Borel sets satisfying the above conditions (1)–(7).

Proof. We construct partitions by using the definition of a tactic in Section 1. Since $\Gamma(0) = o$ P-a.e., we have $\{\Gamma(0) = o\} = \{X(o) \in R\}$ P-a.e. Since $\mathcal{F}_{\Gamma(0)}^{\mathbf{X}} = \mathcal{F}_o^{\mathbf{X}}$ and $\Gamma(1)$ is $\mathcal{F}_{\Gamma(0)}^{\mathbf{X}}$-measurable, for each i there exists a set $B_o^{e_i} \in \mathcal{B}(R)$ such that

$$\{\Gamma(1) = e_i\} = \{X(o) \in B_o^{e_i}\} \quad P\text{-a.e.}$$

Because of

$$\{\Gamma(1) = e_i\} \cap \{\Gamma(1) = e_k\} = \emptyset$$

for $i \neq k$, and

$$\bigcup_{i=1}^d \{\Gamma(1) = e_i\} = \Omega,$$

we may assume that

$$B_o^{e_i} \cap B_o^{e_k} \neq \emptyset \quad \text{and} \quad \bigcup_{i=1}^d B_o^{e_i} = R.$$

If it is necessary, we may set

$$B_o^{e_k} := B_o^{e_k} \cap \bigcap_{i=1}^{k-1} (B_o^{e_i})^C \quad \text{and} \quad B_o^{e_d} := \bigcap_{i=1}^{d-1} (B_o^{e_i})^C.$$

Therefore we have the assertions (1) and (8.1).

Because the assertions (2) are the special cases of (3), we prove only (3). Since $\Gamma(k+1)$ is $\mathcal{F}_{\Gamma(k)}^{\mathbf{X}}$-measurable, for each z $(|z| = k)$, $w \in d(z)$, there exists a set $B_z^w \in \mathcal{B}(R^{I(z)})$ such that

$$\{\Gamma(k+1) = w, \Gamma(k) = z\} = \{\overrightarrow{X(z)} \in B_z^w\} \quad P\text{-a.e.}$$

For $w, w' \in d(z)$ $(w \neq w')$,

$$\{\Gamma(k+1) = w, \Gamma(k) = z\} \cap \{\Gamma(k+1) = w', \Gamma(k) = z\} = \emptyset,$$

which follows

$$\{\overrightarrow{X(z)} \in B_z^w\} \cap \{\overrightarrow{X(z)} \in B_z^{w'}\} = \emptyset.$$

And also

$$\bigcup_{w\in d(z)} \{\Gamma(k+1) = w, \Gamma(k) = z\} = \{\Gamma(k) = z\} = \bigcup_{s\in p(z)} \{\Gamma(k) = z, \Gamma(k-1) = s\},$$

which follows

$$\bigcup_{w\in d(z)} \{\overrightarrow{X(z)} \in B_z^w\} = \bigcup_{s\in p(z)} \{\overrightarrow{X(s)} \in B_s^z\},$$

and then

$$\left\{\overrightarrow{X(z)} \in \bigcup_{w\in d(z)} B_z^w\right\} = \left\{\overrightarrow{X(z)} \in \bigcup_{s\in p(z)} \Pi_{z,s}^{-1}\bigl(B_s^z\bigr)\right\}.$$

Hence, if it is necessary, by redefining the sets as stated above, we have the assertions (3.1) and (8.2).

For $z \neq w$ $(|z| = |w|)$, $r \in d(z)$, $s \in d(w)$,

$$\bigl\{\Gamma(k+1) = r, \Gamma(k) = z\bigr\} \cap \bigl\{\Gamma(k+1) = s, \Gamma(k) = w\bigr\} = \emptyset,$$

which follows

$$\bigl\{\overrightarrow{X(t)} \in \Pi_{t,z}^{-1}\bigl(B_z^r\bigr)\bigr\} \cap \bigl\{\overrightarrow{X(t)} \in \Pi_{t,w}^{-1}\bigl(B_w^s\bigr)\bigr\} = \emptyset,$$

hence, we have the assertion (3.2). And also

$$\Omega = \bigcup_{z\in I,\,|z|=k} \bigl\{\Gamma(k) = z\bigr\} = \bigcup_{z\in I,\,|z|=k}\bigcup_{r\in d(z)} \bigl\{\Gamma(k+1) = r, \Gamma(k) = z\bigr\} = \bigcup_{z\in I,\,|z|=k}\bigcup_{s\in p(z)} \bigl\{\Gamma(k) = z, \Gamma(k-1) = s\bigr\},$$

which follows

$$\bigl\{\overrightarrow{X(t)} \in R^I\bigr\} = \left\{\overrightarrow{X(t)} \in \bigcup_{z\in I,\,|z|=k}\bigcup_{s\in p(z)} \Pi_{t,s}^{-1}\bigl(B_s^z\bigr)\right\},$$

then we have the assertion (3.3).

Since $\{\tau = 0\} \in \mathcal{F}^{\mathbf{X}}_{\Gamma(0)}$, there exists a set $A_o^0 \in \mathcal{B}(R)$ such that $\{\tau = 0\} = \{X(o) \in A_o^0\}$ P-a.e., which follows (4) and (8.3).

Because the assertions (5) are the special cases of (6), we prove only (6). Since $\{\tau = k\} \in \mathcal{F}^{\mathbf{X}}_{\Gamma(k)}$, we have $\{\tau = k, \Gamma(k) = z\} \in \mathcal{F}^{\mathbf{X}}_z$, and there exists a set $A_z^k \in \mathcal{B}(R^{I(z)})$ such that

$$\bigl\{\tau = k, \Gamma(k) = z\bigr\} = \bigl\{\overrightarrow{X(z)} \in A_z^k\bigr\} = \bigl\{\overrightarrow{X(t)} \in \Pi_{t,z}^{-1}\bigl(A_z^k\bigr)\bigr\} \quad P\text{-a.e.}$$

Since $\{\tau = k, \Gamma(k) = z\} \subseteq \{\tau = k\}$, we have

$$\bigl\{\overrightarrow{X(z)} \in A_z^k\bigr\} \subseteq \left\{\overrightarrow{X(z)} \in \bigcup_{r\in d(z)} B_z^r\right\},$$

which follows (6.1). Since, for z, w $(|z| = |w| = k)$,

$$\bigl\{\tau = k, \Gamma(k) = z\bigr\} \cap \bigl\{\tau = k, \Gamma(k) = w\bigr\} = \emptyset,$$

we have

$$\bigl\{\overrightarrow{X(t)} \in \Pi_{t,z}^{-1}\bigl(A_z^k\bigr)\bigr\} \cap \bigl\{\overrightarrow{X(t)} \in \Pi_{t,w}^{-1}\bigl(A_w^k\bigr)\bigr\} = \emptyset,$$

which follows (6.2). Since $\{\tau = k\} \cap \bigcup_{\ell=0}^{k-1}\{\tau = \ell\} = \emptyset$, we have

$$\left\{\overrightarrow{X(t)} \in \bigcup_{z\in I,\,|z|=k} \Pi_{t,z}^{-1}\bigl(A_z^k\bigr)\right\} \cap \left\{\overrightarrow{X(t)} \in \bigcup_{\ell=0}^{k-1}\bigcup_{r\in I,\,|r|=\ell} \Pi_{t,r}^{-1}\bigl(A_r^\ell\bigr)\right\} = \emptyset,$$

which follows (6.3).

Since $\mathcal{F}^{\mathbf{X}}_{\Gamma(|t|)} = \mathcal{F}^{\mathbf{X}}_t$ and $\{\tau = |t|\} \in \mathcal{F}^{\mathbf{X}}_{\Gamma(|t|)}$, there exists a set $A_t^{|t|} \in \mathcal{B}(R^I)$ such that

$$\bigl\{\tau = |t|\bigr\} = \bigl\{\overrightarrow{X(t)} \in A_t^{|t|}\bigr\} \quad P\text{-a.e.}$$

Since $\{\tau = |t|\} \cap \bigcup_{\ell=0}^{|t|-1}\{\tau = \ell\} = \emptyset$ and $\bigcup_{\ell=0}^{|t|}\{\tau = \ell\} = \Omega$, we have

$$\bigl\{\overrightarrow{X(t)} \in A_t^{|t|}\bigr\} \cap \left\{\overrightarrow{X(t)} \in \bigcup_{\ell=0}^{|t|-1}\bigcup_{r\in I,\,|r|=\ell} \Pi_{t,r}^{-1}\bigl(A_r^\ell\bigr)\right\} = \emptyset$$

and

$$\left\{\overrightarrow{X(t)} \in A_t^{|t|}\right\} \cup \left\{\overrightarrow{X(t)} \in \bigcup_{\ell=0}^{|t|-1} \bigcup_{r \in I,\, |r|=\ell} \Pi_{t,r}^{-1}\left(A_r^{\ell}\right)\right\} = \left\{\overrightarrow{X(t)} \in R^I\right\} \quad P\text{-a.e.},$$

which follows (7.2) and (7.3).

Conversely, when the families of Borel sets satisfying the above conditions (1)–(7) are given, we can construct a tactic $(\{\Phi(k),\ 0 \leqslant k \leqslant |t|\}, \mu)$ adapted to the stochastic process $\mathbf{X}$ as follows:

$$\Phi(0) = o,$$
$$\Phi(1) = e_i \quad \text{on } \left\{X(o) \in B_o^{e_i}\right\},$$
$$\Phi(2) = e_i + e_j \quad \text{on } \bigcup_{s \in p(e_i+e_j)} \left\{\overrightarrow{X(s)} \in B_s^{e_i+e_j}\right\},$$
$$\Phi(k) = z \quad \text{on } \bigcup_{s \in p(z)} \left\{\overrightarrow{X(s)} \in B_s^{z}\right\} \ \left(|z| = k\right),$$
$$\Phi\left(|t|\right) = t,$$
$$\mu = 0 \quad \text{on } \left\{X(o) \in A_o^0\right\},$$
$$\mu = 1 \quad \text{on } \bigcup_{|z|=1} \left\{\overrightarrow{X(z)} \in A_z^1\right\},$$
$$\mu = k \quad \text{on } \bigcup_{|z|=k} \left\{\overrightarrow{X(z)} \in A_z^k\right\},$$
$$\mu = |t| \quad \text{on } \left\{\overrightarrow{X(t)} \in A_t^{|t|}\right\}.$$

The proof is completed. □

3. Lower semicontinuity

Theorem 3.1. *Let* $\mathbf{C}$ *be a family of integrable d-parameter stochastic processes whose negative parts are uniformly integrable. Then* V *is lower semicontinuous with respect to the topology of convergence in distribution. That is, if* $\mathbf{X}^n, \mathbf{X} \in \mathbf{C}$, $\mathbf{X}^n \overset{D}{\Longrightarrow} \mathbf{X}$, *then*

$$\liminf_{n\to\infty} V\left(\mathbf{X}^n\right) \geqslant V(\mathbf{X}).$$

Proof. Let $\mathbf{X}^n = \{X^n(z),\ z \in I\}$ and $\mathbf{X} = \{X(z),\ z \in I\} \in \mathbf{C}$ be defined on $(\Omega^n, \mathcal{F}^n, P^n, \{\mathcal{F}_z^{\mathbf{X}^n},\ z \in I\})$ and $(\Omega, \mathcal{F}, P, \{\mathcal{F}_z^{\mathbf{X}},\ z \in I\})$ respectively. From $\mathbf{X} \in \mathbf{C}$, there exists a tactic with respect to $\{\mathcal{F}_z^{\mathbf{X}^n},\ z \in I\}$, $(\{\Gamma(k)\}, \tau)$ such that $V[\mathbf{X}] = E[X(\Gamma(\tau))]$.

By Lemma 2.1, there exist families of Borel sets $\{B_o^{e_i},\ i = 1, 2, \ldots, d\}$, $\{B_{e_i}^{z},\ z \in d(e_i)\}$ $(i = 1, 2, \ldots, d)$, $\ldots$, $\{B_z^{w},\ w \in d(z)\}$ $(z \in I - \{t\})$, and $\{A_o^0\}, \{A_{e_i}^1,\ i = 1, 2, \ldots, d\}, \ldots, \{A_z^k,\ |z| = k\}$ $(1 \leqslant k \leqslant |t| - 1), \{A_t^{|t|}\}$ satisfying the conditions (1)–(8) in Lemma 2.1.

For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\max_{z \in I} \int_S |X(z)|\,dP < \frac{\varepsilon}{\prod_{i=1}^d (1+t_i)}$ whenever $P(S) < \delta$, because I is finite.

By the approximation theorem, for all $z \in I - \{t\}$ and $w \in d(z)$, there exists $C_z^w \in B(R^{I(z)})$ such that

$$P\left(\overrightarrow{X(z)} \in B_z^w \triangle C_z^w\right) < \alpha\delta \quad \text{and} \quad P\left(\overrightarrow{X(z)} \in \partial C_z^w\right) = 0,$$

and for all k and $z \in I$ $(|z| = k)$, there exists $D_z^k \in B(R^{I(z)})$ such that

$$P\left(\overrightarrow{X(z)} \in A_z^k \triangle D_z^k\right) < \beta\delta \quad \text{and} \quad P\left(\overrightarrow{X(z)} \in \partial D_z^k\right) = 0,$$

where $\bar{d} = \max\{d, \prod_{i=1}^d (1+t_i)\}$, $\alpha = \frac{1}{2|t|d^{|t|}(3\bar{d})^{|t|}\prod_{i=1}^d (1+t_i)}$ and $\beta = \frac{1}{(3\bar{d})^{|t|}\prod_{i=1}^d (1+t_i)}$.

We define $\{E_z^w,\ z \in I - \{t\},\ w \in d(z)\}$ as follows:

$$E_o^{e_1} = C_o^{e_1},$$
$$E_o^{e_2} = C_o^{e_2} \cap \left(C_o^{e_1}\right)^C,$$
$$E_o^{e_i} = C_o^{e_i} \cap \bigcap_{\ell=1}^{i-1} \left(C_o^{e_\ell}\right)^C,$$
$$E_o^{e_d} = \bigcap_{\ell=1}^{d-1} \left(C_o^{e_\ell}\right)^C,$$

and for k and $i^k_j \in \alpha(e_k)$ $(j = 1, 2, \dots, m)$,

$$E_{e_k}^{e_k+e_{i^k_1}} = \Pi^{-1}_{e_k,o}\big(E^{e_k}_o\big) \cap C_{e_k}^{e_k+e_{i^k_1}},$$
$$E_{e_k}^{e_k+e_{i^k_2}} = \Pi^{-1}_{e_k,o}\big(E^{e_k}_o\big) \cap C_{e_k}^{e_k+e_{i^k_2}} \cap \big(C_{e_k}^{e_k+e_{i^k_1}}\big)^C,$$
$$E_{e_k}^{e_k+e_{i^k_j}} = \Pi^{-1}_{e_k,o}\big(E^{e_k}_o\big) \cap C_{e_k}^{e_k+e_{i^k_j}} \cap \bigcap_{\ell=1}^{j-1}\big(C_{e_k}^{e_k+e_{i^k_\ell}}\big)^C,$$
$$E_{e_k}^{e_k+e_{i^k_m}} = \Pi^{-1}_{e_k,o}\big(E^{e_k}_o\big) \cap \bigcap_{\ell=1}^{m-1}\big(C_{e_k}^{e_k+e_{i^k_\ell}}\big)^C,$$

and for z and $i^z_j \in \alpha(z)$ $(j = 1, 2, \dots, n)$,

$$E_z^{z+e_{i^z_1}} = \bigcup_{s\in p(z)} \Pi^{-1}_{z,s}\big(E^z_s\big) \cap C_z^{z+e_{i^z_1}},$$
$$E_z^{z+e_{i^z_2}} = \bigcup_{s\in p(z)} \Pi^{-1}_{z,s}\big(E^z_s\big) \cap C_z^{z+e_{i^z_2}} \cap \big(C_z^{z+e_{i^z_1}}\big)^C,$$
$$E_z^{z+e_{i^z_j}} = \bigcup_{s\in p(z)} \Pi^{-1}_{z,s}\big(E^z_s\big) \cap C_z^{z+e_{i^z_j}} \cap \bigcap_{\ell=1}^{j-1}\big(C_z^{z+e_{i^z_\ell}}\big)^C,$$
$$E_z^{z+e_{i^z_n}} = \bigcup_{s\in p(z)} \Pi^{-1}_{z,s}\big(E^z_s\big) \cap \bigcap_{\ell=1}^{n-1}\big(C_z^{z+e_{i^z_\ell}}\big)^C,$$

and we set $E_t = R^I$.

The family $\{E^{e_i}_o, i\}$ is a partition of R, and we have

$$\begin{aligned}
P\big(X(o) \in B^{e_1}_o \triangle E^{e_1}_0\big) &= P\big(X(o) \in B^{e_1}_o \triangle C^{e_1}_0\big) < \alpha\delta < d\alpha\delta < \delta,\\
P\big(X(o) \in B^{e_2}_o \triangle E^{e_2}_0\big) &= P\big(X(o) \in \big(B^{e_2}_o \cap \big(B^{e_1}_o\big)^C\big) \triangle \big(C^{e_2}_0 \cap \big(C^{e_1}_o\big)^C\big)\big)\\
&\leqslant P\big(X(o) \in \big(B^{e_2}_o \triangle C^{e_2}_0\big) \cup \big(\big(B^{e_1}_o\big)^C \triangle \big(C^{e_1}_o\big)^C\big)\big)\\
&\leqslant P\big(X(o) \in B^{e_2}_o \triangle C^{e_2}_0\big) + P\big(X(o) \in B^{e_1}_o \triangle C^{e_1}_o\big)\\
&< 2\alpha\delta < d\alpha\delta < \delta,\\
P\big(X(o) \in B^{e_i}_o \triangle E^{e_i}_0\big) &= P\left(X(o) \in \left(B^{e_i}_o \cap \bigcap_{\ell=1}^{i-1}\big(B^{e_\ell}_o\big)^C\right) \triangle \left(C^{e_i}_o \cap \bigcap_{\ell=1}^{i-1}\big(C^{e_\ell}_o\big)^C\right)\right)\\
&\leqslant P\left(X(o) \in \big(B^{e_i}_o \triangle C^{e_i}_0\big) \cup \bigcup_{\ell=1}^{i-1}\big(\big(B^{e_\ell}_o\big)^C \triangle \big(C^{e_\ell}_o\big)^C\big)\right)\\
&\leqslant P\big(X(o) \in B^{e_i}_o \triangle C^{e_i}_0\big) + \sum_{\ell=1}^{i-1} P\big(X(o) \in B^{e_\ell}_o \triangle C^{e_\ell}_o\big)\\
&< i\alpha\delta < d\alpha\delta < \delta,\\
P\big(X(o) \in B^{e_d}_o \triangle E^{e_d}_0\big) &= P\left(X(o) \in \left(\bigcap_{\ell=1}^{d-1}\big(B^{e_\ell}_o\big)^C\right) \triangle \left(\bigcap_{\ell=1}^{d-1}\big(C^{e_\ell}_o\big)^C\right)\right)\\
&\leqslant P\left(X(o) \in \bigcup_{\ell=1}^{d-1}\big(\big(B^{e_\ell}_o\big)^C \triangle \big(C^{e_\ell}_o\big)^C\big)\right)\\
&\leqslant \sum_{\ell=1}^{d-1} P\big(X(o) \in B^{e_\ell}_o \triangle C^{e_\ell}_o\big)\\
&< (d-1)\alpha\delta < d\alpha\delta < \delta.
\end{aligned}$$

For k, the family $\{E_{e_k}^{e_k+e_{i^k_j}}, j\}$ is a partition of $\Pi^{-1}_{e_k,o}(E^{e_k}_o)$, and satisfies the conditions (2.1) and (2.3) in Lemma 2.1(2). We have

$$\begin{aligned}
P\big(\overrightarrow{X(e_k)} \in B_{e_k}^{e_k+e_{i_j^k}} \triangle E_{e_k}^{e_k+e_{i_j^k}}\big) &= P\left(\overrightarrow{X(e_k)} \in \left(\Pi_{e_k,o}^{-1}\big(B_o^{e_k}\big) \cap B_{e_k}^{e_k+e_{i_j^k}} \cap \bigcap_{\ell=1}^{j-1}\big(B_{e_k}^{e_k+e_{i_\ell^k}}\big)^C\right)\right.\\
&\qquad \left.\triangle \left(\Pi_{e_k,o}^{-1}\big(E_o^{e_k}\big) \cap C_{e_k}^{e_k+e_{i_j^k}} \cap \bigcap_{\ell=1}^{j-1}\big(C_{e_k}^{e_k+e_{i_\ell^k}}\big)^C\right)\right)\\
&\leqslant P\big(\overrightarrow{X(e_k)} \in \Pi_{e_k,o}^{-1}\big(B_o^{e_k}\big) \triangle \Pi_{e_k,o}^{-1}\big(E_o^{e_k}\big)\big) + P\big(\overrightarrow{X(e_k)} \in B_{e_k}^{e_k+e_{i_j^k}} \triangle C_{e_k}^{e_k+e_{i_j^k}}\big)\\
&\quad + \sum_{\ell=1}^{j-1} P\big(\overrightarrow{X(e_k)} \in \big(B_{e_k}^{e_k+e_{i_\ell^k}}\big)^C \triangle \big(C_{e_k}^{e_k+e_{i_\ell^k}}\big)^C\big)\\
&\leqslant P\big(X(o) \in B_o^{e_k} \triangle E_o^{e_k}\big) + P\big(\overrightarrow{X(e_k)} \in B_{e_k}^{e_k+e_{i_j^k}} \triangle C_{e_k}^{e_k+e_{i_j^k}}\big) + \sum_{\ell=1}^{j-1} P\big(\overrightarrow{X(e_k)} \in B_{e_k}^{e_k+e_{i_\ell^k}} \triangle C_{e_k}^{e_k+e_{i_\ell^k}}\big)\\
&< k\alpha\delta + \alpha\delta + (j-1)\alpha\delta < 2d\alpha\delta < \delta,
\end{aligned}$$

$$\begin{aligned}
P\big(\overrightarrow{X(e_k)} \in B_{e_k}^{e_k+e_{i_m^k}} \triangle E_{e_k}^{e_k+e_{i_m^k}}\big) &= P\left(\overrightarrow{X(e_k)} \in \left(\Pi_{e_k,o}^{-1}\big(B_o^{e_k}\big) \cap \bigcap_{\ell=1}^{m-1}\big(B_{e_k}^{e_k+e_{i_\ell^k}}\big)^C\right) \triangle \left(\Pi_{e_k,o}^{-1}\big(E_o^{e_k}\big) \cap \bigcap_{\ell=1}^{m-1}\big(C_{e_k}^{e_k+e_{i_\ell^k}}\big)^C\right)\right)\\
&\leqslant P\big(\overrightarrow{X(e_k)} \in \Pi_{e_k,o}^{-1}\big(B_o^{e_k}\big) \triangle \Pi_{e_k,o}^{-1}\big(E_o^{e_k}\big)\big) + \sum_{\ell=1}^{m-1} P\big(\overrightarrow{X(e_k)} \in \big(B_{e_k}^{e_k+e_{i_\ell^k}}\big)^C \triangle \big(C_{e_k}^{e_k+e_{i_\ell^k}}\big)^C\big)\\
&\leqslant P\big(X(o) \in B_o^{e_k} \triangle E_o^{e_k}\big) + \sum_{\ell=1}^{m-1} P\big(\overrightarrow{X(e_k)} \in B_{e_k}^{e_k+e_{i_\ell^k}} \triangle C_{e_k}^{e_k+e_{i_\ell^k}}\big)\\
&< k\alpha\delta + (m-1)\alpha\delta < 2d\alpha\delta < \delta.
\end{aligned}$$

For z, the family $\{E_z^{z+e_{i_j^z}}, j\}$ is a partition of $\bigcup_{s\in p(z)} \Pi_{z,s}^{-1}(E_s^z)$, and satisfies the conditions (3.1) and (3.3) in Lemma 2.1(3). We have

$$\begin{aligned}
P\big(\overrightarrow{X(z)} \in B_z^{z+e_{i_j^z}} \triangle E_z^{z+e_{i_j^z}}\big) &= P\left(\overrightarrow{X(z)} \in \left(\bigcup_{s\in p(z)} \Pi_{z,s}^{-1}\big(B_s^z\big) \cap B_z^{z+e_{i_j^z}} \cap \bigcap_{\ell=1}^{j-1}\big(B_z^{z+e_{i_\ell^z}}\big)^C\right)\right.\\
&\qquad \left.\triangle \left(\bigcup_{s\in p(z)} \Pi_{z,s}^{-1}\big(E_s^z\big) \cap C_z^{z+e_{i_j^z}} \cap \bigcap_{\ell=1}^{j-1}\big(C_z^{z+e_{i_\ell^z}}\big)^C\right)\right)\\
&\leqslant P\left(\overrightarrow{X(z)} \in \bigcup_{s\in p(z)} \Pi_{z,s}^{-1}\big(B_s^z\big) \triangle \bigcup_{s\in p(z)} \Pi_{z,s}^{-1}\big(E_s^z\big)\right) + P\big(\overrightarrow{X(z)} \in B_z^{z+e_{i_j^z}} \triangle C_z^{z+e_{i_j^z}}\big)\\
&\quad + \sum_{\ell=1}^{j-1} P\big(\overrightarrow{X(z)} \in \big(B_z^{z+e_{i_\ell^z}}\big)^C \triangle \big(C_z^{z+e_{i_\ell^z}}\big)^C\big)\\
&\leqslant \sum_{s\in p(z)} P\big(\overrightarrow{X(z)} \in \Pi_{z,s}^{-1}\big(B_s^z\big) \triangle \Pi_{z,s}^{-1}\big(E_s^z\big)\big) + P\big(\overrightarrow{X(z)} \in B_z^{z+e_{i_j^z}} \triangle C_z^{z+e_{i_j^z}}\big)\\
&\quad + \sum_{\ell=1}^{j-1} P\big(\overrightarrow{X(z)} \in B_z^{z+e_{i_\ell^z}} \triangle C_z^{z+e_{i_\ell^z}}\big)\\
&< d\big(2d^{|z|-1} + d^{|z|-2} + \cdots + d\big)\alpha\delta + \alpha\delta + (d-1)\alpha\delta < \delta,
\end{aligned}$$

$$\begin{aligned}
P\big(\overrightarrow{X(z)} \in B_z^{z+e_{i_n^z}} \triangle E_z^{z+e_{i_n^z}}\big) &= P\left(\overrightarrow{X(z)} \in \left(\bigcup_{s\in p(z)} \Pi_{z,s}^{-1}\big(B_s^z\big) \cap \bigcap_{\ell=1}^{n-1}\big(B_z^{z+e_{i_\ell^z}}\big)^C\right) \triangle \left(\bigcup_{s\in p(z)} \Pi_{z,s}^{-1}\big(E_s^z\big) \cap \bigcap_{\ell=1}^{n-1}\big(C_z^{z+e_{i_\ell^z}}\big)^C\right)\right)\\
&\leqslant P\left(\overrightarrow{X(z)} \in \bigcup_{s\in p(z)} \Pi_{z,s}^{-1}\big(B_s^z\big) \triangle \bigcup_{s\in p(z)} \Pi_{z,s}^{-1}\big(E_s^z\big)\right) + \sum_{\ell=1}^{n-1} P\big(\overrightarrow{X(z)} \in \big(B_z^{z+e_{i_\ell^z}}\big)^C \triangle \big(C_z^{z+e_{i_\ell^z}}\big)^C\big)\\
&\leqslant \sum_{s\in p(z)} P\big(\overrightarrow{X(z)} \in \Pi_{z,s}^{-1}\big(B_s^z\big) \triangle \Pi_{z,s}^{-1}\big(E_s^z\big)\big) + \sum_{\ell=1}^{n-1} P\big(\overrightarrow{X(z)} \in B_z^{z+e_{i_\ell^z}} \triangle C_z^{z+e_{i_\ell^z}}\big)\\
&< d\big(2d^{|z|-1} + d^{|z|-2} + \cdots + d\big)\alpha\delta + (d-1)\alpha\delta < \delta.
\end{aligned}$$

Furthermore we have $\Pi_{t,z}^{-1}(\bigcup_{r\in d(z)} E_z^r) = \Pi_{t,w}^{-1}(\bigcup_{r\in d(w)} E_w^r)$ for all z, w $(z \nleqslant w)$.

Next we define $\{F_z^k,\ k, z \in I,\ |z| = k\}$ as follows:

$$F_o^0 = D_o^0,$$
$$F_z^1 = \bigcup_{r\in d(z)} E_z^r \cap D_z^1 \cap \Pi_{z,o}^{-1}\big(\big(D_o^0\big)^C\big),$$
$$F_z^k = \bigcup_{r\in d(z)} E_z^r \cap D_z^k \cap \bigcap_{s\leqslant z,\, s\neq z} \Pi_{z,s}^{-1}\big(\big(F_s^{|s|}\big)^C\big),$$
$$F_t^{|t|} = \bigcap_{z\neq t} \Pi_{t,z}^{-1}\big(\big(F_z^{|z|}\big)^C\big).$$

By the construction of F_z^k, this family satisfies the conditions (4), (5.1), (5.2), (5.3), (6.1), (6.2), (7.1), (7.2) and (7.3) in Lemma 2.1, and we have for z $(|z| = k)$,

$$\Pi_{t,z}^{-1}\big(F_z^k\big) \subseteq \bigcap_{s\leqslant z,\, s\neq z} \Pi_{t,s}^{-1}\big(\big(F_s^{|s|}\big)^C\big) = \bigcap_{s\leqslant z,\, s\neq z} \Pi_{t,s}^{-1}\big(\big(F_s^{|s|}\big)^C\big) \cap \bigcap_{|s|\leqslant k-1,\, s\nleqslant z} \Pi_{t,s}^{-1}\big(\big(F_s^{|s|}\big)^C\big) = \bigcap_{\ell=0}^{k-1}\bigcap_{|r|=\ell} \Pi_{t,r}^{-1}\big(\big(F_r^{\ell}\big)^C\big),$$

which follows the condition (6.3) in Lemma 2.1. We have

$$P\big(X(o) \in A_o^0 \vartriangle F_o^0\big) = P\big(X(o) \in A_o^0 \vartriangle D_o^0\big) < \beta\delta < \delta,$$

$$\begin{aligned}
P\big(\overrightarrow{X(e_i)} \in A_{e_i}^1 \vartriangle F_{e_i}^1\big) &= P\bigg(\overrightarrow{X(e_i)} \in \bigg(\bigcup_{r\in d(e_i)} B_{e_i}^r \cap A_{e_i}^1 \cap \Pi_{e_i,o}^{-1}\big(\big(A_o^0\big)^C\big)\bigg) \vartriangle \bigg(\bigcup_{r\in d(e_i)} E_{e_i}^r \cap D_{e_i}^1 \cap \Pi_{e_i,o}^{-1}\big(\big(D_o^0\big)^C\big)\bigg)\bigg)\\
&\leqslant P\bigg(\overrightarrow{X(e_i)} \in \bigg(\bigcup_{r\in d(e_i)} B_{e_i}^r\bigg) \vartriangle \bigg(\bigcup_{r\in d(e_i)} E_{e_i}^r\bigg)\bigg) + P\big(\overrightarrow{X(e_i)} \in A_{e_i}^1 \vartriangle D_{e_i}^1\big)\\
&\quad + P\big(\overrightarrow{X(e_i)} \in \Pi_{e_i,o}^{-1}\big(\big(A_o^0\big)^C\big) \vartriangle \Pi_{e_i,o}^{-1}\big(\big(D_o^0\big)^C\big)\big)\\
&\leqslant \sum_{r\in d(e_i)} P\big(\overrightarrow{X(e_i)} \in \big(B_{e_i}^r \vartriangle E_{e_i}^r\big)\big) + P\big(\overrightarrow{X(e_i)} \in A_{e_i}^1 \cap D_{e_i}^1\big) + P\big(X(o) \in A_o^0 \vartriangle D_o^0\big)\\
&< d\beta\delta + \beta\delta + \beta\delta < \frac{1}{(3\bar{d})^{|t|-1}\prod_{i=1}^d (1+t_i)} < \delta,
\end{aligned}$$

$$\begin{aligned}
P\big(\overrightarrow{X(z)} \in A_z^k \vartriangle F_z^k\big) &= P\bigg(\overrightarrow{X(z)} \in \bigg(\bigcup_{r\in d(z)} B_z^r \cap A_z^k \cap \bigcap_{s\leqslant z,\, s\neq z} \Pi_{z,s}^{-1}\big(\big(A_s^{|s|}\big)^C\big)\bigg) \vartriangle \bigg(\bigcup_{r\in d(z)} E_z^r \cap D_z^k \cap \bigcap_{s\leqslant z,\, s\neq z} \Pi_{z,s}^{-1}\big(\big(F_s^{|s|}\big)^C\big)\bigg)\bigg)\\
&\leqslant \sum_{r\in d(z)} P\big(\overrightarrow{X(z)} \in \big(B_z^r \vartriangle E_z^r\big)\big) + P\big(\overrightarrow{X(z)} \in A_z^k \cap D_z^k\big) + \sum_{s\leqslant z,\, s\neq z} P\big(\overrightarrow{X(z)} \in \Pi_{z,s}^{-1}\big(A_s^{|s|}\big) \vartriangle \Pi_{z,s}^{-1}\big(F_s^{|s|}\big)\big)\\
&< d\beta\delta + \beta\delta + \bigg(\prod_{i=1}^d (1+z_i) - 1\bigg) \frac{1}{(3\bar{d})^{|t|-(k-1)}\prod_{i=1}^d (1+t_i)}\delta\\
&< \frac{1}{(3\bar{d})^{|t|-k}\prod_{i=1}^d (1+t_i)} < \delta,
\end{aligned}$$

$$\begin{aligned}
P\big(\overrightarrow{X(t)} \in A_t^{|t|} \vartriangle F_t^{|t|}\big) &= P\bigg(\overrightarrow{X(z)} \in \bigcap_{z\neq t} \Pi_{t,z}^{-1}\big(\big(A_z^{|z|}\big)^C\big) \vartriangle \bigcap_{z\neq t} \Pi_{t,z}^{-1}\big(\big(F_z^{|z|}\big)^C\big)\bigg)\\
&\leqslant \sum_{z\neq t} P\big(\overrightarrow{X(z)} \in \Pi_{t,z}^{-1}\big(\big(A_z^{|z|}\big)^C\big) \vartriangle \Pi_{t,z}^{-1}\big(\big(F_z^{|z|}\big)^C\big)\big)\\
&< \bigg(\prod_{i=1}^d (1+t_i)\bigg) \sum_{k=1}^{|t|} \frac{1}{(3\bar{d})^{|t|-(k-1)}}\delta < \prod_{i=1}^d (1+t_i) \frac{1}{(3\bar{d})^{|t|-k}\prod_{i=1}^d (1+t_i)}\delta < \delta.
\end{aligned}$$

By our definition, $E_z^{z+e_{i_j^z}} = \bigcup_{s\in p(z)} \Pi_{z,s}^{-1}(E_s^z) \cap C_z^{z+e_{i_j^z}} \cap \bigcap_{\ell=1}^{j-1} (C_z^{z+e_{i_\ell^z}})^C$ for z and $i_j^z \in \alpha(z)$ $(j = 1, 2, \ldots, n)$, we have

$$\partial E_z^{z+e_{i_j^z}} \subseteq \partial\bigg(\bigcup_{s\in p(z)} \Pi_{z,s}^{-1}\big(E_s^z\big)\bigg) \cup \partial C_z^{z+e_{i_j^z}} \cup \partial\bigg(\bigcap_{\ell=1}^{j-1} \big(C_z^{z+e_{i_\ell^z}}\big)^C\bigg)$$

$$\subseteq \bigcup_{s\in p(z)} \partial \Pi_{z,s}^{-1}\big(E_s^z\big) \cup \partial C_z^{z+e_{i_j^z}} \cup \bigcup_{\ell=1}^{j-1} \partial\big(C_z^{z+e_{i_\ell^z}}\big)^C$$

$$\subseteq \bigcup_{s\in p(z)} \Pi_{z,s}^{-1}\big(\partial E_s^z\big) \cup \partial C_z^{z+e_{i_j^z}} \cup \bigcup_{\ell=1}^{j-1} \partial C_z^{z+e_{i_\ell^z}},$$

therefore

$$P\big(\overrightarrow{X(z)} \in \partial E_z^{z+e_{i_j^z}}\big) \leqslant \sum_{s\in p(z)} P\big(\overrightarrow{X(z)} \in \Pi_{z,s}^{-1}\big(\partial E_s^z\big)\big) + P\big(\overrightarrow{X(z)} \in \partial C_z^{z+e_{i_j^z}}\big) + \sum_{\ell=1}^{j-1} P\big(\overrightarrow{X(z)} \in \partial C_z^{z+e_{i_\ell^z}}\big).$$

And also we have

$$P\big(X(o) \in \partial E_o^{e_i}\big) \leqslant P\big(X(o) \in \partial C_o^{e_i}\big) + \sum_{\ell} P\big(X(o) \in \partial C_o^{e_\ell}\big) = 0,$$

$$\begin{aligned} P\big(\overrightarrow{X(e_k)} \in \partial E_{e_k}^{e_k+e_i}\big) &\leqslant P\big(\overrightarrow{X(e_k)} \in \Pi_{e_k,o}^{-1}\big(\partial E_o^{e_k}\big)\big) + P\big(\overrightarrow{X(e_k)} \in \partial C_{e_k}^{e_k+e_i}\big) + \sum_{\ell} P\big(\overrightarrow{X(e_k)} \in \partial C_{e_k}^{e_k+e_\ell}\big) \\ &\leqslant P\big(X(o) \in \partial E_o^{e_k}\big) + P\big(\overrightarrow{X(e_k)} \in \partial C_{e_k}^{e_k+e_i}\big) + \sum_{\ell} P\big(\overrightarrow{X(e_k)} \in \partial C_{e_k}^{e_k+e_\ell}\big) \\ &= 0. \end{aligned}$$

Inductively we have $P(\overrightarrow{X(z)} \in \partial E_z^w) = 0$ for $w \in d(z)$.

By our definition $F_z^k = \bigcup_{r\in d(z)} E_z^r \cap D_z^k \cap \bigcap_{s\leqslant z,\, s\neq z} \Pi_{z,s}^{-1}((F_s^{|s|})^C)$, we have

$$P\big(\overrightarrow{X(z)} \in \partial F_z^k\big) \leqslant \sum_{r\in d(z)} P\big(\overrightarrow{X(z)} \in E_z^r\big) + P\big(\overrightarrow{X(z)} \in D_z^k\big) + \sum_{s\leqslant z,\, s\neq z} P\big(\overrightarrow{X(s)} \in F_s^{|s|}\big),$$

moreover

$$P\big(X(o) \in \partial F_o^0\big) = P\big(X(o) \in \partial D_o^0\big) = 0,$$

$$\begin{aligned} P\big(\overrightarrow{X(e_k)} \in \partial F_{e_k}^1\big) &\leqslant \sum_{r\in d(e_k)} P\big(\overrightarrow{X(e_k)} \in \partial E_{e_k}^r\big) + P\big(\overrightarrow{X(e_k)} \in \partial D_{e_k}^1\big) + P\big(\overrightarrow{X(e_k)} \in \partial F_o^0\big) \\ &\leqslant \sum_{r\in d(e_k)} P\big(\overrightarrow{X(e_k)} \in \partial E_{e_k}^r\big) + P\big(\overrightarrow{X(e_k)} \in \partial D_{e_k}^1\big) + P\big(\overrightarrow{X(e_k)} \in \partial D_o^0\big) \\ &= 0. \end{aligned}$$

Inductively we obtain $P(\overrightarrow{X(z)} \in \partial F_z^{|z|}) = 0$.

Now, using $\{E_z^w,\ z \in I - \{t\},\ w \in d(z)\}$, $\{F_z^k,\ k, z \in I,\ |z| = k\}$ and $\mathbf{X}^n$, we define $(\{\Phi^n(k),\ 0 \leqslant k \leqslant |t|\}, \mu^n)$ by

$$\Phi^n(0) = o,$$

$$\Phi^n(1) = e_i \quad \text{on } \big\{X^n(o) \in E_o^{e_i}\big\},$$

$$\Phi^n(2) = e_i + e_j \quad \text{on } \bigcup_{s\in p(e_i+e_j)} \big\{\overrightarrow{X^n(s)} \in E_s^{e_i+e_j}\big\},$$

$$\Phi^n(k) = z \quad \text{on } \bigcup_{s\in p(z)} \big\{\overrightarrow{X^n(s)} \in E_s^z\big\} \ \big(|z| = k\big),$$

$$\Phi^n\big(|t|\big) = t,$$

$$\mu^n = 0 \quad \text{on } \big\{X^n(o) \in F_o^0\big\},$$

$$\mu^n = 1 \quad \text{on } \bigcup_{|z|=1} \big\{\overrightarrow{X^n(z)} \in F_z^1\big\},$$

$$\mu^n = k \quad \text{on } \bigcup_{|z|=k} \big\{\overrightarrow{X^n(z)} \in F_z^k\big\},$$

$$\mu^n = |t| \quad \text{on } \big\{\overrightarrow{X^n(t)} \in F_t^{|t|}\big\}.$$

Then $(\{\Phi^n(k),\ 0 \leqslant k \leqslant |t|\}, \mu^n)$ is a tactic.

Now we estimate the optimal value.

$$\begin{aligned}
V[\mathbf{X}] = E\big[X\big(\Gamma(\tau)\big)\big] &= \sum_{k=0}^{|t|}\sum_{|z|=k} E\big[X(z)1_{\{\Gamma(k)=z,\tau=k\}}\big] = \sum_{k=0}^{|t|}\sum_{|z|=k} E\big[X(z)1_{\{\overrightarrow{X(z)}\in A_z^k\}}\big] \\
&= \sum_{k=0}^{|t|}\sum_{|z|=k}\bigg\{\int\limits_{\{\overrightarrow{X(z)}\in F_z^k\}} X(z)\,dP + \int\limits_{\{\overrightarrow{X(z)}\in A_z^k-F_z^k\}} X(z)\,dP - \int\limits_{\{\overrightarrow{X(z)}\in F_z^k-A_z^k\}} X(z)\,dP\bigg\} \\
&\leqslant \sum_{k=0}^{|t|}\sum_{|z|=k}\int\limits_{\{\overrightarrow{X(z)}\in F_z^k\}} X(z)\,dP + \prod_{i=1}^{d}(1+t_i)\frac{\varepsilon}{\prod_{i=1}^d(1+t_i)}.
\end{aligned}$$

By our assumption on $\mathbf{C}$, there exists $\lambda > 0$ such that

$$\sup_{z,n}\int\limits_{\{X^n(z)\leqslant-\lambda\}}\big|X^n(z)\big|\,dP^n < \frac{\varepsilon}{\prod_{i=1}^d(1+t_i)} \quad\text{and}\quad \max_z\int\limits_{\{X(z)\geqslant\lambda\}} X(z)\,dP < \frac{\varepsilon}{\prod_{i=1}^d(1+t_i)}.$$

Let f be the bounded continuous function defined by

$$f(t) = \begin{cases} t & (|t|\leqslant\lambda),\\ \lambda\,\mathrm{sgn}(t) & (|t|>\lambda).\end{cases}$$

For $\vec{x} = (x_w,\ w\in I)\in R^I$ and $C\in\mathcal{B}(R^I)$, we set $h_z(\vec{x}) = f(x_z)1_C(\vec{x})$. Then h_z is bounded measurable and the set of discontinuous points of h_z is ∂C.

Since $P(\overrightarrow{X(z)}\in\partial F_z^{|z|}) = 0$, we obtain, for $C = \Pi_{t,z}^{-1}(F_z^{|z|})$,

$$\int\limits_{R^I} h_z(\vec{x})\,P\big(\overrightarrow{X(t)}\in d\vec{x}\big) = \int\limits_{\Omega} f\big(X(z)\big)1_{\Pi_{t,z}^{-1}(F_z^{|z|})}\big(\overrightarrow{X(t)}\big)\,P(d\omega) = \int\limits_{\{\overrightarrow{X(z)}\in F_z^{|z|}\}} f\big(X(z)\big)\,P(d\omega).$$

Then, from $\mathbf{X}^n \overset{D}{\Longrightarrow} \mathbf{X}$, we obtain

$$\lim_{n\to\infty}\int\limits_{\{\overrightarrow{X^n(z)}\in F_z^{|z|}\}} f\big(X^n(z)\big)\,P^n(d\omega) = \int\limits_{\{\overrightarrow{X(z)}\in F_z^{|z|}\}} f\big(X(z)\big)\,P(d\omega).$$

Therefore,

$$\begin{aligned}
\liminf_{n\to\infty}\int\limits_{\{\overrightarrow{X^n(z)}\in F_z^{|z|}\}} X^n(z)\,P^n(d\omega) &= \liminf_{n\to\infty}\bigg\{\int\limits_{\{\overrightarrow{X^n(z)}\in F_z^{|z|}\}} f\big(X^n(z)\big)\,P^n(d\omega) + \int\limits_{\{\overrightarrow{X^n(z)}\in F_z^{|z|}\}\cap\{X^n(z)>\lambda\}}\big(X^n(z)-\lambda\big)\,P^n(d\omega) \\
&\qquad + \int\limits_{\{\overrightarrow{X^n(z)}\in F_z^{|z|}\}\cap\{X^n(z)<-\lambda\}}\big(X^n(z)+\lambda\big)\,P^n(d\omega)\bigg\} \\
&\geqslant \lim_{n\to\infty}\bigg\{\int\limits_{\{\overrightarrow{X^n(z)}\in F_z^{|z|}\}} f\big(X^n(z)\big)\,P^n(d\omega) - \frac{\varepsilon}{\prod_{i=1}^d(1+t_i)}\bigg\} \\
&= \int\limits_{\{\overrightarrow{X(z)}\in F_z^{|z|}\}} f\big(X(z)\big)\,P(d\omega) - \frac{\varepsilon}{\prod_{i=1}^d(1+t_i)} \\
&= \int\limits_{\{\overrightarrow{X(z)}\in F_z^{|z|}\}} X(z)\,P(d\omega) + \int\limits_{\{\overrightarrow{X(z)}\in F_z^{|z|}\}\cap\{X(z)>\lambda\}}\big(\lambda - X(z)\big)\,P(d\omega) \\
&\qquad + \int\limits_{\{\overrightarrow{X(z)}\in F_z^{|z|}\}\cap\{X(z)<-\lambda\}}\big(-\lambda - X(z)\big)\,P(d\omega) - \frac{\varepsilon}{\prod_{i=1}^d(1+t_i)} \\
&\geqslant \int\limits_{\{\overrightarrow{X(z)}\in F_z^{|z|}\}} X(z)\,P(d\omega) + \int\limits_{\{\overrightarrow{X(z)}\in F_z^{|z|}\}\cap\{X(z)>\lambda\}}\big(-X(z)\big)\,P(d\omega) - \frac{\varepsilon}{\prod_{i=1}^d(1+t_i)} \\
&= \int\limits_{\{\overrightarrow{X(z)}\in F_z^{|z|}\}} X(z)\,P(d\omega) - 2\frac{\varepsilon}{\prod_{i=1}^d(1+t_i)},
\end{aligned}$$

$$V[\mathbf{X}^n] \geqslant E^n[X^n(\Phi^n(\mu^n))] = \sum_{k=0}^{|t|}\sum_{|z|=k} E^n[X^n(z)1_{\{\Phi^n(k)=z,\mu^n=k\}}] = \sum_{k=0}^{|t|}\sum_{|z|=k} E^n[X^n(z)1_{\{\overrightarrow{X^n(z)}\in A_z^k\}}],$$

$$\begin{aligned}\liminf_{n\to\infty} V[\mathbf{X}^n] &\geqslant \sum_{k=0}^{|t|}\sum_{|z|=k} \liminf_{n\to\infty} E^n[X^n(z)1_{\{\overrightarrow{X^n(z)}\in A_z^k\}}] \geqslant \sum_{k=0}^{|t|}\sum_{|z|=k} E[X(z)1_{\{\overrightarrow{X(z)}\in A_z^k\}}] - 2\prod_{i=1}^{d}(1+t_i)\frac{\varepsilon}{\prod_{i=1}^d(1+t_i)}\\ &\geqslant V[\mathbf{X}] - 3\varepsilon.\end{aligned}$$

Since ε is arbitrary, we have $\liminf_{n\to\infty} V(\mathbf{X}^n) \geqslant V(\mathbf{X})$. The proof is completed. □

4. Examples

In this section we shall state two examples studied in Elton [5]. In the theory of multiparameter optimal stopping problems in the case where $d=2$, it is known that if the filtration $\{\mathcal{F}_z,\ z\in I\}$ satisfies the conditional independence property, that is, $\mathcal{F}_z$ and $\mathcal{F}_w$ are conditionally independent given $\mathcal{F}_{z\wedge w}$ for each $z, w \in I$, then all $\{\mathcal{F}_z\}$-stopping points are accessible. In general, this fact fails to hold in case of the higher dimension, that is, $d \geqslant 3$. However, considering the multiparameter optimal stopping problems in the class of all accessible stopping points as stated in Section 1, we can use the method of backwards induction and calculate the optimal values without the conditional independence property.

Example 4.1. Even when we are restricted to a family of uniformly bounded stochastic processes, V is not continuous.

Let $d=2$, $t=(1,1)$ and $\mathbf{X}^n = \{X^n(0,0), X^n(1,0), X^n(0,1), X^n(1,1)\}$ have range

$$\left\{\left(0,\frac{1}{2},0,0\right),\left(\frac{1}{2},0,0,0\right),\left(\frac{1}{2}+\frac{1}{n},1,0,0\right),\left(1,\frac{1}{2}+\frac{1}{n},0,0\right)\right\}$$

with probability $\frac{1}{4}$ and $\mathbf{X} = \{X(0,0), X(1,0), X(0,1), X(1,1)\}$ have range

$$\left\{\left(0,\frac{1}{2},0,0\right),\left(\frac{1}{2},0,0,0\right),\left(\frac{1}{2},1,0,0\right),\left(1,\frac{1}{2},0,0\right)\right\}$$

with probability $\frac{1}{4}$.

Then $\mathbf{C} = \{\mathbf{X}^n, \mathbf{X}\}$ is a family of uniformly bounded stochastic processes and $\mathbf{X}^n \overset{D}{\Longrightarrow} \mathbf{X}$, and we have $V(\mathbf{X}^n) = \frac{3}{4}$ and $V(\mathbf{X}) = \frac{3}{8}$. This shows that V is not continuous.

Example 4.2. When we are restricted to a family of i.i.d. stochastic processes, V is not lower semicontinuous without the assumption of the uniformly integrability.

Let $d=3$, $t=(1,1,1)$ and $\mathbf{X}^n = \{X^n(0,0,0), X^n(1,0,0), X^n(0,1,0), X^n(0,0,1), X^n(1,1,0), X^n(1,0,1), X^n(0,1,1), X^n(1,1,1)\}$ be an independent sequence defined by

$$X^n(z) = \begin{cases} 0 & \text{with probability } 1-\frac{1}{\sqrt{n}},\\ -n^2 & \text{with probability } \frac{1}{\sqrt{n}},\end{cases}$$

and $\mathbf{X} = \{X(0,0,0), X(1,0,0), X(0,1,0), X(0,0,1), X(1,1,0), X(1,0,1), X(0,1,1), X(1,1,1)\}$ be defined by $X(z)=0$ with probability 1.

Then $\mathbf{C} = \{\mathbf{X}^n, \mathbf{X}\}$ is a family of integrable stochastic processes, which fails to satisfy the uniformly integrability, and $\mathbf{X}^n \overset{D}{\Longrightarrow} \mathbf{X}$. We have $V(\mathbf{X}^n) = -1$ and $V(\mathbf{X}) = 0$ and therefore V is not lower semicontinuous.

5. Application to multiparameter prophet inequalities

The prophet inequalities for one-parameter optimal stopping problems have been studied by many authors. In this section we shall formulate a prophet inequality for a multiparameter optimal stopping problem.

Let $\mathbf{X} = \{X(z),\ z\in I\}$ be an integrable d-parameter stochastic process. We term multiparameter prophet inequality any inequality which compares $E[\max(\mathbf{X})] = E[\max_{z\in I} X(z)]$ to $V[\mathbf{X}] = \sup_{T\in A(\mathbf{X})} E[X(T)]$.

Theorem 5.1. *Let* $\mathbf{C}$ *be a tight and closed family of integrable d-parameter stochastic processes which is uniformly integrable. Then*

$$\sup\{E^\lambda[\max(\mathbf{X}^\lambda)] - V[\mathbf{X}^\lambda]:\ \mathbf{X}^\lambda \in \mathbf{C}\}$$

is attained on $\mathbf{C}$ *and for each* $y \in \{E^\lambda[\max(\mathbf{X}^\lambda)]\colon \mathbf{X}^\lambda \in \mathbf{C}\}$,

$$\inf\{V[\mathbf{X}^\lambda]\colon \mathbf{X}^\lambda \in \mathbf{C},\ E^\lambda[\max(\mathbf{X}^\lambda)] = y\}$$

is attained on $\mathbf{C}$.

Proof. By Prohorov's theorem and our assumption on closedness, $\mathbf{C}$ is compact. The map $\mathbf{X}^\lambda \to E^\lambda[\max(\mathbf{X}^\lambda)]$ is continuous, because of our assumption of the uniformly integrability and the mapping theorem on the convergence of weak convergence. By Theorem 3.1 and our assumption of theorem, $-V$ is upper semicontinuous and hence $E^\lambda[\max(\mathbf{X}^\lambda)] - V[\mathbf{X}^\lambda]$ is upper semicontinuous. This completes the proof. □

Acknowledgments

The author is very grateful to an anonymous referee and one of associate editors for their carefully reading the manuscript and their valuable comments.

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