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Pullback attractors for a class of extremal solutions of the 3D Navier–Stokes system

O.V. Kapustyan [a], P.O. Kasyanov [b], J. Valero [c,*]

[a] Kyiv National Taras Shevchenko University, Kyiv, Ukraine
[b] Institute of Applied System Analysis, Kyiv, Ukraine
[c] Universidad Miguel Hernandez de Elche, Centro de Investigación Operativa, Avda. Universidad s/n, Elche, Spain

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ABSTRACT

In this paper we construct a dynamical process (in general, multivalued) generated by the set of solutions of an optimal control problem for the three-dimensional Navier–Stokes system. We prove the existence of a pullback attractor for such multivalued process. Also, we establish the existence of a uniform global attractor containing the pullback attractor. Moreover, under the unproved assumption that strong globally defined solutions of the three-dimensional Navier–Stokes system exist, which guaranties the existence of a global attractor for the corresponding multivalued semiflow, we show that the pullback attractor of the process coincides with the global attractor of the semiflow.

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1. Introduction

The study of the asymptotic behavior of the weak solutions of the three-dimensional (3D for short) Navier–Stokes system is a challenging problem which is still far to be solved in a satisfactory way. In particular, the existence of a global attractor in the strong topology is an open problem for which only some partial or conditional results are given (see [3,4,6,15, 17–20,31]). Concerning the existence of trajectory attractors some results are proved in [13,23,29]. The main difficulty in this problem (but not the only one!) is to prove the asymptotic compactness of solutions (see [2] for a review on these questions).

With respect to the attractor in the weak topology some results are proved in [15,20]. Also, the Kneser property (that is, the compactness and connectedness of the attainability set for the weak solutions) in both the weak and strong topologies is studied in [21,22].

In this paper we consider an optimal control problem associated with the 3D Navier–Stokes system which, in our point of view, could give some light on all these questions. Namely, let us consider the problem

$$\begin{cases} \dfrac{\partial y}{\partial t} - \nu \Delta y + (u \cdot \nabla) y = -\nabla p + f, \\ div\, y = 0, \\ y|_{\partial \Omega} = 0, \quad y(\tau) = u_\tau, \end{cases} \tag{1}$$

* Corresponding author.
E-mail addresses: alexkap@univ.kiev.ua (O.V. Kapustyan), kasyanov@i.ua (P.O. Kasyanov), jvalero@umh.es (J. Valero).

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where $\Omega \subset \mathbb{R}^3$ is a bounded open subset with smooth boundary, $\nu > 0$ and u is a control function belonging to a suitable set $\mathbb{U}_\tau$ (see Section 3). Then we will solve the following optimality problem: to find a pair $\{u, y\}$ such that y is a solution of (1) associated to u and an appropriate functional $J_\tau(u, y)$ attaches its infimum at (u, y).

We note first that this problem is non-autonomous. Also, we cannot guarantee uniqueness of the problem for a given initial data y_τ, although for a given u the solution to (1) is unique. The reason is that more than one pair $\{u, y\}$ can exist as a solution of the optimality problem. Hence, in order to study the asymptotic behavior of solutions of such problem we use the theory of pullback attractors for multivalued processes developed in [9] (see also [1,7,8]). This theory generalizes the theory of pullback attractors for single-valued processes and cocycles, which has been studied intensively in the last years (see e.g. [5,10,11,16,12,24,25,28,32], among many others).

We prove first that the optimal problem has at least one solution for every initial data in the phase space H, and then we construct a multivalued process associated to these solutions. The main theorem of the paper states the existence of a strictly invariant pullback attractor for this process. Moreover, we prove the existence of a uniform global attractor for the process in the sense of [27] (see also [14] for the single-valued case), which contains the whole pullback attractor.

Finally, in the last section, we study the relationship of the pullback attractor of the optimal control problem with the global attractor of the 3D Navier–Stokes systems under the unproved condition that globally defined strong solutions exist for any initial data in V. In particular, we prove that in this case the global attractor of the Navier–Stokes systems coincides with the pullback attractor. This result shows that there exists a close relation between the dynamics of the solutions of the optimal control problem and the dynamics of the solutions of the 3D Navier–Stokes system. Therefore, we hope that the optimal control problem will help us in the future to gain an insight into the problem on the existence of the global attractor for the 3D Navier–Stokes system.

2. Pullback attractors for multivalued processes

In this section we will recall and extend some well-known general results on pullback attractors for multivalued processes [9].

Let X be a complete metric space with the metric ρ, $P(X)$ be the set of all non-empty subsets of X, and $\beta(X)$ be the set of all non-empty, bounded subsets of X. Put $R_d = \{(t, s) \in \mathbb{R}^2: t \geqslant s\}$.

Definition 1. $U : R_d \times X \mapsto P(X)$ is called a multivalued process (m-process for short) if:

1. $U(\tau, \tau, x) = x$, $\forall \tau \in \mathbb{R}$, $\forall x \in X$;
2. $U(t, \tau, x) \subseteq U(t, s, U(s, \tau, x))$, $\forall t \geqslant s \geqslant \tau$, $\forall x \in X$.

U is called strict if in 2. a strict equality holds.

For $t \in \mathbb{R}$, $B \in \beta(X)$ we define the omega-limit set

$$\omega(t, B) = \bigcap_{s \leqslant t} \overline{\bigcup_{\tau \leqslant s} U(t, \tau, B)}.$$

Let $dist(A, B) = \sup_{y \in A} \inf_{x \in B} \rho(y, x)$. We shall denote by $O_\varepsilon(B) = \{y \in X: dist(y, B) < \varepsilon\}$ an ε-neighborhood of B.

Definition 2. The family of compact sets $\{\Theta(t)\}_{t\in\mathbb{R}}$ is called a pullback attractor if:

1. For any $t \in \mathbb{R}$ the set $\Theta(t)$ attracts every $B \in \beta(X)$ in the pullback sense, that is,
$$dist\big(U(t, \tau, B), \Theta(t)\big) \to 0, \quad \text{as } \tau \to -\infty; \tag{2}$$
2. $\Theta(t) \subseteq U(t, s, \Theta(s))$, $\forall t \geqslant s$ (negatively semi-invariance);
3. $\Theta(t)$ is the minimal closed pullback attracting set for all $t \in \mathbb{R}$.

The pullback attractor is strictly invariant if $\Theta(t) = U(t, s, \Theta(s))$, $\forall t \geqslant s$.

We shall extend now a general result on pullback attractors proved in [9].

Theorem 1. *Let us suppose that there exists a family of compact sets $\{K(t)\}_{t\in\mathbb{R}}$ satisfying* (2) *and that the map $x \mapsto U(t, \tau, x)$ has closed graph for all $t \geqslant \tau$. Then there exists a pullback attractor $\{\Theta(t)\}_{t\in\mathbb{R}}$, $\Theta(t) \subset K(t)$, $\forall t \in \mathbb{R}$, defined by*

$$\Theta(t) = \overline{\bigcup_{B \in \beta(X)} \omega(t, B)}.$$

Moreover, if there exists a closed set $B_0 \in \beta(X)$ such that for all $B \in \beta(X)$,

$$\sup_{\tau\in\mathbb{R}} dist\big(U(s+\tau,\tau,B),B_0\big)\to 0,\quad as\ s\to+\infty,$$

then $\Theta(t)=\omega(t,B_0)\subset B_0$.

In addition, if U *is strict, then* $\Theta(t)=U(t,s,\Theta(s))$, *for any* $t\geqslant s$, *i.e.* $\Theta(t)$ *is invariant.*

Proof. The existence of the pullback attractor is proved in [9, Theorem 18]. Further, we can write

$$\omega(t,B)=\bigcap_{T\geqslant 0}\overline{\bigcup_{s\geqslant T}U(t,t-s,B)}.$$

In view of [9, Theorem 6] the set $\omega(t,B)$ is non-empty, compact and attracts B in the pullback sense. Also, $\omega(t,B)\subset U(t,s,\omega(s,B))$, for all $t\geqslant s$ [9, p. 160].

So, putting $\tau_s=t-s$, from

$$\bigcup_{s\geqslant T}U(t,t-s,B)=\bigcup_{s\geqslant T}U(\tau_s+s,\tau_s,B)\subset O_\varepsilon(B_0),\quad \forall T\geqslant T(B,\varepsilon),$$

for an arbitrary small $\varepsilon>0$, it follows that $\omega(t,B)\subset B_0$ and, hence, $\Theta(t)\subset B_0$.

On the other hand,

$$\omega(t,B)\subset U\big(t,s,\omega(s,B)\big)\subset U(t,s,B_0)\to\omega(t,B_0),\quad \text{as } s\to-\infty.$$

So, $\omega(t,B)\subset\omega(t,B_0)$ and it follows that $\Theta(t)=\omega(t,B_0)$.

If U is strict, then from

$$\omega(t,B_0)\subset U\big(t,s,\omega(s,B_0)\big),\quad \forall t\geqslant s,$$

we obtain

$$U\big(p,t,\omega(t,B_0)\big)\subset U\big(p,t,U\big(t,s,\omega(s,B_0)\big)\big)\subset U(p,s,B_0).$$

Since $U(p,s,B_0)\to\omega(p,B_0)$, as $s\to-\infty$, we obtain

$$U\big(p,t,\omega(t,B_0)\big)\subset\omega(p,B_0),\quad \forall p\geqslant t,$$

and then

$$\Theta(p)=U\big(p,t,\Theta(t)\big),\quad \forall p\geqslant t.\qquad\square$$

3. Setting of the problem and main results

Let $\Omega\subset\mathbb{R}^3$ be a bounded open subset with smooth boundary. We shall define the usual function spaces

$$\mathcal{V}=\big\{u\in\big(C_0^\infty(\Omega)\big)^3:\ div\,u=0\big\},$$
$$H=cl_{(L^2(\Omega))^3}\mathcal{V},\qquad V=cl_{(H_0^1(\Omega))^3}\mathcal{V},$$

where cl_X denotes the closure in the space X. It is well known that H,V are separable Hilbert spaces and identifying H and its dual H^* we have $V\subset H\subset V^*$ with dense and continuous injections. We denote by $(\cdot,\cdot)$, $|\cdot|$ and $((\cdot,\cdot))$, $\|\cdot\|$ the inner product and norm in H and V, respectively. $\langle\cdot,\cdot\rangle$ will denote pairing between V and V^*. We set $\mathbb{L}^4(\Omega)=(L^4(\Omega))^3$ with the norm denoted by $\|\cdot\|_{\mathbb{L}^4}$. We will denote by B_R a closed ball with radius R and centered at 0 in the space H.

For $u,v,w\in V$ we put

$$b(u,v,w)=\int_\Omega\sum_{i,j=1}^{3}u_i\frac{\partial v_j}{\partial x_i}w_j\,dx.$$

It is known [30] that b is a trilinear continuous form on V and $b(u,v,v)=0$, if $u\in V$, $v\in(H_0^1(\Omega))^3$. As usual, for $u,v\in V$ we denote by $B(u,v)$ the element of V^* defined by $\langle B(u,v),w\rangle=b(u,v,w)$, for all $w\in V$.

We consider also the Stokes operator $A:D(A)\to H$, $D(A)=(H^2(\Omega))^3\cap V$, where $Au=-P\Delta u$, P is the Helmholtz–Leray projector and Δ is the Laplacian operator (see e.g. [19] for more details).

We consider the 3D controlled Navier–Stokes system

$$\begin{cases}\dfrac{dy}{dt}+Ay+B(u,y)=f,\\ y(\tau)=y_\tau\in H,\end{cases}\tag{3}$$

where $f \in H$ and

$$u(\cdot) \in \mathbb{U}_\tau = \begin{cases} u \in L^\infty(\tau, +\infty; H) \cap L^2_{loc}(\tau, +\infty; V) \cap L^\infty_{loc}(\tau, +\infty; \mathbb{L}^4(\Omega)), \\ \int_\tau^{+\infty} \|u(p)\|^2 e^{-\delta p}\, dp < \infty,\ |u(p)| \leqslant R_0,\ \text{for a.a. } p \geqslant \tau, \\ \|u(t)\|_{\mathbb{L}^4} \leqslant \alpha,\ \text{for a.a. } t > \tau, \end{cases} \tag{4}$$

$$J_\tau(u, y) = \int_\tau^{+\infty} \left\| y(p) - u(p) \right\|^2 e^{-\delta p}\, dp \to \inf, \tag{5}$$

with $\delta = \lambda_1 \nu$, $R_0 = \frac{|f|}{\nu \lambda_1}$, and where $\alpha > 0$ is some constant.

We have two aims. The first one is to prove that the solutions of the optimal control problem (3)–(5) generate a multivalued process U, which has a pullback attractor. The second aim is to prove that under the unproved assumption about strong global solvability of the 3D Navier–Stokes system the pullback attractor coincides with the global attractor of the multivalued semiflow for the 3D Navier–Stokes system given in [20].

By using standard Galerkin approximations (see [30]) it is easy to show that for any $y_\tau \in H$ and $u(\cdot) \in U_\tau$ there exists a unique weak solution $y(\cdot) \in L^\infty(\tau, +\infty; H) \cap L^2_{loc}(\tau, +\infty; V)$ of (3), that is,

$$\frac{d}{dt}(y, v) + \nu((y, v)) + b(u, y, v) = \langle f, v \rangle, \quad \text{for all } v \in V. \tag{6}$$

Moreover, by the inequality

$$\left| b(u, y, v) \right| = \left| b(u, v, y) \right| \leqslant c_1 \|u\|_{\mathbb{L}^4} \|v\| \|y\|_{\mathbb{L}^4} \leqslant c_2 c_1 \|u\|_{\mathbb{L}^4} \|v\| \|y\|, \quad \forall u, y, v \in V,$$

and (4) we have $B(u(\cdot), y(\cdot)) \in L^2_{loc}(\tau, +\infty; V^*)$, so $\frac{dy}{dt} \in L^2_{loc}(\tau, +\infty; V^*)$ as well. Hence, it follows that $y(\cdot) \in C([\tau, +\infty); H)$ (so the initial condition $y(\tau) = y_\tau$ makes sense for any $y_\tau \in H$) and standard arguments imply that for all $t \geqslant s \geqslant \tau$,

$$F\big(y(t)\big) := \left(\left| y(t) \right|^2 - R_0^2 \right) e^{\delta t} \leqslant F\big(y(s)\big), \tag{7}$$

$$V_\tau\big(y(t)\big) := \frac{1}{2} \left| y(t) \right|^2 + \nu \int_\tau^t \left\| y(p) \right\|^2 dp - \int_\tau^t \big(f, y(p)\big)\, dp \leqslant V_\tau\big(y(s)\big), \tag{8}$$

$$\left| y(t) \right|^2 + \nu \int_\tau^t \left\| y(p) \right\|^2 dp \leqslant |y_\tau|^2 + \frac{|f|^2}{\nu \lambda_1}(t - \tau). \tag{9}$$

So, for all $n \geqslant 0$,

$$\begin{aligned} \int_{\tau+n}^{\tau+(n+1)} \left\| y(p) - u(p) \right\|^2 e^{-\delta p}\, dp &\leqslant 2e^{-\delta(n+\tau)} \int_{\tau+n}^{\tau+(n+1)} \left\| y(p) \right\|^2 dp + 2 \int_{\tau+n}^{\tau+(n+1)} \left\| u(p) \right\|^2 e^{-\delta p}\, dp \\ &\leqslant \frac{2}{\nu} e^{-\delta(n+\tau)} \left(|y_\tau|^2 + \frac{|f|^2}{\nu \lambda_1} \right) + 2 \int_{\tau+n}^{\tau+(n+1)} \left\| u(p) \right\|^2 e^{-\delta p}\, dp. \end{aligned}$$

From this

$$\begin{aligned} J_\tau(u, y) &= \sum_{n=0}^\infty \int_{\tau+n}^{\tau+(n+1)} \left\| y(p) - u(p) \right\|^2 e^{-\delta p}\, dp \\ &\leqslant \frac{2e^{-\delta\tau}}{\nu} \left(|y_\tau|^2 + \frac{|f|^2}{\nu \lambda_1} \right) \sum_{n=0}^\infty e^{-\delta n} + 2 \sum_{n=0}^\infty \int_{\tau+n}^{\tau+(n+1)} \left\| u(p) \right\|^2 e^{-\delta p}\, dp < \infty. \end{aligned}$$

Therefore, the functional J_τ and the optimal control problem (3)–(5) is correctly defined.

Lemma 1. *For any $\tau \in \mathbb{R}$ and $y_\tau \in H$ the optimal control problem* (3)–(5) *has at least one solution $\{y(\cdot), u(\cdot)\}$, and, moreover, $\frac{dy}{dt} \in L^2_{loc}(\tau, +\infty; V^*)$, $y(\cdot) \in C([\tau, +\infty); H)$ and* (7)–(9) *hold.*

Proof. Let $\{y_n, u_n\}$ be a minimizing sequence such that

$$\int_{\tau}^{+\infty} \|y_n(p) - u_n(p)\|^2 e^{-\delta p}\, dp \leqslant d + \frac{1}{n}, \quad \forall n \geqslant 1,$$

where $d = \inf J_\tau(u, y)$. Thus, for all $T > \tau$,

$$\int_{\tau}^{T} \|y_n(p) - u_n(p)\|^2 e^{-\delta p}\, dp \leqslant d + \frac{1}{n},$$

$$\int_{\tau}^{T} \|y_n(p) - u_n(p)\|^2 dp \leqslant \left(d + \frac{1}{n}\right) e^{\delta T}. \tag{10}$$

From (7)–(9) we obtain that $\{y_n\}$ is bounded in $L^\infty(\tau, T; H) \cap L^2(\tau, T; V)$. Hence, (10) implies that $\{u_n\}$ is bounded in $L^2(\tau, T; V)$ and from the definition of $\mathbb{U}_\tau$ it follows that

$$\begin{aligned} &|u_n(p)| \leqslant R_0, \quad \forall p \geqslant \tau, \\ &\|u_n(p)\|_{\mathbb{L}^4} \leqslant \alpha, \quad \text{for a.a. } p > \tau. \end{aligned}$$

Therefore, there exist $u \in L^\infty(\tau, T; H) \cap L^2(\tau, T; V) \cap L^\infty(\tau, T; \mathbb{L}^4(\Omega))$ and $y \in L^\infty(\tau, T; H) \cap L^2(\tau, T; V)$ such that

$$\begin{aligned} &u_n \to u \quad \text{weakly in } L^2(\tau, T; V), \\ &u_n \to u \quad *\text{-weakly in } L^\infty(\tau, T; H), \\ &u_n \to u \quad *\text{-weakly in } L^\infty\big(\tau, T; \mathbb{L}^4(\Omega)\big), \\ &y_n \to y \quad \text{weakly in } L^2(\tau, T; V), \\ &y_n \to y \quad *\text{-weakly in } L^\infty(\tau, T; H). \end{aligned} \tag{11}$$

Moreover, $\|B(u_n, y_n)\|_{V^*} \leqslant c_1 \|y_n\| \|u_n\|_{\mathbb{L}^4}$. Hence, $\frac{dy_n}{dt}$ is bounded in $L^2(\tau, T; V^*)$. From this using standard arguments, we obtain that $y(\cdot) \in C([\tau, T]; H)$ is the solution of (3) with control $u(\cdot)$, $y(\cdot)$ satisfies (7)–(9), and for this control the following relations hold:

$$\begin{aligned} &|u(p)| \leqslant R_0, \quad \text{for a.a. } p \geqslant \tau, \\ &\|u(p)\|_{\mathbb{L}^4} \leqslant \alpha, \quad \text{for a.a. } p > \tau, \\ &u \in L^2(\tau, T; V), \\ &\int_{\tau}^{T} \|y(p) - u(p)\|^2 e^{-\delta p}\, dp \leqslant d. \end{aligned}$$

By using a standard diagonal procedure we can claim that $y(\cdot)$ and $u(\cdot)$ are defined on $[\tau, +\infty)$, $y_n \to y$, $u_n \to u$ in the previous sense on every $[\tau, T]$, and

$$\int_{\tau}^{+\infty} \|y(p) - u(p)\|^2 e^{-\delta p}\, dp \leqslant d. \tag{12}$$

By (9), arguing as before,

$$\int_{\tau}^{+\infty} \|y(p)\|^2 e^{-\delta p}\, dp < \infty,$$

and from (12) we have

$$\int_{\tau}^{+\infty} \|u(p)\|^2 e^{-\delta p}\, dp < \infty.$$

It follows that $u(\cdot) \in \mathbb{U}_\tau$ and from (12) we obtain that $\{y(\cdot), u(\cdot)\}$ is an optimal pair of problem (3)–(5). $\square$

Now we are ready to construct a multivalued process associated to the solutions of problem (3)–(5).

For every $y_\tau \in H$, $\tau \in \mathbb{R}$, and $t \geqslant \tau$ we put

$$U(t,\tau,y_\tau) = \{\tilde{y}(t)\colon \tilde{y}(\cdot) \text{ is a solution of the optimal problem (3)–(5)}, \ \tilde{y}(\tau) = y_\tau\}. \tag{13}$$

Lemma 2. *The multivalued map U defined by* (13) *is a strict multivalued process.*

Proof. It is obvious that $U(\tau,\tau,y_\tau) = y_\tau$.

Let $\xi \in U(t,\tau,y_\tau)$. Thus, $\xi = \tilde{y}(t)$, where $\{\tilde{y}(\cdot), \tilde{u}(\cdot)\}$ is an optimal pair of problem (3)–(5) with $\tilde{y}(\tau) = y_\tau$, $\tilde{u}(\cdot) \in \mathbb{U}_\tau$. Then, of course, $\tilde{y}(s) \in U(s,\tau,y_\tau)$, for all $s \in (\tau,t)$. We should prove that $\tilde{y}(t) \in U(t,s,\tilde{y}(s))$, that is, Bellman's principle of optimality. Let

$$\tilde{J}_\tau = J_\tau(\tilde{y},\tilde{u}) = \int_\tau^{+\infty} \|\tilde{y}-\tilde{u}\|^2 e^{-\delta p}\,dp = \int_\tau^{s} \|\tilde{y}-\tilde{u}\|^2 e^{-\delta p}\,dp + \int_s^{+\infty} \|\tilde{y}-\tilde{u}\|^2 e^{-\delta p}\,dp.$$

We consider the problem (3)–(5) on the interval $[s,+\infty)$ (formally s is instead of τ) with the set of controls $\mathbb{U}_s$ and the initial data $(s,\tilde{y}(s))$. It is easy to verify that $\{\tilde{y}(\cdot), \tilde{u}(\cdot)\}$ on $[s,+\infty)$ is a solution of the optimal control problem. Indeed, we note that $\tilde{u}(\cdot) \in \mathbb{U}_s$ and that $\tilde{y}(\cdot)$ is the unique solution of (3) corresponding to $\tilde{u}(\cdot)$. Let $\{\hat{y}(\cdot), \hat{u}(\cdot)\}$ be an optimal pair of this problem. Suppose that

$$\int_s^{+\infty} \|\hat{y}-\hat{u}\|^2 e^{-\delta p}\,dp < \int_s^{+\infty} \|\tilde{y}-\tilde{u}\|^2 e^{-\delta p}\,dp.$$

Let us consider the control

$$u(t) = \begin{cases} \tilde{u}(t), & t \in [\tau,s), \\ \hat{u}(t), & t \in [s,+\infty). \end{cases}$$

We claim that $u(\cdot) \in \mathbb{U}_\tau$. Indeed, it is clear that $|u(p)| \leqslant R_0$ for a.a. $p \geqslant \tau$, $\|u(t)\|_{\mathbb{L}^4} \leqslant \alpha$ for a.a. $t > \tau$,

$$u(\cdot) \in L^\infty(\tau,+\infty;H) \cap L^2_{loc}(\tau,+\infty;V),$$
$$\int_\tau^{+\infty} \|u(p)\|^2 e^{-\delta p}\,dp < \infty.$$

Then $u(\cdot) \in \mathbb{U}_\tau$ and

$$y(t) = \begin{cases} \tilde{y}(t), & t \in [\tau,s), \\ \hat{y}(t), & t \in [s,+\infty) \end{cases}$$

is the solution of problem (3) which corresponds to the control $u(\cdot)$ (because of uniqueness of the solution of problem (3) for a fixed $u(\cdot)$).

Finally,

$$\begin{aligned} J_\tau(u,y) &= \int_\tau^{s} \|\tilde{y}-\tilde{u}\|^2 e^{-\delta p}\,dp + \int_s^{+\infty} \|\hat{y}-\hat{u}\|^2 e^{-\delta p}\,dp \geqslant \tilde{J}_\tau \\ &= \int_\tau^{s} \|\tilde{y}-\tilde{u}\|^2 e^{-\delta p}\,dp + \int_s^{+\infty} \|\tilde{y}-\tilde{u}\|^2 e^{-\delta p}\,dp, \end{aligned}$$

which is a contradiction. Hence, $\{\tilde{y}(\cdot), \tilde{u}(\cdot)\}$ is an optimal pair on $[s,+\infty)$ and then $\tilde{y}(t) \in U(t,s,\tilde{y}(s))$, so that U is a multivalued process.

Let us prove that it is strict. Let $\xi \in U(t,s,U(s,\tau,y_\tau))$. Then $\xi = \tilde{y}_2(t)$ and $\{\tilde{y}_2(\cdot), \tilde{u}_2(\cdot)\}$ is an optimal pair of problem (3)–(5) with $\tilde{y}_2(s) = y_s$, $\tilde{u}_2(\cdot) \in \mathbb{U}_s$, and $y_s = \tilde{y}_1(s)$, where $\{\tilde{y}_1(\cdot), \tilde{u}_1(\cdot)\}$ is an optimal pair of problem (3)–(5) with $\tilde{y}_1(\tau) = y_\tau$, $\tilde{u}_1(\cdot) \in \mathbb{U}_\tau$. Let us consider the control

$$u(t) = \begin{cases} \tilde{u}_1(t), & t \in [\tau,s), \\ \tilde{u}_2(t), & t \in [s,+\infty). \end{cases}$$

As before $u \in \mathbb{U}_\tau$ and

$$y(t)=\begin{cases}\tilde y_1(t), & t\in[\tau,s),\\ \tilde y_2(t), & t\in[s,+\infty)\end{cases}$$

is the solution of problem (3) which corresponds to the control $u(\cdot)$. Also, it is clear, as $\tilde y_1(\cdot)$ is a solution of problem (3) on $[s,+\infty)$ corresponding to $\tilde u_1(\cdot)$, that

$$\begin{aligned}J_\tau(u,y)&=\int_\tau^s\|\tilde y_1-\tilde u_1\|^2e^{-\delta p}\,dp+\int_s^{+\infty}\|\tilde y_2-\tilde u_2\|^2e^{-\delta p}\,dp\\ &\leqslant\int_\tau^s\|\tilde y_1-\tilde u_1\|^2e^{-\delta p}\,dp+\int_s^{+\infty}\|\tilde y_1-\tilde u_1\|^2e^{-\delta p}\,dp.\end{aligned}$$

Hence, $\xi=y(t)\in U(t,\tau,y_\tau)$. $\square$

Theorem 2. *For the multivalued process U, given by* (13), *there exists a strictly invariant pullback attractor $\{\Theta(t)\}_{t\in\mathbb{R}}$ such that $\Theta(t)\subset B_{R_0}$, for all $t\in\mathbb{R}$.*

Proof. First of all, from (7) for every $R>R_0$, $y_\tau\in H$ such that $|y_\tau|\leqslant R$ it holds

$$\begin{aligned}&\big|y(s+\tau)\big|^2-R_0^2\leqslant e^{-\delta(s+\tau)}\big(|y_\tau|^2-R_0^2\big)e^{\delta\tau},\\ &\big|y(s+\tau)\big|^2\leqslant e^{-\delta s}\big(R^2-R_0^2\big)+R_0^2.\end{aligned}$$

So,

$$\sup_{\tau\in\mathbb{R}} dist\big(U(s+\tau,\tau,B_R),B_{R_0}\big)\to 0,\quad \text{as } s\to+\infty. \tag{14}$$

In virtue of Theorem 1 and

$$U(t,s,B_R)\subset U\big(t,t-1,U(t-1,s,B_R)\big)\subset U(t,t-1,B_{R_0+1}),$$

where the last inclusion follows from (14) by taking a sufficiently small s, we only need to prove that the set $K(t):=\overline{U(t,t-1,B_{R_0+1})}$ is compact and that the map $x\mapsto U(t,\tau,x)$ has closed graph. These two properties are true, if the following statements hold for all $t\geqslant\tau$:

$(U1)$ If $\eta_n\to\eta$ weakly in H and $\xi_n\in U(t,\tau,\eta_n)$, then the sequence $\{\xi_n\}$ is pre-compact in H;
$(U2)$ If $\eta_n\to\eta$ strongly in H and $\xi_n\in U(t,\tau,\eta_n)$, then up to a subsequence $\xi_n\to\xi\in U(t,\tau,\eta)$.

Let $\xi_n\in U(t,\tau,\eta_n)$, where $\eta_n\to\eta$ weakly in H. Then $\xi_n=\tilde y_n(t)$, $\tilde y_n(\tau)=\eta_n$, where $\{\tilde y_n(\cdot),\tilde u_n(\cdot)\}$ is an optimal pair of problem (3)–(5), $\tilde u_n(\cdot)\in\mathbb{U}_\tau$. We have that $\{\tilde y_n(\cdot)\}$ satisfy (7)–(9). If we consider the control $u(\cdot)\equiv 0\in\mathbb{U}_\tau$ and the corresponding solution of (3), $y_n(\cdot)$, with $y_n(\tau)=\eta_n$, then

$$J_\tau(\tilde u_n,\tilde y_n)\leqslant J_\tau(0,y_n)=\int_\tau^{+\infty}\big\|y_n(p)\big\|^2e^{-\delta p}\,dp=\sum_{k=0}^{\infty}\int_{\tau+k}^{\tau+k+1}\big\|y_n(p)\big\|^2e^{-\delta p}\,dp.$$

But $y_n(\cdot)$ satisfies (7)–(9), so

$$\int_{\tau+k}^{\tau+k+1}\big\|y_n(p)\big\|^2e^{-\delta p}\,dp\leqslant\frac{e^{-\delta(k+\tau)}}{\nu}\bigg(|\eta_n|^2+\frac{|f|^2}{\nu\lambda_1}\bigg).$$

Thus $J_\tau(\tilde u_n,\tilde y_n)\leqslant\tilde C$, where $\tilde C$ does not depend on n. Then in the same way as in Lemma 1 we obtain (up to a subsequence) that $\tilde u_n\to\tilde u\in\mathbb{U}_\tau$, $\tilde y_n\to\tilde y$ in the sense of (11) on any interval (τ,T), where $\tilde y\in C([\tau,+\infty);H)$ is the solution of problem (3) with control $\tilde u(\cdot)$. Moreover, in a standard way (see e.g. the proof of Lemma 11 in [20]) one can prove that $\tilde y_n(s)\to\tilde y(s)$ strongly in H for all $s>\tau$. Therefore, $(U1)$ holds.

Assume now additionally that $\eta_n\to\eta$ strongly in H. Let us prove that $\{\tilde y(\cdot),\tilde u(\cdot)\}$ is an optimal pair.

Fix an arbitrary $u(\cdot)\in\mathbb{U}_\tau$. Let $y_n(\cdot)$ be the solution of problem (3) with control $u(\cdot)$ and initial data $y_n(\tau)=\eta_n$. Then, of course, $y_n(\cdot)\to y(\cdot)$ in the sense of (11), where $y(\cdot)$ is the solution of problem (3) with control $u(\cdot)\in\mathbb{U}_\tau$ and initial data $y(\tau)=\eta$. Also, one can prove that $y_n\to y$ strongly in $L^2(\tau,T;V)$ for all $\tau<T$. Indeed, in a standard way we obtain

$$\frac{1}{2}\frac{d}{dt}|y_n-y|^2+\nu\|y_n-y\|^2+B(u,y_n-y,y_n-y)=0.$$

As $B(u, y_n - y, y_n - y) = 0$, we have

$$\left|y_n(T) - y(T)\right|^2 + 2\nu \int_\tau^T \|y_n - y\|^2\, ds = |\eta_n - \eta|^2 \to 0, \quad \text{as } n \to \infty. \tag{15}$$

Further, since $\{\tilde{y}_n(\cdot), \tilde{u}_n(\cdot)\}$ is an optimal process, we have

$$J_\tau(\tilde{u}_n, \tilde{y}_n) \leqslant J_\tau(y_n, u),$$

so that, for all $T > \tau$,

$$\int_\tau^T \|\tilde{y}_n - \tilde{u}_n\|^2 e^{-\delta p}\, dp \leqslant \int_\tau^T \|y_n - u\|^2 e^{-\delta p}\, dp + \int_T^{+\infty} \|y_n - u\|^2 e^{-\delta p}\, dp - \int_T^{+\infty} \|\tilde{y}_n - \tilde{u}_n\|^2 e^{-\delta p}\, dp. \tag{16}$$

As we showed before

$$J_\tau(\tilde{u}_n, \tilde{y}_n) \leqslant \tilde{C},$$

where $\tilde{C}$ does not depend on n. Moreover, since $\{\tilde{y}_n(\cdot), \tilde{u}_n(\cdot)\}$ is an optimal pair on $[T, +\infty)$ (see Lemma 2), if for any $T > \tau$ we consider the problem (3)–(5) with initial data $(T, \tilde{y}_n(T))$, the control $u(\cdot) \equiv 0 \in \mathbb{U}_T$ and the corresponding solution of (3) $z_n(\cdot)$ with $z_n(T) = \tilde{y}_n(T)$, then

$$\int_T^{+\infty} \|\tilde{y}_n - \tilde{u}_n\|^2 e^{-\delta p}\, dp \leqslant \int_T^{+\infty} \|z_n\|^2 e^{-\delta p}\, dp.$$

But $z_n(\cdot)$ satisfies (7)–(9), so that for any $\varepsilon > 0$ there exists $T_1(\varepsilon)$ such that for any $T \geqslant T_1(\varepsilon)$, $n \geqslant 1$, we have

$$\int_T^{+\infty} \|\tilde{y}_n - \tilde{u}_n\|^2 e^{-\delta p}\, dp < \frac{\varepsilon}{2}.$$

The functions $\{y_n(\cdot)\}$ from (16) also satisfy (7)–(9), and $u(\cdot) \in \mathbb{U}_\tau$. Hence

$$\int_T^{+\infty} \|y_n - u\|^2 e^{-\delta p}\, dp \leqslant 2 \int_T^{+\infty} \|y_n\|^2 e^{-\delta p}\, dp + 2 \int_T^{+\infty} \|u\|^2 e^{-\delta p}\, dp$$

and for any $\varepsilon > 0$ there exists $T_2(\varepsilon, u)$ such that for all $T \geqslant T_2(\varepsilon, u)$, $n \geqslant 1$,

$$\int_T^{+\infty} \|y_n - u\|^2 e^{-\delta p}\, dp < \frac{\varepsilon}{2}.$$

Hence, for any $\varepsilon > 0$ there exists $T(\varepsilon, u)$ such that for all $T \geqslant T(\varepsilon, u)$, $n \geqslant 1$,

$$\int_\tau^T \|\tilde{y}_n - \tilde{u}_n\|^2 e^{-\delta p}\, dp \leqslant \int_\tau^T \|y_n - u\|^2 e^{-\delta p}\, dp + \varepsilon. \tag{17}$$

Then by (15), passing to the limit in (17), we obtain

$$\begin{aligned} \int_\tau^T \|\tilde{y} - \tilde{u}\|^2 e^{-\delta p}\, dp &\leqslant \liminf_{n\to\infty} \int_\tau^T \|\tilde{y}_n - \tilde{u}_n\|^2 e^{-\delta p}\, dp \\ &\leqslant \int_\tau^T \|y - u\|^2 e^{-\delta p}\, dp + \varepsilon, \quad \forall T \geqslant T(\varepsilon, u). \end{aligned}$$

Thus, letting $T \to +\infty$ we obtain

$$J_\tau(\tilde{u}, \tilde{y}) \leqslant J_\tau(u, y) + \varepsilon, \quad \forall u \in \mathbb{U}_\tau,\ \forall \varepsilon > 0,$$

and $\{\tilde{y}, \tilde{u}\}$ is an optimal pair.

The property $\Theta(t) \subset B_{R_0}$ follows from Theorem 1. Also, by Lemma 2 and Theorem 1 we obtain that $\Theta(t) = U(t, s, \Theta(s))$, for all $t \geqslant s$. $\square$

Further we shall obtain that a uniform global attractor exists and that it contains the pullback attractor $\Theta(t)$.

Definition 3. The compact set $\mathcal{A}$ is called a uniform global attractor for the multivalued process U if

$$\lim_{t \to +\infty} \sup_{\tau \in \mathbb{R}} dist\big(U(t+\tau, \tau, B), \mathcal{A}\big) = 0, \quad \forall B \in \beta(H), \tag{18}$$

and it is the minimal closed set satisfying this property.

Let us define now the multivalued map $G : \mathbb{R}^+ \times H \to P(H)$ given by

$$G(t, y_0) = \bigcup_{\tau \in \mathbb{R}} U(t+\tau, \tau, y_0).$$

Lemma 3. *G is a multivalued semiflow in the sense of* [26].

Proof. It is clear that $G(0, \cdot) = Id$. Let $\xi \in G(t+s, y_0)$. Then for some $\tau \in \mathbb{R}$,

$$\begin{aligned} \xi &\in U(t+s+\tau, \tau, y_0) \subset U\big(t+s+\tau, s+\tau, U(s+\tau, \tau, y_0)\big) \\ &\subset G\big(t, G(s, y_0)\big). \end{aligned}$$

Hence, $G(t+s, y_0) \subset G(t, G(s, y_0))$. $\square$

Theorem 3. *For the multivalued process U there exists a uniform global attractor $\mathcal{A}$. Moreover, $\Theta(t) \subset \mathcal{A}$, for all $t \in \mathbb{R}$.*

Proof. In view of (14) we have that for all $B \in \beta(H)$ there exists $T(B)$ such that $G(s, B) \subset B_{R_0+1}$ if $s \geqslant T$. Hence,

$$G(t, B) \subset G\big(1, G(t-1, B)\big) \subset G(1, B_{R_0+1}), \quad \forall t \geqslant T(B). \tag{19}$$

We will show that the set $G(1, B_{R_0+1})$ is pre-compact. If $\xi_n \in G(1, B_{R_0+1})$, then there exist $\tau_n \in \mathbb{R}$, $\eta_n \in B_{R_0+1}$ and optimal pair $\{\tilde{y}_n(\cdot), \tilde{u}_n(\cdot)\}$ of problem (3)–(5) with $\tilde{u}_n(\cdot) \in \mathbb{U}_{\tau_n}$ such that $\xi_n = \tilde{y}_n(\tau_n + 1)$, $\tilde{y}_n(\tau_n) = \eta_n$. We have that $\{\tilde{y}_n(\cdot)\}$ satisfy (7)–(9), so

$$\begin{aligned} &\sup_{s \in [\tau_n, \tau_n+1]} \big|\tilde{y}_n(s)\big| \leqslant C_1, \\ &\int_{\tau_n}^{\tau_n+1} \big\|\tilde{y}_n(s)\big\|^2 ds \leqslant C_2, \quad \text{for all } n. \end{aligned}$$

Also, by $\|B(\tilde{u}_n, \tilde{y}_n)\|_{V^*} \leqslant c_1 \|\tilde{y}_n\| \|\tilde{u}_n\|_{\mathbb{L}^4} \leqslant c_1 \alpha \|\tilde{y}_n\|$, we have

$$\int_{\tau_n}^{\tau_n+1} \left\| \frac{d\tilde{y}_n}{dt} \right\|_{V^*}^2 ds \leqslant C_3.$$

Arguing as in Theorem 2 we obtain also that

$$\int_{\tau_n}^{\tau_n+1} \big\|\tilde{u}_n(s)\big\|^2 ds \leqslant C_4.$$

Indeed,

$$J_{\tau_n}(\tilde{u}_n, \tilde{y}_n) \leqslant J_{\tau_n}(0, z_n) \leqslant \sum_{k=0}^{\infty} \int_{\tau_n+k}^{\tau_n+k+1} \big\|z_n(p)\big\|^2 e^{-\delta p}\, dp \leqslant e^{-\delta \tau_n} \tilde{C},$$

where the constant $\tilde{C}$ does not depend on n and $z_n(\cdot)$ is the solution of (3) corresponding to $u \equiv 0 \in U_{\tau_n}$ and $z_n(\tau_n) = \eta_n$. So

$$e^{-\delta(\tau_n+1)}\int_{\tau_n}^{\tau_n+1}\left\|\tilde{u}_n(p)\right\|^2dp\leqslant\int_{\tau_n}^{\tau_n+1}\left\|\tilde{u}_n(p)\right\|^2e^{-\delta p}\,dp$$
$$\leqslant 2\int_{\tau_n}^{\tau_n+1}\left\|\tilde{y}_n(p)\right\|^2e^{-\delta p}\,dp+2\int_{\tau_n}^{\tau_n+1}\|\tilde{y}_n-\tilde{u}_n\|^2e^{-\delta p}\,dp\leqslant 2e^{-\delta\tau_n}(C_2+\tilde{C}),$$

and we obtain the required estimate.

Let us define the functions $y_n(t)=\tilde{y}_n(t+\tau_n)$, $u_n(t)=\tilde{u}_n(t+\tau_n)$, $t\geqslant 0$. The functions $y_n(\cdot)$ are the unique solutions of problem (3) with initial data $y_n(0)=\eta_n$ and control $u_n(\cdot)$. By the previous estimates there exist functions $y(\cdot)$ and $u(\cdot)$ for which the convergences (11) hold in the interval $(0,1)$. Also, $u\in\mathbb{U}_0$ and

$$\frac{dy_n}{dt}\to\frac{dy}{dt}\quad\text{weakly in } L^2\big(0,1;V^*\big).$$

Hence, by standard arguments we obtain that $y\in C([0,1],H)$ and that it is the solution of (3) with control $u(\cdot)$ and initial data $y(0)=\eta$. Then in a standard way (see e.g. the proof of Lemma 11 in [20]) one can prove that $y_n(s)\to y(s)$ strongly in H for all $s\in(0,1]$. Hence, $\xi_n=\tilde{y}_n(\tau_n+1)=y_n(1)$ is convergent and $G(1,B_{R_0+1})$ is pre-compact.

By Theorem 1 in [26] the omega-limit set of any $B\in\beta(H)$ given by $\omega(B)=\bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s}G(t,B)}$ is non-empty, compact and attracts B, i.e.

$$dist\big(G(t,B),\omega(B)\big)\to 0,\quad\text{as } t\to\infty.$$

Since $G(p+s,B)\subset G(p,B_{R_0+1})$, for all $p\geqslant 0$, $s\geqslant T(R_0)$, we deduce that $\omega(B)\subset\omega(B_{R_0+1})$ for all $B\in\beta(H)$. Then the set

$$\mathcal{A}=\bigcup_{B\in\beta(H)}\omega(B)=\omega(B_{R_0+1})$$

is compact and attracts every $B\in\beta(H)$. Hence

$$\lim_{t\to+\infty}\sup_{\tau\in\mathbb{R}}dist\big(U(t+\tau,\tau,B),\mathcal{A}\big)=0. \tag{20}$$

The set $\mathcal{A}$ is the minimal closed set satisfying (20). Indeed, let $\mathcal{C}$ be a closed set satisfying (20) with $B=B_{R_0+1}$ and such that $\mathcal{A}\not\subset\mathcal{C}$. Then there exists $y\in\mathcal{A}$ such that $y\notin\mathcal{C}$. We take sequences y_n, t_n such that $y_n\in G(t_n,B_{R_0+1})$, so that $y_n\in U(t_n+\tau_n,\tau_n,B_{R_0+1})$ for some $\tau_n\in\mathbb{R}$, and converging to y as $n\to\infty$. Since

$$dist(y_n,\mathcal{C})\to 0,\quad\text{as } n\to\infty,$$

we have $y\in\mathcal{C}$, which is a contradiction.

Thus, $\mathcal{A}$ is a uniform global attractor.

Finally, since $\Theta(\tau)\subset B_{R_0}$, for all $\tau\in\mathbb{R}$, we obtain that

$$\Theta(t)\subset U\big(t,t-s,\Theta(t-s)\big)\subset U(t,t-s,B_{R_0})=U(\tau_s+s,\tau_s,B_{R_0})$$
$$\subset G(s,B_{R_0})\to\mathcal{A},\quad\text{as } s\to+\infty.$$

Hence, $\Theta(t)\subset\mathcal{A}$, for all $t\in\mathbb{R}$. □

4. Relationship with the attractor of the 3D Navier–Stokes system

Consider now the three-dimensional (3D) Navier–Stokes system

$$\begin{cases}\dfrac{dy}{dt}+Ay+B(y,y)=f,\\ y(\tau)=y_\tau\in H.\end{cases} \tag{21}$$

Our aim now is to study the relation between the pullback attractor of the optimal control problem (3)–(5) and the global attractor for the multivalued semiflow generated by (21). We recall first some conditional results proved in [20].

Assumption 1. Assume that for any $\tau\in\mathbb{R}$, $y_\tau\in V$ there exists a unique globally defined strong solution $y(\cdot)$ of the 3D Navier–Stokes system, that is,

$$y(\cdot)\in C\big([\tau,+\infty);V\big)\cap L^2_{loc}\big(\tau,+\infty;D(A)\big).$$

Then following [20] one can correctly define the map $G : \mathbb{R}^+ \times H \mapsto P(H)$ by

$$G(t, y_0) = \{y(t)\colon\ y(\cdot) \text{ is a weak solution of (21) with } y(0) = y_0 \text{ such that } y(\cdot) \text{ satisfies (8)}\}. \tag{22}$$

We state a result about existence of regular solutions.

Theorem 4. *(See [20, Theorem 5].) Let Assumption* 1 *hold. Then for any $R > 0$ and $y_0 \in H$ such that $|y_0| < R$ there exists at least one weak solution of* (21) *such that*

$$y(\cdot) \in C([0, +\infty), H), \tag{23}$$

$$y(\cdot) \in L^\infty([s, T]; \mathbb{L}^4(\Omega)), \quad \textit{for all } 0 < s < T, \tag{24}$$

$$\|y(t)\|_{\mathbb{L}^4(\Omega)} \leqslant G(R, T, \delta), \tag{25}$$

for all $T > 0$, $0 < \delta < T$, and for a.a. $t \in (\delta, T)$, where $R \mapsto G(R, T, \delta)$, $T \mapsto G(R, T, \delta)$ are non-decreasing functions. Moreover, (8) *holds.*

Recall that a bounded complete trajectory of (21) is a weak solution defined on $(-\infty, \infty)$, which satisfies (8), and such that $|y(t)| \leqslant C$, for all $t \in (-\infty, \infty)$.

Theorem 5. *(See [20, pp. 261–262].) Under Assumption* 1 *the multivalued map* (22) *is a multivalued semiflow which has a strictly invariant global attractor A, consisting of all bounded complete trajectories, that is,*

$$A = \{\varphi(t)\colon\ \varphi(\cdot) \text{ is a bounded weak solution of (21) satisfying (8) and defined on } (-\infty, +\infty)\}, \tag{26}$$

where $t \in \mathbb{R}$ can be chosen arbitrarily.

Now we are ready to prove that the global attractor A of (21) and the pullback attractor $\Theta(t)$ of (3)–(5) coincide.

Theorem 6. *Under Assumption* 1 *there exists $C(R_0)$ such that if $\alpha \geqslant C(R_0)$ in* (4)*, then $A = \Theta(t)$, $\forall t \in \mathbb{R}$.*

Proof. First, we shall check that $A \subset \Theta(t)$, for any $t \in \mathbb{R}$. We know from [20, p. 259] that the ball $B_\delta = \{y \in H\colon\ |y| \leqslant R_0 + \delta\}$ is absorbing for G for any $\delta > 0$, and then it is clear that $A \subset B_{R_0}$. Also, it follows from Theorem 4 the existence of $C > 0$ such that $\|\xi\|_{\mathbb{L}^4} \leqslant C$, for any $\xi \in A$. Indeed, since $A = G(1, A)$, we can choose a weak solution $y(\cdot)$ of the Navier–Stokes system (21) satisfying (8) and such that $y(0) \in A$, $y(1) = \xi$. This solution is unique in the class of weak solutions satisfying (8) [20, p. 262]. Then (25) implies that

$$\|y(t)\|_{\mathbb{L}^4(\Omega)} \leqslant G\left(R_0, 1, \frac{1}{2}\right), \quad \text{for all } \frac{1}{2} \leqslant t \leqslant 1.$$

Choosing $C = G(R_0, 1, \frac{1}{2})$ we obtain the desired property.

Let us take $\alpha \geqslant C$ in (4). Then for any $\xi \in A$, $t \geqslant \tau$, there exist $\eta \in A$ and a weak solution $y(\cdot)$ of the 3D Navier–Stokes system (21) satisfying (8) such that $\xi = y(t)$, $\eta = y(\tau)$, $y(p) \in A$, for all $p \geqslant \tau$. So, $\|y(p)\|_{\mathbb{L}^4} \leqslant C$ and $y(\cdot)$ can be taken as a control, that is, $y(\cdot) \in \mathbb{U}_\tau$. Thus, $y(\cdot)$ is in fact the optimal control, because for $\tilde{u}(\cdot) \equiv y(\cdot)$, $\tilde{y}(\cdot) \equiv y(\cdot)$ we have $J_\tau(\tilde{u}, \tilde{y}) = 0$. So, $\xi = y(t) \in U(t, \tau, \eta) \subset U(t, \tau, A)$ and taking $\tau \to -\infty$ we deduce that $\xi \in \Theta(t)$.

Further, we shall prove that $\Theta(t) \subset A$, for any $t \in \mathbb{R}$. Let $\xi \in \Theta(t)$. Since $\Theta(t) = U(t, s, \Theta(s))$, for all $t > s$, there exists an optimal pair $\{\tilde{y}(\cdot), \tilde{u}(\cdot)\}$ of problem (3)–(5) with $\tilde{y}(s) = y_s \in \Theta(s)$, $\tilde{u}(\cdot) \in \mathbb{U}_s$, such that $\tilde{y}(t) = \xi$.

We take $s < t - 2$. In view of (9) and $|y_s| \leqslant R_0$, there exists $s < s_0 < t - 1$ such that

$$\|\tilde{y}(s_0)\|^2 \leqslant \frac{R_0^2}{\nu} + \frac{|f|^2}{\nu^2 \lambda_1} = R_1^2. \tag{27}$$

It is clear that $\tilde{y}(s_0) \in V \cap \Theta(s_0)$. From the arguments in Lemma 2 $\tilde{u}(\cdot) \in \mathbb{U}_{s_0}$ and the pair $\{\tilde{y}(\cdot), \tilde{u}(\cdot)\}$ is also an optimal pair of (3)–(5) in $[s_0, +\infty)$. But by Assumption 1 for the initial data $\tilde{y}(s_0)$ there exists a unique globally defined strong solution $y_0(\cdot)$ of the Navier–Stokes system (21). We shall check that $\{y_0(\cdot), y_0(\cdot)\}$ is an optimal process of the problem (3)–(5) with $y_0(s_0) = \tilde{y}(s_0)$, $y_0(\cdot) \in \mathbb{U}_{s_0}$.

First, we note that any strong solution of (3) satisfies (7). Using this inequality and $|\tilde{y}(s_0)| \leqslant R_0$ it is obvious that $|y_0(r)| \leqslant R_0$ for all $r \geqslant s_0$.

It is known [31, p. 382] that

$$\sup_{\|y_{s_0}\| \leqslant R_1, t \in [s_0, T]} \|y(t)\| = K(R_1, T - s_0) < +\infty, \tag{28}$$

where $y(\cdot)$ is the unique strong solution of system (3) corresponding to y_{s_0}. We note that the function K is non-decreasing with respect to both variables. Then (27) and (28) imply

$$\left\| y_0(r) \right\| \leqslant K(R_1, 2), \quad \text{for all } s_0 \leqslant r \leqslant s_0 + 2.$$

Since $y_0(\cdot)$ also satisfies (9), one can choose $\widetilde{s}_0 \in (s_0 + 1, s_0 + 2)$ such that $y_0(\widetilde{s}_0)$ satisfies (27). Hence, using again (28) we obtain that $\|y_0(r)\| \leqslant K(R_1, 2)$ for all $s_0 \leqslant r \leqslant s_0 + 3$. Repeating the same arguments inductively we obtain that $\|y_0(r)\| \leqslant K(R_1, 2)$ for all $r \geqslant s_0$. Therefore,

$$\left\| y_0(r) \right\|_{\mathbb{L}^4} \leqslant cK(R_1, 2), \quad \text{for all } r \geqslant s_0.$$

Finally, $\int_{s_0}^{\infty} \|y_0(r)\|^2 e^{-\delta r}\, dr < \infty$ follows from (9).

We choose $\alpha \geqslant \max\{G(R_0, 1, \frac{1}{2}), cK(R_1, 2)\}$. Then $y_0(\cdot) \in \mathbb{U}_{s_0}$. Since $J_{s_0}(y_0, y_0) = 0$, the pair $\{y_0(\cdot), y_0(\cdot)\}$ is an optimal process.

It follows that $J_{s_0}(\tilde{y}, \tilde{u}) = J_{s_0}(y_0, y_0) = 0$, so that $\tilde{y}(\cdot) = \tilde{u}(\cdot)$ and $\tilde{y}(\cdot)$ is a weak solution of the Navier–Stokes system (21). As $\tilde{y}(\cdot)$ is unique in the class of weak solutions satisfying $\tilde{y}(\cdot) \in L^8_{loc}(s_0, +\infty; \mathbb{L}^4(\Omega))$ [30, pp. 297–298], we have $y_0 = \tilde{y}$ on $[s_0, +\infty)$ and then $y_0(t) = \xi$. We note that $\Theta(s) = U(s, s_0, \Theta(s_0))$, for all $s \geqslant s_0$, implies that $y_0(s) \in \Theta(s)$, for all $t \geqslant s \geqslant s_0$. In the same way for some $s_1 < s_0 - 2$ we can define a weak solution (in fact strong) $y_1(\cdot)$ such that $y_1(s_0) = y_0(s_0)$, $y_1(s) \in \Theta(s)$, for all $s_1 \leqslant s \leqslant s_0$. The same can be done for some sequence $s_0 > s_1 > s_2 > \cdots > s_n \to -\infty$. Concatenating the functions $y_k(\cdot)$ we obtain a weak solution $y(\cdot)$ of (21) defined on $(-\infty, t]$ and such that $y(t) = \xi$, $y(s) \in \Theta(s)$, for all $s \leqslant t$. It is easy to see that y satisfies (8). On the other hand, in the interval $[t, +\infty)$ we take an optimal process $\{\tilde{y}(\cdot), \tilde{u}(\cdot)\}$ of (3)–(5) such that $\tilde{y}(t) = \xi$ and in the same way one can check that $\tilde{y}(\cdot)$ is the unique strong solution of (21) with $\tilde{y}(t) = \xi$. Hence, we put $y(s) = \tilde{y}(s)$ for $s \geqslant t$. The invariance property $\Theta(s) = U(s, t, \Theta(t))$ implies that $y(s) \in \Theta(s)$, for all $s \geqslant t$. By Theorem 2 we have $\Theta(s) \subset B_{R_0}$, so that the function $y(\cdot)$ is bounded on $\mathbb{R}$. It follows from (26) that $\xi = y(t) \in A$.

Therefore, $A = \Theta(t)$, for all $t \in \mathbb{R}$. $\square$

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