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Γ-convergence of multiscale periodic functionals depending on the time-derivative and the curl operator

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ABSTRACT

We study the Γ-convergence of a family of multiscale periodic quadratic integral functionals defined in a product space, whose densities depend on the time-derivative and on the curl of solenoidal fields, through the multiscale convergence in time–space and the multiscale Young measures in time–space associated with relevant sequences of pairs. An explicit representation of the Γ-limit density is given by means of an infinite dimensional minimization problem.

1. Introduction

In this work we are interested in study the asymptotic behaviour, as the parameter ε goes to 0, of the family of integral functionals F_ε defined by

$$F_\varepsilon(B,H)=\int_0^T\int_\Omega f\left(t,x,\frac{t}{\varepsilon},\frac{x}{\varepsilon^2},\partial_t B(t,x),\operatorname{curl} H(t,x)\right)dx\,dt, \tag{1.1}$$

where the density $f:(0,T)\times\Omega\times\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^{3\times 2}\to\mathbb{R}$ is a Carathéodory function satisfying the hypotheses:

(H1) $f(t,x,\cdot,y,\lambda,\rho)$ is $(0,1)$-periodic in $\mathbb{R}$;
(H2) $f(t,x,\tau,\cdot,\lambda,\rho)$ is Q-periodic in $\mathbb{R}^3$, with $Q=(0,1)^3$;
(H3) $f(t,x,\tau,y,\cdot,\cdot)$ is convex in $\mathbb{R}^{3\times 2}$;
(H4) there exist constants $C>c>0$ such that

$$c\left(|\lambda|^2+|\rho|^2-1\right)\leqslant f(t,x,\tau,y,\lambda,\rho)\leqslant C\left(|\lambda|^2+|\rho|^2+1\right),$$

for a.e. $(t,x)\in(0,T)\times\Omega$ and every $(\tau,y,\lambda,\rho)\in\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^{3\times 2}$,

with $T>0$ and Ω an open bounded set in $\mathbb{R}^3$. The functional F_ε is well defined for pairs (B,H) such that B is a vector field in $L^2(\Omega;H^1(0,T)^3)$ while H is a vector field in $L^2(0,T;X(\Omega))$. We define the space of functions $X(\Omega)$ by

$$X(\Omega)=\left\{w\in L^2\left(\Omega;\mathbb{R}^3\right):\ \operatorname{curl} w\in L^2\left(\Omega;\mathbb{R}^3\right),\operatorname{div} w=0\text{ in }\Omega,\ w\cdot\nu=0\text{ on }\partial\Omega\right\},$$

where ν stands for the outer normal vector to $\partial\Omega$, so that $X(\Omega)$ is a Hilbert space respect to the norm

$$\|w\|^2_{X(\Omega)}=\|w\|^2_{L^2(\Omega;\mathbb{R}^3)}+\|\operatorname{curl} w\|^2_{L^2(\Omega;\mathbb{R}^3)}.$$

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It turns out that $X(\Omega)$ is continuously embedded into the Sobolev space $H^1(\Omega;\mathbb{R}^3)$ if Ω is either of class $C^{1,1}$ (for definition, see [4, Notation 2.1]) or a convex polyhedron, see [4,11,13]. Thus, the space $L^2(0,T;X(\Omega))$ is defined as the space of all vector fields $H:(0,T)\times\Omega\to\mathbb{R}^3$ such that, for a.e. $t\in(0,T)$, the solenoidal field $H(t,\cdot)$ is in $X(\Omega)$ and its norm is finite:

$$\|H\|^2_{L^2(0,T;X(\Omega))}=\int_0^T\big\|H(t,\cdot)\big\|^2_{X(\Omega)}\,dt=\int_0^T\int_\Omega\left(\left|H(t,x)\right|^2+\left|\operatorname{curl}H(t,x)\right|^2\right)dx\,dt<\infty.$$

The space $L^2(0,T;X(\Omega))$ is a Hilbert space respect to the previous norm $\|.\|_{L^2(0,T;X(\Omega))}$.

Our aim is to study the asymptotic behaviour of the sequence $\{F_\varepsilon\}$ through the Γ-convergence, i.e. a variational convergence for sequences of functionals defined in metric spaces, see [7,9]. In this way, we look for the characterization of the limit functional, in the sense of Γ-convergence, of the sequence of functionals F_ε, as the parameter ε goes to 0, as an integral functional whose density may be defined through the density f. Since Young measures turn out to be a useful tool to represent weak limits of sequences composed with nonlinear functions, we intend to apply this concept to study the Γ-convergence of the sequence of functionals F_ε, following the ideas introduced in [18] for multiscale periodic functionals depending on gradient fields (see also [3,19]). Here we are interested in multiscale periodic functionals defined for pairs (B,H) and depending on the time-derivative $\partial_t B$ as well as on the curl of H. Thus, we generalize the notion of multiscale Young measure associated with sequences of time-independent functions to sequences of time-dependent functions, and develop the concept of multiscale Young measure in time–space associated with sequences of pairs of type $\{(\partial_t B_\varepsilon,\operatorname{curl}H_\varepsilon)\}$.

A motivation to study the Γ-convergence of sequences of functionals of type (1.1) is the homogenization of linear problems of type

$$\begin{cases}-\partial_t(b(t,\frac{t}{\varepsilon})\partial_t B_\varepsilon)+\operatorname{curl}(A(t,x,\frac{t}{\varepsilon},\frac{x}{\varepsilon^2})\operatorname{curl}H_\varepsilon)=\operatorname{curl}g(t,x,\frac{t}{\varepsilon},\frac{x}{\varepsilon^2}) & \text{in } (0,T)\times\Omega,\\ \operatorname{div}B_\varepsilon=\operatorname{div}H_\varepsilon=0 & \text{in } (0,T)\times\Omega,\\ H_\varepsilon\cdot\nu=0 & \text{on } (0,T)\times\partial\Omega,\\ (A\operatorname{curl}H_\varepsilon)\times\nu=g\times\nu & \text{on } (0,T)\times\partial\Omega,\\ B_\varepsilon(0,x)=B_0(x) & \text{in } \Omega,\\ \partial_t B_\varepsilon(0,x)=B_1(x) & \text{in } \Omega,\end{cases}\tag{1.2}$$

when the positive parameter ε goes to 0, see [6,21]. We assume that $b(t,\tau):(0,T)\times\mathbb{R}\to\mathbb{R}^+$ is a Carathéodory function and, for a.e. $t\in(0,T)$, $b(t,\cdot)$ is $(0,1)$-periodic in $\mathbb{R}$; the 3×3-matrix-valued function $A(t,x,\tau,y)$ is symmetric, there exist $\beta>\alpha>0$ such that $\alpha|\xi|^2\leqslant\xi^TA\xi\leqslant\beta|\xi|^2$ for every $\xi\in\mathbb{R}^3$, $A(t,x,\cdot,y)$ is $(0,1)$-periodic in $\mathbb{R}$, and $A(t,x,\tau,\cdot)$ is Q-periodic in $\mathbb{R}^3$, and the function $g(t,x,\tau,y):(0,T)\times\Omega\times\mathbb{R}\times\mathbb{R}^3\to\mathbb{R}^3$ is also $(0,1)$-periodic in the third variable and Q-periodic in the last one. Notice that, in this initial boundary value problem the quadratic coefficients b and A, and particularly the source term g, oscillate in separated time and space length scales. We would like to investigate whether the interactions between the microscopic oscillatory behaviour in both time and space variables of each coefficient may determine the macroscopic behaviour of the source term.

The initial boundary value problem (1.2) may be considered as the first-order optimality condition associated with the minimization problem in $L^2(\Omega;H^1(0,T)^3)\times L^2(0,T;X(\Omega))$ of quadratic energies of type

$$E_\varepsilon(B,H)=\int_0^T\int_\Omega\left(\frac{b_\varepsilon(t)}{2}|\partial_t B|^2+\frac{A_\varepsilon(t,x)}{2}\operatorname{curl}H\cdot\operatorname{curl}H-g_\varepsilon(t,x)\cdot\operatorname{curl}H\right)dx\,dt,\tag{1.3}$$

where the quadratic coefficients $b_\varepsilon(t)$ and $A_\varepsilon(t,x)$, and the linear coefficient $g_\varepsilon(t,x)$, are defined by

$$b_\varepsilon(t)=b\left(t,\frac{t}{\varepsilon}\right),\qquad A_\varepsilon(t,x)=A\left(t,x,\frac{t}{\varepsilon},\frac{x}{\varepsilon^2}\right),\qquad g_\varepsilon(t,x)=g\left(t,x,\frac{t}{\varepsilon},\frac{x}{\varepsilon^2}\right),$$

respectively. Thus, for each $\varepsilon>0$, if the pair $(B_\varepsilon,H_\varepsilon)$ minimizes the energy E_ε in $L^2(\Omega;H^1(0,T)^3)\times L^2(0,T;X(\Omega))$, it turns out that it is a solution of the system (1.2). In this way, we may study the homogenization of initial boundary value problems of type (1.2) through the Γ-convergence of the sequence of its associated energies of type (1.3), because once the sequence $\{E_\varepsilon\}$ Γ-converges to the quadratic functional E, we may deduce that the sequence of optimal pairs $\{(B_\varepsilon,H_\varepsilon)\}$ solution of (1.2) converges weakly to the optimal pair $\{(B,H)\}$ which minimizes the limit energy E.

This work is divided into 5 sections as follows. In Section 2, we study the multiscale convergence in time-scale of sequences of pairs of type $\{(\partial_t B_\varepsilon,\operatorname{curl}H_\varepsilon)\}$, which turns out to be fundamental to prove the main result on Γ-convergence of sequences of functionals of type (1.1). In Section 3, we introduce the definition of multiscale Young measure in time–space associated with sequences of time-dependent functions, and present the main properties which will be applied lately in Section 5 to prove the main result Theorem 4.1, which is presented in Section 4.

2. Multiscale convergence in time–space

In this section, we extend the notion of two spacial scale convergence introduced by Nguetseng in [15] (see also [1,2]) to two-scale convergence in time and space, as it is proposed by Holmbom, Svanstedt and Wellander in [14] (see also [16,22]).

Consider two separated length scales $l_1(\varepsilon)$ and $l_2(\varepsilon)$, that is two smooth functions $l_1, l_2 : (0, \varepsilon_0) \to (0, +\infty)$ such that $\lim_{\varepsilon \searrow 0} l_i(\varepsilon) = 0$ $(i = 1, 2)$, and $\lim_{\varepsilon \searrow 0} \frac{l_2(\varepsilon)}{l_1(\varepsilon)} = 0$. In order to study the oscillatory behaviour of a sequence of functions u_ε with respect to a slower time variable and a faster space variable, we test it against sequences of oscillatory functions of type $\varphi(t, x, \frac{t}{l_1(\varepsilon)}, \frac{x}{l_2(\varepsilon)})$, according to the following definition.

Definition 2.1. We say that a sequence $\{u_\varepsilon\} \subset L^2((0,T) \times \Omega)^3$ multiscale converges in time–space, with respect to two temporal and two spatial scales, to the function u_0 in $L^2((0,T) \times \Omega \times (0,1) \times Q)^3$, i.e. $u_\varepsilon \stackrel{s}{\rightharpoonup} u_0$, if

$$\lim_{\varepsilon \searrow 0} \int_0^T \int_\Omega u_\varepsilon(t,x) \cdot \varphi\left(t, x, \frac{t}{l_1(\varepsilon)}, \frac{x}{l_2(\varepsilon)}\right) dx\, dt = \int_0^T \int_\Omega \int_0^1 \int_Q u_0(t,x,\tau,y) \cdot \varphi(t,x,\tau,y)\, dy\, d\tau\, dx\, dt$$

for every φ in $L^2((0,T) \times \Omega; C_{per}((0,1) \times Q)^3)$, where $l_1(\varepsilon)$ and $l_2(\varepsilon)$ are two separated length scales.

The following compactness result ensures the existence of a multiscale convergent subsequence in time–space for every bounded sequence in $L^2((0,T) \times \Omega)^3$, see for instance [14,16].

Theorem 2.1 *(Compactness Theorem). If $\{u_\varepsilon\}$ is a bounded sequence in $L^2((0,T) \times \Omega)^3$, then there exist a subsequence $\{u_{\varepsilon_k}\}$ and a function u_0 in $L^2((0,T) \times \Omega \times (0,1) \times Q)^3$ such that $\{u_{\varepsilon_k}\}$ multiscale converges in time–space to u_0.*

Nevertheless, in this work we are mainly interested in study the multiscale convergence in time–space of sequences of pairs of type $\{(\partial_t B_\varepsilon, \operatorname{curl} H_\varepsilon)\}$, as the parameter goes to 0, when the sequence $\{B_\varepsilon\}$ is bounded in $L^2(\Omega; H^1(0,T)^3)$, and the sequence of solenoidal fields H_ε is bounded in the Hilbert space $L^2(0,T; X(\Omega))$ defined previously. Let us analyze firstly the multiscale convergence in time–space of a sequence of time-derivatives $\{\partial_t B_\varepsilon\}$.

Theorem 2.2. *Let $\{B_\varepsilon\}$ be a bounded sequence in $L^2(\Omega; H^1(0,T)^3)$. Then, there exists a subsequence $\{B_{\varepsilon_k}\}$ and a function $B \in L^2(\Omega; H^1(0,T)^3)$ such that*

$$B_{\varepsilon_k} \to B \quad \text{strongly in } L^2\big((0,T) \times \Omega\big)^3.$$

There exists a function $B_1 \in L^2((0,T) \times \Omega; H^1_{per}(0,1)^3)$, and a function $V_2 \in L^2((0,T) \times \Omega; L^2_{per}(Q)^3)$ satisfying

$$\int_Q V_2(t,x,y)\, dy = 0,$$

so that $\{\partial_t B_{\varepsilon_k}\}$ multiscale converges in time–space:

$$\partial_t B_{\varepsilon_k} \stackrel{s}{\rightharpoonup} \partial_t B(t,x) + \partial_\tau B_1(t,x,\tau) + V_2(t,x,y) \quad \text{in } L^2\big((0,T) \times \Omega \times (0,1) \times Q\big)^3.$$

Proof. There exists a subsequence $\{B_{\varepsilon_k}\}$ converging strongly to a function B in $L^2((0,T) \times \Omega)^3$, provided $L^2(\Omega; H^1(0,T)^3)$ is compactly embedded into $L^2((0,T) \times \Omega)^3$.

Consider the sequence $\{\partial_t B_{\varepsilon_k}\}$ bounded in $L^2((0,T) \times \Omega)^3$. It comes from [12, Theorem 1.2] that, for a.e. $x \in \Omega$, the sequence $\{\partial_t B_{\varepsilon_k}\}$ two-scale converges in time to $\partial_t B(t,x) + \partial_\tau B_1(t,x,\tau)$ in $L^2((0,T) \times \Omega \times (0,1))^3$, for some function B_1 in $L^2((0,T) \times \Omega; H^1_{per}(0,1))$.

Moreover, the sequence $\{\partial_t B_{\varepsilon_k}\}$ two-scale converges in space to $\partial_t B(t,x) + V_2(t,x,y)$ in $L^2((0,T) \times \Omega \times Q)^3$, for some function V_2 in $L^2((0,T) \times \Omega; L^2_{per}(Q)^3)$ such that

$$\int_Q V_2(t,x,y)\, dy = 0.$$

In this way, it follows that the sequence $\{\partial_t B_{\varepsilon_k}\}$ multiscale converges in time–space to $\partial_t B(t,x) + \partial_\tau B_1(t,x,\tau) + V_2(t,x,y)$ in $L^2((0,T) \times \Omega \times (0,1) \times Q)^3$. $\square$

Now, we focus on the sequence of solenoidal fields $\{\operatorname{curl} H_\varepsilon\}$ under the constraint that each field H_ε is itself divergence-free. The next theorem gives a characterization of the multiscale limit in time–space of such sequence.

Theorem 2.3. *Let $\{H_\varepsilon\}$ be a bounded sequence in $L^2(0,T;X(\Omega))$. Then, there exists a subsequence $\{H_{\varepsilon_k}\}$ and a function $H \in L^2(0,T;X(\Omega))$ such that*

$$H_{\varepsilon_k} \to H \quad \textit{strongly in } L^2\big((0,T)\times\Omega\big)^3.$$

There exists a function H_1 in $L^2((0,T)\times\Omega;X(Q))$ for which $\operatorname{curl}_y H_1(t,x,\cdot)$ *is Q-periodic, and a function W_2 in $L^2((0,T)\times(0,1);X(\Omega))$ for which* $\operatorname{curl} W_2(t,x,1) = \operatorname{curl} W_2(t,x,0)$,

$$\int_Q \operatorname{curl}_y H_1(t,x,y)\,dy = 0, \qquad \int_0^1 \operatorname{curl} W_2(t,x,\tau)\,d\tau = 0,$$

so that $\{\operatorname{curl} H_{\varepsilon_k}\}$ *multiscale converges in time–space*:

$$\operatorname{curl} H_{\varepsilon_k} \xrightarrow{S} \operatorname{curl} H(t,x) + \operatorname{curl}_y H_1(t,x,y) + \operatorname{curl} W_2(t,x,\tau) \quad \textit{in } L^2\big((0,T)\times\Omega\times(0,1)\times Q\big)^3.$$

Proof. Since the space $L^2(0,T;X(\Omega))$ is compactly embedded into $L^2((0,T)\times\Omega)^3$, there exists a function H in $L^2(0,T;X(\Omega))$ and a subsequence $\{H_{\varepsilon_k}\}$ such that $\{H_{\varepsilon_k}\}$ converges strongly to H in $L^2((0,T)\times\Omega)^3$.

Let us consider the sequence $\{\operatorname{curl} H_{\varepsilon_k}\}$ bounded in $L^2((0,T)\times\Omega)^3$. Applying [12, Theorem 1.2] it follows that, for a.e. $t\in(0,T)$, $\{\operatorname{curl} H_{\varepsilon_k}(t,\cdot)\}$ two-scale converges in space to $\operatorname{curl} H(t,x) + \operatorname{curl}_y H_1(t,x,y)$, for some function H_1 in $L^2((0,T)\times\Omega;X(Q))$ such that $\operatorname{curl}_y H_1(t,x,\cdot)$ is Q-periodic, and

$$\int_Q \operatorname{curl}_y H_1(t,x,y)\,dy = 0.$$

On the other hand, for a.e. $x\in\Omega$, the sequence $\{\operatorname{curl} H_{\varepsilon_k}(\cdot,x)\}$ two-scale converges in time to $\operatorname{curl} H(t,x) + U_2(t,x,\tau)$, for some function U_2 in $L^2((0,T)\times\Omega;L^2_{per}(0,1)^3)$ so that

$$\int_0^1 U_2(t,x,\tau)\,d\tau = 0.$$

If we consider test functions $\phi \in L^2((0,T);C_0^\infty(\Omega)\times C_{per}(0,1))$, then

$$\lim_{\varepsilon\searrow 0}\int_0^T\int_\Omega \operatorname{curl} H_{\varepsilon_k}(t,x)\cdot\nabla\phi\left(t,x,\frac{t}{\varepsilon}\right)dx\,dt = \int_0^T\int_\Omega\int_0^1 \big(\operatorname{curl} H(t,x) + U_2(t,x,\tau)\big)\cdot\nabla\phi(t,x,\tau)\,d\tau\,dx\,dt.$$

Integrating by parts over the variable x, we get

$$\int_0^T\int_0^1\int_{\partial\Omega} (\operatorname{curl} H + U_2)\cdot\nu\phi\,dS\,d\tau\,dt - \int_0^T\int_\Omega\int_0^1 \operatorname{div}\big(\operatorname{curl} H(t,x) + U_2(t,x,\tau)\big)\phi(t,x,\tau)\,d\tau\,dx\,dt = 0.$$

So, since $\operatorname{div} U_2 = 0$ in Ω, there exists a field W_2 such that $U_2(t,x,\tau) = \operatorname{curl} W_2(t,x,\tau)$.

Thus, the sequence $\{\operatorname{curl} H_{\varepsilon_k}\}$ multiscale converges in time–space (with respect to two temporal and two spatial scales) to $\operatorname{curl} H(t,x) + \operatorname{curl}_y H_1(t,x,y) + \operatorname{curl} W_2(t,x,\tau)$, i.e. for every $\varphi \in L^2((0,T)\times\Omega;C_{per}((0,1)\times Q))$,

$$\lim_{\varepsilon\searrow 0}\int_0^T\int_\Omega \operatorname{curl} H_{\varepsilon_k}(t,x)\cdot\varphi\left(t,x,\frac{t}{l_1(\varepsilon)},\frac{x}{l_2(\varepsilon)}\right)dx\,dt$$
$$= \int_0^T\int_\Omega\int_0^1\int_Q \big(\operatorname{curl} H(t,x) + \operatorname{curl}_y H_1(t,x,y) + \operatorname{curl} W_2(t,x,\tau)\big)\cdot\varphi(t,x,\tau,y)\,dy\,d\tau\,dx\,dt. \qquad \square$$

The previous characterization of the multiscale limit in time–space of the sequence of pairs $\{(\partial_t B_\varepsilon, \operatorname{curl} H_\varepsilon)\}$ is the key point, together with the properties of the multiscale Young measures in time–space introduced below, to achieve a full representation of the density of the Γ-limit of the sequence of functionals F_ε of type (1.1).

3. Multiscale Young measures in time–space

Here, we extend the definition of two-scale Young measure associated with sequences of time-independent functions introduced by Pedregal in [18] to sequences of time-dependent functions. Firstly, let us recall that if the length scale $l_2(\varepsilon)$ is faster than $l_1(\varepsilon)$, then the Young measure associated with the sequence of pairs

$$\left\{\left(\left\langle\frac{t}{l_1(\varepsilon)}\right\rangle, \left\langle\frac{x}{l_2(\varepsilon)}\right\rangle\right)\right\},$$

where $\langle y\rangle \in [0,1)^l$ stands for the fractional part of $y \in \mathbb{R}^l$ ($l = 1$ or $l = 3$), is the product of two Lebesgue measures $d\tau \otimes dy$ over $(0,1) \times Q$, see [17,23].

Definition 3.1. A family of probability measures $\{\mu_{t,x,\tau,y}\}_{t\in(0,T),x\in\Omega,\tau\in(0,1),y\in Q}$ supported on $\mathbb{R}^d$ is said to be the multiscale Young measure in time–space associated with the sequence of functions $u_\varepsilon : (0,T) \times \Omega \to \mathbb{R}^d$ (at the separated length scales $l_1(\varepsilon)$ and $l_2(\varepsilon)$) if the joint Young measure $\theta = \{\theta_{t,x}\}_{t\in(0,T),x\in\Omega}$ associated with the sequence of pairs

$$\left\{\left(u_\varepsilon(t,x), \left\langle\frac{t}{l_1(\varepsilon)}\right\rangle, \left\langle\frac{x}{l_2(\varepsilon)}\right\rangle\right)\right\}$$

may be decomposed, for a.e. $(t,x) \in (0,T) \times \Omega$ and $(\tau, y) \in (0,1) \times Q$, as

$$\theta_{t,x}(\lambda, \tau, y) = \mu_{t,x,\tau,y}(\lambda) \otimes d\tau \otimes dy.$$

Therefore, if $\{\mu_{t,x,\tau,y}\}_{t\in(0,T),x\in\Omega,\tau\in(0,1),y\in Q}$ is the multiscale Young measure in time–space associated with the sequence $\{u_\varepsilon\}$, we know that, for any Carathéodory function $\psi : (0,T) \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ such that the sequence $\{\psi(t,x,\langle\frac{t}{l_1(\varepsilon)}\rangle, \langle\frac{x}{l_2(\varepsilon)}\rangle, u_\varepsilon(t,x))\}$ converges weakly in $L^1((0,T)\times\Omega)$, it follows

$$\lim_{\varepsilon\searrow 0}\int_0^T\int_\Omega \psi\left(t,x,\left\langle\frac{t}{l_1(\varepsilon)}\right\rangle, \left\langle\frac{x}{l_2(\varepsilon)}\right\rangle, u_\varepsilon(t,x)\right) dt\,dx = \int_0^T\int_\Omega\int_0^1\int_Q\int_{\mathbb{R}^d} \psi(t,x,\tau,y,\lambda)\,d\mu_{t,x,\tau,y}(\lambda)\,dy\,d\tau\,dx\,dt.$$

Nevertheless, when the weak convergence in $L^1((0,T)\times\Omega)$ is not ensured, we always have a lower estimate for the lower limit given through such multiscale Young measure, as follows in the next proposition.

Proposition 3.2. *If $\{\mu_{t,x,\tau,y}\}_{t\in(0,T),x\in\Omega,\tau\in(0,1),y\in Q}$ is the multiscale Young measure associated with the sequence $\{u_\varepsilon\}$ (at the separated length scales $l_1(\varepsilon)$ and $l_2(\varepsilon)$), then*

$$\liminf_{\varepsilon\searrow 0}\int_0^T\int_\Omega \psi\left(t,x,\left\langle\frac{t}{l_1(\varepsilon)}\right\rangle, \left\langle\frac{x}{l_2(\varepsilon)}\right\rangle, u_\varepsilon(t,x)\right) dx\,dt \geqslant \int_0^T\int_\Omega\int_0^1\int_Q\int_{\mathbb{R}^d} \psi(t,x,\tau,y,\lambda)\,d\mu_{t,x,\tau,y}(\lambda)\,dy\,d\tau\,dx\,dt,$$

for every Carathéodory function $\psi : (0,T) \times \Omega \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^d \to \mathbb{R}$ bounded from below.

Notice that, the definition of multiscale Young measures in time–space is intimately related to the multiscale convergence in time–space of its associated sequence. Namely, the barycenter of a multiscale Young measure in time–space may be characterized as the multiscale limit of the associated sequence of functions.

Proposition 3.3. *Let $\{u_\varepsilon\}$ be a multiscale convergent sequence in time–space, and u_0 be its multiscale limit. If $\{\mu_{t,x,\tau,y}\}_{t\in(0,T),x\in\Omega,\tau\in(0,1),y\in Q}$ is the multiscale Young measure associated with $\{u_\varepsilon\}$, then $u_0 : (0,T) \times \Omega \times (0,1) \times Q \to \mathbb{R}^d$ is the first moment of such multiscale Young measure given by*

$$u_0(t,x,\tau,y) = \int_{\mathbb{R}^d} \lambda\,d\mu_{t,x,\tau,y}(\lambda).$$

4. The Γ-convergence result

Γ-convergence is a variational convergence for sequences of functionals introduced by De Giorgi and Franzoni in [10] as a useful tool to study the asymptotic behaviour of minimizing problems depending on a parameter (see the monographs [7,9]). The notion of Γ-convergence is based on two conditions: the lower-bound estimate for the lower-limit of the sequence of functionals, and the existence of a recovering sequence for which the lower estimate is attained.

Let us consider the sequence of integral functionals F_ε defined in the Hilbert space $L^2(\Omega; H^1(0,T)^3) \times L^2(0,T; X(\Omega))$ by

$$F_\varepsilon(B,H) = \int_0^T \int_\Omega f\left(t, x, \frac{t}{\varepsilon}, \frac{x}{\varepsilon^2}, \partial_t B(t,x), \operatorname{curl} H(t,x)\right) dx\, dt,$$

where the density $f : (0,T) \times \Omega \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^{3\times 2} \to \mathbb{R}$ is a Carathéodory function satisfying the hypotheses (H1), (H2), (H3) and (H4), with $T > 0$ and Ω an open bounded set in $\mathbb{R}^3$.

Definition 4.1. We say that the sequence $\{F_\varepsilon\}$ defined in $L^2(\Omega; H^1(0,T)^3) \times L^2(0,T; X(\Omega))$ Γ-converges, with respect to the weak topology, to the functional F if, for every B in $L^2(\Omega; H^1(0,T)^3)$ and every H in $L^2(0,T; X(\Omega))$:

(1) for any $B_\varepsilon \rightharpoonup B$ weakly in $L^2(\Omega; H^1(0,T)^3)$, and $H_\varepsilon \rightharpoonup H$ weakly in $L^2(0,T; X(\Omega))$, we have

$$\liminf_{\varepsilon \searrow 0} F_\varepsilon(B_\varepsilon, H_\varepsilon) \geqslant F(B,H);$$

(2) there exist $B_\varepsilon \rightharpoonup B$ weakly in $L^2(\Omega; H^1(0,T)^3)$, and $H_\varepsilon \rightharpoonup H$ weakly in $L^2(0,T; X(\Omega))$, such that

$$\lim_{\varepsilon \searrow 0} F_\varepsilon(B_\varepsilon, H_\varepsilon) = F(B,H).$$

Our main result here is the following explicit representation of the Γ-limit of the sequence $\{F_\varepsilon\}$ as an integral functional whose density is defined by means of an infinite dimensional minimization problem. This result is proved in the next section.

Theorem 4.1. *The sequence of integral functionals $\{F_\varepsilon\}$ defined in* (1.1) *Γ-converges, with respect to the weak topology in $L^2(\Omega; H^1(0,T)^3) \times L^2(0,T; X(\Omega))$, to the functional I defined by*

$$F(B,H) = \int_0^T \int_\Omega f_{hom}\big(t, x, \partial_t B(t,x), \operatorname{curl} H(t,x)\big)\, dx\, dt,$$

where the homogenized density $f_{hom} : (0,T) \times \Omega \times \mathbb{R}^{3\times 2} \to \mathbb{R}$ is given by

$$\begin{aligned}
&f_{hom}(t,x,\lambda,\rho) \\
&\quad = \inf\left\{ \int_0^1 \int_Q f\big(t, x, \tau, y, \lambda + \partial_\tau B_1(t,x,\tau) + V_2(t,x,y), \rho + \operatorname{curl}_y H_1(t,x,y) + \operatorname{curl} W_2(t,x,\tau)\big)\, dy\, d\tau : \right. \\
&\qquad B_1 \in L^2\big((0,T) \times \Omega; H^1_{per}(0,1)^3\big),\ V_2 \in L^2\big((0,T) \times \Omega; L^2_{per}(Q)^3\big), \\
&\qquad H_1 \in L^2\big((0,T) \times \Omega; X(Q)\big),\ W_2 \in L^2\big((0,T) \times (0,1); X(\Omega)\big), \\
&\qquad \operatorname{curl}_y H_1(t,x,\cdot) \text{ is } Q\text{-periodic},\ \operatorname{curl} W_2(t,x,\cdot) \text{ is } (0,1)\text{-periodic}, \\
&\qquad \left. \int_Q \operatorname{curl}_y H_1(t,x,y)\, dy = 0,\ \int_0^1 \operatorname{curl} W_2(t,x,\tau)\, d\tau = 0,\ \int_Q V_2(t,x,y)\, dy = 0 \right\}.
\end{aligned}$$

Notice that, the functionals F_ε are defined in the product space $L^2(\Omega; H^1(0,T)^3) \times L^2(0,T; X(\Omega))$ so that the vector field H is assumed to be divergence-free. The Γ-convergence of sequences of (time-independent) two-scale periodic integral functionals under a divergence-free constraint was studied previously in [5,8,12]. In the non-periodic context, the Γ-convergence of a particular sequence of (time-independent) non-periodic functionals under a divergence-free constraint was treated in [20] by means of div-Young measures.

As commented in the Introduction, an application to the above Γ-convergence result is the homogenization of initial boundary value problems of type (1.2). Indeed, we may deduce the convergence of solutions of a family of boundary value problems from the Γ-convergence of the families of their associated energies. If the sequence of energies F_ε Γ-converges to the functional F, and if each energy F_ε has an optimal pair $(B_\varepsilon, H_\varepsilon)$, then the sequence of minimizers $\{(B_\varepsilon, H_\varepsilon)\}$ converges weakly to a minimizer of the Γ-limit F, as follows.

Proposition 4.2. *(See [7].) If $\{F_\varepsilon\}$ is a sequence of equicoercive functionals in $L^2(\Omega; H^1(0,T)^3) \times L^2(0,T; X(\Omega))$ such that it Γ-converges, with respect to the weak topology, to F, then*

$$\min_{B,H} F(B,H) = \lim_{\varepsilon\searrow 0} \inf_{B,H} F_\varepsilon(B,H).$$

Moreover, if the sequence $\{B_\varepsilon\}$ converges weakly in $L^2(\Omega; H^1(0,T)^3)$, $\{H_\varepsilon\}$ converges weakly in $L^2(0,T; X(\Omega))$, and $\lim_{\varepsilon\searrow 0} F_\varepsilon(B_\varepsilon, H_\varepsilon) = \lim_{\varepsilon\searrow 0} \inf_{(B,H)} F_\varepsilon(B,H)$, then the weak limit of $\{(B_\varepsilon, H_\varepsilon)\}$ is a minimum of F.

5. Proof of Theorem 4.1

This section is entirely dedicated to prove our main result Theorem 4.1. For such purpose, firstly we will prove the lower limit inequality, and then prove the existence of a recovering sequence, according to Definition 4.1.

Let $B \in L^2(\Omega; H^1(0,T)^3)$ and $H \in L^2(0,T; X(\Omega))$. Consider any sequence $\{B_\varepsilon\} \subset L^2(\Omega; H^1(0,T)^3)$ converging weakly to B in $L^2(\Omega; H^1(0,T)^3)$, and any sequence $\{H_\varepsilon\} \subset L^2(0,T; X(\Omega))$ converging weakly to H in $L^2(0,T; X(\Omega))$. Thus, the sequence of pairs $\{(\partial_t B_\varepsilon, \operatorname{curl} H_\varepsilon)\}$ converges weakly to $\{(\partial_t B, \operatorname{curl} H)\}$ in $L^2((0,T)\times\Omega)^6$.

Let $\{\mu_{t,x,\tau,y}\}_{t\in(0,T),x\in\Omega,\tau\in(0,1),y\in Q}$ be the multiscale Young measure in time–space associated with the sequence of pairs $\{(\partial_t B_\varepsilon, \operatorname{curl} H_\varepsilon)\}$ with support on $\mathbb{R}^3\times\mathbb{R}^3$. Then, it comes from Proposition 3.2 that

$$\begin{aligned}
&\liminf_{\varepsilon\searrow 0} \int_0^T \int_\Omega f\left(t, x, \frac{t}{\varepsilon}, \frac{x}{\varepsilon^2}, \partial_t B_\varepsilon(t,x), \operatorname{curl} H_\varepsilon(t,x)\right) dx\,dt \\
&\quad \geqslant \int_0^T \int_\Omega \int_0^1 \int_Q \int_{\mathbb{R}^3\times\mathbb{R}^3} f(t,x,\tau,y,\lambda,\rho)\, d\mu_{t,x,\tau,y}(\lambda,\rho)\, dy\, d\tau\, dx\, dt.
\end{aligned}$$

Since the density $f(t,x,\tau,y,\cdot,\cdot)$ is convex in $\mathbb{R}^3\times\mathbb{R}^3$, for a.e. $(t,x)\in(0,T)\times\Omega$ and every $(\tau,y)\in(0,1)\times Q$, we may apply Jensen's inequality so that

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} f(t,x,\tau,y,\lambda,\rho)\, d\mu_{t,x,\tau,y}(\lambda,\rho) \geqslant f\left(t,x,\tau,y, \int_{\mathbb{R}^3\times\mathbb{R}^3} \lambda\mu_{t,x,\tau,y}(\lambda,\rho), \int_{\mathbb{R}^3\times\mathbb{R}^3} \rho\mu_{t,x,\tau,y}(\lambda,\rho)\right).$$

Notice that, the sequence $\{\partial_t B_\varepsilon\}$ multiscale converges in time–space to $\partial_t B(t,x) + \partial_\tau B_1(t,x,\tau) + V_2(t,x,y)$ in $L^2((0,T)\times\Omega\times(0,1)\times Q)^3$, for some functions B_1 in $L^2((0,T)\times\Omega; H^1_{per}(0,1))$ and V_2 in $L^2((0,T)\times\Omega; L^2_{per}(Q)^3)$ satisfying

$$\int_Q V_2(t,x,y)\, dy = 0,$$

according to Theorem 2.2. Thus, the multiscale limit of $\{\partial_t B_\varepsilon\}$ may be written as

$$\partial_t B(t,x) + \partial_\tau B_1(t,x,\tau) + V_2(t,x,y) = \int_{\mathbb{R}^3\times\mathbb{R}^3} \lambda\mu_{t,x,\tau,y}(\lambda,\rho)$$

for a.e. $(t,x)\in(0,T)\times\Omega$ and $(\tau,y)\in(0,1)\times Q$, according to Proposition 3.3.

In a similar way, provided the sequence $\{\operatorname{curl} H_\varepsilon\}$ converges weakly to $\operatorname{curl} H$ in $L^2((0,T)\times\Omega)^3$, it follows from Theorem 2.3 that $\{\operatorname{curl} H_\varepsilon\}$ multiscale converges in time–space to $\operatorname{curl} H(t,x) + \operatorname{curl}_y H_1(t,x,y) + \operatorname{curl} W_2(t,x,\tau)$ in $L^2((0,T)\times\Omega\times(0,1)\times Q)^3$, for some functions H_1 in $L^2((0,T)\times\Omega; X(Q))$ and W_2 in $L^2((0,T)\times(0,1); X(\Omega))$ such that $\operatorname{curl}_y H_1(t,x,\cdot)$ is Q-periodic, $\operatorname{curl} W_2(t,x,\cdot)$ is $(0,1)$-periodic, and

$$\int_Q \operatorname{curl}_y H_1(t,x,y)\, dy = 0, \qquad \int_0^1 \operatorname{curl} W_2(t,x,\tau)\, d\tau = 0.$$

Moreover, this multiscale limit may also be characterized through the multiscale Young measure in time–space $\{\mu_{t,x,\tau,y}\}$ in the following way

$$\operatorname{curl} H(t,x) + \operatorname{curl}_y H_1(t,x,y) + \operatorname{curl} W_2(t,x,\tau) = \int_{\mathbb{R}^3\times\mathbb{R}^3} \rho\mu_{t,x,\tau,y}(\lambda,\rho),$$

for a.e. $(t,x)\in(0,T)\times\Omega$ and every $(\tau,y)\in(0,1)\times Q$, according to Proposition 3.3.

Therefore, we conclude that there exist functions H_1, B_1, V_2 and W_2 such that

$$\begin{aligned}
&\liminf_{\varepsilon\searrow 0}\int_0^T\int_\Omega f\left(t,x,\frac{t}{\varepsilon},\frac{x}{\varepsilon^2},\partial_t B_\varepsilon(t,x),\operatorname{curl} H_\varepsilon(t,x)\right)dx\,dt\\
&\quad\geqslant \int_0^T\int_\Omega\int_0^1\int_Q f\big(t,x,\tau,y,\partial_t B(t,x)+\partial_\tau B_1(t,x,\tau)+V_2(t,x,y),\\
&\qquad\qquad \operatorname{curl} H(t,x)+\operatorname{curl}_y H_1(t,x,y)+\operatorname{curl} W_2(t,x,\tau)\big)\,dy\,d\tau\,dx\,dt.
\end{aligned}$$

If we take the infimum over such functions, we achieve the desired lower inequality:

$$\liminf_{\varepsilon\searrow 0}\int_0^T\int_\Omega f\left(t,x,\frac{t}{\varepsilon},\frac{x}{\varepsilon^2},\partial_t B_\varepsilon(t,x),\operatorname{curl} H_\varepsilon(t,x)\right)dx\,dt \geqslant \int_0^T\int_\Omega f_{hom}\big(t,x,\partial_t B(t,x),\operatorname{curl} H(t,x)\big)\,dx\,dt,$$

where f_{hom} is the homogenized density defined in Theorem 4.1.

Now, let us prove there exists a sequence of pairs $\{(B_\varepsilon,H_\varepsilon)\}$ converging weakly to (B,H) in $L^2(\Omega;H^1(0,T)^3)\times L^2(0,T;X(\Omega))$ for which the lower limit inequality above is indeed an equality. For fixed B in $L^2(\Omega;H^1(0,T)^3)$ and H in $L^2(0,T;X(\Omega))$, assume there exist minimizers H_1, B_1, V_2 and W_2 such that, for a.e. $(t,x)\in(0,T)\times\Omega$, we have

$$\begin{aligned}
&f_{hom}\big(t,x,\partial_t B(t,x),\operatorname{curl} H(t,x)\big)\\
&\quad=\int_0^1\int_Q f\big(t,x,\tau,y,\partial_t B(t,x)+\partial_\tau B_1(t,x,\tau)+V_2(t,x,y),\\
&\qquad\qquad \operatorname{curl} H(t,x)+\operatorname{curl}_y H_1(t,x,y)+\operatorname{curl} W_2(t,x,\tau)\big)\,dy\,d\tau\,dx\,dt.
\end{aligned}$$

Let us start by defining the sequence of functions $B_\varepsilon:(0,T)\times\Omega\to\mathbb{R}^3$ by putting

$$B_\varepsilon(t,x)=B(t,x)+\varepsilon B_1\left(t,x,\frac{t}{\varepsilon}\right)+\int_0^t V_2\left(s,x,\frac{x}{\varepsilon^2}\right)ds.$$

Then the sequence of time-derivatives $\partial_t B_\varepsilon:(0,T)\times\Omega\to\mathbb{R}^3$ given by

$$\partial_t B_\varepsilon(t,x)=\partial_t B(t,x)+\varepsilon\partial_t B_1\left(t,x,\frac{t}{\varepsilon}\right)+\partial_\tau B_1\left(t,x,\frac{t}{\varepsilon}\right)+V_2\left(t,x,\frac{x}{\varepsilon^2}\right)$$

converges weakly to $\partial_t B$ in $L^2((0,T)\times\Omega)^3$, and it multiscale converges in time–space to $\partial_t B(t,x)+\partial_\tau B_1(t,x,\tau)+V_2(t,x,y)$ in $L^2((0,T)\times\Omega\times(0,1)\times Q)^3$.

If we define the sequence of fields $H_\varepsilon:(0,T)\times\Omega\to\mathbb{R}^3$ by

$$H_\varepsilon(t,x)=H(t,x)+\varepsilon^2 H_1\left(t,x,\frac{x}{\varepsilon^2}\right)+W_2\left(t,x,\frac{t}{\varepsilon}\right),$$

so that

$$\operatorname{curl} H_\varepsilon(t,x)=\operatorname{curl} H(t,x)+\varepsilon^2\operatorname{curl} H_1\left(t,x,\frac{x}{\varepsilon^2}\right)+\operatorname{curl}_y H_1\left(t,x,\frac{x}{\varepsilon^2}\right)+\operatorname{curl} W_2\left(t,x,\frac{t}{\varepsilon}\right)$$

for a.e. $(t,x)\in(0,T)\times\Omega$, the sequence $\{\operatorname{curl} H_\varepsilon\}$ converges weakly to $\operatorname{curl} H$ in $L^2((0,T)\times\Omega)^3$, and it multiscale converges in time–space to $\operatorname{curl} H(t,x)+\operatorname{curl}_y H_1(t,x,y)+\operatorname{curl} W_2(t,x,\tau)$ in $L^2((0,T)\times\Omega\times(0,1)\times Q)^3$.

Therefore, if we consider the sequence of the pairs $\{\partial_t B_\varepsilon,\operatorname{curl} H_\varepsilon\}$ just defined above, it follows that

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\int_0^T\int_\Omega f\left(t,x,\frac{t}{\varepsilon},\frac{x}{\varepsilon^2},\partial_t B_\varepsilon(t,x),\operatorname{curl} H_\varepsilon(t,x)\right)dx\,dt\\
&\quad=\int_0^T\int_\Omega\int_0^1\int_Q f\big(t,x,\tau,y,\partial_t B(t,x)+\partial_\tau B_1(t,x,\tau)+V_2(t,x,y),
\end{aligned}$$

$$\operatorname{curl} H(t,x) + \operatorname{curl}_y H_1(t,x,y) + \operatorname{curl} W_2(t,x,\tau)\big)\, dy\, d\tau\, dx\, dt$$

$$= \int_0^T \int_\Omega f_{hom}\big(t, x, \partial_t B(t,x), \operatorname{curl} H(t,x)\big)\, dx\, dt,$$

which concludes the proof of Theorem 4.1.

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