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A note on Liouvillian integrability

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A B S T R A C T

There was the belief that if a planar polynomial differential system has a Liouvillian first integral, then it has some finite invariant algebraic curves. In this note we provide a counterexample to this belief.

1. Introduction

Let $\mathbb{C}[x, y]$ be the ring of all polynomials in the variables x and y with coefficients in $\mathbb{C}$.

By definition a *complex planar polynomial differential system* or simply a *polynomial system* is a differential system of the form

$$\frac{dx}{dt} = \dot{x} = P(x, y), \qquad \frac{dy}{dt} = \dot{y} = Q(x, y), \tag{1}$$

where the dependent variables x and y are complex, and the independent one (the *time*) t can be real or complex, and $P, Q \in \mathbb{C}[x, y]$. We denote by $m = \max\{\deg P, \deg Q\}$ the *degree* of the polynomial system.

Alternatively, we can view the polynomial system (1) as defining a complex polynomial foliation with singularities on $\mathbb{CP}^2$ of degree n. Such a foliation has an invariant line at infinity and, conversely, any such foliation can be brought to a polynomial system of the form (1).

Let $f = f(x, y) = 0$ be an algebraic curve in $\mathbb{C}^2$. We say that it is *invariant* or that it is a *finite invariant algebraic curve* by the polynomial system (1) if

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = kf, \tag{2}$$

for some polynomial $k = k(x, y) \in \mathbb{C}[x, y]$, called the *cofactor* of the algebraic curve $f = 0$. Note that the degree of the polynomial k is at most $m - 1$. From (2) it is immediate to check that the algebraic curve $f = 0$ is formed by trajectories of the polynomial system (1).

Let $h, g \in \mathbb{C}[x, y]$ and assume that h and g are relatively prime in the ring $\mathbb{C}[x, y]$. Then the function $\exp(g/h)$ is called an *exponential factor* of the polynomial system (1) if for some polynomial $k \in \mathbb{C}[x, y]$ of degree at most $m - 1$ it satisfies equation

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$$P\frac{\partial \exp(g/h)}{\partial x}+Q\frac{\partial \exp(g/h)}{\partial y}=k\exp(g/h). \tag{3}$$

If $\exp(g/h)$ is an exponential factor it is easy to show that $h=0$ is an invariant algebraic curve. An exponential factor appears when an invariant algebraic curve has in some sense multiplicity larger than 1. An exponential factor plays the same role than an invariant algebraic curve in order to obtain a first integral for a polynomial system. For more details on exponential factors see [4].

Let U be an open subset of $\mathbb{C}^2$. We say that a non-constant function $H: U \to \mathbb{C}$ is a *first integral* of the polynomial system (1) in U if H is constant on the trajectories of the polynomial system (1) contained in U; i.e. if

$$P\frac{\partial H}{\partial x}+Q\frac{\partial H}{\partial y}=0,$$

in the points $(x,y)\in U$.

We say that a non-constant function $R: U \to \mathbb{C}$ is an *integrating factor* of the polynomial system (1) in U if R satisfies that

$$\frac{\partial (RP)}{\partial x}+\frac{\partial (RQ)}{\partial y}=0,$$

in the points $(x,y)\in U$.

Due to the algebraic structure of a polynomial system there is a close relationship between its integrability (a topological phenomena) and its exact algebraic solutions. Darboux [5] in 1878 proved that if a polynomial system of degree m has at least $m(m+1)/2+1$ invariant algebraic curves, then it has a first integral which can be expressed by means of these invariant algebraic curves in the following way:

$$H(x,y)=\prod_{i=1}^{q} f_i^{\lambda_i}(x,y),$$

where $\lambda_i \in \mathbb{C}$ and $f_i(x,y)=0$ are invariant algebraic curves of system (1). This kind of first integral is called a *Darboux first integral*.

Jouanolou [10] in 1979 showed that if the number of invariant algebraic curves is at least $[m(m+1)/2]+2$, then the first integral is rational, and consequently all the trajectories of the vector field are contained in invariant algebraic curves.

Prelle and Singer [11] demonstrated in 1983 that if a polynomial system has an elementary first integral, then this integral can be computed using the invariant algebraic curves of the system; in particular they showed that this polynomial system admits an integrating factor R such that R^N, with $N\in\mathbb{Z}$, is a rational function with coefficients in $\mathbb{C}$, see [11]. Later on, in 1992 Singer [12] showed that if a polynomial system has a Liouvillian first integral then the system has an integrating factor of the form

$$R(x,y)=\exp\left(\int_{(x_0,y_0)}^{(x,y)} U(x,y)\,dx+V(x,y)\,dy\right), \tag{4}$$

where U and V are rational functions which verify $\partial U/\partial y=\partial V/\partial x$. In 1999 Christopher [2] showed that the integrating factor (4) can be written into the form

$$R=\exp(g/h)\prod f_i^{\lambda_i}, \tag{5}$$

where g, h and f_i are polynomials and $\lambda_i\in\mathbb{C}$. For another proof of the explicit expression for the inverse integrating factor, see page 26 of [3]. This condition guarantees the existence of a first integral that can be expressed by quadratures of elementary functions (Liouville function). This type of integrability is known since then as *Liouville integrability theory*. The cited works [11,12] give a criterion for knowing when a polynomial system has an elementary first integral or a Liouvillian first integral. An important fact following from those results is that invariant algebraic curves and exponential factors play a key role in these criteria.

Non-algebraic invariant curves with polynomial cofactor can also be used in order to find a first integral for a system. This observation permits the generalization of the Darboux integrability theory given in [7–9] where a new kind of first integrals, not only the Liouvillian ones, is described.

There was the belief that a Liouvillian integrable system has always an invariant algebraic curve in $\mathbb{C}^2$. Moreover, this claim was proved under certain hypotheses, see [13]. The aim of this paper is to prove that there exist Liouvillian integrable polynomial systems without any finite invariant algebraic curve. To obtain this result we provide a Liouvillian integrable planar polynomial system of degree 2 in $\mathbb{C}^2$ without finite invariant algebraic curves.

2. The construction of the example

We note that $f_i = 0$ and $h = 0$ in (5) are invariant algebraic curves and $\exp(g/h)$ is an exponential factor for system (1), for more details see [1]. Therefore if there exist an example of a Liouvillian integrable planar polynomial system without finite invariant algebraic curves, then it must have an integrating factor of the form $R = \exp(g(x, y))$, where g is a polynomial. Note that $g = 0$ does not need to be an invariant algebraic curve of system (1). From [4] $\exp(g(x, y))$ is an exponential of the invariant line at infinity.

The first idea for finding our example is looking for it inside the class of linear differential systems. But inside this class of polynomial systems such an example cannot exist as it follows from the next result.

Proposition 1. *Every polynomial system of degree* 1 *has some finite invariant algebraic curve.*

Proof. If the planar linear differential system has more than one singular point, it has an invariant straight line of singular points, and the proposition is proved. So we can assume that the linear differential system has a unique singular point. Without loss of generality we can assume that this singular point is the origin of coordinates. Then the linear system becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix},$$

with determinant of the 2×2 matrix A different from zero. Again without loss of generality we can assume that A is of one of the following two Jordan normal forms

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

with λ_1, λ_2 and λ in $\mathbb{C}$. Clearly in both cases the straight line $y = 0$ is invariant. This completes the proof. □

Consequently, if we want a Liouvillian integrable system without finite invariant algebraic curves, we must consider polynomial systems of degree 2 (quadratic systems) or higher. Starting with quadratic systems, one could hope to find a Liouvillian integrable system without finite invariant algebraic curves with $g(x, y)$ linear, i.e. the case when $g(x, y) = ax + by$. Note that we can consider the case $g(0, 0) = 0$, because the case $g(0, 0) \neq 0$ provides an irrelevant multiplicative constant in the integrating factor R. Proposition 2 shows that this is not possible.

Proposition 2. *There is no quadratic systems with an integrating factor of the form* $\exp(ax + by)$*, without finite invariant algebraic curves.*

Proof. We write the quadratic system as

$$\dot{x} = \sum_{i+j=0}^{2} a_{ij} x^i y^j, \qquad \dot{y} = \sum_{i+j=0}^{2} b_{ij} x^i y^j, \tag{6}$$

with i and j non-negative integers. Then, without loss of generality we can assume that the integrating factor is of the form $\exp(x + by)$. Therefore system (7) becomes

$$\begin{aligned} \dot{x} &= b_{11} - bb_{00} - b_{01} + bb_{10} - 2bb_{20} + (2bb_{20} - bb_{10} - b_{11})x \\ &\quad + (bb_{11} - bb_{01} - 2b_{02})y - bb_{20}x^2 - bb_{11}xy - bb_{02}y^2, \\ \dot{y} &= b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2. \end{aligned} \tag{7}$$

This system has the invariant algebraic curve

$$b_{00} - b_{10} + 2b_{20} + (b_{10} - 2b_{20})x + (b_{01} - b_{11})y + b_{20}x^2 + b_{11}xy + b_{02}y^2 = 0.$$

Hence the proposition is proved. □

If we look for quadratic systems having an integrating factor of the form $\exp(ax^2 + bxy + cx^2)$ and without finite invariant algebraic curves, we can find them, as the following result shows.

Theorem 3. *Consider the quadratic system*

$$\dot{x} = -1 - x(2x + y), \qquad \dot{y} = 2x(2x + y). \tag{8}$$

This system is a Liouvillian integrable and has no finite invariant algebraic curves.

Theorem 3 is proved in the next section.

3. Proof of Theorem 3

3.1. Existence of a Darboux integrating factor

The following result shows that system (8) is Liouvillian integrable.

Proposition 4. *System* (8) *is Liouvillian integrable because it has the integrating factor* $R = e^{-(2x+y)^2/4}$, *and the first integral*

$$H = 2e^{-\frac{1}{4}(2x+y)^2}x - \sqrt{\pi}\,\mathrm{erf}\left(\frac{1}{2}(2x+y)\right), \tag{9}$$

where $\mathrm{erf}(z)$ *is the error function, i.e. the integral of the Gaussian distribution, given by*

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt.$$

Proof. Easy computations show that the function $R = e^{-(2x+y)^2/4}$ verifies the equation

$$\frac{\partial(R(-1-x(2x+y)))}{\partial x} + \frac{\partial(R(2x(2x+y)))}{\partial y} = 0,$$

and therefore R is an integrating factor of system (8), and consequently system (8) is Liouvillian integrable with a first integral H satisfying that

$$\dot{x} = R\big(-1-x(2x+y)\big) = \frac{\partial H}{\partial y}, \qquad \dot{y} = R\big(2x(2x+y)\big) = -\frac{\partial H}{\partial x}.$$

An easy computation shows that this H is the one given in (9). $\square$

3.2. Non-existence of algebraic invariant curves

In this subsection we establish that system (8) has no finite invariant algebraic curves.

Proposition 5. *System* (8) *has no finite invariant algebraic curves.*

Proof. We force that $f(x,y) = 0$, where f is a polynomial of degree $n \geqslant 1$, be an invariant algebraic curve of system (8). So f must satisfy

$$\big(-1-2x^2-xy\big)\frac{\partial f}{\partial x} + 2x(2x+y)\frac{\partial f}{\partial y} = kf, \tag{10}$$

where k, the cofactor of the curve $f = 0$, is a polynomial of degree 1, and we write $k = ax + by + c$.

Now we develop the invariant curve f in homogeneous terms into the form

$$f = f_n(x,y) + f_{n-1}(x,y) + \cdots + f_0(x,y), \tag{11}$$

where $f_i = f_i(x,y)$ is a homogeneous polynomial of degree i, for $i = 0, 1, \ldots, n$. We substitute the expression of f given by (11) into Eq. (10), and we consider the homogeneous term of largest degree which is of degree $n+1$ and that is given by

$$-x(2x+y)\frac{\partial f_n}{\partial x} + 2x(2x+y)\frac{\partial f_n}{\partial y} = (ax+by)f_n.$$

The general solution of this partial differential equation is

$$f_n(x,y) = \exp\left(\frac{(2b-a)x}{2x+y}\right)x^{-b}g(2x+y),$$

where g is an arbitrary C^1 function. Since $f_n(x,y)$ must be a homogeneous polynomial of degree n, first we have that $a = 2b$, and after that we obtain that

$$f_n(x,y) = c_n x^{-b}(2x+y)^{n+b},$$

with b an integer in $\{-n, -(n-1), \ldots, 0\}$, and c_n a non-zero constant.

Now we consider the terms of degree n of (10), i.e.

$$-x(2x+y)\frac{\partial f_{n-1}}{\partial x}+2x(2x+y)\frac{\partial f_{n-1}}{\partial y}=(ax+by)f_{n-1}+cf_n.$$

The general solution of this partial differential equation is

$$f_{n-1}(x,y)=x^{-b}\big(g(2x+y)-cc_n(2x+y)^{b+n-1}\log x\big),$$

where g is an arbitrary C^1 function. Since $c_n\neq 0$ and $f_{n-1}(x,y)$ must be a homogeneous polynomial of degree $n-1$, we get that $c=0$, and

$$f_{n-1}(x,y)=c_{n-1}x^{-b}(2x+y)^{n+b-1},$$

with b an integer in $\{-(n-1),-(n-2),\dots,0\}$ if c_{n-1} is a non-zero constant.

The terms of degree $n-1$ of (10) are

$$-x(2x+y)\frac{\partial f_{n-2}}{\partial x}+2x(2x+y)\frac{\partial f_{n-2}}{\partial y}=b(2x+y)f_{n-2}+\frac{\partial f_n}{\partial x}.$$

The general solution of this partial differential equation is

$$f_{n-2}(x,y)=x^{-b-1}\big(xg(2x+y)-c_n(2x+y)^{b+n-2}\big(b(2x+y)+2(b+n)x\log(x)\big)\big),$$

where g is an arbitrary C^1. Since $f_{n-2}(x,y)$ must be a homogeneous polynomial of degree $n-2$, we get that $b=-n$. Substituting $b=-n$ in $f_{n-1}(x,y)$ we obtain $c_{n-1}=0$. Therefore

$$f_{n-2}(x,y)=x^n g(2x+y)+nc_nx^{n-1}(2x+y)^{-1}.$$

In contradiction with the fact that $f_{n-2}(x,y)$ must be a homogeneous polynomial of degree $n-2$ and $c_n\neq 0$. This completes the proof of the proposition. □

Proof of Theorem 3. It follows directly from Propositions 4 and 5. □

4. Phase portrait of system (8)

In this section we study the phase portrait of the quadratic system (8). Clearly this system has no finite singularities.

Let $X\in\mathcal{P}_n(\mathbb{R}^2)$ be a planar vector field of degree n. The *Poincaré compactified vector field $p(X)$ corresponding to X* is an analytic vector field induced on $\mathbb{S}^2$ as follows (see for instance Chapter 5 of [6]). Let $\mathbb{S}^2=\{y=(y_1,y_2,y_3)\in\mathbb{R}^3\colon\ y_1^2+y_2^2+y_3^2=1\}$ (the *Poincaré sphere*) and $T_y\mathbb{S}^2$ be the tangent space to $\mathbb{S}^2$ at point y. Consider the central projection $f:T_{(0,0,1)}\mathbb{S}^2\to\mathbb{S}^2$. This map defines two copies of X, one in the northern hemisphere and the other in the southern hemisphere. Denote by X' the vector field $Df\circ X$ defined on $\mathbb{S}^2$ except on its equator $\mathbb{S}^1=\{y\in\mathbb{S}^2\colon\ y_3=0\}$. Clearly $\mathbb{S}^1$ is identified to the *infinity* of $\mathbb{R}^2$. In order to extend X' to a vector field on $\mathbb{S}^2$ (including $\mathbb{S}^1$) it is necessary that X satisfies suitable conditions. In the case that $X\in\mathcal{P}_n(\mathbb{R}^2)$, $p(X)$ is the only analytic extension of $y_3^{n-1}X'$ to $\mathbb{S}^2$. On $\mathbb{S}^2\backslash\mathbb{S}^1$ there are two symmetric copies of X, and knowing the behaviour of $p(X)$ around $\mathbb{S}^1$, we know the behaviour of X at infinity. The projection of the closed northern hemisphere of $\mathbb{S}^2$ on $y_3=0$ under $(y_1,y_2,y_3)\mapsto(y_1,y_2)$ is called the *Poincaré disc*, and it is denoted by $\mathbf{D}^2$. The Poincaré compactification has the property that $\mathbb{S}^1$ is invariant under the flow of $p(X)$.

As $\mathbb{S}^2$ is a differentiable manifold, for computing the expression for $p(X)$, we can consider the six local charts $U_i=\{y\in\mathbb{S}^2\colon\ y_i>0\}$, and $V_i=\{y\in\mathbb{S}^2\colon\ y_i<0\}$ where $i=1,2,3$; and the diffeomorphisms $F_i:U_i\to\mathbb{R}^2$ and $G_i:V_i\to\mathbb{R}^2$ for $i=1,2,3$ are the inverses of the central projections from the planes tangent at the points $(1,0,0)$, $(-1,0,0)$, $(0,1,0)$, $(0,-1,0)$, $(0,0,1)$ and $(0,0,-1)$ respectively. If we denote by $z=(z_1,z_2)$ the value of $F_i(y)$ or $G_i(y)$ for any $i=1,2,3$ (so z represents different things according to the local charts under consideration), then some easy computations give for $p(X)$ the following expressions:

$$z_2^n\Delta(z)\left(Q\left(\frac{1}{z_2},\frac{z_1}{z_2}\right)-z_1P\left(\frac{1}{z_2},\frac{z_1}{z_2}\right),-z_2P\left(\frac{1}{z_2},\frac{z_1}{z_2}\right)\right)\quad\text{in } U_1,$$
$$z_2^n\Delta(z)\left(P\left(\frac{z_1}{z_2},\frac{1}{z_2}\right)-z_1Q\left(\frac{z_1}{z_2},\frac{1}{z_2}\right),-z_2Q\left(\frac{z_1}{z_2},\frac{1}{z_2}\right)\right)\quad\text{in } U_2,$$
$$\Delta(z)\big(P(z_1,z_2),Q(z_1,z_2)\big)\quad\text{in } U_3,$$

where $\Delta(z)=(z_1^2+z_2^2+1)^{-\frac{1}{2}(n-1)}$. The expression for V_i is the same as that for U_i except for a multiplicative factor $(-1)^{n-1}$. In these coordinates for $i=1,2$, $z_2=0$ always denotes the points of $\mathbb{S}^1$. Rescaling the independent variable we can omit the factor $\Delta(z)$ by rescaling the vector field $p(X)$. Thus we obtain a polynomial vector field in each local chart.

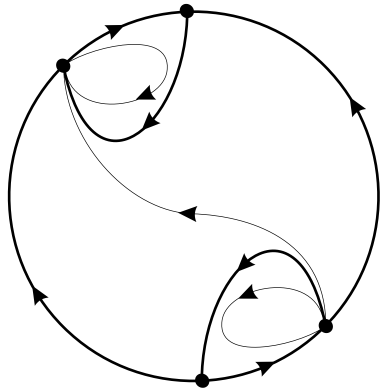

Fig. 1. Phase portrait of the quadratic system (8) in the Poincaré disc.

This phase portrait has two pairs of points at infinity. Two are saddles and the other two are highly degenerate singular points formed by two elliptic and parabolic sectors.

Using the Poincaré compactification we obtain the phase portrait of Fig. 1 in the Poincaré disc for the quadratic system (8).

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