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On the exponential decay of the Euler–Bernoulli beam with boundary energy dissipation ☆

Barbara Lazzari *, Roberta Nibbi

Dipartimento di Matematica, Alma Mater Studiorum – Università di Bologna, Piazza di Porta S. Donato 5, 40126 Bologna, Italy

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ABSTRACT

We study the asymptotic behavior of the Euler–Bernoulli beam which is clamped at one end and free at the other end. We apply a boundary control with memory at the free end of the beam and prove that the "exponential decay" of the memory kernel is a necessary and sufficient condition for the exponential decay of the energy.

1. Introduction

In this paper we study the long time behavior of the Euler–Bernoulli beam clamped at one end and free at the other end (cantilever beam). Here, for simplicity and without loss of generality, we assume the length, the density and the flexural rigidity of the beam equal to 1.

The dynamic problem in $(0,1)\times\mathbb{R}^+$ is therefore described by the well-known equation of motion

$$u_{tt}(x,t)+u_{xxxx}(x,t)=0, \tag{1.1}$$

together with the boundary conditions

$$u(0,t)=u_x(0,t)=0 \tag{1.2}$$

and

$$u_{xx}(1,t)=\beta(t), \qquad u_{xxx}(1,t)=\Gamma(t),$$

where $\beta(t)$ and $\Gamma(t)$ are boundary control terms applied to the free end of the beam.

The boundary feedback stabilization problem of this model, that is the problem of finding boundary controls capable to guarantee the exponential stability, has been studied at length (see [1–5] and references therein).

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* Corresponding author. Fax: +390512094490.
E-mail addresses: lazzari@dm.unibo.it (B. Lazzari), nibbi@dm.unibo.it (R. Nibbi).

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In this paper we choose the following boundary control with memory

$$u_{xx}(1,t)=0, \qquad u_{xxx}(1,t)=\gamma_0 u_t(1,t)+\int_0^\infty \lambda(s)u_t(1,t-s)\,\mathrm{d}s, \tag{1.3}$$

where $\gamma_0\in\mathbb{R}^+$ and the memory kernel $\lambda:\mathbb{R}^+\to\mathbb{R}$ belongs to $L^1(\mathbb{R}^+)\cap H^2(\mathbb{R}^+)$.

This control has been already proposed in [6] and [7] where, in presence of further structural dampings, it has been proved that the energy has the same rate of decay (exponential or polynomial) of the memory kernel.

Here, generalizing energy estimates obtained in [1] for the boundary control

$$u_{xx}(1,t)=0, \qquad u_{xxx}(1,t)=\gamma_0 u_t(1,t), \quad \gamma_0>0, \tag{1.4}$$

we prove that, whenever the memory kernel decays exponentially, so does the energy of the system.

It is interesting to observe that boundary condition (1.4) ensures the exponential decay of the energy for the cantilever beam, while, in the presence of a memory term at the boundary, such a result is not assured. Indeed, we shall show that the condition

$$\int_0^\infty e^{\delta t}\lambda(t)\,\mathrm{d}t<\infty \tag{1.5}$$

for some $\delta>0$, turns out to be necessary for the exponential decay of the solution.

Finally, we observe that this control can be also seen as a variation of the case of a mass attached to the free end of the beam [8,9]. If, in fact, we choose an exponential function as memory kernel, by differentiating (1.3) with respect to time, we obtain

$$u_{xxx}(1,t)-mu_{tt}(1,t)=\alpha u_t(1,t)-\beta u_{xxxt}(1,t).$$

The outline of the paper is the following.

In Section 2 we prove existence, uniqueness and regularity of solutions for the related initial boundary problem via semigroup theory. In Section 3, after developing the needed estimates, we prove that the exponential decay of the memory kernel turns out to be a necessary and sufficient condition for the exponential decay of the energy.

2. Well posedness

Let us consider problem (1.1)–(1.3) together with the initial conditions

$$u(x,0)=u_0(x), \qquad u_t(x,0)=v_0(x), \quad x\in(0,1). \tag{2.1}$$

From now on, whenever no ambiguity arises, we shall drop the x variable and we shall refer to problem (1.1)–(1.3) and (2.1) as to *problem P*.

The aim of this section is the proof of the well-posedness of *problem P* via semigroup theory. To this end we introduce the *past history*

$$w(1,t-s)=u(1,t-s)-u(1,t)$$

and rewrite the boundary control term in (1.3) as follows

$$u_{xxx}(1,t)=\gamma_0 u_t(1,t)+\int_0^\infty \lambda'(s)w(1,t-s)\,\mathrm{d}s. \tag{2.2}$$

Evolution problems presenting boundary controls of memory type similar to (2.2) have been studied in several fields, making use of the concepts of dissipative boundary and boundary energy (see [10] and references therein), and the related solutions have been usually found among those with "finite energy". It should however be noted that in presence of memory, the energy-type functional turns out to be non-unique.

Following this approach, (2.2) is compatible with the definition of dissipative boundary if the memory kernel satisfies

$$\omega\int_0^\infty \lambda'(s)\sin(\omega s)\,\mathrm{d}s<0, \quad \omega\neq 0. \tag{2.3}$$

If (2.3) holds, reasoning as in [10], it is possible to prove the well-posedness of *problem P* in the past histories space $\mathcal{H}_1$, for which the energy functional

$$\frac{1}{2}\int_0^\infty\int_0^\infty \frac{\partial^2\lambda(|s_1-s_2|)}{\partial s_1\partial s_2} w(1,t-s_1)w(1,t-s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2 \tag{2.4}$$

is finite. On the other hand this result is not satisfactory, because $\mathcal{H}_1$ does not contain even all the bounded histories (indeed [11] sinusoidal histories do not belong to $\mathcal{H}_1$) and it is therefore desirable to obtain well-posedness results in wider past histories spaces.

Certainly the space $\mathcal{H}_2$ of the past histories for which

$$-\frac{1}{2}\int_0^\infty \lambda'(s)\big|w(1,t-s)\big|^2\,\mathrm{d}s<\infty \tag{2.5}$$

contains at least all the bounded histories. However, to obtain well-posedness results in this space the memory kernel must satisfy the following more restrictive hypotheses

$$\lambda'(s)<0,\qquad \lambda''(s)\geqslant 0,\quad s\in\mathbb{R}^+. \tag{2.6}$$

As observed in [12], it is not necessary to know the past history w at all times, because two different histories w_1 and w_2 satisfying

$$\int_0^\infty \lambda'(\tau+s)w_1(1,t-\tau)\,\mathrm{d}\tau=\int_0^\infty \lambda'(\tau+s)w_2(1,t-\tau)\,\mathrm{d}\tau\quad \forall s\in\mathbb{R}^+,$$

lead to the same boundary control term in (2.2). It is therefore convenient a formulation which relies only on the minimal information required to determine the boundary control. To this end, here we study *problem P* in terms of the new variable

$$\breve{a}^t(1,s)=-\int_0^\infty \lambda'(\tau+s)w(1,t-\tau)\,\mathrm{d}\tau,\quad s\in\mathbb{R}^+, \tag{2.7}$$

so that (2.2) takes the form

$$u_{xxx}(1,t)-\gamma_0u_t(1,t)=-\breve{a}^t(1,0). \tag{2.8}$$

By introducing the functional

$$\psi_b(t)=-\frac{1}{2}\int_0^\infty \frac{1}{\lambda'(s)}\left|\frac{\partial\breve{a}^t(1,s)}{\partial s}\right|^2\mathrm{d}s, \tag{2.9}$$

we are able to achieve well-posedness results in a space wider than $\mathcal{H}_2$, leaving unchanged the hypotheses (2.6) on the kernel. In fact the following proposition holds.

Proposition 2.1. *Let $w\in\mathcal{H}_2$ and $\breve{a}^t$ be defined in terms of w through* (2.7)*, then the functional* (2.9) *is finite.*

Proof. Rewriting (2.9) in terms of w and using classical inequalities, it follows that

$$\begin{aligned}\psi_b(t)&=-\frac{1}{2}\int_0^\infty \frac{1}{\lambda'(s)}\left|\int_0^\infty \lambda''(\tau+s)w(1,t-\tau)\,\mathrm{d}\tau\right|^2\mathrm{d}s\\&\leqslant-\frac{1}{2}\int_0^\infty \frac{\int_0^\infty\lambda''(\tau+s)\,\mathrm{d}\tau}{\lambda'(s)}\left(\int_0^\infty \lambda''(\tau+s)\big|w(1,t-\tau)\big|^2\,\mathrm{d}\tau\right)\mathrm{d}s\\&=\frac{1}{2}\int_0^\infty\int_0^\infty \lambda''(\tau+s)\big|w(1,t-\tau)\big|^2\,\mathrm{d}\tau\,\mathrm{d}s=-\frac{1}{2}\int_0^\infty \lambda'(\tau)\big|w(1,t-\tau)\big|^2\,\mathrm{d}\tau.\end{aligned}$$

The thesis follows immediately from the hypothesis $w\in\mathcal{H}_2$. $\square$

Moreover, the functional ψ_b satisfies

$$\dot{\psi}_b(t) = -\breve{a}^t(1,0)u_t(1,t) - \frac{1}{2}\int_0^\infty \frac{\lambda''(s)}{[\lambda'(s)]^2}\left|\frac{\partial \breve{a}^t(1,s)}{\partial s}\right|^2 \mathrm{d}s + \frac{1}{2}\frac{1}{\lambda'(0)}\left|\frac{\partial \breve{a}^t(1,0)}{\partial s}\right|^2, \tag{2.10}$$

since the derivative of $\breve{a}^t$ with respect to t is given by

$$\frac{\partial}{\partial s}\breve{a}^t(1,s) - \lambda(s)v(1,t).$$

Finally we observe that

$$\begin{aligned}\left|\breve{a}^t(1,s)\right|^2 &= \left|-\int_0^\infty \frac{\partial \breve{a}^t(1,s+\tau)}{\partial \tau}\,\mathrm{d}\tau\right|^2 \\ &\leqslant \int_0^\infty -\lambda'(s+\tau)\,\mathrm{d}\tau \int_0^\infty -\frac{1}{\lambda'(s+\tau)}\left|\frac{\partial \breve{a}^t(1,s+\tau)}{\partial \tau}\right|^2 \mathrm{d}\tau \leqslant 2\lambda(s)\psi_b(t);\end{aligned} \tag{2.11}$$

in particular (2.8) and (2.11) yield

$$\left|u_{xxx}(1,t)\right|^2 \leqslant 2\gamma_0^2\left|u_t(1,t)\right|^2 + 4\lambda(0)\psi_b(t). \tag{2.12}$$

Let us now define $v = u_t$ and introduce the state $\sigma = (v, u_{xx}, \breve{a}^t)$ and the total energy

$$\psi(t) = \underbrace{\frac{1}{2}\int_0^1 \left[\left|v(t)\right|^2 + \left|u_{xx}(t)\right|^2\right]\mathrm{d}x}_{\psi_\Omega(t)} \underbrace{- \frac{1}{2}\int_0^\infty \frac{1}{\lambda'(s)}\left|\frac{\partial \breve{a}^t(1,s)}{\partial s}\right|^2 \mathrm{d}s}_{\psi_b(t)}. \tag{2.13}$$

This energy coincides with the energy proposed in [8] when the memory kernel is an exponential function.

We rewrite the *problem P* as an abstract first-order Cauchy problem as follows:

$$\begin{cases}\dot{\sigma}(t) = A\sigma(t),\\ \sigma(0) = \sigma_0\end{cases} \tag{2.14}$$

with $\sigma_0 = (v_0, u_{0xx}, \breve{a}^0(1,\cdot))$ and

$$A\sigma(t) = \left(-u_{xxxx}(t), v_{xx}(t), \frac{\partial \breve{a}^t(1,s)}{\partial s} - \lambda(s)v(1,t)\right). \tag{2.15}$$

As said before, the natural setting in which to look for existence and uniqueness of solutions for problem (2.14) is the *admissible states space* $\mathcal{K}$, consisting in those states σ for which the total energy (2.13) is finite. We endow $\mathcal{K}$ with the inner product

$$\left\langle \sigma_1(t), \sigma_2(t)\right\rangle = \int_0^1 \left[v_1(t)v_2(t) + u_{1xx}(t)u_{2xx}(t)\right]\mathrm{d}x - \int_0^\infty \frac{1}{\lambda'(s)}\frac{\partial \breve{a}_1^t(1,s)}{\partial s}\frac{\partial \breve{a}_2^t(1,s)}{\partial s}\,\mathrm{d}s,$$

where $\sigma_i(t) = (v_i(t), u_{ixx}(t), \breve{a}_i^t(1,\cdot))$, for $i = 1, 2$, so that

$$\left\langle \sigma(t), \sigma(t)\right\rangle = \left\|\sigma(t)\right\|^2 = 2\psi(t).$$

We denote by $\mathcal{D}(A)$ the domain of the operator A, namely

$$\mathcal{D}(A) = \left\{\sigma \in \mathcal{K};\ A\sigma \in \mathcal{K} \text{ and boundary conditions (1.2) and (2.8) hold}\right\}$$

and claim that the operator A is dissipative.

In fact if $\sigma \in \mathcal{D}(A)$, we have

$$\left\langle A\sigma(t), \sigma(t)\right\rangle = \int_0^1 \left[-u_{xxxx}(t)v(t) + u_{xx}(t)v_{xx}(t)\right]\mathrm{d}x - \int_0^\infty \frac{1}{\lambda'(s)}\frac{\partial}{\partial s}\left[\frac{\partial \breve{a}^t(1,s)}{\partial s} - \lambda(s)v(1,t)\right]\frac{\partial \breve{a}^t(1,s)}{\partial s}\,\mathrm{d}s$$

$$= -u_{xxx}(1,t)v(1,t) - \int_0^\infty \frac{1}{\lambda'(s)} \frac{\partial^2 \breve{a}^t(1,s)}{\partial s^2} \frac{\partial \breve{a}^t(1,s)}{\partial s}\, \mathrm{d}s - v(1,t)\breve{a}^t(1,0)$$

$$= -\gamma_0 \big|v(1,t)\big|^2 - \frac{1}{2}\int_0^\infty \frac{\lambda''(s)}{[\lambda'(s)]^2} \left|\frac{\partial \breve{a}^t(1,s)}{\partial s}\right|^2 \mathrm{d}s + \frac{1}{2}\frac{1}{\lambda'(0)} \left|\frac{\partial \breve{a}^t(1,0)}{\partial s}\right|^2 \leqslant 0.$$

We now proceed to show that also $\widetilde{A}$, the adjoint of A, is dissipative so that, thanks to the Lumer–Phillips theorem [13], A generates a C_0-semigroup.

Let $\widetilde{\sigma} = (\widetilde{v}, \widetilde{u}_{xx}, \widetilde{a}^t)$ be in $\mathcal{K}$ and consider the boundary conditions

$$\widetilde{u}(0,t) = \widetilde{u}_x(0,t) = \widetilde{u}_{xx}(1,t) = 0, \qquad \widetilde{u}_{xxx}(1,t) = -\gamma_0 \widetilde{v}(1,t) - \widetilde{a}^t(1,0). \tag{2.16}$$

Denoting by H the Heaviside function and introducing $j[\widetilde{a}^t(1,\cdot)]$ such that

$$\frac{\partial}{\partial s} j\big[\widetilde{a}^t(1,\cdot)\big](s) = -\lambda'(s)\frac{\partial}{\partial s}\left(\frac{H(s)}{\lambda'(s)}\right)\frac{\partial}{\partial s}\widetilde{a}^t(1,s),$$

we claim that

$$\widetilde{A}\widetilde{\sigma}(t) = \left(\widetilde{u}_{xxxx}(t), -\widetilde{v}_{xx}(t), -\frac{\partial}{\partial s}\widetilde{a}^t(1,s) + \lambda(s)\widetilde{v}(1,t) + j\big[\widetilde{a}^t(1,\cdot)\big](s)\right)$$

and that the domain of $\widetilde{A}$ is

$$\mathcal{D}(\widetilde{A}) = \big\{\widetilde{\sigma} \in \mathcal{K};\ \widetilde{A}\widetilde{\sigma} \in \mathcal{K} \text{ and the boundary conditions (2.16) hold}\big\}.$$

Let us now compute $\langle A\sigma, \widetilde{\sigma}\rangle$, where $\sigma \in \mathcal{D}(A)$ and $\widetilde{\sigma} \in \mathcal{D}(\widetilde{A})$:

$$\big\langle A\sigma(t), \widetilde{\sigma}(t)\big\rangle = \int_0^1 \big[-u_{xxxx}(t)\widetilde{v}(t) + v_{xx}(t)\widetilde{u}_{xx}(t)\big]\,\mathrm{d}x - \int_0^\infty \frac{1}{\lambda'(s)}\frac{\partial}{\partial s}\left(\frac{\partial \breve{a}^t(1,s)}{\partial s} - \lambda(s)v(1,t)\right)\frac{\partial \widetilde{a}^t(1,s)}{\partial s}\,\mathrm{d}s$$

$$= \int_0^1 \big[v(t)\widetilde{u}_{xxxx}(t) - \widetilde{v}_{xx}(t)u_{xx}(t)\big]\,\mathrm{d}x - v(1,t)\big[\widetilde{u}_{xxx}(1,t) + \gamma_0\widetilde{v}(1,t) + \widetilde{a}^t(1,0)\big]$$

$$+ \int_0^\infty \frac{1}{\lambda'(s)}\frac{\partial \breve{a}^t(1,s)}{\partial s}\frac{\partial}{\partial s}\left(\frac{\partial \widetilde{a}^t(1,s)}{\partial s} - \lambda(s)\widetilde{v}(1,t)\right)\mathrm{d}s$$

$$+ \int_0^\infty \frac{\partial}{\partial s}\left(\frac{1}{\lambda'(s)}\right)\frac{\partial \breve{a}^t(1,s)}{\partial s}\frac{\partial \widetilde{a}^t(1,s)}{\partial s}\,\mathrm{d}s + \frac{1}{\lambda'(0)}\frac{\partial \breve{a}^t(1,0)}{\partial s}\frac{\partial \widetilde{a}^t(1,0)}{\partial s},$$

so that, if $\widetilde{\sigma}$ satisfies the boundary conditions (2.16), we have

$$\big\langle A\sigma(t), \widetilde{\sigma}(t)\big\rangle = \big\langle \sigma(t), -A\widetilde{\sigma}(t)\big\rangle + \int_0^\infty \frac{\partial}{\partial s}\left(\frac{H(s)}{\lambda'(s)}\right)\frac{\partial \breve{a}^t(1,s)}{\partial s}\frac{\partial \widetilde{a}^t(1,s)}{\partial s}\,\mathrm{d}s = \big\langle \sigma(t), \widetilde{A}\widetilde{\sigma}(t)\big\rangle.$$

Now observe that, for $\widetilde{\sigma} \in \mathcal{D}(\widetilde{A})$, we have

$$\big\langle \widetilde{A}\widetilde{\sigma}(t), \widetilde{\sigma}(t)\big\rangle = -\gamma_0\big|\widetilde{v}(1,t)\big|^2 - \frac{1}{2}\int_0^\infty \frac{\lambda''(s)}{(\lambda'(s))^2}\left|\frac{\partial \widetilde{a}^t(1,s)}{\partial s}\right|^2 \mathrm{d}s + \frac{1}{2}\frac{1}{\lambda'(0)}\left|\frac{\partial \widetilde{a}^t(1,0)}{\partial s}\right|^2 \leqslant 0.$$

Finally, making use of well-known results on the semigroup theory [14], it is possible to state the following theorem establishing the well-posedness of *problem P*:

Theorem 2.1. *If $\sigma_0 \in \mathcal{D}(A)$, then problem* (2.14) *admits one and only one strict solution $\sigma \in C^1(\mathbb{R}^+; \mathcal{K}) \cap C(\mathbb{R}^+; \mathcal{D}(A))$.*

3. Exponential decay

In order to show that an exponential decay of the energy (2.13) occurs over time, it is necessary to impose further conditions. More precisely, we shall assume that $\gamma_0 > 0$ and that there exists $k_0 > 0$ such that

$$\lambda''(s) + k_0\lambda'(s) \geqslant 0, \quad s \in \mathbb{R}^+. \tag{3.1}$$

It should be remarked that (3.1) is in some sense a hypothesis of exponential decay on the memory kernel λ, in the sense that it easily yields

$$\left|\lambda'(s)\right| = -\lambda'(s) \leqslant c_0 e^{-\kappa s}, \quad s \in \mathbb{R}^+,$$

for a suitable positive constant c_0.

The main result of this section is the following

Theorem 3.1. *Let σ be a solution of* (2.14)*. If $\gamma_0 > 0$ and the memory kernel satisfies* (2.6) *and* (3.1)*, then there exist two positive constants c_1 and c_2 such that*

$$\psi(t) \leqslant c_2 e^{-c_1 t}\psi(0).$$

Proof. Thanks to the semigroup properties proved in the preceding section, in order to obtain the exponential decay of the total energy it is sufficient to show that (see, for instance, Theorem 4.1 in [13])

$$\psi(t) \leqslant \frac{h_1}{(t+h_2)}. \tag{3.2}$$

To this aim we introduce the functional

$$\mathcal{L}_{t_0}(t) = (t+t_0)\psi(t) + \int_0^1 x u_t(t) u_x(t)\,\mathrm{d}x$$

and prove that, for t_0 sufficiently large, it is monotonically non-increasing for every solution of (2.14).

In fact, if σ is a solution of (2.14), then it is easy to show that

$$\dot\psi(t) = -\gamma_0\left|u_t(1,t)\right|^2 + \frac{1}{2}\frac{1}{\lambda'(0)}\left|\frac{\partial\breve{a}^t(1,0)}{\partial s}\right|^2 - \frac{1}{2}\int_0^\infty \frac{\lambda''(s)}{[\lambda'(s)]^2}\left|\frac{\partial\breve{a}^t(1,s)}{\partial s}\right|^2 \mathrm{d}s$$

and (3.1) yields

$$\dot\psi(t) \leqslant -\gamma_0\left|u_t(1,t)\right|^2 - k_0\psi_b(t). \tag{3.3}$$

Moreover,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 x u_t(t)u_x(t)\,\mathrm{d}x\right) &= -\psi_\Omega(t) - \int_0^1 \left|u_{xx}(t)\right|^2 \mathrm{d}x + \frac{1}{2}\left|u_t(1,t)\right|^2 - u_x(1,t)u_{xxx}(1,t)\\ &\leqslant -\psi_\Omega(t) - \int_0^1 \left|u_{xx}(t)\right|^2 \mathrm{d}x + \frac{1}{2}\left|u_t(1,t)\right|^2 + \left|u_x(1,t)\right|^2 + \frac{1}{4}\left|u_{xxx}(1,t)\right|^2.\end{aligned}$$

Recalling the boundary conditions (2.8) and the inequality (2.12), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 x u_t(t)u_x(t)\,\mathrm{d}x\right) \leqslant -\psi_\Omega(t) + \frac{1}{2}\left|u_t(1,t)\right|^2 + \frac{1}{2}\left[\gamma_0^2\left|u_t(1,t)\right|^2 + 2\lambda(0)\psi_b(t)\right]. \tag{3.4}$$

Finally, thanks to (3.3) and (3.4),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{t_0}(t) &= (t+t_0)\dot\psi(t) + \psi(t) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 x u_t(t)u_x(t)\,\mathrm{d}x\right)\\ &\leqslant \left[\frac{1}{2}\left(1+\gamma_0^2\right) - \gamma_0(t+t_0)\right]\left|u_t(1,t)\right|^2 + \left[\lambda(0) - k_0(t+t_0)\right]\psi_b(t).\end{aligned} \tag{3.5}$$

On the other hand, using the classical inequalities of Cauchy–Schwarz and Poincaré, it is easy to prove that

$$\left| \int_0^1 x u_t(t) u_x(t)\,\mathrm{d}x \right| \leqslant \psi_\Omega(t) \leqslant \psi(t),$$

so that

$$\mathcal{L}_{t_0}(t) - \mathcal{L}_{t_0}(0) \geqslant (t + t_0 - 1)\psi(T) - (t_0 + 1)\psi(0). \tag{3.6}$$

Finally, choosing

$$t_0 \geqslant \max\left\{ \frac{1}{2\gamma_0}\big(1 + \gamma_0^2\big), \frac{\lambda(0)}{k_0}, 1 \right\},$$

it follows that

$$0 \geqslant \mathcal{L}_{t_0}(t) - \mathcal{L}_{t_0}(0) \geqslant (t + t_0 - 1)\psi(t) - (t_0 + 1)\psi(0)$$

or, equivalently,

$$\psi(t) \leqslant \frac{(t_0 + 1)}{(t + t_0 - 1)}\psi(0),$$

which coincides with (3.2) by putting $h_1 = (t_0 + 1)\psi(0)$ and $h_2 = (t_0 - 1)$. □

The previous theorem guarantees the exponential decay not only of the internal energy of the beam but also of ψ_b. Therefore, thanks to estimate (2.11), also $\breve{a}^t(1, 0)$ decays exponentially but we cannot conclude the same for $u_t(1, \cdot)$ and $u_{xxx}(1, \cdot)$ separately. However, as already noted in [8], if the initial data are sufficiently smooth (for example $\sigma_0 \in \mathcal{D}(A)$) we obtain also the exponential decay of $u_t(1, \cdot)$ and $u_{xxx}(1, \cdot)$.

We close this section showing that the "exponential decay" of the control memory kernel turns out to be a necessary condition for the "exponential decay" of the solution of *problem P*. To be more precise, we give the following definition.

Definition 3.1. A function u *decays exponentially* if there exists a positive constant δ such that

$$\int_0^\infty e^{\delta t} \big|u(t)\big|\,\mathrm{d}t < \infty.$$

We will obtain the exponential decay of the memory kernel as a consequence of the following result (see [15, Theorem 2]):

Lemma 3.1. *A non-negative function $\lambda \in L^1(\mathbb{R}^+)$ decays exponentially if there exist a neighborhood $U \subset \mathbb{C}$ of 0 and a holomorphic function $f : U \to \mathbb{C}$ such that the Laplace transform of λ coincides with f in $U \cap \mathbb{C}^+$, where $\mathbb{C}^+ = \{z \in \mathbb{C};\ \Re\{z\} \geqslant 0\}$.*

Theorem 3.2. *Let u be a solution of problem P with null sources and such that*

$$\int_0^\infty e^{2\delta t}\big(\big|u_t(1, t)\big|^2 + \big|u_{xxx}(1, t)\big|^2\big)\,\mathrm{d}t < \infty \tag{3.7}$$

for some $\delta > 0$, then λ decays exponentially.

Proof. First of all we observe that if u is a solution of *problem P* satisfying (3.7) then the Laplace transforms

$$\hat{u}_t(1, z) = \int_0^\infty e^{-zt} u_t(1, t)\,\mathrm{d}t, \qquad \hat{u}_{xxx}(1, z) = \int_0^\infty e^{-zt} u_{xxx}(1, t)\,\mathrm{d}t$$

are holomorphic functions in $\mathcal{D}_\delta = \{z \in \mathrm{C};\ \Re\{z\} > -\delta\}$ and, since $\lambda \in L^1(\mathbb{R}^+)$, the Laplace transform of the boundary condition (1.3)

$$\hat{u}_{xxx}(1, z) = \big(\gamma_0 + \hat{\lambda}(z)\big)\hat{u}_t(1, z) \tag{3.8}$$

is well defined for $z \in \mathbb{C}^+$.

Reasoning as in the proof of Theorem 3.1, it is easy to show that the (constant in time) energy ψ_Ω, associated to the cantilever beam problem (1.1)–(1.2) with a vanishing boundary control term $\Gamma(t)$ at the free end, satisfies

$$(t-2)\psi_\Omega(0) \leqslant \int_0^t \left|u_t(1,\tau)\right|^2 \mathrm{d}\tau;$$

therefore, if we give the additional boundary condition $u_t(1,t) \stackrel{t}{\equiv} 0$, the cantilever beam problem admits only the trivial solution.

Consequently, if u is a non-trivial solution of *problem P* satisfying (3.7), there exists a non-negative integer k such that

$$\frac{\partial^k}{\partial z^k}\hat{u}_t(1,0) \neq 0. \tag{3.9}$$

If k_0 is the first integer for which (3.9) holds, then

$$\hat{u}_t(1,z) = z^{k_0-1} g(z) \tag{3.10}$$

with g holomorphic on $\mathcal{D}_\delta$ and $g(0) \neq 0$.

Similarly, by virtue of (3.8), there exists G, holomorphic in $\mathcal{D}_\delta$, such that

$$\hat{u}_{xxx}(1,z) = z^{k_0-1} G(z). \tag{3.11}$$

Therefore, we conclude that

$$\hat{\lambda}(z) = \frac{G(z)}{g(z)} - \gamma_0, \quad z \in \mathbb{C}^+,$$

where the right-hand side is a holomorphic function in a neighborhood $\mathcal{U}$ of 0, since $g(0) \neq 0$. $\square$

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