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Classical solutions for the compressible liquid crystal flows with nonnegative initial densities

Shixiang Ma

School of Mathematical Sciences, South China Normal University, Guang Zhou 510631, China

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ABSTRACT

In this paper, we study the Cauchy problem of the simplified Ericksen–Leslie system modeling compressible nematic liquid crystal flows in R^3. We first prove the local existence of the strong solutions provided that the data satisfies a natural compatibility condition. Then by deriving the smoothing effect of the solution in $t > 0$, we conclude that it is indeed classical on some time interval $[0, T_0]$. Here we do not need the condition that initial density is bounded below away from zero.

1. Introduction

The time evolution of the materials under the influence of both the flow velocity field and the macroscopic description of the microscopic orientation configurations of rod-like liquid crystals is governed by the Ericksen–Leslie system [1,2]. When the viscous fluid is incompressible, Lin [3] first derived a simplified Ericksen–Leslie system modeling liquid crystal flows. Then Lin and Liu [4,5] study the existence of weak and strong solutions and the partial regularity of suitable solutions, of the simplified Ericksen–Leslie system, under the assumption that the liquid crystal director field is of varying length by Leslie's terminology or variable degree of orientation by Ericksen's terminology. Recently, Lin et al. [6] have established the global weak solutions, which are smooth away from at most finitely many singular times, in any bounded smooth domain of R^2. However, once the fluid is allowed to be compressible, the Ericksen–Leslie system becomes more complicated. On the hydrodynamics of the compressible nematic liquid crystals, under the influence of temperature gradient or electromagnetic forces, there have been both modeling study [7] and numerical study [8], and yet very few analytic works seem to be available.

In this paper, we consider the Cauchy problem in R^3 for the simplified version of the Ericksen–Leslie system

$$\begin{cases} \varrho_t + \operatorname{div}(\varrho u) = 0, \\ (\varrho u)_t + \operatorname{div}(\varrho u \otimes u) + Lu + \nabla p = -\gamma \operatorname{div}\left(\nabla n \odot \nabla n - \dfrac{|\nabla n|^2}{2} Id\right), \\ n_t + (u \cdot \nabla) n = \theta(\Delta n + |\nabla n|^2 n), \end{cases} \tag{1.1}$$

with the initial data

$$(\varrho, u, n)|_{t=0} = (\varrho_0, u_0, n_0) \quad \text{in } R^3, \tag{1.2}$$

and the far field behavior

$$(\varrho, u, \nabla n) \to (\varrho^\infty, 0, 0) \quad \text{as } |x| \to \infty, \tag{1.3}$$

E-mail address: mashx822@gmail.com.

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for some constant vector $(\varrho^{\infty}, 0, 0)$ satisfying $\varrho^{\infty} \geq 0$. This system was derived in the recent preprint paper [9] based on energetic-variational approaches. Here $\varrho \in R^1, u \in R^3, p = p(\varrho) \in R^1$ and $n \in S^2$ denote the density, the velocity, the pressure, and the macroscopic average of the nematic liquid crystal orientation field, respectively, and

$$Lu = -\mu \Delta u - (\lambda + \mu)\nabla(\operatorname{div} u).$$

The constant coefficients λ, μ are the bulk viscosity and shear viscosity coefficients, γ is the competition between kinetic and potential energy, and θ is the microscopic elastic relaxation time. λ and μ satisfy the physical restrictions

$$\mu > 0, \quad 2\mu + 3\lambda \geq 0.$$

There have been some important studies on the well-posedness to (1.1). In dimension one, Ding et al. [10,11] have obtained the global existence for the weak and strong solutions. However, for dimensions at least two, it is reasonable to believe that the local strong solutions may cease to exist globally. In fact, there exist finite time singularities of the (transported) heat flow of harmonic maps $(1.1)_3$ in dimensions two or higher [12]. In dimension three, Huang et al. [13,14] have obtained the local existence of the strong solutions and some blow up criterions, suppose that the initial data (ϱ_0, u_0, n_0) satisfies the regularity condition

$$\begin{aligned} &\varrho_0 \geq 0, \quad \varrho_0 \in W^{1,q} \cap H^1 \cap L^1 \text{ for some } q \in (3, 6], \\ &u_0 \in D_0^1 \cap D^2, \qquad \nabla n_0 \in H^2 \quad \text{with } |n_0| = 1, \end{aligned}$$

and the compatibility condition

$$Lu_0 + \nabla p(\varrho_0) + \gamma \nabla n_0 \cdot \Delta n_0 = \sqrt{\varrho_0} g_0 \quad \text{for some } g_0 \in L^2.$$

Here motivated by the results in [15] for the compressible Navier–Stokes equations, we aim to look for the classical solutions (ϱ, u, n) to (1.1)–(1.3).

2. Main results

Theorem 2.1. *Assume that*

$$\varrho_0 - \varrho^{\infty} \in H^3, \quad \varrho_0 \geq 0, \qquad u_0 \in D_0^1 \cap D^3, \qquad \nabla n_0 \in H^3 \quad \text{with } |n_0| = 1, \tag{2.1}$$

$$p = p(\cdot) \in C^3(\overline{R_+}). \tag{2.2}$$

Assume further that the data (ϱ_0, u_0, n_0) satisfies the compatibility condition

$$Lu_0 + \nabla p(\varrho_0) + \gamma \operatorname{div}\left(\nabla n_0 \odot \nabla n_0 - \frac{|\nabla n_0|^2}{2} Id\right) = \varrho_0 g \tag{2.3}$$

for some $g \in D_0^1$ with $\sqrt{\varrho_0} g \in L^2$. Then there exist a small time T_0 and a unique strong solution (ϱ, u, n) to the problem (1.1)–(1.3) such that

$$\varrho - \varrho^{\infty} \in C([0, T_0], H^3), \qquad \varrho_t \in L^{\infty}(0, T_0; H^2), \qquad \sqrt{\varrho} u_t \in L^{\infty}(0, T_0; L^2), \tag{2.4}$$

$$u \in C([0, T_0]; D_0^1 \cap D^3) \cap L^2(0, T_0; D^4), \qquad u_t \in L^{\infty}(0, T_0; D_0^1) \cap L^2(0, T_0; D^2), \tag{2.5}$$

$$\nabla n \in C([0, T_0]; H^3) \cap L^2(0, T_0; D^4), \tag{2.6}$$

$$n_t \in C([0, T_0]; H^2) \cap L^2(0, T_0; D^3), \qquad n_{tt} \in L^{\infty}(0, T_0; L^2) \cap L^2(0, T_0; D^1). \tag{2.7}$$

Theorem 2.2. *In addition to (2.4)–(2.7), the solution (ρ, u, n) in Theorem 2.1 satisfies the following regularity*

$$t^{\frac{1}{2}} u \in L^{\infty}(0, T_0; D^4), \qquad t^{\frac{1}{2}} u_t \in L^{\infty}(0, T_0; D^2), \tag{2.8}$$

$$t^{\frac{1}{2}} u_{tt} \in L^2(0, T_0; D_0^1), \qquad t^{\frac{1}{2}} \sqrt{\rho} u_{tt} \in L^{\infty}(0, T_0; L^2), \tag{2.9}$$

$$t u_t \in L^{\infty}(0, T_0; D^3), \qquad t u_{tt} \in L^{\infty}(0, T_0; D_0^1) \cap L^2(0, T_0; D^2), \tag{2.10}$$

$$t \sqrt{\rho} u_{ttt} \in L^{\infty}(0, T_0; L^2), \qquad t^{\frac{3}{2}} u_{tt} \in L^{\infty}(0, T_0; D^2), \tag{2.11}$$

$$t^{\frac{3}{2}} \sqrt{\rho} u_{ttt} \in L^{\infty}(0, T_0; L^2), \qquad t^{\frac{3}{2}} u_{ttt} \in L^2(0, T_0; D_0^1), \tag{2.12}$$

$$t^{\frac{1}{2}} n_t \in L^{\infty}(0, T_0; D^3), \qquad t^{\frac{1}{2}} n_{tt} \in L^{\infty}(0, T_0; D^1), \qquad t n_{tt} \in L^{\infty}(0, T_0; D^2), \tag{2.13}$$

which imply that (ρ, u, n) is indeed a classical solution to (1.1)–(1.3).

Remark 2.3. By a more simple proof, we can show that similar results hold for the case that the spatial domain Ω is bounded with smooth boundary and the boundary condition is $(u, \partial n/\partial \nu)|_{\partial\Omega} = 0$. Here ν is the unit outer normal vector of $\partial\Omega$. The main difficulty of this paper is that vacuum may appear. To get the regularity estimates for u, we need deduce some good estimates for n.

Notation. In this paper, we adopt the following notations for the standard homogeneous and inhomogeneous Sobolev spaces

$$D^{k,r} = \{v \in L^1_{loc}(\Omega) : \|v\|_{D^{k,r}} \equiv \|\nabla^k v\|_{L^r} < \infty\},$$

$$D^1_0 = \{v \in L^6(\Omega) : \|v\|_{D^1_0} \equiv \|\nabla v\|_{L^2} < \infty \text{ and } v = 0 \text{ on } \partial\Omega\},$$

$$W^{k,r} = L^r \cap D^{k,r}, \qquad H^k = W^{k,2}, \qquad D^k = D^{k,2}, \qquad H^1_0 = L^2 \cap D^1_0.$$

And for matrices $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{m\times n}$, $A : B \equiv \sum_{i=1}^m \sum_{j=1}^n a_{ij} b_{ij}$.

3. Linear equations

To prove Theorems 2.1–2.2, we first consider the following linearized equations

$$\varrho_t + \operatorname{div}(\varrho v) = 0, \tag{3.1}$$

$$\varrho u_t + Lu + \nabla p = -\varrho(v\cdot\nabla)v - \gamma \operatorname{div}\left(\nabla n \odot \nabla n - \frac{|\nabla n|^2}{2} Id\right), \tag{3.2}$$

$$n_t + (v\cdot\nabla)n = \theta(\Delta n + |\nabla m|^2 n), \tag{3.3}$$

with $(t, x) \in (0, T) \times \Omega$, and

$$(\varrho, u, n)|_{t=0} = (\varrho_0, u_0, n_0) \quad \text{in } \Omega, \tag{3.4}$$

$$\left(u, \frac{\partial n}{\partial \nu}\right) = (0, 0) \quad \text{on } (0, T) \times \partial\Omega, \tag{3.5}$$

$$(\varrho, u, \nabla n) \to (\varrho^\infty, 0, 0) \quad \text{as } |x| \to \infty. \tag{3.6}$$

When $\Omega \subset R^3$ is a bounded domain (or the whole space), the condition (3.6) at infinity (or the boundary condition in (3.5), respectively) is unnecessary and should be neglected. Here v, m are known vector fields in $(0, T) \times \Omega$ such that

$$v \in C([0, T]; D^1_0 \cap D^3) \cap L^2(0, T; D^4), \qquad v_t \in L^\infty(0, T; D^1_0) \cap L^2(0, T; D^2), \tag{3.7}$$

$$\nabla m \in C([0, T]; H^3) \cap L^2(0, T; D^4), \tag{3.8}$$

$$m_t \in C([0, T]; H^2) \cap L^2(0, T; D^3), \qquad m_{tt} \in L^\infty(0, T; L^2) \cap L^2(0, T_0; D^1). \tag{3.9}$$

4. A priori estimates for the linearized problem

Since (3.1) is a linear hyperbolic equation for ρ, (3.2) and (3.3) are linear parabolic systems for u and n, respectively, by the standard existence and uniqueness theory for the linear system, we have the following results.

Lemma 4.1. *Let Ω be a bounded domain in R^3 with smooth boundary. In addition to (2.1)–(2.2) and (3.7)–(3.9), we assume that $\varrho_0 \geq \delta$ in Ω for some $\delta > 0$ and $v(0)\cdot\nabla v(0) + \varrho_0^{-1}\{Lu_0 + \nabla p(\varrho_0) + \gamma \operatorname{div}(\nabla n_0 \odot \nabla n_0 - \frac{|\nabla n_0|^2}{2} Id)\} \in H^1_0(\Omega)$. Then there exist $T > 0$ and a unique solution (ϱ, u, n) to the linearized problem (3.1)–(3.5) such that*

$$\varrho - \varrho^\infty \in C([0, T], H^3(\Omega)), \qquad \varrho_t \in C([0, T], H^2(\Omega)), \tag{4.1}$$

$$u \in C([0, T]; H^1_0(\Omega) \cap H^3(\Omega)) \cap L^2(0, T; H^4(\Omega)), \tag{4.2}$$

$$u_t \in C([0, T]; H^1_0(\Omega)) \cap L^2(0, T; H^2(\Omega)), \qquad u_{tt} \in L^2(0, T; L^2(\Omega)), \tag{4.3}$$

$$n \in C([0, T]; H^4(\Omega)) \cap L^2(0, T; H^5(\Omega)), \tag{4.4}$$

$$n_t \in C([0, T]; H^2(\Omega)) \cap L^2(0, T; H^3(\Omega)), \tag{4.5}$$

$$n_{tt} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \quad \text{and} \quad \varrho \geq \underline{\delta}, \tag{4.6}$$

for some constant $\underline{\delta} > 0$.

Our purpose is to show that for $\rho_0 \geq 0$, (1.1)–(1.3) has a smooth solution up to some time T_0. This will follow from the following key estimates on (ρ, u, n).

Proposition 4.2. *Suppose the conditions in Lemma 4.1 hold. Let $c_0 > 1$ be a given constant such that*

$$1 + \varrho^\infty + \|\varrho_0 - \varrho^\infty\|_{H^3} + \|u_0\|_{D_0^1} + \|\nabla n_0\|_{H^3} + \|\sqrt{\varrho_0} g\|_{L^2} + \|g\|_{D_0^1} < c_0. \tag{4.7}$$

Then there exist $T_* > 0$, $T_* \in [0, T]$ and constants $c_i (i = 1, 2, 3, 4)$ with $1 < c_0 \le c_1 \le c_2 \le c_3 \le c_4$, which are independent of δ and the size of Ω, such that if

$$\begin{cases} \|v(0)\|_{D_0^1 \cap D^3} + \|\nabla m(0)\|_{H^3} \le 1 + c_1, \\ \sup\limits_{0\le t\le T_*} (\|v(t)\|_{D_0^1} + \|m_t(t)\|_{L^2} + \|\nabla m(t)\|_{H^1}) + \displaystyle\int_0^{T_*} (\|v(t)\|_{D^2}^2 + \|m_t(t)\|_{D^1}^2 + \|m(t)\|_{D^3}^2) dt \le 1 + c_2, \\ \sup\limits_{0\le t\le T_*} (\|v(t)\|_{D^2} + \|m_t(t)\|_{D^1} + \|m(t)\|_{D^3}) \\ \quad + \displaystyle\int_0^{T_*} (\|v_t(t)\|_{D_0^1}^2 + \|v(t)\|_{D^3}^2 + \|m_{tt}(t)\|_{L^2}^2 + \|m_t(t)\|_{D^2}^2 + \|m(t)\|_{D^4}^2) dt \le 1 + c_3, \\ \operatorname*{ess\,sup}\limits_{0\le t\le T_*} (\|v_t(t)\|_{D_0^1} + \|v(t)\|_{D^3} + \|m_{tt}(t)\|_{L^2} + \|m_t(t)\|_{D^2} + \|m(t)\|_{D^4}) \\ \quad + \displaystyle\int_0^{T_*} (\|v_t(t)\|_{D^2}^2 + \|v(t)\|_{D^4}^2 + \|m_{tt}(t)\|_{D^1}^2 + \|m_t(t)\|_{D^3}^2 + \|m(t)\|_{D^5}^2) dt \le 1 + c_4, \end{cases} \tag{4.8}$$

then the solution (ϱ, u, n) satisfies

$$\begin{cases} \|u(0)\|_{D_0^1 \cap D^3} + \|\nabla n(0)\|_{H^3} \le c_1, \\ \sup\limits_{0\le t\le T_*} (\|u(t)\|_{D_0^1} + \|n_t(t)\|_{L^2} + \|\nabla n(t)\|_{H^1}) + \displaystyle\int_0^{T_*} (\|u(t)\|_{D^2}^2 + \|n_t(t)\|_{D^1}^2 + \|n(t)\|_{D^3}^2) dt \le c_2, \\ \sup\limits_{0\le t\le T_*} (\|u(t)\|_{D^2} + \|n_t(t)\|_{D^1} + \|n(t)\|_{D^3}) \\ \quad + \displaystyle\int_0^{T_*} (\|u_t(t)\|_{D_0^1}^2 + \|u(t)\|_{D^3}^2 + \|n_{tt}(t)\|_{L^2}^2 + \|n_t(t)\|_{D^2}^2 + \|n(t)\|_{D^4}^2) dt \le c_3, \\ \operatorname*{ess\,sup}\limits_{0\le t\le T_*} (\|u_t(t)\|_{D_0^1} + \|u(t)\|_{D^3} + \|n_{tt}(t)\|_{L^2} + \|n_t(t)\|_{D^2} + \|n(t)\|_{D^4}) \\ \quad + \displaystyle\int_0^{T_*} (\|u_t(t)\|_{D^2}^2 + \|u(t)\|_{D^4}^2 + \|n_{tt}(t)\|_{D^1}^2 + \|n_t(t)\|_{D^3}^2 + \|n(t)\|_{D^5}^2) dt \le c_4, \\ \operatorname*{ess\,sup}\limits_{0\le t\le T_*} (\|\varrho(t) - \varrho^\infty\|_{H^3} + \|\varrho_t(t)\|_{H^2} + \|\sqrt{\varrho} u_t(t)\|_{L^2}) + \displaystyle\int_0^{T_*} \|\sqrt{\varrho} u_{tt}(t)\|_{L^2}^2 dt \le c_4. \end{cases} \tag{4.9}$$

The proof of this proposition occupies the rest of this section. We denote by C a generic positive constant depending only on the fixed constants $\mu, \lambda, \gamma, \theta, T$ and $\|p\|_{C^3(\overline{R_+})}$.

Lemma 4.3 (*[15, Lemma 5]*)**.**

$$\|\varrho_t(t)\|_{L^\infty} + \|\varrho(t) - \varrho^\infty\|_{H^3} \le Cc_0, \qquad \|p(t) - p^\infty\|_{H^3} \le M(c_0),$$

$$\|\varrho_t(t)\|_{H^1} \le Cc_3^2, \qquad \int_0^t \|\varrho_{tt}(s)\|_{L^2}^2 ds \le Cc_3^8,$$

$$\|p_t(t)\|_{H^1} \le M(c_0)c_3^2, \qquad \int_0^t \|p_{tt}(s)\|_{L^2}^2 ds \le M(c_0)c_3^8,$$

$$\|\varrho_t(t)\|_{H^2} \le Cc_4^2, \qquad \|p_t(t)\|_{H^2} \le M(c_0)c_4^2, \quad \text{and} \quad \inf_\Omega \varrho(t) \ge C^{-1}\delta,$$

for $0 \le t \le \min(T_*, T_1)$, where $T_1 = (1 + c_4)^{-1}$ and $p^\infty = p(\varrho^\infty)$.

Lemma 4.4.

$$\|n_t(t)\|_{H^1} + \|\nabla n(t)\|_{H^1} + \int_0^t (\|n_{tt}(s)\|_{L^2}^2 + \|n_t(s)\|_{D^2}^2 + \|n(s)\|_{D^4}^2) ds \le M(c_1), \tag{4.10}$$

$$\|n(t)\|_{D^3} \le M(c_1) c_2^{\frac{3}{2}} c_3^{\frac{1}{2}}, \tag{4.11}$$

$$\|u(t)\|_{D_0^1} + \int_0^t (\|\sqrt{\varrho} u_t(s)\|_{L^2}^2 + \|u(s)\|_{D^2}^2) ds \le M(c_1), \tag{4.12}$$

for $0 \le t \le \min(T_*, T_2)$, where $T_2 = (1 + c_4)^{-4}$.

Proof. Multiplying Eq. (3.3) by n_t and integrating the resulting equation with respect to x over Ω, we have

$$\begin{aligned}\|n_t\|_{L^2}^2+\frac{\theta}{2}\frac{d}{dt}\|\nabla n\|_{L^2}^2 &= \int\{-v\cdot\nabla n+\theta|\nabla m|^2 n\}\cdot n_t dx\\ &\le \frac{1}{2}\|n_t\|_{L^2}^2+C(\|v\|_{L^\infty}^2\|\nabla n\|_{L^2}^2+\|\nabla m\|_{L^6}^2\|\nabla m\|_{L^3}^2\|n\|_{L^\infty}^2)\\ &\le \frac{1}{2}\|n_t\|_{L^2}^2+C(\|v\|_{D_0^1\cap D^2}^2+\|\nabla m\|_{H^1}^4)\|\nabla n\|_{H^1}^2.\end{aligned}$$

This gives

$$\frac{d}{dt}\|\nabla n\|_{L^2}^2\le Cc_3^4\|n\|_{D^1\cap D^2}^2. \tag{4.13}$$

We differentiate (3.3) with respect to x, then multiply the resulting system by $\nabla\Delta n$ and integrate over Ω to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla^2 n\|_{L^2}^2+\theta\|\nabla\Delta n\|_{L^2}^2 &= \int\{\nabla(v\cdot\nabla n)-\theta\nabla(|\nabla m|^2 n)\}\cdot\nabla\Delta n dx\\ &\le \frac{\theta}{2}\|\nabla\Delta n\|_{L^2}^2+C(\|v\|_{D_0^1\cap D^2}^2+\|\nabla m\|_{H^2}^4)\|\nabla n\|_{H^1}^2\\ &\le \frac{\theta}{2}\|\nabla\Delta n\|_{L^2}^2+Cc_3^4\|\nabla n\|_{H^1}^2,\end{aligned}$$

which deduces

$$\frac{d}{dt}\|\nabla^2 n\|_{L^2}^2+\|\nabla\Delta n\|_{L^2}^2\le Cc_3^4\|\nabla n\|_{H^1}^2. \tag{4.14}$$

Therefore adding the inequalities (4.13) and (4.14) and using Gronwall's inequality, we conclude that

$$\|\nabla n(t)\|_{H^1}^2+\int_0^t\|\nabla\Delta n(s)\|_{L^2}^2 ds\le M(c_0), \tag{4.15}$$

for $0\le t\le\min(T_*,T_2)$.

Differentiate (3.3) with respect to t to derive

$$n_{tt}+(v\cdot\nabla n)_t=\theta(\Delta n_t+(|\nabla m|^2 n)_t). \tag{4.16}$$

Multiplying (4.16) by n_t and integrating the resulting system with respect to x over Ω lead to

$$\frac{1}{2}\frac{d}{dt}\|n_t\|_{L^2}^2+\theta\|\nabla n_t\|_{L^2}^2=\int\{-(v\cdot\nabla n)_t+\theta(|\nabla m|^2 n)_t\}\cdot n_t dx. \tag{4.17}$$

Due to Sobolev's inequalities, we have

$$\begin{aligned}-\int(v\cdot\nabla n)_t\cdot n_t dx &\le C(\|v_t\|_{D_0^1}\|\nabla n\|_{H^1}\|n_t\|_{L^2}+\|v\|_{D_0^1\cap D^2}\|\nabla n_t\|_{L^2}\|n_t\|_{L^2})\\ &\le \frac{\theta}{2}\|\nabla n_t\|_{L^2}^2+C\|v_t\|_{D_0^1}^2\|\nabla n\|_{H^1}^2+C(1+\|v\|_{D_0^1\cap D^2}^2)\|n_t\|_{L^2}^2\\ &\le \frac{\theta}{2}\|\nabla n_t\|_{L^2}^2+M(c_0)c_4^2+Cc_3^2\|n_t\|_{L^2}^2,\end{aligned}$$

and

$$\begin{aligned}\theta\int(|\nabla m|^2 n)_t\cdot n_t dx &\le C\|\nabla m_t\|_{L^2}\|\nabla m\|_{L^\infty}\|n\|_{L^\infty}\|n_t\|_{L^2}+C\|\nabla m\|_{L^\infty}^2\|n_t\|_{L^2}^2\\ &\le C\|\nabla m\|_{H^2}^2\|n_t\|_{L^2}^2+C\|m_t\|_{D^1}^2\|\nabla n\|_{H^1}^2\\ &\le Cc_3^2\|n_t\|_{L^2}^2+M(c_0)c_3^2.\end{aligned}$$

Thus one obtains

$$\frac{d}{dt}\|n_t\|_{L^2}^2+\|\nabla n_t\|_{L^2}^2\le Cc_3^2\|n_t\|_{L^2}^2+M(c_0)c_3^2. \tag{4.18}$$

On the other hand, since $n_t\in C([0,T];H_0^1\cap H^2)$ and

$$n_t(0)=-v(0)\cdot\nabla n_0+\theta(\Delta n_0+|\nabla m(0)|^2 n_0) \tag{4.19}$$

we have

$$\|n_t(0)\|_{L^2} \le M(c_0)c_1^2. \tag{4.20}$$

It follows from Gronwall's inequality that

$$\|n_t(t)\|_{L^2}^2 + \int_0^t \|n_t(s)\|_{D^1}^2 ds \le M(c_1), \tag{4.21}$$

for $0 \le t \le \min(T_*, T_2)$.

Multiply (4.16) by n_{tt} and integrate over Ω to give

$$\begin{aligned}\int |n_{tt}|^2 dx + \frac{\theta}{2}\frac{d}{dt}\int |\nabla n_t|^2 dx &= \int \{-(v\cdot\nabla n)_t + \theta(|\nabla m|^2 n)_t\}\cdot n_{tt} dx \\ &\le \frac{1}{2}\|n_{tt}\|_{L^2}^2 + C(\|v\|_{D_0^1\cap D^2}^2 \|\nabla n_t\|_{L^2}^2 + \|v_t\|_{D_0^1}^2 \|\nabla n\|_{H^1}^2) \\ &\quad + C(\|\nabla m\|_{H^2}^4 \|n_t\|_{L^2}^2 + \|\nabla m\|_{H^2}^2 \|m_t\|_{D^1}^2 \|\nabla n\|_{H^1}^2) \\ &\le \frac{1}{2}\|n_{tt}\|_{L^2}^2 + Cc_3^2\|\nabla n_t\|_{L^2}^2 + M(c_1)c_4^4.\end{aligned} \tag{4.22}$$

According to (4.19),

$$\|n_t(0)\|_{D^1} \le C(\|v(0)\|_{D_0^1} + 1 + \|\nabla m(0)\|_{H^2}^2)\|\nabla n_0\|_{H^2} \le M(c_0)c_1^2. \tag{4.23}$$

Thus

$$\|n_t(t)\|_{D^1}^2 + \int_0^t \|n_{tt}(s)\|_{L^2}^2 ds \le M(c_1), \tag{4.24}$$

for $0 \le t \le \min(T_*, T_2)$.

Moreover, since for each $t \in (0, T)$, $n = n(t)$ is a solution of the elliptic system

$$\theta\Delta n = v\cdot\nabla n - \theta|\nabla m|^2 n + n_t, \tag{4.25}$$

it follows from the standard elliptic regularity estimates that

$$\begin{aligned}\|n\|_{D^3} &\le C(\|v\cdot\nabla n - \theta|\nabla m|^2 n + n_t\|_{D^1}) \\ &\le C(\|\nabla v\|_{L^3}\|\nabla n\|_{L^6} + \|v\|_{L^\infty}\|\nabla^2 n\|_{L^2} + \|\nabla m\|_{L^\infty}\|\nabla^2 m\|_{L^2}\|n\|_{L^\infty} + \|\nabla m\|_{L^6}^2\|\nabla n\|_{L^6} + \|\nabla n_t\|_{L^2}) \\ &\le C\{(\|v\|_{D_0^1}^{\frac{1}{2}}\|v\|_{D^2}^{\frac{1}{2}} + \|m\|_{D^2}^{\frac{3}{2}}\|m\|_{D^3}^{\frac{1}{2}} + \|m\|_{D^2}^2)\|\nabla n\|_{H^1} + \|\nabla n_t\|_{L^2}\} \\ &\le M(c_1)c_2^{\frac{3}{2}}c_3^{\frac{1}{2}},\end{aligned} \tag{4.26}$$

$$\begin{aligned}\|n_t\|_{D^2} &\le C\|n_{tt}\|_{L^2} + C(\|v\|_{D_0^1\cap D^2} + \|m\|_{D^2}^2)\|\nabla n_t\|_{L^2} + C(\|v_t\|_{D_0^1} + \|m_t\|_{D^1}\|\nabla m\|_{H^2})\|\nabla n\|_{H^1} \\ &\le C\|n_{tt}\|_{L^2} + M(c_1)c_4^2,\end{aligned} \tag{4.27}$$

and

$$\begin{aligned}\|n\|_{D^4} &\le C\|n_t\|_{D^2} + C\|v\|_{D_0^1\cap D^2}\|n\|_{D^3} + \|\nabla m\|_{H^2}^2\|\nabla n\|_{H^1} \\ &\le C\|n_t\|_{D^2} + M(c_1)c_3^3.\end{aligned} \tag{4.28}$$

Thus we obtain

$$\int_0^t (\|n_t(s)\|_{D^2}^2 + \|n(s)\|_{D^4}^2) ds \le M(c_1), \tag{4.29}$$

for $0 \le t \le \min(T_*, T_2)$. Up to now, (4.10)–(4.11) are proved.

Multiply (3.2) by u_t and integrate over Ω to deduce

$$\begin{aligned}&\int \varrho|u_t|^2 dx + \frac{1}{2}\frac{d}{dt}\int (\mu|\nabla u|^2 + (\lambda+\mu)(\mathrm{div} u)^2) dx \\ &\quad = -\int \nabla p\cdot u_t dx - \int \varrho(v\cdot\nabla v)\cdot u_t dx - \gamma\int \mathrm{div}\left(\nabla n\odot\nabla n - \frac{|\nabla n|^2}{2}Id\right)\cdot u_t dx\end{aligned}$$

$$= \int (p - p^{\infty})\mathrm{div}u_t dx - \int \varrho(v\cdot\nabla v)\cdot u_t dx + \gamma \int \left(\nabla n \odot \nabla n - \frac{|\nabla n|^2}{2} Id\right) : \nabla u_t dx$$
$$= \frac{d}{dt}\left\{\int (p - p^{\infty})\mathrm{div}u dx + \gamma \int \left(\nabla n \odot \nabla n - \frac{|\nabla n|^2}{2} Id\right) : \nabla u dx\right\}$$
$$- \int p_t \mathrm{div}u dx - \int \varrho(v\cdot\nabla v)\cdot u_t dx - \gamma \int \left(\nabla n \odot \nabla n - \frac{|\nabla n|^2}{2} Id\right)_t : \nabla u dx. \tag{4.30}$$

We estimate the terms on the right hand side of (4.30) separately as follows.

$$\int (p - p^{\infty})\mathrm{div}u dx \le \frac{\mu}{8}\|\nabla u\|_{L^2}^2 + C\|p - p^{\infty}\|_{L^2}^2 \le \frac{\mu}{8}\|\nabla u\|_{L^2}^2 + M(c_0), \tag{4.31}$$

$$\gamma \int \left(\nabla n \odot \nabla n - \frac{|\nabla n|^2}{2} Id\right) : \nabla u dx \le \frac{\mu}{8}\|\nabla u\|_{L^2}^2 + C\|\nabla n\|_{H^1}^4 \le \frac{\mu}{8}\|\nabla u\|_{L^2}^2 + M(c_0), \tag{4.32}$$

$$- \int p_t \mathrm{div}u dx \le \|\nabla u\|_{L^2}^2 + C\|p_t\|_{L^2}^2 \le \|\nabla u\|_{L^2}^2 + M(c_0)c_3^4, \tag{4.33}$$

$$- \int \varrho(v\cdot\nabla v)\cdot u_t dx \le \|\varrho\|_{L^\infty}^{\frac{1}{2}}\|v\|_{D_0^1\cap D^2}^2\|\sqrt{\varrho}u_t\|_{L^2} \le \frac{1}{2}\|\sqrt{\varrho}u_t\|_{L^2}^2 + Cc_3^4, \tag{4.34}$$

and

$$- \gamma \int \left(\nabla n \odot \nabla n - \frac{|\nabla n|^2}{2} Id\right)_t : \nabla u dx \le \|\nabla n\|_{H^2}\|\nabla n_t\|_{L^2}\|\nabla u\|_{L^2}$$
$$\le \|\nabla u\|_{L^2}^2 + C\|\nabla n\|_{H^2}^2\|\nabla n_t\|_{L^2}^2 \le \|\nabla u\|_{L^2}^2 + M(c_1)c_3^4. \tag{4.35}$$

Integrate (4.30) with respect to t and use the estimates (4.31)–(4.35) to get

$$\int_0^t \int \varrho|u_t|^2(s) dx ds + \|\nabla u(t)\|_{L^2}^2 \le \int_0^t \|\nabla u(s)\|_{L^2}^2 ds + M(c_1)c_3^4 t.$$

It follows from Gronwall's inequality that

$$\|u(t)\|_{D_0^1}^2 + \int_0^t \int \varrho|u_t|^2(s) dx ds \le M(c_1), \tag{4.36}$$

for $0 \le t \le \min(T_*, T_2)$.

Using the regularity estimates for the elliptic system, one has

$$\|u\|_{D^2} \le C(\|\varrho\|_{L^\infty}^{\frac{1}{2}}\|\sqrt{\varrho}u_t\|_{L^2} + \|\nabla p\|_{L^2} + \|\varrho\|_{L^\infty}\|v\|_{D_0^1}^{\frac{3}{2}}\|v\|_{D^2}^{\frac{1}{2}} + \|n\|_{D^2}^{\frac{3}{2}}\|n\|_{D^3}^{\frac{1}{2}})$$
$$\le M(c_1)(\|\sqrt{\varrho}u_t\|_{L^2} + c_2^{\frac{3}{2}}c_3^{\frac{1}{2}}), \tag{4.37}$$

and thus

$$\int_0^t \|u(s)\|_{D^2}^2 ds \le M(c_1)\left(\int_0^t \|\sqrt{\varrho}u_t(s)\|_{L^2}^2 ds + c_3^4 t\right) \le M(c_1), \tag{4.38}$$

for $0 \le t \le \min(T_*, T_2)$. □

Lemma 4.5.

$$\|\sqrt{\varrho}u_t(t)\|_{L^2} + \|u(t)\|_{D^2} + \int_0^t (\|u_t(s)\|_{D_0^1}^2 + \|u(s)\|_{D^3}^2) ds \le M(c_1)c_2^{\frac{3}{2}}c_3^{\frac{1}{2}}, \tag{4.39}$$

for $0 \le t \le \min(T_*, T_3)$, *where* $T_3 = (1 + c_4)^{-9} \le T_2$.

Proof. Differentiate (3.2) with respect to t to give

$$\varrho u_{tt} + Lu_t + \nabla p_t = -\varrho_t u_t - (\varrho v\cdot\nabla v)_t - \gamma\,\mathrm{div}\left(\nabla n \odot \nabla n - \frac{|\nabla n|^2}{2} Id\right)_t. \tag{4.40}$$

Multiplying this by u_t and integrating over Ω, one has

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int \varrho|u_t|^2dx + \int(\mu|\nabla u_t|^2+(\lambda+\mu)(\mathrm{div}u_t)^2)dx\\ = -\int \nabla p_t\cdot u_t dx - \frac{1}{2}\int \varrho_t|u_t|^2dx\\ -\int\left\{\varrho_t(v\cdot\nabla v)+\varrho(v\cdot\nabla v)_t+\gamma\,\mathrm{div}\left(\nabla n\odot\nabla n-\frac{|\nabla n|^2}{2}Id\right)_t\right\}\cdot u_t dx.\end{aligned} \tag{4.41}$$

Next we estimate each term on the right hand side of (4.41). First,

$$-\int\nabla p_t\cdot u_t dx=\int p_t\mathrm{div}u_t dx\le\frac{\mu}{10}\|\nabla u_t\|_{L^2}^2+C\|p_t\|_{L^2}^2\le\frac{\mu}{10}\|\nabla u_t\|_{L^2}^2+M(c_0)c_3^4. \tag{4.42}$$

Due to (3.1), we obtain

$$\begin{aligned}-\frac{1}{2}\int\varrho_t|u_t|^2dx=\int\mathrm{div}(\varrho v)\left(\frac{1}{2}|u_t|^2\right)dx\le\int\varrho|v||u_t||\nabla u_t|dx\\ \le\|\varrho\|_{L^\infty}^{\frac{3}{4}}\|v\|_{D_0^1}\|\sqrt{\varrho}u_t\|_{L^2}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}^{\frac{3}{2}}\le\frac{\mu}{10}\|\nabla u_t\|_{L^2}^2+Cc_2^7\|\sqrt{\varrho}u_t\|_{L^2}^2.\end{aligned} \tag{4.43}$$

By Sobolev's inequality and Young's inequality, the third and fourth terms are computed as follows.

$$-\int\varrho_t(v\cdot\nabla v)\cdot u_t dx\le\|\varrho_t\|_{H^1}\|v\|_{D_0^1\cap D^2}^2\|\nabla u_t\|_{L^2}\le\frac{\mu}{10}\|\nabla u_t\|_{L^2}^2+Cc_3^8, \tag{4.44}$$

and

$$\begin{aligned}-\int\varrho(v\cdot\nabla v)_t\cdot u_t dx\le\|\varrho\|_{L^\infty}^{\frac{3}{4}}\|v_t\|_{D_0^1}\|v\|_{D_0^1}\|\sqrt{\varrho}u_t\|_{L^2}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}^{\frac{1}{2}}\\ \le\frac{\mu}{10}\|\nabla u_t\|_{L^2}^2+\eta\|v_t\|_{D_0^1}^2+C\eta^{-2}c_2^7\|\sqrt{\varrho}u_t\|_{L^2}^2,\end{aligned} \tag{4.45}$$

where $\eta\in(0,1)$ is a small constant to be determined. Finally,

$$-\gamma\int\mathrm{div}\left(\nabla n\odot\nabla n-\frac{|\nabla n|^2}{2}Id\right)_t\cdot u_t dx\le C\|\nabla n\|_{L^3}\|\nabla n_t\|_{L^6}\|\nabla u_t\|_{L^2}\le\frac{\mu}{10}\|\nabla u_t\|_{L^2}^2+M(c_1)\|n_t\|_{D^2}^2. \tag{4.46}$$

Collecting the estimates (4.42)–(4.46) and Taking $\eta=(1+c_3)^{-1}$, we get

$$\frac{d}{dt}\int\varrho|u_t|^2dx+\mu\int|\nabla u_t|^2dx\le M(c_1)(\|n_t\|_{D^2}^2+c_3^8)+2(1+c_3)^{-1}\|v_t\|_{D_0^1}^2+Cc_3^9\|\sqrt{\varrho}u_t\|_{L^2}^2. \tag{4.47}$$

Since

$$u_t\in C([0,T];H_0^1)\quad\text{and}\quad u_t(0)=-v(0)\cdot\nabla v(0)-g_2, \tag{4.48}$$

then we deduce that

$$\|\sqrt{\varrho}u_t(0)\|_{L^2}+\|u_t(0)\|_{D_0^1}\le Cc_1^3. \tag{4.49}$$

Integrate (4.47) to give

$$\|\sqrt{\varrho}u_t(t)\|_{L^2}^2+\int_0^t\|\nabla u_t(s)\|_{L^2}^2ds\le M(c_1)(1+c_3^8t)+Cc_3^9\int_0^t\|\sqrt{\varrho}u_t(s)\|_{L^2}^2ds.$$

Use Gronwall's inequality to yield

$$\|\sqrt{\varrho}u_t(t)\|_{L^2}^2+\int_0^t\|u_t(s)\|_{D_0^1}^2ds\le M(c_1), \tag{4.50}$$

for $0\le t\le\min(T_*,T_3)$.

Applying the elliptic regularity result again, we have

$$\|u\|_{D^2} \le M(c_1)(1 + \|v\|_{D_0^1}^{\frac{3}{2}}\|v\|_{D^2}^{\frac{1}{2}} + \|n\|_{D^2}^{\frac{3}{2}}\|n\|_{D^3}^{\frac{1}{2}} + \|\sqrt{\varrho}u_t(t)\|_{L^2}) \le M(c_1)c_2^{\frac{3}{2}}c_3^{\frac{1}{2}}, \tag{4.51}$$

and

$$\|u\|_{D^3} \le M(c_1)(1 + \|v(s)\|_{D_0^1\cap D^2}^2 + \|n(s)\|_{D^3}^2 + \|u_t(s)\|_{D_0^1}), \tag{4.52}$$

which gives

$$\int_0^t \|u(s)\|_{D^3}^2 ds \le M(c_1)\int_0^t \|u_t(s)\|_{D_0^1}^2 ds + M(c_1)c_3^4 t \le M(c_1), \tag{4.53}$$

for $0 \le t \le \min(T_*, T_3)$. □

Lemma 4.6.

$$\|n_t(t)\|_{D^2} + \int_0^t (\|n_{tt}(s)\|_{D^1}^2 + \|n_t(s)\|_{D^3}^2 + \|n(s)\|_{D^5}^2)ds \le M(c_1), \tag{4.54}$$

$$\|n(t)\|_{D^4} \le M(c_1)c_3^3, \tag{4.55}$$

$$\|u_t(t)\|_{D_0^1}^2 + \|u(t)\|_{D^3}^2 + \int_0^t (\|\sqrt{\varrho}u_{tt}(s)\|_{L^2}^2 + \|u_t(s)\|_{D^2}^2 + \|u(s)\|_{D^4}^2)ds \le M(c_1)c_3^{12}, \tag{4.56}$$

for $0 \le t \le \min(T_*, T_3)$.

Proof. Differentiate (4.16) with respect to x, multiply the resulting system by ∇n_{tt} and integrate over Ω to get

$$\begin{aligned}
\frac{\theta}{2}\frac{d}{dt}\int |\nabla^2 n_t|^2 dx + \int |\nabla n_{tt}|^2 dx &= \int \{-\nabla(v\cdot\nabla n)_t + \theta\nabla(|\nabla m|^2 n)_t\}\cdot\nabla n_{tt}dx \\
&\le \frac{1}{2}\|\nabla n_{tt}\|_{L^2}^2 + C(\|\nabla n\|_{H^2}^2\|v_t\|_{D_0^1}^2 + \|v\|_{D_0^1\cap D^2}^2\|n_t\|_{D^2}^2) \\
&\quad + C\|\nabla m\|_{H^2}^2(\|m_t\|_{D^2}^2\|\nabla n\|_{H^1}^2 + \|\nabla m\|_{H^2}^2\|n_t\|_{D^1}^2) \\
&\le \frac{1}{2}\|\nabla n_{tt}\|_{L^2}^2 + c_3^2\|n_t\|_{D^2}^2 + M(c_1)c_4^6,
\end{aligned}$$

which gives

$$\frac{d}{dt}\|n_t\|_{D^2}^2 + \|n_{tt}\|_{D^1}^2 \le c_3^2\|n_t\|_{D^2}^2 + M(c_1)c_4^6. \tag{4.57}$$

Since

$$\|n_t(0)\|_{D^2} \le C(\|v(0)\|_{D_0^1\cap D^2} + \|\nabla m(0)\|_{H^2}^2)\|\nabla n_0\|_{H^2} + \theta\|n_0\|_{D^4} \le M(c_0)c_1^2, \tag{4.58}$$

applying Gronwall's inequality to (4.57), we have

$$\|n_t(t)\|_{D^2}^2 + \int_0^t \|n_{tt}(s)\|_{D^1}^2 ds \le M(c_1), \tag{4.59}$$

for $0 \le t \le \min(T_*, T_3)$.

Due to (4.28) and (4.59), we obtain

$$\|n\|_{D^4} \le M(c_1)c_3^3. \tag{4.60}$$

It follows from the regularity results for the elliptic system that

$$\begin{aligned}
\int_0^t \|n_t(s)\|_{D^3}^2 ds &\le C\int_0^t \|(v\cdot\nabla n)_t - \theta(|\nabla m|^2 n)_t + n_{tt}\|_{D^1}^2 ds \\
&\le M(c_1)c_4^6 t + C\int_0^t \|n_{tt}\|_{D^1}^2 ds \le M(c_1),
\end{aligned} \tag{4.61}$$

and

$$\begin{aligned}\int_0^t \|n(s)\|_{D^5}^2 ds &\le C\int_0^t \|v\cdot\nabla n - \theta|\nabla m|^2 n + n_t\|_{D^3}^2 ds\\ &\le C\int_0^t (\|v\|_{D_0^1\cap D^3}^2\|\nabla n\|_{H^3}^2 + \|\nabla m\|_{H^3}^4\|\nabla n\|_{H^2}^2 + \|n_t\|_{D^3}^2)ds\\ &\le M(c_1)c_4^8 t + C\int_0^t \|n_t\|_{D^3}^2 ds \le M(c_1),\end{aligned}\tag{4.62}$$

for $0 \le t \le \min(T_*, T_3)$.

Multiply (4.40) by u_{tt} and integrate over Ω to give

$$\begin{aligned}&\int \varrho|u_{tt}|^2dx + \frac{1}{2}\frac{d}{dt}\int(\mu|\nabla u_t|^2 + (\lambda+\mu)(\mathrm{div}u_t)^2)dx = -\int \nabla p_t\cdot u_{tt}dx - \int \varrho_t u_t\cdot u_{tt}dx\\ &\quad - \int\left\{\varrho_t(v\cdot\nabla v) + \varrho(v\cdot\nabla v)_t + \gamma\,\mathrm{div}\left(\nabla n\odot\nabla n - \frac{|\nabla n|^2}{2}Id\right)_t\right\}\cdot u_{tt}dx\\ &= \frac{d}{dt}\int\left\{p_t\mathrm{div}u_t dx - \frac{1}{2}\varrho_t|u_t|^2 - \varrho_t(v\cdot\nabla v)\cdot u_t + \gamma\left(\nabla n\odot\nabla n - \frac{|\nabla n|^2}{2}Id\right)_t : \nabla u_t\right\}dx\\ &\quad - \int p_{tt}\mathrm{div}u_t dx + \frac{1}{2}\int \varrho_{tt}|u_t|^2dx + \int\varrho_{tt}(v\cdot\nabla v)\cdot u_t dx + \int \varrho_t(v\cdot\nabla v)_t\cdot u_t dx\\ &\quad - \int\varrho(v\cdot\nabla v)_t\cdot u_{tt}dx - \gamma\int\left(\nabla n\odot\nabla n - \frac{|\nabla n|^2}{2}Id\right)_{tt} : \nabla u_t dx.\end{aligned}\tag{4.63}$$

Using Eq. (3.1) and Sobolev's inequality, one has

$$\begin{aligned}\frac{1}{2}\int\varrho_{tt}|u_t|^2dx &\le \int(|\varrho_t||v| + |\varrho||v_t|)|u_t||\nabla u_t|dx\\ &\le C(\|\varrho_t\|_{H^1}\|v\|_{D_0^1\cap D^2}\|\nabla u_t\|_{L^2}^2 + \|\varrho\|_{L^\infty}^{\frac{3}{4}}\|v_t\|_{D_0^1}\|\sqrt{\varrho}u_t\|_{L^2}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}^{\frac{3}{2}})\\ &\le Cc_3^3\|\nabla u_t\|_{L^2}^2 + C(1+c_3)^{-1}\|v_t\|_{D_0^1}^2(\|\sqrt{\varrho}u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2),\end{aligned}\tag{4.64}$$

$$\int\varrho_{tt}(v\cdot\nabla v)\cdot u_t dx \le C\|\varrho_{tt}\|_{L^2}\|v\|_{D_0^1\cap D^2}^2\|\nabla u_t\|_{L^2} \le \|\nabla u_t\|_{L^2}^2 + Cc_3^4\|\varrho_{tt}\|_{L^2}^2,\tag{4.65}$$

$$\int\varrho_t(v\cdot\nabla v)_t\cdot u_t dx \le C\|\varrho_t\|_{H^1}\|v\|_{D_0^1\cap D^2}\|v_t\|_{D_0^1}\|\nabla u_t\|_{D_0^1} \le \|\nabla u_t\|_{L^2}^2 + Cc_3^6\|v_t\|_{D_0^1}^2,\tag{4.66}$$

$$-\int\varrho(v\cdot\nabla v)_t\cdot u_{tt}dx \le C\|\varrho\|_{L^\infty}^{\frac{1}{2}}\|v\|_{D_0^1\cap D^2}\|v_t\|_{D_0^1}\|\sqrt{\varrho}u_{tt}\|_{L^2} \le \frac{1}{2}\|\sqrt{\varrho}u_{tt}\|_{L^2}^2 + Cc_0c_3^2\|v_t\|_{D_0^1}^2,\tag{4.67}$$

and

$$\begin{aligned}-\gamma\int\left(\nabla n\odot\nabla n - \frac{|\nabla n|^2}{2}Id\right)_{tt} : \nabla u_t dx &\le C\int(|\nabla n_{tt}||\nabla n| + |\nabla n_t|^2)|\nabla u_t|dx\\ &\le \|\nabla u_t\|_{L^2}^2 + C(\|\nabla n\|_{H^2}^2\|n_{tt}\|_{D^1}^2 + \|\nabla n_t\|_{H^1}^4)\\ &\le \|\nabla u_t\|_{L^2}^2 + M(c_1)c_3^4\|\nabla n_{tt}\|_{L^2}^2 + M(c_1).\end{aligned}\tag{4.68}$$

Collecting the estimates in (4.64)–(4.68), we have

$$\begin{aligned}&\int\varrho|u_{tt}|^2dx + \frac{d}{dt}\int(\mu|\nabla u_t|^2 + (\lambda+\mu)(\mathrm{div}u_t)^2)dx\\ &\le \frac{d}{dt}\int\left\{2p_t\mathrm{div}u_t dx - \varrho_t|u_t|^2 - 2\varrho_t(v\cdot\nabla v)\cdot u_t + 2\gamma\left(\nabla n\odot\nabla n - \frac{|\nabla n|^2}{2}Id\right)_t : \nabla u_t\right\}dx\\ &\quad + M(c_1)(\|p_{tt}\|_{L^2}^2 + c_3^4\|\varrho_{tt}\|_{L^2}^2 + c_3^6\|v_t\|_{D_0^1}^2 + c_3^3\|u_t\|_{D_0^1}^2 + c_3^4\|n_{tt}\|_{D^1}^2 + 1)\\ &\quad + C(1+c_3)^{-1}\|v_t\|_{D_0^1}^2\|u_t\|_{D_0^1}^2.\end{aligned}\tag{4.69}$$

Denote

$$\Lambda(t) = \int(\mu|\nabla u_t|^2 + (\lambda+\mu)(\mathrm{div}u_t)^2)dx$$

$$-\int\left\{2p_t\mathrm{div}u_t dx-\varrho_t|u_t|^2-2\varrho_t(v\cdot\nabla v)\cdot u_t+2\gamma\left(\nabla n\odot\nabla n-\frac{|\nabla n|^2}{2}Id\right)_t:\nabla u_t\right\}dx.$$

It follows from (4.43) to (4.44) that

$$\begin{aligned}\Lambda(t)&\geq\frac{\mu}{2}\int|\nabla u_t|^2dx-C(\|p_t\|_{L^2}^2+\|\nabla n\|_{H^2}^2\|n_t\|_{D_0^1}^2)-Cc_3^8\\&\geq\frac{\mu}{2}\int|\nabla u_t|^2dx-M(c_1)c_3^8,\end{aligned}\tag{4.70}$$

and

$$\Lambda(0)\leq C\|\nabla u_t(0)\|_{L^2}^2+Cc_3^8\leq M(c_1)c_3^8.\tag{4.71}$$

Using (4.70)–(4.71) and integrating (4.69), we get

$$\|u_t(t)\|_{D_0^1}^2+\int_0^t\|\sqrt{\varrho}u_{tt}(s)\|_{L^2}^2ds\leq M(c_1)c_3^{12}+C(1+c_3)^{-1}\int_0^t\|v_t(s)\|_{D_0^1}^2\|u_t(s)\|_{D_0^1}^2ds.$$

In view of Gronwall's inequality we conclude that

$$\|u_t(t)\|_{D_0^1}^2+\int_0^t\|\sqrt{\varrho}u_{tt}(s)\|_{L^2}^2ds\leq M(c_1)c_3^{12}.\tag{4.72}$$

By (4.52) and (4.72), we have

$$\|u\|_{D^3}^2\leq M(c_1)c_3^{12}.\tag{4.73}$$

It follows from the regularity for the elliptic system again that

$$\begin{aligned}\int_0^t\|u_t(s)\|_{D^2}^2ds&\leq C\int_0^t\left\|\nabla p_t+(\varrho u_t)_t+(\varrho v\cdot\nabla v)_t+\gamma\left(\nabla n\odot\nabla n-\frac{|\nabla n|^2}{2}Id\right)_t\right\|_{L^2}^2ds\\&\leq M(c_1)\int_0^t(c_3^4+\|\sqrt{\varrho}u_{tt}\|_{L^2}^2+c_3^4\|u_t\|_{D_0^1}^2+c_3^2\|v_t\|_{D_0^1}^2)ds\leq M(c_1)c_3^{12},\end{aligned}\tag{4.74}$$

and

$$\begin{aligned}\int_0^t\|u(s)\|_{D^4}^2ds&\leq C\int_0^t\left\|\nabla p+\varrho u_t+\varrho v\cdot\nabla v+\gamma\left(\nabla n\odot\nabla n-\frac{|\nabla n|^2}{2}Id\right)\right\|_{D^2}^2ds\\&\leq M(c_1)\int_0^t(\|u_t\|_{D^2}^2+c_4^8)ds\leq M(c_1)c_3^{12},\end{aligned}\tag{4.75}$$

for $0\leq t\leq\min(T_*,T_3)$. We complete the proof of Lemma 4.6. In summary, Lemmas 4.3–4.6 yield that

$$\sup_{0\leq t\leq T_*}(\|u(t)\|_{D_0^1}+\|n_t(t)\|_{L^2}+\|\nabla n(t)\|_{H^1})+\int_0^{T_*}(\|u(t)\|_{D^2}^2+\|n_t(t)\|_{D^1}^2+\|n(t)\|_{D^3}^2)dt\leq M(c_1),$$

$$\sup_{0\leq t\leq T_*}(\|u(t)\|_{D^2}+\|n_t(t)\|_{D^1}+\|n(t)\|_{D^3})+\int_0^{T_*}(\|u_t(t)\|_{D_0^1}^2+\|u(t)\|_{D^3}^2+\|n_t(t)\|_{D^2}^2+\|n(t)\|_{D^4}^2)dt\leq M(c_1)c_2^{\frac{3}{2}}c_3^{\frac{1}{2}},$$

$$\begin{aligned}&\operatorname*{ess\,sup}_{0\leq t\leq T_*}(\|u_t(t)\|_{D_0^1}+\|u(t)\|_{D^3}+\|n_{tt}(t)\|_{L^2}+\|n_t(t)\|_{D^2}+\|n(t)\|_{D^4})\\&\quad+\int_0^{T_*}(\|u_t(t)\|_{D^2}^2+\|u(t)\|_{D^4}^2+\|n_{tt}(t)\|_{D^1}^2+\|n_t(t)\|_{D^3}^2+\|n(t)\|_{D^5}^2)dt\leq M(c_1)c_3^{12},\end{aligned}$$

$$\operatorname*{ess\,sup}_{0\leq t\leq T_*}(\|\varrho(t)-\varrho^\infty\|_{H^3}+\|\varrho_t(t)\|_{H^2}+\|\sqrt{\varrho}u_t(t)\|_{L^2})+\int_0^{T_*}\|\sqrt{\varrho}u_{tt}(t)\|_{L^2}^2dt\leq M(c_1)c_3^{12},$$

for $0\leq t\leq\min(T_*,T_3)$. Here $M=M(\cdot)$ is a fixed increasing continuous function on $[1,+\infty)$ which depends only on the parameters of C. Therefore, setting

$$c_1=M(c_0),\qquad c_2=M(c_1),\qquad c_3=c_2^5,\qquad c_4=c_2c_3^{12},$$

and

$$T_*=\min(T,T_3),\quad T_3=(1+c_4)^{-9},$$

we complete the proof of Proposition 4.2. □

5. Proof of Theorem 2.1

Let (ϱ_0, u_0, n_0) be a given data satisfying the hypotheses of Theorem 2.1. To prove the existence, we construct a sequence $\{(\varrho^k, u^k, n^k)\}_{k\ge 1}$ of approximate solutions solving the linearized problem (3.1)–(3.4) and (3.6) successively. First, we solve the heat equation

$$\begin{cases} F_t - \Delta F = 0 & \text{in } (0,\infty)\times R^3, \\ F|_{t=0} = F(0) \equiv -\nabla p(\varrho_0) + \varrho_0 g - \gamma \operatorname{div}\left(\nabla n_0 \odot \nabla n_0 - \dfrac{|\nabla n_0|^2}{2} Id\right) & \text{in } R^3. \end{cases} \tag{5.1}$$

From the compatibility condition, we have $F(0) \in H^1(R^3)$, and thus there exists a unique solution $F(t,x) \in C([0,\infty); H^1) \cap L^2(0,\infty; H^2)$.

Since $u_0 \in D_0^1 \cap D^3$ and $Lu_0 - F(0) = 0 \in D_0^1$, we can get a unique solution $u^0 \in C([0,\infty); D_0^1 \cap D^3) \cap L^2(0,\infty; D^4)$ to the following linear parabolic system

$$\begin{cases} u_t^0 + Lu^0 = F & \text{in } (0,\infty)\times R^3, \\ u^0|_{t=0} = u^0(0) = u_0 & \text{in } R^3. \end{cases} \tag{5.2}$$

Define c_0 as

$$c_0 = 2 + \varrho^\infty + \|\varrho_0 - \varrho^\infty\|_{H^3} + \|u_0\|_{D_0^1} + \|\nabla n_0\|_{H^3} + \|\sqrt{\varrho_0} g\|_{L^2} + \|g\|_{D_0^1}. \tag{5.3}$$

It follows from the elliptic regularity result that

$$\|u_0\|_{D_0^1\cap D^3} \le C(\|F(0)\|_{H^1} + \|u_0\|_{D_0^1}) \le M(c_0). \tag{5.4}$$

Then by energy estimates for (5.2), one has

$$\sup_{0\le t\le T_*} (\|u^0(t)\|_{D_0^1\cap D^3} + \|u_t^0(t)\|_{D_0^1}) + \int_0^{T_*} (\|u_t^0(t)\|_{D^2}^2 + \|u^0(t)\|_{D^4}^2) dt \le M(c_0). \tag{5.5}$$

Let n^0 be the solution of the following problem

$$\begin{cases} n_t^0 - \Delta n^0 = 0 & \text{in } (0,\infty)\times R^3, \\ n^0|_{t=0} = n^0(0) = n_0 & \text{in } R^3, \\ \nabla n^0 \to 0, & \text{as } |x| \to \infty. \end{cases} \tag{5.6}$$

Then we have $\nabla n^0 \in C([0,\infty); H^3) \cap L^2(0,\infty; H^4)$ and

$$\sup_{0\le t\le T_*} (\|\nabla n^0(t)\|_{H^3} + \|n_t^0(t)\|_{H^2} + \|n_{tt}^0(t)\|_{L^2}) + \int_0^{T_*} (\|n_{tt}^0(t)\|_{D^1}^2 + \|n_t^0(t)\|_{D^3}^2 + \|n^0(t)\|_{D^5}^2) dt \le M(c_0). \tag{5.7}$$

Up to now, (u^0, n^0) is established. To determine (ϱ^k, u^k, n^k) inductively, we need the following lemma.

Lemma 5.1. *Suppose that the spatial domain is the whole space R^3. Let (v, m) be a known vector field satisfying the regularity (3.7)–(3.8) with T replaced by T_*. Assume further that*

$$\begin{cases} \|v(0)\|_{D_0^1\cap D^3} + \|\nabla m(0)\|_{H^3} \le c_1, \\ \sup\limits_{0\le t\le T_*} (\|v(t)\|_{D_0^1} + \|m_t(t)\|_{L^2} + \|\nabla m(t)\|_{H^1}) + \displaystyle\int_0^{T_*} (\|v(t)\|_{D^2}^2 + \|m_t(t)\|_{D^1}^2 + \|m(t)\|_{D^3}^2) dt \le c_2, \\ \sup\limits_{0\le t\le T_*} (\|v(t)\|_{D^2} + \|m_t(t)\|_{D^1} + \|m(t)\|_{D^3}) \\ \quad + \displaystyle\int_0^{T_*} (\|v_t(t)\|_{D_0^1}^2 + \|v(t)\|_{D^3}^2 + \|m_t(t)\|_{D^2}^2 + \|m(t)\|_{D^4}^2) dt \le c_3, \\ \operatorname*{ess\,sup}\limits_{0\le t\le T_*} (\|v_t(t)\|_{D_0^1} + \|v(t)\|_{D^3} + \|m_t(t)\|_{D^2} + \|m(t)\|_{D^4}) \\ \quad + \displaystyle\int_0^{T_*} (\|v_t(t)\|_{D^2}^2 + \|v(t)\|_{D^4}^2 + \|m_t(t)\|_{D^3}^2 + \|m(t)\|_{D^5}^2) dt \le c_4. \end{cases} \tag{5.8}$$

Then there exists a unique solution (ϱ, u, n) to the problem (3.1)–(3.4) and (3.6) such that the estimate (4.9) holds, and

$$\begin{aligned} &\varrho - \varrho^\infty \in C([0,T_*], H^3(R^3)), \qquad u \in C([0,T_*]; D_0^1(R^3) \cap D^3(R^3)) \cap L^2(0,T_*; D^4(R^3)), \\ &u_t \in L^\infty(0,T_*; D_0^1(R^3)) \cap L^2(0,T_*; D^2(R^3)), \qquad \sqrt{\varrho} u_t \in L^\infty(0,T_*; L^2(R^3)), \\ &\nabla n \in C([0,T_*]; H^3(R^3)) \cap L^2(0,T_*; D^4(R^3)), \\ &n_t \in C([0,T_*]; H^2(R^3)) \cap L^2(0,T_*; D^3(R^3)), \\ &n_{tt} \in L^\infty(0,T_*; L^2(R^3)) \cap L^2(0,T_*; D^1(R^3)). \end{aligned} \tag{5.9}$$

Proof. We first approximate the initial data. Let R be a sufficiently large number and $\varphi \in C_0^\infty(B_1)$ satisfy

$$\varphi = 1 \text{ in } B_{\frac{1}{2}}, \quad 0 \le \varphi(x) \le 1.$$

We define

$$\varphi^R(x) = \varphi\left(\frac{x}{R}\right), \qquad g^R(x) = \varphi^R(x)g(x), \qquad v^R(t,x) = \varphi^R(x)v(t,x),$$
$$\varrho_0^R(x) = \varrho_0(x) + R^{-3}, \qquad m^R(t,x) = \varphi^R(x)m(t,x), \qquad n_0^R(x) = \varphi^R(x)n_0(x), \tag{5.10}$$

for $(t,x) \in [0,T_*] \times \Omega$.

Let $u_0^R \in H_0^1(B_R) \cap H^3(B_R)$ be the solution to the elliptic boundary value problem

$$\begin{cases} Lu_0^R = F_0^R \equiv -\nabla p(\varrho_0^R) + \varrho_0^R g^R - \gamma \operatorname{div}\left(\nabla n_0^R \odot \nabla n_0^R - \dfrac{|\nabla n_0^R|^2}{2} Id\right), & \text{in } B_R, \\ u_0^R|_{\partial B_R} = 0, \end{cases} \tag{5.11}$$

and then extend u_0^R to R^3 by defining zero outside B_R. We claim that

$$u_0^R \to u_0 \text{ strongly in } D_0^1(R^3), \quad \text{as } R \to \infty. \tag{5.12}$$

Recall that, by the compatibility condition

$$Lu_0 = F_0 \equiv -\nabla p(\varrho_0) + \varrho_0 g - \gamma \operatorname{div}\left(\nabla n_0 \odot \nabla n_0 - \frac{|\nabla n_0|^2}{2} Id\right), \tag{5.13}$$

we have $L(u_0^R - u_0) = F_0^R - F_0$ on B_R. It follows that

$$\begin{aligned} &\int_{B_R} (\mu|\nabla u_0^R|^2 + (\lambda+\mu)|\operatorname{div} u_0^R|^2)dx - \int_{B_R} (\mu \nabla u_0 : \nabla u_0^R + (\lambda+\mu)\operatorname{div} u_0 \operatorname{div} u_0^R)dx \\ &\quad = \int_{B_R} (F_0^R - F_0)\cdot u_0^R dx. \end{aligned} \tag{5.14}$$

We compute the right hand side of (5.14) as follows.

$$\begin{aligned} \int_{B_R} (F_0^R - F_0)\cdot u_0^R dx &= \int_{B_R} (p(\varrho_0^R) - p(\varrho_0)) \operatorname{div} u_0^R dx + R^{-3}\int_{B_R} g^R \cdot u_0^R dx + \int_{B_R} \varrho_0 (g^R - g)\cdot u_0^R dx \\ &\quad + \gamma \int_{B_R} \left\{ \nabla n_0^R \odot \nabla n_0^R - \frac{|\nabla n_0^R|^2}{2} Id - \left(\nabla n_0 \odot \nabla n_0 - \frac{|\nabla n_0|^2}{2} Id\right)\right\} : \nabla u_0^R dx \\ &\le C\int_{B_R} |\varrho_0^R - \varrho_0||\nabla u_0^R| dx + R^{-3}\int_{B_R} |g||u_0^R| dx + \int_{B_R} \varrho_0 g(\varphi^R - 1)\cdot u_0^R dx \\ &\quad + \gamma \int_{B_R} \left\{ \nabla n_0^R \odot \nabla n_0^R - \frac{|\nabla n_0^R|^2}{2} Id - \left(\nabla n_0 \odot \nabla n_0 - \frac{|\nabla n_0|^2}{2} Id\right)\right\} : \nabla u_0^R dx \\ &= \sum_{i=1}^4 I_i, \end{aligned}$$

where

$$I_1 \le CR^{-3}\int_{B_R} |\nabla u_0^R| dx \le CR^{-\frac{3}{2}} \|\nabla u_0^R\|_{L^2}, \tag{5.15}$$

$$I_2 \le CR^{-3}\|g\|_{L^6(B^R)} \|u_0^R\|_{L^6(B^R)} |B_R|^{\frac{2}{3}} \le CR^{-1}\|g\|_{D^1}\|\nabla u_0^R\|_{L^2}, \tag{5.16}$$

$$\begin{aligned} I_3 &= \int_{B_R} \left(Lu_0 + \nabla p(\varrho_0) + \gamma \operatorname{div}\left(\nabla n_0 \odot \nabla n_0 - \frac{|\nabla n_0|^2}{2} Id\right)\right)(\varphi^R - 1)\cdot u_0^R dx \\ &\le C\int_{B_R} (|\nabla u_0| + |p(\varrho_0) - p(\varrho^\infty)| + |\nabla n_0|^2)(|\nabla \varphi^R||u_0^R| + |\varphi^R - 1||\nabla u_0^R|)dx \\ &\le C\left\{ \|\nabla u_0\|_{L^2\left(B_R \setminus B_{\frac{R}{2}}\right)} + \|\varrho_0 - \varrho^\infty\|_{L^2\left(B_R \setminus B_{\frac{R}{2}}\right)} + \|\nabla n_0\|^{\frac{1}{2}}_{L^2\left(B_R \setminus B_{\frac{R}{2}}\right)} \|\nabla n_0\|^{\frac{3}{2}}_{L^6\left(B_R \setminus B_{\frac{R}{2}}\right)} \right\} \|\nabla u_0^R\|_{L^2}, \end{aligned} \tag{5.17}$$

and

$$\begin{aligned}
I_4 &\le \gamma \int_{B_R} ((\varphi^R)^2 - 1)\left(\nabla n_0 \odot \nabla n_0 - \frac{|\nabla n_0|^2}{2} Id\right) : \nabla u_0^R dx \\
&\quad + C \int_{B_R} \left(|\nabla \varphi^R|^2 |n_0|^2 + |\varphi^R||\nabla \varphi^R||n_0||\nabla n_0|\right) |\nabla u_0^R| dx \\
&\le C \int_{B_R \setminus B_{\frac{R}{2}}} \left(|\nabla n_0|^2 + \frac{1}{R^2} + \frac{1}{R}|\nabla n_0|\right) |\nabla u_0^R| dx \\
&\le C \left(\|\nabla n_0\|_{L^2\left(B_R\setminus B_{\frac{R}{2}}\right)}^{\frac{1}{2}} \|\nabla n_0\|_{L^6\left(B_R\setminus B_{\frac{R}{2}}\right)}^{\frac{3}{2}} + R^{-\frac{1}{2}} + R^{-1}\|\nabla n_0\|_{L^2} \right) \|\nabla u_0^R\|_{L^2}.
\end{aligned} \tag{5.18}$$

Substituting the estimates (5.15)–(5.18) into (5.14) and choosing R large enough, we have

$$\|\nabla u_0^R\|_{D_0^1(R^3)} \le \|\nabla u_0\|_{D_0^1(R^3)} + o(1) \quad \text{and} \quad \int_{R^3} (F_0^R - F_0)\cdot u_0^R dx = o(1), \tag{5.19}$$

where $o(1)$ denotes a function of R which tends to zero as $R \to \infty$. This implies that there exist a sequence $\{R_j\}$, $R_j \to \infty$, and some vector u_0^∞ such that

$$u_0^{R_j} \to u_0^\infty \text{ weakly in } D_0^1(R^3), \quad \text{as } j \to \infty,$$

and

$$Lu_0^{R_j} \to Lu_0^\infty \text{ in } D^{-1}(R^3), \quad \text{as } j \to \infty.$$

On the other hand, similar argument as (5.19) can show that

$$\int_{R^3} (F_0^R - F_0)\cdot \Psi dx \to 0, \quad \text{as } R \to \infty, \forall \Psi \in D_0^1(R^3),$$

which means

$$Lu_0^{R_j} \to Lu_0 \text{ in } D^{-1}(R^3), \quad \text{as } j \to \infty.$$

Therefore $Lu_0^\infty = Lu_0$. So we have $u_0^\infty = u_0$, and

$$u_0^{R_j} \to u_0 \text{ weakly in } D_0^1(R^3), \quad \text{as } j \to \infty.$$

It follows from (5.19) again that

$$\|\nabla u_0^{R_j}\|_{D_0^1(R^3)} \to \|\nabla u_0\|_{D_0^1(R^3)}, \quad \text{as } j \to \infty.$$

Thus we obtain

$$u_0^{R_j} \to u_0, \text{ strongly in } D_0^1(R^3), \quad \text{as } j \to \infty.$$

Since the above argument also shows that every subsequence of $\{u_0^R\}$ has a subsequence converging in $D_0^1(R^3)$ to the same limit u_0, we conclude that the whole sequence $\{u_0^R\}$ converges to u_0 in $D_0^1(R^3)$ as $R \to \infty$. The claim is proved.

We consider the following approximate problem in $(0, T) \times B_R$

$$\begin{cases}
\varrho_t^R + \operatorname{div}(\varrho^R v^R) = 0, \\
\varrho^R u_t^R + Lu^R + \nabla p^R = -\varrho^R (v^R\cdot\nabla)v^R - \gamma \operatorname{div}\left(\nabla n^R \odot \nabla n^R - \dfrac{|\nabla n^R|^2}{2} Id\right), \\
n_t^R + (v^R\cdot\nabla)n^R = \theta(\Delta n^R + |\nabla m^R|^2 n^R), \\
(\varrho^R, u^R, n^R)\big|_{t=0} = (\varrho_0^R, u_0^R, n_0^R) & \text{in } B_R, \\
\left(u^R, \dfrac{\partial n^R}{\partial R}\right) = (0, 0) & \text{on } (0, T)\times \partial B_R,
\end{cases} \tag{5.20}$$

where $p^R = p(\varrho^R)$. Since $\varrho_0^R \ge R^{-3} > 0$, it follows from Lemma 4.1 that there exists a unique strong solution (ϱ^R, u^R, n^R) to (5.20) satisfying (4.1)–(4.6) with $T = T_*$, $\Omega = B_R$.

By construction, as $R \to \infty$,

$$\|v^R - v\|_{C([0,T_*];D_0^1\cap D^3)\cap L^2(0,T_*;D^4)} + \|v_t^R - v_t\|_{L^\infty(0,T_*;D_0^1)\cap L^2(0,T_*;D^2)} + \|\nabla m^R - \nabla m\|_{C([0,T_*];H^3)\cap L^2(0,T_*;D^4)} + \|m_t^R - m_t\|_{L^\infty(0,T_*;H^2)\cap L^2(0,T_*;D^3)} \to 0, \tag{5.21}$$

and

$$\left\|\sqrt{\varrho_0^R}g^R - \sqrt{\varrho_0}g\right\|_{L^2} + \|g^R - g\|_{D_0^1} \to 0. \tag{5.22}$$

Combining (5.3), (5.8), (5.12), (5.21) and (5.22), we deduce that there exists a large number $R_1 > 1$, such that for all $R > R_1$,

$$1 + (\varrho^\infty + R^{-3}) + \|\varrho_0^R - (\varrho^\infty + R^{-3})\|_{H^3(B_R)} + \|u_0^R\|_{D_0^1(B_R)} + \|\nabla n_0^R\|_{H^3(B_R)} + \left\|\sqrt{\varrho_0^R}g^R\right\|_{L^2(B_R)} + \|g^R\|_{D_0^1(B_R)} \le c_0, \tag{5.23}$$

and (v^R, m^R) satisfies the assumption of Proposition 4.2, so the a priori estimate (4.9) holds for the solution (ϱ^R, u^R, n^R) with the spatial domain being B_R. Since these estimates are uniform in R, there exist a subsequence $\{R_j\}$ and some vector (ϱ, u, n) such that

$$(\varrho^{R_j}, u^{R_j}, n^{R_j}) \to (\varrho, u, n) \text{ weakly or weak-}* \quad \text{as } j \to \infty \text{ in the corresponding spaces.}$$

Furthermore, since (ϱ, u, n) also satisfies (4.9) with domain being B_R for each $R > R_1$, we have the regularities in (5.9).

We will show that (ϱ, u, n) is a solution to (3.1)–(3.4) and (3.6). First, from (4.9), it is clear that $(\varrho, u, \nabla n) \to (\varrho^\infty, 0, 0)$ as $|x| \to \infty$. For any fixed $R > R_1$, $(\varrho^{R_j}, u^{R_j}, n^{R_j})$ satisfies (4.9) on the ball B_R, thus the standard compactness argument shows that

$$(\varrho^{R_j}, u^{R_j}) \to (\rho, u) \text{ in } C([0, T_*]; H^1(B_R)), \quad \text{as } j \to \infty,$$
$$n^{R_j} \to n \text{ in } C([0, T_*]; H^2(B_R)), \quad \text{as } j \to \infty. \tag{5.24}$$

It follows from this, (5.10) and (5.12) that (ϱ, u, n) solves (3.1)–(3.5) on B_R. Since R is arbitrary, (ϱ, u, n) is a solution to (3.1)–(3.4) and (3.6).

The uniqueness of the strong solution to (3.1)–(3.4) and (3.6) in the class (4.9) is trivial as follows from the standard energy estimates. Finally, by the Lions–Aubin Lemma we have

$$u \in C([0, T_*]; D_0^1 \cap D^3), \qquad \nabla n \in C([0, T_*]; H^3), \qquad n_t \in C([0, T_*]; H^2),$$

then from the transport equation $(3.1)_1$, we deduce that

$$\varrho \in C([0, T_*]; H^3).$$

Lemma 5.1 is proved. □

Now we are ready to prove Theorem 2.1. We have established (u^0, n^0) satisfying (5.8), then we can define (ϱ^1, u^1, n^1) by solving (3.1)–(3.4) and (3.6) with $v = u^0$ and $m = n^0$, and (ϱ^1, u^1, n^1) satisfies (4.9). That is, (u^1, n^1) satisfies (5.8). Then inductively we can obtain (ϱ^k, u^k, n^k) satisfying (4.9). Thus passing to the limit (ϱ, u, n), we can show that (ϱ, u, n) satisfies the Cauchy problem (1.1)–(1.3) with the regularity (2.4)–(2.7).

6. Proof of Theorem 2.2

To prove Theorem 2.2, we follow the same methods as in the proof of Theorem 2.1. Hence we consider the linearized problem (3.1)–(3.6). In addition to the conditions (3.7)–(3.9), we assume further the known vector (v, m) satisfies

$$t^{\frac{1}{2}} v \in L^\infty(0, T_0; D^4), \qquad t^{\frac{1}{2}} v_t \in L^\infty(0, T_0; D^2), \qquad t^{\frac{1}{2}} v_{tt} \in L^2(0, T_0; D_0^1), \tag{6.1}$$

$$t v_t \in L^\infty(0, T_0; D^3), \qquad t v_{tt} \in L^\infty(0, T_0; D^1) \cap L^2(0, T_0; D^2), \tag{6.2}$$

$$t^{\frac{3}{2}} v_{tt} \in L^\infty(0, T_0; D^2), \qquad t^{\frac{3}{2}} v_{ttt} \in L^2(0, T_0; D_0^1), \tag{6.3}$$

$$t^{\frac{1}{2}} m_t \in L^\infty(0, T_0; D^3), \qquad t^{\frac{1}{2}} m_{tt} \in L^\infty(0, T_0; D^1), \qquad t m_{tt} \in L^\infty(0, T_0; D^2). \tag{6.4}$$

Then the key point for the proof of Theorem 2.2 is the following estimates.

Proposition 6.1. *In addition to (4.8), we assume further*

$$\operatorname*{ess\,sup}_{0\le t\le T_*} t^{\frac{1}{2}}(\|v_t\|_{D^2}+\|v\|_{D^4}+\|m_{tt}\|_{D^1}+\|m_t\|_{D^3})+\int_0^{T_*} t\|v_{tt}\|^2_{D_0^1}dt\le 1+c_5, \tag{6.5}$$

$$\operatorname*{ess\,sup}_{0\le t\le T_*} t(\|v_{tt}\|_{D_0^1}+\|v_t\|_{D^3}+\|m_{tt}\|_{D^2})+\int_0^{T_*} t^2\|v_{tt}\|^2_{D^2}dt\le 1+c_6, \tag{6.6}$$

$$\operatorname*{ess\,sup}_{0\le t\le T_*} t^{\frac{3}{2}}\|v_{tt}\|_{D^2}+\int_0^{T_*} s^3\|v_{ttt}(s)\|^2_{D_0^1}ds\le 1+c_6, \tag{6.7}$$

for some constants c_5 and c_6 satisfying $c_4\le c_5\le c_6$. Then (ϱ,u,n) satisfies (4.9) and

$$\operatorname*{ess\,sup}_{0\le t\le T_*} t^{\frac{1}{2}}(\|u_t\|_{D^2}+\|u\|_{D^4}+\|n_{tt}\|_{D^1}+\|n_t\|_{D^3})+\int_0^{T_*} t\|u_{tt}\|^2_{D_0^1}dt\le c_5, \tag{6.8}$$

$$\operatorname*{ess\,sup}_{0\le t\le T_*} t(\|u_{tt}\|_{D_0^1}+\|u_t\|_{D^3}+\|n_{tt}\|_{D^2})+\int_0^{T_*} t^2\|u_{tt}\|^2_{D^2}dt\le c_6, \tag{6.9}$$

$$\operatorname*{ess\,sup}_{0\le t\le T_*} t^{\frac{3}{2}}(\|\sqrt{\rho}u_{ttt}\|_{L^2}+\|u_{tt}\|_{D^2})+\int_0^{T_*} s^3\|u_{ttt}(s)\|^2_{D_0^1}ds\le c_6. \tag{6.10}$$

Lemma 6.2 (*[15, Lemma 11]*)**.**

$$\|\varrho_{tt}(t)\|_{L^2}\le Cc_4^4,\qquad \int_0^t\|\varrho_{tt}(s)\|^2_{H^1}ds\le Cc_4^8,$$

$$\|p_{tt}(t)\|_{L^2}\le M(c_0)c_4^4,\qquad \int_0^t\|p_{tt}(s)\|^2_{H^1}ds\le M(c_0)c_4^8,$$

$$\int_0^t s\|\varrho_{ttt}(s)\|^2_{L^2}\le Cc_5^{12},\qquad \int_0^t s\|p_{ttt}(s)\|^2_{L^2}\le M(c_0)c_5^{12},$$

for $0\le t\le\min(T_*,T_1)$, where $T_1=(1+c_4)^{-1}$ and $p^\infty=p(\varrho^\infty)$.

Lemma 6.3.

$$t(\|\sqrt{\rho}u_{tt}(t)\|^2_{L^2}+\|u_t(t)\|^2_{D^2}+\|u(t)\|^2_{D^4})+\int_0^t s\|u_{tt}(s)\|^2_{D_0^1}dt\le M(c_1)c_4^{12}, \tag{6.11}$$

for $0\le t\le\min(T_*,T_4)$, where $T_4=(1+c_5)^{-9}\le T_3$.

Proof. We differentiate (4.40) with respect to t to derive

$$\begin{aligned}\varrho u_{ttt}+Lu_{tt}&=-\nabla p_{tt}-\varrho(v\cdot\nabla v)_{tt}-2\varrho_t(u_t+v\cdot\nabla v)_t\\&\quad-\varrho_{tt}(u_t+v\cdot\nabla v)-\gamma\,\mathrm{div}\left(\nabla n\odot\nabla n-\frac{|\nabla n|^2}{2}Id\right)_{tt}.\end{aligned}\tag{6.12}$$

Multiplying (6.12) by u_{tt} and integrating over Ω, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\varrho|u_{tt}|^2dx+\int\mu|\nabla u_{tt}|^2+(\lambda+\mu)(\mathrm{div}u_{tt})^2dx\\&=\int p_{tt}\mathrm{div}u_{tt}dx-\int\varrho(v\cdot\nabla v)_{tt}\cdot u_{tt}dx-2\int\varrho_t(v\cdot\nabla v)_t\cdot u_{tt}dx\\&\quad-\int\varrho_{tt}(v\cdot\nabla)v\cdot u_{tt}dx-\frac{3}{2}\int\varrho_t|u_{tt}|^2dx-\int\varrho_{tt}u_t\cdot u_{tt}dx-\gamma\int\mathrm{div}\left(\nabla n\odot\nabla n-\frac{|\nabla n|^2}{2}Id\right)_{tt}\cdot u_{tt}dx.\end{aligned}\tag{6.13}$$

Now we estimate each term of the right hand side of (6.13) as follows.

$$\int p_{tt}\mathrm{div}u_{tt}dx\le\frac{\mu}{12}\|\nabla u_{tt}\|^2_{L^2}+C\|p_{tt}(s)\|^2_{L^2},$$

$$
\begin{aligned}
-\int \varrho(v\cdot\nabla v)_{tt}\cdot u_{tt}dx &\le C\|\varrho\|_{L^\infty}^{\frac12}(\|v\|_{D_0^1\cap D^2}\|v_{tt}\|_{D_0^1}+\|v_t\|_{D_0^1}\|v_t\|_{D_0^1\cap D^2})\|\sqrt{\varrho}u_{tt}\|_{L^2}\\
&\le \eta\|v_{tt}\|_{D_0^1}^2+C\eta^{-1}c_3^3\|\sqrt{\varrho}u_{tt}\|_{L^2}^2+Cc_4^2\|v_t\|_{D_0^1\cap D^2}^2,
\end{aligned}
$$

$$
-2\int \varrho_t(v\cdot\nabla v)_t\cdot u_{tt}dx\le C\|\varrho_t\|_{L^3}\|v\|_{D_0^1\cap D^2}\|v_t\|_{D_0^1}\|\nabla u_{tt}\|_{L^2}\le \frac{\mu}{12}\|\nabla u_{tt}\|_{L^2}^2+Cc_4^8,
$$

$$
-\int \varrho_{tt}(v\cdot\nabla)v\cdot u_{tt}dx\le C\|\varrho_{tt}\|_{L^2}\|v\|_{D_0^1\cap D^2}^2\|\nabla u_{tt}\|_{L^2}\le \frac{\mu}{12}\|\nabla u_{tt}\|_{L^2}^2+Cc_3^4\|\varrho_{tt}\|_{L^2}^2,
$$

$$
-\frac32\int \varrho_t|u_{tt}|^2dx\le 3\int \varrho|v||u_{tt}||\nabla u_{tt}|dx\le \frac{\mu}{12}\|\nabla u_{tt}\|_{L^2}^2+Cc_3^3\|\sqrt{\varrho}u_{tt}\|_{L^2}^2,
$$

$$
\begin{aligned}
-\int \varrho_{tt}u_t\cdot u_{tt}dx &\le \int(|\varrho_t||v|+\varrho|v_t|)(|u_t||\nabla u_{tt}|+|\nabla u_t||u_{tt}|)dx\\
&\le C\|\varrho_t\|_{L^3}\|v\|_{D_0^1\cap D^2}\|u_t\|_{D_0^1}\|\nabla u_{tt}\|_{L^2}+C\|\varrho\|_{L^\infty}^{\frac34}\|v_t\|_{D_0^1}\|\sqrt{\varrho}u_t\|_{L^2}^{\frac12}\|u_t\|_{D_0^1}^{\frac12}\|\nabla u_{tt}\|_{L^2}\\
&\quad +C\|\varrho\|_{L^\infty}^{\frac34}\|v_t\|_{D_0^1}\|u_t\|_{D_0^1}\|\sqrt{\varrho}u_{tt}\|_{L^2}^{\frac12}\|\nabla u_{tt}\|_{L^2}^{\frac12}\\
&\le C\|\varrho_t\|_{L^3}^2\|v\|_{D_0^1\cap D^2}^2\|u_t\|_{D_0^1}^2+C\|\varrho\|_{L^\infty}^{\frac32}\|v_t\|_{D_0^1}^2\|\sqrt{\varrho}u_t\|_{L^2}\|u_t\|_{D_0^1}\\
&\quad +C\|\varrho\|_{L^\infty}^{\frac32}\|v_t\|_{D_0^1}^2\|u_t\|_{D_0^1}^2+C\|\sqrt{\varrho}u_{tt}\|_{L^2}^2+\frac{\mu}{12}\|\nabla u_{tt}\|_{L^2}^2\\
&\le Cc_4^7\|u_t\|_{D_0^1}^2+\|\sqrt{\varrho}u_t\|_{L^2}^2+C\|\sqrt{\varrho}u_{tt}\|_{L^2}^2+\frac{\mu}{12}\|\nabla u_{tt}\|_{L^2}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
&-\gamma\int \operatorname{div}\left(\nabla n\odot\nabla n-\frac{|\nabla n|^2}{2}Id\right)_{tt}\cdot u_{tt}dx\le C\int(|\nabla n_t|^2+|\nabla n||\nabla n_{tt}|)|\nabla u_{tt}|dx\\
&\le C(\|\nabla n_t\|_{H^1}^2+\|\nabla n\|_{H^2}\|n_{tt}\|_{D^1})\|\nabla u_{tt}\|_{L^2}\le \frac{\mu}{12}\|\nabla u_{tt}\|_{L^2}^2+M(c_1)+M(c_1)c_3^4\|n_{tt}\|_{D^1}^2.
\end{aligned}
$$

Substituting all the estimates into (6.13) and taking $\eta=(1+c_5)^{-1}$, we have

$$
\begin{aligned}
\frac12\frac{d}{dt}\int \varrho|u_{tt}|^2dx+\frac{\mu}{2}\int|\nabla u_{tt}|^2dx &\le M(c_1)\{\|p_{tt}\|_{L^2}^2+c_4^2\|v_t\|_{D_0^1\cap D^2}^2+c_4^8+c_3^4\|\varrho_{tt}\|_{L^2}^2+c_3^4\|n_{tt}\|_{D^1}^2\}\\
&\quad +C(c_4^7\|u_t\|_{D_0^1}^2+\|\sqrt{\varrho}u_t\|_{L^2}^2)+(1+c_5)^{-1}\|v_{tt}\|_{D_0^1}^2+Cc_5^4\|\sqrt{\varrho}u_{tt}\|_{L^2}^2. \qquad (6.14)
\end{aligned}
$$

Multiplying (6.14) by t and integrating over $(\tau,\bar t)$, we obtain

$$
\bar t\|\sqrt{\varrho}u_{tt}(\bar t)\|_{L^2}^2+\mu\int_\tau^{\bar t}t\|\nabla u_{tt}(t)\|_{L^2}^2dt\le M(c_1)c_4^{12}+\tau\|\sqrt{\varrho}u_{tt}(\tau)\|_{L^2}^2+Cc_5^4\int_\tau^{\bar t}t\|\sqrt{\varrho}u_{tt}(t)\|_{L^2}^2dt,
$$

for $0<\tau\le\bar t\le\min(T_*,T_3)$. By virtue of Gronwall's inequality, we deduce that

$$
t\|\sqrt{\varrho}u_{tt}(t)\|_{L^2}^2+\int_\tau^t s\|\nabla u_{tt}(s)\|_{L^2}^2ds\le M(c_1)(c_4^{12}+\tau\|\sqrt{\varrho}u_{tt}(\tau)\|_{L^2}^2), \qquad (6.15)
$$

for $0<\tau\le t\le\min(T_*,T_4)$, where $T_4=(1+c_5)^{-9}\le T_3$. Since $\sqrt{\varrho}u_{tt}\in L^2(0,T;L^2)$, there is a sequence $\{\tau_k\}$ of positive times such that

$$
\tau_k\to 0\quad\text{and}\quad \tau_k\|\sqrt{\varrho}u_{tt}(\tau_k)\|_{L^2}^2\to 0 \text{ as } k\to\infty.
$$

Let $\tau=\tau_k\to 0$ in (6.15). We conclude that

$$
t\|\sqrt{\varrho}u_{tt}(t)\|_{L^2}^2+\int_0^t s\|\nabla u_{tt}(s)\|_{L^2}^2ds\le M(c_1)c_4^{12},
$$

for $0\le t\le\min(T_*,T_4)$. It follows from the elliptic regularity result that

$$
t\|u_t(t)\|_{D^2}^2+t\|u(t)\|_{D^4}^2\le M(c_1)c_4^{12},
$$

for $0\le t\le\min(T_*,T_4)$. This completes the proof of Lemma 6.3. □

Lemma 6.4.

$$t\|n_{tt}(t)\|_{D^1}^2 + t\|n_t(t)\|_{D^3}^2 + \int_0^t s(\|n_{ttt}(s)\|_{L^2}^2 + \|n_{tt}(s)\|_{D^2}^2)ds \le M(c_1)c_4^7c_5, \tag{6.16}$$

for $0 \le t \le \min(T_*, T_3)$.

Proof. We differentiate (4.16) with respect to t to derive

$$n_{ttt} + (v \cdot \nabla n)_{tt} = \theta(\Delta n_{tt} + (|\nabla m|^2 n)_{tt}). \tag{6.17}$$

Multiplying (6.17) by n_{ttt} and integrating over Ω, we have

$$\begin{aligned}\frac{\theta}{2}\frac{d}{dt}\int |\nabla n_{tt}|^2dx + \|n_{ttt}\|_{L^2}^2 &= \int (\theta(|\nabla m|^2 n)_{tt} - (v \cdot \nabla n)_{tt}) \cdot n_{ttt}dx \\ &\le C\{(\|\nabla m\|_{H^2}\|m_{tt}\|_{D^1} + \|m_t\|_{D^2}^2)\|\nabla n\|_{H^1} + \|m\|_{D^2}\|m_t\|_{D^2}\|n_t\|_{D^1} \\ &\quad + \|m\|_{D^2}^2\|n_{tt}\|_{D^1} + \|\nabla n\|_{H^1}\|v_{tt}\|_{D_0^1} + \|\nabla n_t\|_{H^1}\|v_t\|_{D_0^1} + \|n_{tt}\|_{D^1}\|v\|_{D_0^1}\}\|n_{ttt}\|_{L^2} \\ &\le \frac{1}{2}\|n_{ttt}\|_{L^2}^2 + M(c_1)(c_3^2\|m_{tt}\|_{D^1}^2 + c_4^8 + c_3^4\|n_{tt}\|_{D^1}^2 + \|v_{tt}\|_{D_0^1}^2).\end{aligned} \tag{6.18}$$

Multiplying (6.18) by t and integrating over $(\tau, \bar{t})$, we obtain

$$\begin{aligned}\bar{t}\|n_{tt}(\bar{t})\|_{D^1}^2 + \int_\tau^{\bar{t}} t\|n_{ttt}(t)\|_{L^2}^2dt &\le \tau\|n_{tt}(\tau)\|_{D^1}^2 + M(c_1)\left(c_4^8 + \int_\tau^{\bar{t}} t\|v_{tt}(t)\|_{D_0^1}^2dt\right) \\ &\le \tau\|n_{tt}(\tau)\|_{D^1}^2 + M(c_1)c_4^7c_5,\end{aligned}$$

for $0 < \tau \le \bar{t} \le \min(T_*, T_3)$. It follows from this and

$$\tau_k\|n_{tt}(\tau_k)\|_{D^1}^2 \to 0 \quad \text{for some } \tau_k \to 0$$

that

$$t\|n_{tt}(t)\|_{D^1}^2 + \int_0^t s\|n_{ttt}(s)\|_{L^2}^2ds \le M(c_1)c_4^7c_5,$$

for $0 \le t \le \min(T_*, T_3)$. Then in view of the elliptic regularity result, we complete the proof of Lemma 6.4. □

Lemma 6.5.

$$t^2\|n_{tt}(t)\|_{D^2}^2 + \int_0^t s^2\|n_{ttt}(s)\|_{D^1}^2ds \le M(c_1)c_5^{16}, \tag{6.19}$$

for $0 \le t \le \min(T_*, T_5)$, *where* $T_5 = (1 + c_6)^{-9} \le T_4$.

Proof. We differentiate (6.17) with respect to x, multiplying the resulting equation by ∇n_{ttt} and integrating over Ω, we get

$$\begin{aligned}\frac{\theta}{2}\frac{d}{dt}\int |\nabla^2 n_{tt}|^2dx + \|n_{ttt}\|_{D^1}^2 &= \int \nabla(\theta(|\nabla m|^2 n)_{tt} - (v \cdot \nabla n)_{tt}) \cdot \nabla n_{ttt}dx \\ &\le C\{(\|\nabla m\|_{H^2}\|m_{tt}\|_{D^2} + \|\nabla m_t\|_{H^2}\|m_t\|_{D^2})\|\nabla n\|_{H^1} + \|\nabla m\|_{H^2}^2\|n_{tt}\|_{D^1} \\ &\quad + \|\nabla m\|_{H^2}\|\nabla m_t\|_{H^2}\|n_t\|_{D^1} + \|\nabla n\|_{H^2}\|v_{tt}\|_{D_0^1} + \|\nabla n_t\|_{H^2}\|v_t\|_{D_0^1} \\ &\quad + \|n_{tt}\|_{D^2}\|v\|_{D_0^1\cap D^2}\}\|n_{ttt}\|_{D^1} \\ &\le \frac{1}{2}\|n_{ttt}\|_{D^1}^2 + M(c_1)(c_3^6\|\nabla m_t\|_{H^2}^2 + c_3^4\|n_{tt}\|_{D^1}^2 + c_3^4\|v_{tt}\|_{D_0^1}^2 + c_4^2\|\nabla n_t\|_{H^2}^2) \\ &\quad + Cc_3^2\|m_{tt}\|_{D^2}^2 + Cc_3^2\|n_{tt}\|_{D^2}^2.\end{aligned} \tag{6.20}$$

Multiplying (6.20) by t^2 and integrating over $(\tau, \bar{t})$, we obtain

$$\begin{aligned}&\bar{t}^2\|n_{tt}(\bar{t})\|_{D^2}^2 + \int_\tau^{\bar{t}} t^2\|n_{ttt}(t)\|_{D^1}^2dt \\ &\quad \le \tau^2\|n_{tt}(\tau)\|_{D^2}^2 + 2\int_\tau^{\bar{t}} t\|n_{tt}(t)\|_{D^2}^2dt + M(c_1)c_5^{16} + Cc_6^4\bar{t} + Cc_3^2\int_\tau^{\bar{t}} t^2\|n_{tt}(t)\|_{D^2}^2dt \\ &\quad \le \tau^2\|n_{tt}(\tau)\|_{D^2}^2 + M(c_1)c_5^{16} + Cc_3^2\int_\tau^{\bar{t}} t^2\|n_{tt}(t)\|_{D^2}^2dt,\end{aligned}$$

for $0<\tau\le\bar t\le\min(T_*,T_5)$, where $T_5=(1+c_6)^{-9}\le T_4\le T_3$. It follows from this and Gronwall's inequality that

$$\bar t^2\|n_{tt}(\bar t)\|_{D^2}^2+\int_\tau^{\bar t}t^2\|n_{ttt}(t)\|_{D^1}^2dt\le\tau^2\|n_{tt}(\tau)\|_{D^2}^2+M(c_1)c_5^{16},$$

for $0<\tau\le\bar t\le\min(T_*,T_5)$. Using the fact that

$$\tau_k^2\|n_{tt}(\tau_k)\|_{D^2}^2\to 0\quad\text{for some }\tau_k\to 0,$$

we obtain

$$t^2\|n_{tt}(t)\|_{D^2}^2+\int_0^t s^2\|n_{ttt}(s)\|_{D^1}^2ds\le M(c_1)c_5^{16},$$

for $0\le t\le\min(T_*,T_5)$. We complete the proof of Lemma 6.5. □

Lemma 6.6.

$$t^2(\|u_{tt}(t)\|_{D_0^1}^2+\|u_t(t)\|_{D^3}^2)+\int_0^t s^2(\|\sqrt{\rho}u_{ttt}\|_{L^2}^2+\|u_{tt}(s)\|_{D^2}^2)dt\le M(c_1)c_5^{18},\tag{6.21}$$

for $0\le t\le\min(T_*,T_5)$.

Proof. Multiply (6.12) by u_{ttt} and integrate over Ω to give

$$\begin{aligned}&\int\varrho|u_{ttt}|^2dx+\frac12\frac{d}{dt}\int\mu|\nabla u_{tt}|^2+(\lambda+\mu)(\mathrm{div}u_{tt})^2dx\\&=-\int\nabla p_{tt}\cdot u_{ttt}dx-\int\varrho(v\cdot\nabla v)_{tt}\cdot u_{ttt}dx-2\int\varrho_t(u_t+v\cdot\nabla v)_t\cdot u_{ttt}dx\\&\quad-\int\varrho_{tt}(u_t+v\cdot\nabla v)\cdot u_{ttt}dx-\gamma\int\mathrm{div}\left(\nabla n\odot\nabla n-\frac{|\nabla n|^2}{2}Id\right)_{tt}\cdot u_{ttt}dx.\end{aligned}\tag{6.22}$$

First, it follows from the Cauchy inequality and Sobolev's inequality that

$$\begin{aligned}-\int\nabla p_{tt}\cdot u_{ttt}dx&=\frac{d}{dt}\int p_{tt}\mathrm{div}u_{tt}dx-\int p_{ttt}\mathrm{div}u_{tt}dx\\&\le\frac{d}{dt}\int\nabla p_{tt}\mathrm{div}u_{tt}dx+\|u_{tt}\|_{D_0^1}^2+\|p_{ttt}\|_{L^2}^2,\end{aligned}$$

and

$$\begin{aligned}-\int\varrho(v\cdot\nabla v)_{tt}\cdot u_{ttt}dx&\le C\|\varrho\|_{L^\infty}^{\frac12}(\|v\|_{D_0^1\cap D^2}\|v_{tt}\|_{D_0^1}+\|v_t\|_{D_0^1}\|v_t\|_{D_0^1\cap D^2})\|\sqrt{\varrho}u_{ttt}\|_{L^2}\\&\le Cc_4^3(\|v_{tt}\|_{D_0^1}^2+\|v_t\|_{D_0^1\cap D^2}^2)+\frac12\|\sqrt{\varrho}u_{ttt}\|_{L^2}^2.\end{aligned}$$

To estimate the third term, we observe that

$$\begin{aligned}-2\int\varrho_t(u_t+v\cdot\nabla v)_t\cdot u_{ttt}dx&=-\frac{d}{dt}\int\{\varrho_t|u_{tt}|^2+2\varrho_t(v\cdot\nabla v)_t\cdot u_{tt}\}dx+\int\varrho_{tt}|u_{tt}|^2dx\\&\quad+2\int\varrho_{tt}(v\cdot\nabla v)_t\cdot u_{tt}dx+2\int\varrho_t(v\cdot\nabla v)_{tt}\cdot u_{tt}dx,\end{aligned}$$

where

$$\int\varrho_{tt}|u_{tt}|^2dx\le\int(|\varrho_t||v|+\varrho|v_t|)|u_{tt}||\nabla u_{tt}|dx\le Cc_4^3\|u_{tt}\|_{D_0^1}^2+C\|v_t\|_{D_0^1\cap D^2}^2\|\sqrt{\varrho}u_{tt}\|_{L^2}^2,$$

$$\int\varrho_{tt}(v\cdot\nabla v)_t\cdot u_{tt}dx\le C\|\varrho_{tt}\|_{L^2}\|v\|_{D_0^1\cap D^2}\|v_t\|_{D_0^1\cap D^2}\|u_{tt}\|_{D_0^1}\le Cc_4^5(\|v_t\|_{D_0^1\cap D^2}^2+\|u_{tt}\|_{D_0^1}^2),$$

and

$$\begin{aligned}\int\varrho_t(v\cdot\nabla v)_{tt}\cdot u_{tt}dx&\le C\|\varrho_t\|_{L^3}(\|v\|_{D_0^1\cap D^2}\|v_{tt}\|_{D_0^1}+\|v_t\|_{D_0^1}\|v_t\|_{D_0^1\cap D^2})\|u_{tt}\|_{D_0^1}\\&\le Cc_4^3(\|v_{tt}\|_{D_0^1}^2+\|v_t\|_{D_0^1\cap D^2}^2+\|u_{tt}\|_{D_0^1}^2).\end{aligned}$$

Similarly, we can estimate the fourth term as follows.

$$\begin{aligned}-\int \varrho_{tt}(v\cdot\nabla)v\cdot u_{ttt}dx &= -\frac{d}{dt}\int \varrho_{tt}(v\cdot\nabla)v\cdot u_{tt}dx + \int \varrho_{ttt}(v\cdot\nabla)v\cdot u_{tt}dx + \int \varrho_{tt}(v\cdot\nabla v)_t\cdot u_{tt}dx\\ &\le -\frac{d}{dt}\int \varrho_{tt}(v\cdot\nabla)v\cdot u_{tt}dx + C(\|\varrho_{ttt}\|_{L^2}\|v\|^2_{D_0^1\cap D^2} + \|\varrho_{tt}\|_{L^2}\|v\|_{D_0^1\cap D^2}\|v_t\|_{D_0^1\cap D^2})\|u_{tt}\|_{D_0^1}\\ &\le -\frac{d}{dt}\int \varrho_{tt}(v\cdot\nabla)v\cdot u_{tt}dx + Cc_4^5(\|\varrho_{ttt}\|^2_{L^2} + \|v_t\|^2_{D_0^1\cap D^2} + \|u_{tt}\|^2_{D_0^1}),\end{aligned}$$

and

$$\begin{aligned}-\int \varrho_{tt}u_t\cdot u_{ttt}dx &= -\frac{d}{dt}\int \varrho_{tt}u_t\cdot u_{tt}dx + \int(\varrho_{ttt}u_t + \varrho_{tt}u_{tt})\cdot u_{tt}dx\\ &= -\frac{d}{dt}\int \varrho_{tt}u_t\cdot u_{tt}dx - \int \operatorname{div}(\varrho v)_{tt}u_t\cdot u_{tt}dx + \int \varrho_{tt}|u_{tt}|^2dx\\ &\le -\frac{d}{dt}\int \varrho_{tt}u_t\cdot u_{tt}dx + Cc_4^5(\|u_t\|^2_{D_0^1\cap D^2} + \|u_{tt}\|^2_{D_0^1})\\ &\quad + C(\|v_t\|^2_{D_0^1\cap D^2} + \|v_{tt}\|^2_{D_0^1})(\|\sqrt{\varrho}u_t\|^2_{L^2} + \|u_t\|^2_{D_0^1} + \|\sqrt{\varrho}u_{tt}\|^2_{L^2}).\end{aligned}$$

Finally,

$$\begin{aligned}-\gamma\int \operatorname{div}\left(\nabla n\odot\nabla n - \frac{|\nabla n|^2}{2}Id\right)_{tt}\cdot u_{ttt}dx &= \gamma\frac{d}{dt}\int\left(\nabla n\odot\nabla n - \frac{|\nabla n|^2}{2}Id\right)_{tt} : \nabla u_{tt}dx\\ &\quad -\gamma\int\left(\nabla n\odot\nabla n - \frac{|\nabla n|^2}{2}Id\right)_{ttt} : \nabla u_{tt}dx\\ &\le \gamma\frac{d}{dt}\int\left(\nabla n\odot\nabla n - \frac{|\nabla n|^2}{2}Id\right)_{tt} : \nabla u_{tt}dx\\ &\quad + C(\|n_{ttt}\|_{D^1}\|\nabla n\|_{H^2} + \|n_{tt}\|_{D^1}\|\nabla n_t\|_{H^2})\|u_{tt}\|_{D_0^1}\\ &\le \gamma\frac{d}{dt}\int\left(\nabla n\odot\nabla n - \frac{|\nabla n|^2}{2}Id\right)_{tt} : \nabla u_{tt}dx\\ &\quad + M(c_1)c_3^2(\|n_{ttt}\|^2_{D^1} + \|u_{tt}\|^2_{D_0^1}) + c_3^{-2}\|\nabla n_t\|^2_{H^2}\|n_{tt}\|^2_{D^1}.\end{aligned}$$

Substituting all the estimates into (6.22), we have

$$\begin{aligned}\int \varrho|u_{ttt}|^2dx + \frac{\mu}{2}\frac{d}{dt}\int|\nabla u_{tt}|^2dx &\le \frac{d}{dt}\Lambda(t) + Cc_4^5(\|p_{ttt}\|^2_{L^2} + \|\varrho_{ttt}\|^2_{L^2})\\ &\quad + M(c_1)\{c_4^5(\|v_{tt}\|^2_{D_0^1} + \|v_t\|^2_{D_0^1\cap D^2} + \|u_t\|^2_{D_0^1\cap D^2} + \|u_{tt}\|^2_{D_0^1}) + c_3^2\|n_{ttt}\|^2_{D^1}\}\\ &\quad + C(\|v_t\|^2_{D_0^1\cap D^2} + \|v_{tt}\|^2_{D_0^1} + \|n_{tt}\|^2_{D^1})(\|\sqrt{\varrho}u_t\|^2_{L^2} + \|u_t\|^2_{D_0^1} + \|\sqrt{\varrho}u_{tt}\|^2_{L^2} + c_3^{-2}\|\nabla n_t\|^2_{H^2}),\end{aligned} \tag{6.23}$$

where

$$\begin{aligned}\Lambda(t) &= \int p_{tt}\operatorname{div} - \int \varrho_t|u_{tt}|^2dx - \int \varrho_{tt}u_t\cdot u_{tt}dx - 2\int \varrho_t(v\cdot\nabla v)_t\cdot u_{tt}dx\\ &\quad - \int \varrho_{tt}(v\cdot\nabla)v\cdot u_{tt}dx + \gamma\frac{d}{dt}\int\left(\nabla n\odot\nabla n - \frac{|\nabla n|^2}{2}Id\right)_{tt} : \nabla u_{tt}dx\end{aligned}$$

satisfying

$$|\Lambda(t)| \le \frac{\mu}{8}\|u_{tt}\|^2_{D_0^1} + t^{-1}M(c_1)c_5^{18} \quad \text{for } 0 \le t \le \min(T_*, T_5). \tag{6.24}$$

Multiplying (6.27) by t^2 and integrating over $(\tau, \bar{t})$, we obtain

$$\frac{\mu}{2}\bar{t}^2\|\nabla u_{tt}(\bar{t})\|^2_{L^2} + \frac{1}{2}\int_\tau^{\bar{t}} t^2\|\sqrt{\varrho}u_{ttt}(t)\|^2_{L^2}dt \le \tau^2\|\nabla u_{tt}(\tau)\|^2_{L^2} + |\bar{t}^2\Lambda(\bar{t})| + |\tau^2\Lambda(\tau)| + \int_\tau^{\bar{t}} t|\Lambda(t)|dt + M(c_1)c_5^{18},$$

for $0 < \tau \le \bar{t} \le \min(T_*, T_5)$. Using (6.24) and recalling that

$$\int_0^t s\|u_{tt}(s)\|^2_{D_0^1}ds \le M(c_1)c_4^{12} \quad \text{for } 0 \le t \le \min(T_*, T_4)$$

and

$$\tau_k^2\|u_{tt}(\tau_k)\|^2_{D_0^1} \to 0 \quad \text{for some sequence } \{\tau_k\} \text{ with } \tau_k \to 0,$$

we conclude that

$$t^2\|u_{tt}(t)\|^2_{D_0^1} + \int_0^t s^2\|\sqrt{\varrho}u_{ttt}(s)\|^2_{L^2}ds \le M(c_1)c_5^{18},$$

for $0 \le t \le \min(T_*, T_5)$. Due to the elliptic regularity result we establish Lemma 6.6. $\square$

Lemma 6.7.

$$t^3(\|\sqrt{\rho}u_{ttt}(t)\|^2_{L^2} + \|u_{tt}(t)\|^2_{D^2}) + \int_0^t s^3\|u_{ttt}(s)\|^2_{D_0^1}ds \le M(c_1)c_5^{20}, \tag{6.25}$$

for $0 \le t \le \min(T_*, T_5)$.

Proof. Differentiating (6.12) with respect to t again and multiplying the resulting equation by u_{ttt} and integrating over Ω, we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \varrho|u_{ttt}|^2dx + \int \mu|\nabla u_{ttt}|^2 + (\lambda+\mu)(\mathrm{div}u_{ttt})^2dx \\
&\quad = \int p_{ttt}\mathrm{div}u_{ttt}dx - \int \varrho(v\cdot\nabla v)_{ttt}\cdot u_{ttt}dx - 3\int \varrho_t(v\cdot\nabla v)_{tt}\cdot u_{ttt}dx - \frac{5}{2}\int \varrho_t|u_{ttt}|^2dx \\
&\quad -3\int \varrho_{tt}(u_t + v\cdot\nabla v)_t\cdot u_{ttt}dx - \int \varrho_{ttt}(u_t + v\cdot\nabla v)\cdot u_{ttt}dx \\
&\quad -\gamma\int \mathrm{div}\left(\nabla n\odot\nabla n - \frac{|\nabla n|^2}{2}Id\right)_{ttt}\cdot u_{ttt}dx.
\end{aligned} \tag{6.26}$$

We estimate each term of the right hand side of (6.26) as follows.

$$\begin{aligned}
&\int p_{ttt}\mathrm{div}u_{ttt}dx \le \frac{\mu}{18}\|\nabla u_{ttt}\|^2_{L^2} + C\|p_{ttt}\|^2_{L^2},\\
&-\int \varrho(v\cdot\nabla v)_{ttt}\cdot u_{ttt}dx \le C\|\varrho\|^{\frac{1}{2}}_{L^\infty}(\|v\|_{D_0^1\cap D^2}\|v_{ttt}\|_{D_0^1} + \|v_t\|_{D_0^1\cap D^2}\|v_{tt}\|_{D_0^1})\|\sqrt{\varrho}u_{ttt}\|_{L^2}\\
&\qquad \le \eta^{-1}Cc_3^3\|\sqrt{\varrho}u_{ttt}\|^2_{L^2} + \eta\|v_{ttt}\|^2_{D_0^1} + C\|v_t\|^2_{D_0^1\cap D^2}\|v_{tt}\|^2_{D_0^1},\\
&-3\int \varrho_t(v\cdot\nabla v)_{tt}\cdot u_{ttt}dx \le C\|\varrho_t\|_{L^3}(\|v\|_{D_0^1\cap D^2}\|v_{tt}\|_{D_0^1} + \|v_t\|_{D_0^1}\|v_t\|_{D_0^1\cap D^2})\|\nabla u_{ttt}\|_{L^2}\\
&\qquad \le Cc_4^6(\|v_{tt}\|^2_{D_0^1} + \|v_t\|^2_{D_0^1\cap D^2}) + \frac{\mu}{18}\|\nabla u_{ttt}\|^2_{L^2},\\
&-\frac{5}{2}\int \varrho_t|u_{ttt}|^2dx \le C\int \varrho|v\|u_{ttt}\|\nabla u_{ttt}|dx \le Cc_3^3\|\sqrt{\varrho}u_{ttt}\|^2_{L^2} + \frac{\mu}{18}\|\nabla u_{ttt}\|^2_{L^2},\\
&-3\int \varrho_{tt}(v\cdot\nabla v)_t\cdot u_{ttt}dx \le C\|\varrho_{tt}\|_{L^2}\|v\|_{D_0^1\cap D^2}\|v_t\|_{D_0^1\cap D^2}\|\nabla u_{ttt}\|_{L^2}\\
&\qquad \le Cc_4^{10}\|v_t\|_{D_0^1\cap D^2} + \frac{\mu}{18}\|\nabla u_{ttt}\|^2_{L^2},\\
&-3\int \varrho_{tt}u_{tt}\cdot u_{ttt}dx \le C\int(|\varrho_t||v| + \varrho|v_t|)(|u_{tt}||\nabla u_{ttt}| + |\nabla u_{tt}||u_{ttt}|)dx\\
&\qquad \le C\|\varrho_t\|_{L^3}\|v\|_{D_0^1\cap D^2}\|u_{tt}\|_{D_0^1}\|\nabla u_{ttt}\|_{L^2}\\
&\qquad\quad + C\|\varrho\|^{3/4}_{L^\infty}\|v_t\|_{D_0^1}(\|u_{tt}\|^{1/2}_{D_0^1}\|\sqrt{\varrho}u_{tt}\|^{1/2}_{L^2}\|\nabla u_{ttt}\|_{L^2} + \|u_{tt}\|_{D_0^1}\|\sqrt{\varrho}u_{ttt}\|^{1/2}_{L^2}\|\nabla u_{ttt}\|^{1/2}_{L^2})\\
&\qquad \le Cc_4^7\|u_{tt}\|^2_{D_0^1} + \|\sqrt{\varrho}u_{tt}\|^2_{L^2} + C\|\sqrt{\varrho}u_{ttt}\|^2_{L^2} + \frac{\mu}{18}\|\nabla u_{ttt}\|^2_{L^2},\\
&-\int \varrho_{ttt}(v\cdot\nabla)v\cdot u_{ttt}dx \le C\|v\|_{D_0^1\cap D^2}\|\varrho_{ttt}\|_{L^2}\|\nabla u_{ttt}\|_{L^2} \le Cc_3^4\|\varrho_{ttt}\|^2_{L^2} + \frac{\mu}{18}\|\nabla u_{ttt}\|^2_{L^2},
\end{aligned}$$

$$\begin{aligned}
-\int \varrho_{ttt} u_t \cdot u_{ttt} dx &\le C\int (|\varrho_{tt}||v| + |\varrho_t||v_t| + \varrho|v_{tt}|)(|u_t||\nabla u_{ttt}| + |\nabla u_t||u_{ttt}|)dx \\
&\le C(\|\varrho_{tt}\|_{L^2}\|v\|_{D_0^1\cap D^2} + \|\varrho_t\|_{H^1}\|v_t\|_{D_0^1})\|u_t\|_{D_0^1\cap D^2}\|\nabla u_{ttt}\|_{L^2} \\
&\quad + C\|\varrho\|_{L^\infty}^{3/4}\|v_{tt}\|_{D_0^1}(\|u_t\|_{D_0^1}^{1/2}\|\sqrt{\varrho}u_t\|_{L^2}^{1/2}\|\nabla u_{ttt}\|_{L^2} + \|u_t\|_{D_0^1}\|\sqrt{\varrho}u_{ttt}\|_{L^2}^{1/2}\|\nabla u_{ttt}\|_{L^2}^{1/2}) \\
&\le Cc_4^{10}\|u_t\|_{D_0^1\cap D^2}^2 + M(c_1)c_3^{10}\|v_{tt}\|_{D_0^1}^2 + C\|\sqrt{\varrho}u_{ttt}\|_{L^2}^2 + \frac{\mu}{18}\|\nabla u_{ttt}\|_{L^2}^2,
\end{aligned}$$

and finally,

$$\begin{aligned}
-\gamma\int \operatorname{div}\left(\nabla n\odot\nabla n - \frac{|\nabla n|^2}{2}Id\right)_{ttt}\cdot u_{ttt}dx &\le C(\|n_{ttt}\|_{D^1}\|\nabla n\|_{H^2} + \|\nabla n_{tt}\|_{H^1}\|\nabla n_t\|_{H^1})\|\nabla u_{ttt}\|_{L^2} \\
&\le \frac{\mu}{18}\|\nabla u_{ttt}\|_{L^2}^2 + M(c_1)c_3^4\|n_{ttt}\|_{D^1}^2 + M(c_1)\|\nabla n_{tt}\|_{H^1}^2.
\end{aligned}$$

Substituting all the estimates into (6.26) and choosing $\eta = (1+c_6)^{-1}$, we have

$$\begin{aligned}
\frac{d}{dt}\int \varrho|u_{ttt}|^2dx + \mu\int|\nabla u_{ttt}|^2dx &\le Cc_3^4(\|p_{ttt}\|_{L^2}^2 + \|\varrho_{ttt}\|_{L^2}^2) \\
&\quad + M(c_1)c_4^{10}(\|v_t\|_{D_0^1\cap D^2}^2 + \|v_{tt}\|_{D_0^1}^2 + \|v_t\|_{D_0^1\cap D^2}^2\|v_{tt}\|_{D_0^1}^2) \\
&\quad + (1+c_6)^{-1}\|v_{ttt}\|_{D_0^1}^2 + Cc_4^{10}(\|u_{tt}\|_{D_0^1}^2 + \|u_t\|_{D_0^1\cap D^2}^2 + \|\sqrt{\varrho}u_{tt}\|_{L^2}^2) \\
&\quad + Cc_6^4\|\sqrt{\varrho}u_{ttt}\|_{L^2}^2 + M(c_1)c_3^4(\|n_{ttt}\|_{D^1}^2 + \|\nabla n_{tt}\|_{H^1}^2).
\end{aligned} \tag{6.27}$$

Multiplying (6.27) by t^3 and integrating over $(\tau,\bar{t})$, we obtain

$$\begin{aligned}
&\bar{t}^3\|\sqrt{\varrho}u_{ttt}(\bar{t})\|_{L^2}^2 + \mu\int_\tau^{\bar{t}} t^3\|u_{ttt}(t)\|_{D_0^1}^2dt \le \tau^3\|\sqrt{\varrho}u_{ttt}(\tau)\|_{L^2}^2 + 3\int_\tau^{\bar{t}} t^2\|\sqrt{\varrho}u_{ttt}(t)\|_{L^2}^2dt \\
&\quad + M(c_1)c_5^{16} + Cc_6^4\int_\tau^{\bar{t}} t^3\|\sqrt{\varrho}u_{ttt}(t)\|_{L^2}^2dt + M(c_1)c_3^4\left(\int_\tau^{\bar{t}} t^3\|n_{ttt}(t)\|_{D^1}^2dt + \int_\tau^{\bar{t}} t^3\|\nabla n_{tt}(t)\|_{H^1}^2dt\right) \\
&\le \tau^3\|\sqrt{\varrho}u_{ttt}(\tau)\|_{L^2}^2 + M(c_1)c_5^{18} + Cc_6^4\int_\tau^{\bar{t}} t^3\|\sqrt{\varrho}u_{ttt}(t)\|_{L^2}^2dt \\
&\quad + M(c_1)c_3^4\left\{\int_\tau^{\bar{t}} t^2\|n_{ttt}(t)\|_{D^1}^2dt + \int_\tau^{\bar{t}} t\|n_{tt}(t)\|_{D^2}^2dt + \int_\tau^{\bar{t}}\|n_{tt}(t)\|_{D^1}^2dt\right\} \\
&\le \tau^3\|\sqrt{\varrho}u_{ttt}(\tau)\|_{L^2}^2 + M(c_1)c_5^{20} + Cc_6^4\int_\tau^{\bar{t}} t^3\|\sqrt{\varrho}u_{ttt}(t)\|_{L^2}^2dt,
\end{aligned}$$

for $0<\tau\le\bar{t}\le\min(T_*,T_5)$. Recalling that

$$\tau_k^3\|\sqrt{\varrho}u_{ttt}(\tau_k)\|_{L^2}^2 \to 0 \quad \text{for some sequence } \{\tau_k\} \text{ with } \tau_k\to 0,$$

and using Gronwall's inequality we conclude that

$$t^3\|\sqrt{\rho}u_{ttt}(t)\|_{L^2}^2 + \int_0^t s^3\|u_{ttt}(s)\|_{D_0^1}^2ds \le M(c_1)c_5^{20},$$

for $0\le t\le\min(T_*,T_5)$. Due to the elliptic regularity result we establish Lemma 6.7.

Combining all the previous lemmas, we obtain

$$\begin{aligned}
&\sup_{0\le t\le T_*}(\|u(t)\|_{D_0^1} + \|n_t(t)\|_{L^2} + \|\nabla n(t)\|_{H^1}) + \int_0^{T_*}(\|u(t)\|_{D^2}^2 + \|n_t(t)\|_{D^1}^2 + \|n(t)\|_{D^3}^2)dt \le M(c_1), \\
&\sup_{0\le t\le T_*}(\|u(t)\|_{D^2} + \|n_t(t)\|_{D^1} + \|n(t)\|_{D^3}) + \int_0^{T_*}(\|u_t(t)\|_{D_0^1}^2 + \|u(t)\|_{D^3}^2 + \|n_t(t)\|_{D^2}^2 + \|n(t)\|_{D^4}^2)dt \le M(c_1)c_2^{\frac{3}{2}}c_3^{\frac{1}{2}}, \\
&\operatorname*{ess\,sup}_{0\le t\le T_*}(\|u_t(t)\|_{D_0^1} + \|u(t)\|_{D^3} + \|n_{tt}(t)\|_{L^2} + \|n_t(t)\|_{D^2} + \|n(t)\|_{D^4}) \\
&\quad + \int_0^{T_*}(\|u_t(t)\|_{D^2}^2 + \|u(t)\|_{D^4}^2 + \|n_{tt}(t)\|_{D^1}^2 + \|n_t(t)\|_{D^3}^2 + \|n(t)\|_{D^5}^2)dt \le M(c_1)c_3^{12}, \\
&\sup_{0\le t\le T_*}(\|\varrho(t)-\varrho^\infty\|_{H^3} + \|\varrho_t(t)\|_{H^2} + \|\sqrt{\varrho}u_t(t)\|_{L^2}) + \int_0^{T_*}\|\sqrt{\varrho}u_{tt}(t)\|_{L^2}^2dt \le M(c_1)c_3^{12},
\end{aligned}$$

$$\operatorname*{ess\,sup}_{0\le t\le T_*} t^{\frac{1}{2}}(\|u_t\|_{D^2}+\|u\|_{D^4}+\|n_{tt}\|_{D^1}+\|n_t\|_{D^3})+\int_0^{T_*} t\|u_{tt}\|_{D_0^1}^2dt\le M(c_1)c_4^{\frac{23}{2}}c_5^{\frac{1}{2}},$$

$$\operatorname*{ess\,sup}_{0\le t\le T_*} t(\|u_{tt}\|_{D_0^1}+\|u_t\|_{D^3}+\|n_{tt}\|_{D^2})+\int_0^{T_*} t^2\|u_{tt}\|_{D^2}^2dt\le M(c_1)c_5^{18},$$

$$\operatorname*{ess\,sup}_{0\le t\le T_*} t^3(\|\sqrt{\rho}u_{ttt}\|_{L^2}^2+\|u_{tt}\|_{D^2}^2)+\int_0^{t} s^3\|u_{ttt}(s)\|_{D_0^1}^2ds\le M(c_1)c_5^{20},$$

for $0\le t\le \min(T_*,T_5)$. Here $M=M(\cdot)$ is a fixed increasing continuous function on $[1,+\infty)$ which depends only on the parameters of C. Therefore, setting

$$c_1=M(c_0),\qquad c_2=M(c_1),\qquad c_3=c_2^5,\qquad c_4=c_2c_3^{12},\qquad c_5=c_2^2c_4^{23},\qquad c_6=c_2c_5^{20},$$

and

$$T_*=\min(T,T_5),\quad T_5=(1+c_6)^{-9},$$

we complete the proof of Proposition 6.1. □

By virtue of these a priori estimates, we can prove the existence and regularity (2.8)–(2.13) of a unique local solution (ϱ,u,n) to the original nonlinear problem following exactly the same arguments as that in the proof of Theorem 2.1. Then the estimates in (2.7) imply

$$n_t\in C([0,T]\times R^3).$$

By the continuity Eq. $(1.1)_1$ and the estimates in (2.4)–(2.5), we have

$$\varrho_t\in C([0,T];H^2),\qquad \varrho_{tt}\in L^\infty(0,T;L^2)\cap L^2(0,T;H^1),$$

and then we get

$$\varrho_t\in C([0,T]\times R^3).$$

According to (2.5) and (2.10), we deduce that

$$\partial_t(tu_t)=u_t+tu_{tt}\in L^\infty(0,T;D^1)\cap L^2(0,T;D^2);$$

hence

$$tu_t\in C([0,T];D^1\cap D^2).$$

Rewrite the momentum equation as

$$Lu=-\left\{\varrho(u_t+u\cdot\nabla u)+\nabla p+\gamma\operatorname{div}\left(\nabla n\odot\nabla n-\frac{|\nabla n|^2}{2}Id\right)\right\}.$$

Then it follows from (2.10), elliptic regularity results and the Lions–Aubin Lemma

$$L^\infty(0,T;H^1)\cap H^1(0,T;H^{-1})\hookrightarrow C([0,T];L^q),\quad 2\le q<6,$$

we have

$$t\nabla^2u\in C([0,T];W^{1,4}),$$

and thus

$$\nabla^2u\in C((0,T]\times R^3).$$

So (ϱ,u,n) is a classical solution to (1.1)–(1.3).

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