J. Math. Anal. Appl. 400 (2013) 363–376

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Some block Toeplitz composition operators

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ARTICLE INFO

Article history:
Received 8 December 2011
Available online 5 December 2012
Submitted by Lawrence A. Fialkow

Keywords:
Composition operator
Toeplitz operator
Numerical range

ABSTRACT

The composition operators on $H^2(\mathbb{D})$ with minimal polynomial $z^n - 1$ are shown to be block Toeplitz with Toeplitz symbol equal to an $n \times n$ matrix-valued polynomial of degree 1. This result is used to prove that the numerical range of a composition operator on $H^2(\mathbb{D})$ with minimal polynomial $z^3 - 1$ cannot be a circular disk.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$, the set of bounded linear operators on $\mathcal{H}$. The numerical range $W(T)$ of T is the subset of the complex plane defined by

$$W(T) = \{\langle Tv, v\rangle \mid v \in \mathcal{H}, \|v\| = 1\}.$$

The Toeplitz–Hausdorff Theorem [25,12] states that $W(T)$ is always convex.

Other well-known properties of the numerical range [9,13] include the following.

(a) The closure of $W(T)$ contains the spectrum of T.
(b) If T and S are unitarily equivalent, then $W(T) = W(S)$.
(c) If $\mathcal{H}$ is finite-dimensional, then $W(T)$ is closed.
(d) If $\dim \mathcal{H} = 2$, then $W(T)$ is a possibly degenerate elliptic disk with foci equal to the eigenvalues of T.

In [26], Tso and Wu showed that result (d) generalizes to any Hilbert space $\mathcal{H}$ as follows. If T is a quadratic operator on $\mathcal{H}$, i.e., if there is a second-degree polynomial q such that $q(T) = 0$, then $W(T)$ is a possibly degenerate elliptic disk whose foci are the eigenvalues of T. These numerical ranges are always either open or closed; Tso and Wu include norm conditions that determine which case holds. Rodman and Spitkovsky extended these results to generalizations of the numerical range in [21].

In this paper, we will consider the numerical range of T when T is a certain composition operator on $H^2 = H^2(\mathbb{D})$. Recall that H^2 is defined to be all analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{H^2}^2 = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \,\frac{d\theta}{2\pi} < \infty.$$

This space is a Hilbert space with inner product

$$\langle f, g\rangle_{H^2} = \lim_{r\uparrow 1} \int_0^{2\pi} f\left(re^{i\theta}\right)\overline{g\left(re^{i\theta}\right)}\,\frac{d\theta}{2\pi} = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} f(z)\overline{g(z)}\,\frac{dz}{z},$$

where in the latter integral f and g refer to the corresponding functions on the boundary of $\mathbb{D}$.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2012.11.057

For each $\lambda \in \mathbb{D}$, and for each $f \in H^2$, $f(\lambda) = \langle f, k_\lambda \rangle$, where k_λ is the reproducing kernel in H^2 given by $k_\lambda(z) = \frac{1}{1-\bar{\lambda}z}$.

If φ is an analytic self-map of the open unit disk $\mathbb{D}$, the associated composition operator is defined for $f \in H^2$ by $C_\varphi f = f \circ \varphi$. The mapping φ is called the symbol of the composition operator. On H^2, the operator C_φ is bounded for all analytic mappings φ from $\mathbb{D}$ to itself. This and many other results about composition operators appear in [8,22].

Several authors [1,4,5,17,18] have studied the numerical ranges of composition operators. In particular, in [4], Bourdon and Shapiro analyzed the numerical ranges of composition operators with automorphic symbols. This paper is concerned with an open question in [4], so we recall some relevant results.

Any automorphism of the disk can be classified into one of three types: hyperbolic (with two fixed points on $\partial\mathbb{D}$); parabolic (with one fixed point on $\partial\mathbb{D}$); or elliptic (with one fixed point inside $\mathbb{D}$ and one fixed point outside the closure of $\mathbb{D}$). Any elliptic automorphism is conformally conjugate to a mapping of the form $\Phi(z) = e^{i\alpha}z$ for some real number α; the value $e^{i\alpha}$ is called the rotation parameter of the automorphism.

Bourdon and Shapiro showed that if C_φ is a composition operator on H^2 with symbol φ equal to a hyperbolic or parabolic automorphism of $\mathbb{D}$, then $W(C_\varphi)$ is a circular disk centered at the origin. (Throughout the remainder of the paper, the term "disk" implies circular disk.) Matache [18] recently showed that, more generally, all inner functions of hyperbolic or parabolic type have numerical ranges which are disks. If φ is an elliptic automorphism that fixes the origin, then $W(C_\varphi)$ is either a polygon (if the rotation parameter is $e^{i\alpha}$ with α equal to a rational multiple of π) or a disk with some boundary points; see [17,4].

Finally, if the elliptic automorphism φ does not fix the origin and has rotation parameter with argument α not equal to a rational multiple of π, then Bourdon and Shapiro showed that the numerical range of C_φ is a disk centered at the origin. In several of the cases mentioned above, the question of which, if any, boundary points were included in the numerical range is open.

The remaining case discussed by Bourdon and Shapiro involved an elliptic automorphism φ that does not fix the origin where α is equal to a rational multiple of π. The mapping φ has a rotation parameter with argument α equal to a rational multiple of π if and only if there is a positive integer n such that $C_\varphi^n = I$. More precisely, if φ has rotation parameter $e^{i\alpha}$ with α equal to a rational multiple of π, then there exists a positive integer n such that the rotation parameter is a primitive nth root of unity. In this case, the mapping $\Phi(z) = e^{i\alpha}z$ induces a composition operator C_Φ with minimal polynomial equal to $z^n - 1$, and therefore the similar composition operator C_φ also has minimal polynomial $z^n - 1$. Conversely, if C_φ has minimal polynomial $z^n - 1$, then the rotation parameter of φ is a primitive nth root of unity.

Bourdon and Shapiro showed that when C_φ has minimal polynomial $z^2 - 1$, the numerical range of C_φ is an elliptic disk with foci at ± 1; in fact they showed this result holds for any operator T with minimal polynomial $z^2 - 1$, as is consistent with the general results about quadratic operators in [26]. When $n \geq 3$, they conjectured that if C_φ has minimal polynomial $z^n - 1$, then the numerical range of C_φ is not a disk.

A natural question is to what extent the phenomenon that T has minimal polynomial $z^n - 1$ implies that $W(T)$ is not a disk holds when T is not a composition operator. When T is an operator on a finite-dimensional Hilbert space, results in [16] (see also [28]) show that if $W(T)$ is a disk, then the center of the disk is a multiple eigenvalue of T. Since an operator with minimal polynomial $z^n - 1$ has no multiple eigenvalues, such an operator on a finite-dimensional space cannot have a disk as its numerical range.

However, unlike the quadratic case, this type of result does not extend to infinite-dimensional Hilbert spaces; in [11], an operator T on an infinite-dimensional Hilbert space with minimal polynomial equal to $z^3 - 1$ and with numerical range equal to an open disk is constructed.

In this paper, we show that the conjecture of Bourdon and Shapiro does in fact hold for composition operators on H^2 with minimal polynomial equal to $z^3 - 1$: the numerical range of such a composition operator is not a disk. In order to prove this result, we first show that for integers $n \geq 2$ a composition operator C_φ with minimal polynomial $z^n - 1$ is block Toeplitz with respect to a certain orthonormal basis, and the Toeplitz symbol of the operator is an $n \times n$ matrix-valued polynomial of degree 1. A recent result of Bebiano and Spitkovsky [2] about numerical ranges of Toeplitz operators with matrix-valued symbols is then applied to characterize the closure of the numerical range of C_φ as the convex hull of the numerical ranges of the $n \times n$ matrices in the range of the Toeplitz symbol. In the $n = 3$ case, enough specific information about the Toeplitz symbol is known to conclude that the numerical range of C_φ is not a disk.

2. Preliminary calculations

For an analytic self-map φ of the disk and a natural number m, let φ_m denote the function obtained by composing φ with itself m times. Set $\varphi_0(z) = z$ for all $z \in \mathbb{D}$, and note that $\varphi_1 = \varphi$.

Throughout the paper, φ will denote an elliptic automorphism whose rotation parameter is an nth root of unity. That is, $\varphi_n(z) = z$ for all $z \in \mathbb{D}$. Since φ is a disk automorphism, there exist $\eta \in \partial\mathbb{D}$ and $p \in \mathbb{D}$ such that

$$\varphi(z) = \eta \frac{p - z}{1 - \bar{p}z}. \tag{1}$$

As discussed in the introduction, numerical ranges of composition operators with symbols that are rotations fixing the origin were analyzed in [17]. If $p \neq 0$ but $\eta = -1$, then φ is a hyperbolic automorphism. Therefore we may assume without loss of generality that $n \geq 2$ (because the $n = 1$ case leads to the identity mapping and operator), $p \neq 0$ and $\eta \neq -1$. With these assumptions, we will find conditions on p and η that guarantee $\varphi_n(z) = z$ for all z and $\varphi_k(z) \not\equiv z$ for any k satisfying $1 \leq k < n$. To find these iterates, represent the linear fractional transformation φ by the matrix

$$M = \begin{pmatrix} -\eta & \eta p \\ -\overline{p} & 1 \end{pmatrix}.$$

The matrix M has eigenvalues

$$\xi_1 = \frac{1-\eta}{2} + \frac{\sqrt{(1+\eta)^2 - 4\eta|p|^2}}{2} \quad \text{and} \quad \xi_2 = \frac{1-\eta}{2} - \frac{\sqrt{(1+\eta)^2 - 4\eta|p|^2}}{2}. \tag{2}$$

If these eigenvalues were equal, φ would be a parabolic automorphism or the identity mapping, so we know that $\xi_1 \neq \xi_2$. Hence M can be diagonalized as $M = PDP^{-1}$, where D is the 2×2 diagonal matrix with ξ_1 and ξ_2 on the diagonal and P contains the corresponding eigenvectors in its columns:

$$P = \begin{pmatrix} \eta p & \eta p \\ \eta + \xi_1 & \eta + \xi_2 \end{pmatrix}.$$

Consequently,

$$\begin{aligned} M^n &= PD^nP^{-1} \\ &= \begin{pmatrix} \dfrac{\eta p \left\{\eta(\xi_1^n - \xi_2^n) + \xi_1\xi_2(\xi_1^{n-1} - \xi_2^{n-1})\right\}}{\eta p(\xi_2 - \xi_1)} & \dfrac{(\eta p)^2(\xi_2^n - \xi_1^n)}{\eta p(\xi_2 - \xi_1)} \\ \dfrac{(\eta+\xi_2)(\eta+\xi_2)(\xi_1^n - \xi_2^n)}{\eta p(\xi_2 - \xi_1)} & \dfrac{\eta p \left\{\eta(\xi_2^n - \xi_1^n) + \xi_2^{n+1} - \xi_1^{n+1}\right\}}{\eta p(\xi_2 - \xi_1)} \end{pmatrix}. \end{aligned}$$

Since φ_n is the identity map if and only if M^n is a constant multiple of the identity matrix, the following result holds.

Proposition 1. *Let n be an integer with $n \geq 2$. The automorphism of the form (1) with $p \neq 0$ and $\eta \neq -1$ satisfies $\varphi_k(z) = z$ for all z when $k = n$ but for no positive integer smaller than n if and only if*

$$\xi_1^n = \xi_2^n, \quad \textit{and}$$
$$\xi_1^k \neq \xi_2^k \quad \textit{for all integers } k \textit{ with } 1 \leq k < n,$$

where ξ_1 and ξ_2 are given by (2).

When $n = 2$, the conditions in Proposition 1 are $\xi_1 \neq \xi_2$ and $\eta = 1$.
When $n = 3$, the conditions simplify to $\xi_1 \neq \xi_2$ and

$$1 - \eta + \eta^2 - \eta|p|^2 = 0 \quad \text{and} \quad \eta \neq 1. \tag{3}$$

When $n = 4$, the condition becomes $\eta + \overline{\eta} = 2|p|^2$ along with the restriction that the previous conditions for $n = 1$, $n = 2$, and $n = 3$ do not hold.

Fix an integer n with $n \geq 2$. The conditions in Proposition 1 that guarantee $\varphi_n(z) = z$ for all $z \in \mathbb{D}$ are also equivalent to $\varphi_{n-1} = \varphi^{-1}$. Set

$$\lambda_j = \varphi_{j-1}(0) \quad \text{for } j = 1, \ldots, n, \tag{4}$$

so that $\lambda_1 = 0$, $\lambda_2 = \varphi(0) = \eta p, \ldots, \lambda_n = \varphi_{n-1}(0) = \varphi^{-1}(0) = p$. This cyclic condition motivates a choice of orthonormal basis which exhibits the numerical range behavior of the associated composition operator. The basis depends on the Blaschke product of degree n defined below:

$$B(z) = \varphi_1(z)\varphi_2(z)\cdots\varphi_n(z). \tag{5}$$

Note that $B(\lambda_j) = 0$ for $j = 1, \ldots, n$ because $\varphi_{n-j+1}(\lambda_j) = \varphi_n(0) = 0$.
Also,

$$B(\varphi(z)) = \varphi_2(z)\varphi_3(z)\varphi_4(z)\cdots\varphi_n(z)\varphi_1(z) = \varphi_1(z)\varphi_2(z)\varphi_3(z)\cdots\varphi_{n-1}(z)\varphi_n(z) = B(z),$$

which implies that

$$C_\varphi B^N = B^N \tag{6}$$

for all non-negative integers N.

Let $g_1, \ldots, g_n$ be any collection of orthonormal functions in H^2 such that

$$\text{span}\{g_i \mid 1 \leq i \leq j\} = \text{span}\{k_{\lambda_i} \mid 1 \leq i \leq j\}$$

for all $j \in \{1, \ldots, n\}$. Let G denote the subspace of H^2 spanned by $g_1, g_2, \ldots, g_n$. Clearly, H^2 has the orthogonal decomposition $H^2 = G \oplus BH^2$. Furthermore,

$$H^2 = G \oplus BG \oplus B^2G \oplus \cdots = \oplus_{N=0}^{\infty} B^N G,$$

because the summands are pairwise orthogonal, and any function f in H^2 that is orthogonal to all of the summands must satisfy $f(\lambda_i) = f'(\lambda_i) = f''(\lambda_i) = \cdots = f^{(N)}(\lambda_i) = \cdots = 0$ for all $i = 1, \ldots, n$ and for all non-negative integers N. That is, f is identically zero.

In other words, the set

$$\mathcal{E} = \{B^N g_j \mid j = 1, 2, \ldots, n, \text{ and } N = 0, 1, 2, \ldots\} \tag{7}$$

is a complete orthonormal basis for H^2.

A convenient formula for the g_i functions will be motivated by the Malmquist basis. In [10], Guyker used an orthonormal basis to represent certain composition operators in terms of a lower triangular matrix. Bourdon and Shapiro [5] and Abdollahi [1] each used Guyker's basis to study composition operator numerical ranges; Bourdon and Shapiro point out (with credit to Harold Shapiro) that the Guyker basis is a special case of the Malmquist (or Malmquist–Takenaka) basis which is used in interpolation problems.

In general, the Malmquist basis is defined in terms of a sequence of points $\{z_i\}_{i=1}^{\infty}$ in the disk $\mathbb{D}$. If $\sum_{i=1}^{\infty}(1-|z_i|) = \infty$, then the basis defined by $m_1(z) = k_{z_1}(z)/\|k_{z_1}\|$ and

$$m_j(z) = \frac{\sqrt{1-|z_j|^2}}{1-\overline{z_j}z} \prod_{i=1}^{j-1} \frac{z-z_i}{1-\overline{z_i}z}, \quad j = 2, 3, \ldots \tag{8}$$

is a complete orthonormal basis for $H^2(\mathbb{D})$. In addition, if $z_1, \ldots, z_n$ are distinct, then the basis vectors $\{m_1, \ldots, m_n\}$ have the same span as the reproducing kernels $\{k_{z_1}, \ldots, k_{z_n}\}$. See [15,20,24], or [27] for more information.

In most interpolation applications, all of the points $\{z_i\}$ are distinct. However, the Guyker basis is the special case of the Malmquist basis with $z_i = \alpha$ for all $i = 1, 2, 3, \ldots$, where α is the fixed point (assumed to be in $\mathbb{D}$) of the symbol φ of a composition operator.

The basis $\mathcal{E}$ in (7) is (up to unimodular constant multiples) a different case of the Malmquist basis where the sequence $\{z_i\}_{i=1}^{\infty}$ is the repeating sequence

$$\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_1, \lambda_2, \ldots, \lambda_n, \ldots.$$

That is,

$$z_j = \varphi_{j-1}(0) = \varphi_{(j-1)[\text{mod } n]}(0) = \lambda_{j[\text{mod } n]}. \tag{9}$$

Assume that $\eta_1, \ldots, \eta_n$ are unimodular complex constants. For any choice of $g_1, \ldots, g_n$ such that $g_j = \eta_j m_j$ for $j = 1, \ldots, n$ with m_j defined as in (8) for points given by (9), the basis $\mathcal{E}$ from (7) will result in a complete orthonormal basis for H^2, as claimed. Since our computations involve composition with φ, it will be convenient to define the g_j functions directly in terms of iterates of φ. Consequently, let $g_1(z) = m_1(z) = k_{\lambda_1}(z) = 1$, and for $2 \le j \le n$ define

$$g_j = \sqrt{1-|\lambda_j|^2}\, k_{\lambda_j}\, \varphi_n \cdots \varphi_{n-j+2}. \tag{10}$$

The proposition below can be used to prove that these g_j functions are in fact unimodular constant multiples of the corresponding Malmquist basis vectors. The proof of the proposition is a straightforward but lengthy induction argument, so it is omitted.

Proposition 2. *Let n be an integer with $n \ge 2$. Let φ be an elliptic automorphism of the form (1), where η and p satisfy the conditions in Proposition 1. Let φ_j be the jth iterate of φ, and let $\lambda_1, \ldots, \lambda_n$ be defined as in (4). Then*

$$\varphi_j(z) = -\frac{\lambda_{j+1}}{\lambda_{n-j+1}} \left(\frac{z-\lambda_{n-j+1}}{1-\overline{\lambda_{n-j+1}}z} \right) \quad \text{for } j = 1, \ldots, n-1$$

and $\varphi_n(z) = z$.

A consequence of Proposition 2 is that each ratio $\lambda_{j+1}/\lambda_{n-j+1}$ is unimodular; hence the choice of iterates of φ in (10) implies that each g_j is a unimodular multiple of the corresponding Malmquist basis vector.

Next we present some technical results that are used to compute the matrix for C_φ with respect to the basis $\mathcal{E}$ in (7). The proposition below is simply a special case of Cowen's adjoint formula for composition operators with linear fractional symbol; for details, see Theorem 2 in [7] or Theorem 9.2 in [8].

Proposition 3. *Let φ be a disk automorphism of the form (1). Then, for any f in H^2,*

$$\left(C_\varphi^* f\right)(z) = \frac{p(f(\varphi^{(-1)}(z)) - f(0))}{\overline{\eta}z - p} + \frac{f(\varphi^{(-1)}(z))}{1-\overline{p\eta}(z)}. \tag{11}$$

Proof. Assume that $\tau(z) = \frac{az+b}{cz+d}$, with $a, b, c, d \in \mathbb{C}$. Cowen's formula states that if this linear fractional transformation is a self-map of the unit disk, then the corresponding composition operator has adjoint

$$C_\tau^* = T_g C_\sigma T_h^*,$$

where $\sigma(z) = \frac{\bar{a}z-\bar{c}}{-\bar{b}z+\bar{d}}$, $h(z) = cz+d$, and $g(z) = \frac{1}{-\bar{b}z+\bar{d}}$. For a bounded analytic function g, the operator T_g denotes the Toeplitz operator on H^2 with symbol g. That is, T_g is multiplication by g followed by projection onto H^2. The Toeplitz symbols that arise in the formula are bounded whenever τ is a self-map of the disk.

We will apply Cowen's formula directly to $\varphi(z) = \eta\frac{p-z}{1-\bar{p}z}$. In this case, $\sigma(z) = \frac{-\bar{\eta}z+p}{1-\bar{\eta}\bar{p}z} = \varphi^{-1}(z)$, $g(z) = \frac{1}{1-\bar{\eta}\bar{p}z}$, and $h(z) = -\bar{p}z+1$.

Note that $T_h = -\bar{p}S + I$, where S is the unilateral shift (multiplication by z followed by projection onto H^2) and I is the identity operator on H^2. Therefore, $T_h^* = -pS^* + I$, so

$$(T_h^* f)(z) = -p\left(\frac{f(z)-f(0)}{z}\right) + f(z),$$

where the removable singularity makes the right side of the expression above well defined for all $z \in \mathbb{D}$.

Therefore,

$$(C_\sigma T_h^* f)(z) = -p\left(\frac{f(\varphi^{(-1)}(z)) - f(0)}{\varphi^{(-1)}(z)}\right) + f(\varphi^{(-1)}(z)).$$

Finally, since $\varphi^{(-1)}(z) = \frac{p-\bar{\eta}z}{1-\overline{\eta p}z}$,

$$\begin{aligned}(T_g C_\sigma T_h^* f)(z) &= \frac{-p}{1-\overline{\eta p}z}\left(\frac{f(\varphi^{(-1)}(z)) - f(0)}{\varphi^{(-1)}(z)}\right) + \frac{1}{1-\overline{\eta p}z} f(\varphi^{(-1)}(z)) \\ &= -p\left(\frac{f(\varphi^{(-1)}(z)) - f(0)}{p-\bar{\eta}z}\right) + \frac{f(\varphi^{(-1)}(z))}{1-\overline{\eta p}z},\end{aligned}$$

which is equivalent to the statement in the proposition. $\square$

The remaining results in this section will all use the following hypotheses.

$$\left.\begin{array}{l}\text{(i) The value } n \text{ is an integer with } n \ge 2;\\ \text{(ii) the map } \varphi \text{ is a disk automorphism of the form (1);}\\ \text{(iii) the composition operator } C_\varphi \text{ has minimal polynomial } z^n - 1;\\ \text{(iv) the points } \lambda_1, \ldots, \lambda_n \text{ are defined as in (4); and}\\ \text{(v) the functions } B \text{ and } g_1, \ldots, g_n \text{ are defined in (5) and (10), respectively.}\end{array}\right\} \tag{12}$$

We will repeatedly simplify terms by canceling inner functions (including B and all iterates of φ) that appear in both sides of any inner product in H^2.

The following result about $C_\varphi^* f$ evaluated at the points $\lambda_1, \ldots, \lambda_n$ is an immediate consequence of the previous proposition. The special case when $j = 2$ occurs because $\lambda_2 = \eta p$ corresponds to the removable singularity in (11). Recall that $\lambda_0 = \lambda_{0[\text{mod } n]} = \lambda_n$.

Corollary 4. *Assume that the conditions in (12) hold. If $f \in H^2$, then*

$$\langle C_\varphi^* f, k_{\lambda_j}\rangle = \begin{cases} \dfrac{pf'(0)}{-1+|p|^2} + \dfrac{f(0)}{1-|p|^2} & \text{if } j = 2,\\[2ex] \dfrac{p(f(\lambda_{j-1}) - f(0))}{\bar{\eta}\lambda_j - p} + \dfrac{f(\lambda_{j-1})}{1-\overline{p\eta}\lambda_j} & \text{if } 1 \le j \le n,\ j \ne 2.\end{cases}$$

The above corollary allows us to compute $\langle C_\varphi^* f, k_{\lambda_j}\rangle$ even if we only know the values $f(0)$, $f'(0)$, and $f(\lambda_{j-1})$.

Lemma 5. *If the conditions in (12) hold, then*

$$\langle C_\varphi^* f, k_{\lambda_j}\varphi_n\rangle = \begin{cases} \dfrac{\bar{\eta}(\bar{p}f(0) - f'(0))}{1-|p|^2} & \text{if } j = 2.\\[2ex] \dfrac{\bar{\eta}(1-|p|^2)f(\lambda_{j-1})}{(\bar{\eta}\lambda_j - p)(1-\overline{p\eta}\lambda_j)} - \dfrac{\bar{\eta}f(0)}{\bar{\eta}\lambda_j - p} & \text{if } 3 \le j \le n.\end{cases}$$

Proof. In general, if $h \in H^2$ and $z_0 \neq 0$ is in $\mathbb{D}$, then

$$\langle h, k_{z_0}\varphi_n\rangle = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{h(z)\bar{z}}{(1-z_0\bar{z})}\,\frac{dz}{z} = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{h(z)}{z(z-z_0)}\,dz = \frac{h(z_0)-h(0)}{z_0}.$$

When $j = 2$ (that is, $z_0 = \lambda_2 = \eta p$) and $f \in H^2$, the equation above and Corollary 4 show that

$$\langle C_\varphi^* f, k_{\lambda_2}\varphi_n\rangle = \frac{C_\varphi^* f(\lambda_2) - C_\varphi^* f(0)}{\lambda_2} = \frac{\frac{f(0)-pf'(0)}{1-|p|^2} - f(0)}{\lambda_2} = \frac{\overline{\eta}(\overline{p}f(0)-f'(0))}{1-|p|^2}.$$

Finally, when $3 \le j \le n$ and $f \in H^2$,

$$\langle C_\varphi^* f, k_{\lambda_j}\varphi_n\rangle = \frac{C_\varphi^* f(\lambda_j) - C_\varphi^* f(0)}{\lambda_j} = \frac{\left(p\left(\frac{f(\lambda_{j-1})-f(0)}{\overline{\eta}\lambda_j - p}\right) + \frac{f(\lambda_{j-1})}{1-\overline{\eta}\overline{p}\lambda_j}\right) - f(0)}{\lambda_j}.$$

The final term simplifies to the expression in the lemma. $\square$

Lemma 6. *If the assumptions in (12) hold and $1 \le i, j \le n$, then*

$$\langle C_\varphi g_j, g_i\rangle = \begin{cases} 1 & \text{if } j = i = 1,\\ 0 & \text{if } j = 1, i \ge 2,\\ \dfrac{\sqrt{1-|\lambda_i|^2}}{\sqrt{1-|\lambda_j|^2}}\lambda_j & \text{if } j - i = 1,\\ 0 & \text{if } j - i \ge 2,\\ -\eta & \text{if } j = i = 2,\\ \dfrac{-\eta\overline{\lambda_n}\cdots\overline{\lambda_{n-i+3}}\sqrt{1-|\lambda_i|^2}}{\sqrt{1-|p|^2}} & \text{if } j = 2, i > j,\\ -\dfrac{\eta\overline{\lambda_{n-j+3}}\cdots\overline{\lambda_{n-i+3}}\sqrt{1-|\lambda_i|^2}\sqrt{1-|\lambda_j|^2}}{\eta\overline{\lambda_j} - \overline{p}} & \text{if } j > 2, i \ge j. \end{cases} \tag{13}$$

Proof. When $j = 1$, the formula for $\langle C_\varphi g_j, g_i\rangle$ follows from the identity $C_\varphi g_1 = g_1$ and the orthonormality of $g_1, \ldots, g_n$.

When $j - i = 1$ (implying $j \ge 2$ and $i \le n-1$), $g_j = \frac{k_{\lambda_j}}{\|k_{\lambda_j}\|}\varphi_n \cdots \varphi_{n-j+2}$ has at least one factor that is an iterate of φ. In addition, $\varphi(\lambda_i) = \lambda_{i+1} = \lambda_j$ for $i = 1, \ldots, n-1$. Therefore,

$$\begin{aligned}
\langle C_\varphi g_j, g_i\rangle &= \left\langle C_\varphi\left(\frac{k_{\lambda_j}}{\|k_{\lambda_j}\|}\varphi_n\cdots\varphi_{n-j+2}\right), \frac{k_{\lambda_i}}{\|k_{\lambda_i}\|}\varphi_n\cdots\varphi_{n-i+2}\right\rangle\\
&= \left\langle C_\varphi\left(\frac{k_{\lambda_j}}{\|k_{\lambda_j}\|}\right)\varphi_1\varphi_n\cdots\varphi_{n-j+3}, \frac{k_{\lambda_i}}{\|k_{\lambda_i}\|}\varphi_n\cdots\varphi_{n-i+2}\right\rangle\\
&= \left\langle C_\varphi\left(\frac{k_{\lambda_j}}{\|k_{\lambda_j}\|}\right)\varphi_1\varphi_n\cdots\varphi_{n-i+2}, \frac{k_{\lambda_i}}{\|k_{\lambda_i}\|}\varphi_n\cdots\varphi_{n-i+2}\right\rangle\\
&= \left\langle C_\varphi\left(\frac{k_{\lambda_j}}{\|k_{\lambda_j}\|}\right)\varphi_1, \frac{k_{\lambda_i}}{\|k_{\lambda_i}\|}\right\rangle = \sqrt{1-|\lambda_j|^2}\sqrt{1-|\lambda_i|^2}\,\frac{\varphi(\lambda_i)}{1-\overline{\lambda_j}\varphi(\lambda_i)}\\
&= \frac{\sqrt{1-|\lambda_i|^2}}{\sqrt{1-|\lambda_j|^2}}\lambda_j.
\end{aligned}$$

When $j - i \ge 2$ (and thus $n - j + 3 \le n - i + 1$), g_j has at least two factors which are iterates of φ. Consequently,

$$\begin{aligned}
\langle C_\varphi g_j, g_i\rangle &= \left\langle C_\varphi\left(\frac{k_{\lambda_j}}{\|k_{\lambda_j}\|}\varphi_n\cdots\varphi_{n-j+2}\right), \frac{k_{\lambda_i}}{\|k_{\lambda_i}\|}\varphi_n\cdots\varphi_{n-i+2}\right\rangle\\
&= \left\langle C_\varphi\left(\frac{k_{\lambda_j}}{\|k_{\lambda_j}\|}\right)\varphi_1\varphi_n\cdots\varphi_{n-j+3}, \frac{k_{\lambda_i}}{\|k_{\lambda_i}\|}\varphi_n\cdots\varphi_{n-i+2}\right\rangle
\end{aligned}$$

$$= \left\langle C_\varphi \left(\frac{k_{\lambda_j}}{\|k_{\lambda_j}\|} \right) \varphi_1 \varphi_{n-i+1} \cdots \varphi_{n-j+3}, \frac{k_{\lambda_i}}{\|k_{\lambda_i}\|} \right\rangle$$
$$= 0,$$

because $\varphi_{n-i+1}(\lambda_i) = \lambda_{n-i+1+i} = 0$ for $i = 1, \ldots, n-2$.

When $j = 2$ and $i \geq j$, Lemma 5 yields

$$\langle C_\varphi g_2, g_i \rangle = \left\langle C_\varphi \left(\frac{k_{\lambda_2}}{\|k_{\lambda_2}\|} \varphi_n \right), g_i \right\rangle = \left\langle \frac{k_{\lambda_2}}{\|k_{\lambda_2}\|} \varphi_n, C_\varphi^* g_i \right\rangle = \sqrt{1-|p|^2}\, \langle k_{\lambda_2}\varphi_n, C_\varphi^* g_i \rangle$$
$$= \sqrt{1-|p|^2}\, \overline{\langle C_\varphi^* g_i, k_{\lambda_2}\varphi_n \rangle} = \sqrt{1-|p|^2}\, \frac{\eta(p\,\overline{g_i(0)} - \overline{g_i'(0)})}{1-|p|^2} = \frac{\eta(p\overline{g_i(0)} - \overline{g_i'(0)})}{\sqrt{1-|p|^2}}.$$

Since $i \geq j = 2$,

$$g_i(z) = \frac{k_{\lambda_i}(z)}{\|k_{\lambda_i}\|} z \varphi_{n-1}(z) \cdots \varphi_{n-i+2}(z)$$

always contains the factor z. Hence $g_i(0) = 0$ and

$$g_i'(0) = \frac{k_{\lambda_i}(0)}{\|k_{\lambda_i}\|} \varphi_{n-1}(0) \cdots \varphi_{n-i+2}(0) = \begin{cases} \sqrt{1-|p|^2} & \text{if } i = 2, \\ \sqrt{1-|\lambda_i|^2}\lambda_n \cdots \lambda_{n-i+3} & \text{if } i \geq 3. \end{cases}$$

We conclude that, when $i \geq 2$,

$$\langle C_\varphi g_2, g_i \rangle = \begin{cases} -\eta & \text{if } i = 2, \\ \dfrac{-\eta\overline{\lambda_n} \cdots \overline{\lambda_{n-i+3}}\sqrt{1-|\lambda_i|^2}}{\sqrt{1-|p|^2}} & \text{if } i \geq 3. \end{cases}$$

Finally, when $i \geq j \geq 3$, we obtain

$$\langle C_\varphi g_j, g_i \rangle = \left\langle C_\varphi \left(\frac{k_{\lambda_j}}{\|k_{\lambda_j}\|} \right) \varphi_1 \varphi_n \cdots \varphi_{n-j+3}, \frac{k_{\lambda_i}}{\|k_{\lambda_i}\|} \varphi_n \cdots \varphi_{n-i+2} \right\rangle$$
$$= \left\langle C_\varphi \left(\frac{k_{\lambda_j}}{\|k_{\lambda_j}\|} \right) \varphi_1, \frac{k_{\lambda_i}}{\|k_{\lambda_i}\|} \varphi_{n-j+2} \cdots \varphi_{n-i+2} \right\rangle$$
$$= \left\langle \frac{k_{\lambda_j}}{\|k_{\lambda_j}\|} \varphi_n, C_\varphi^* \left(\frac{k_{\lambda_i}}{\|k_{\lambda_i}\|} \varphi_{n-j+2} \cdots \varphi_{n-i+2} \right) \right\rangle$$
$$= \sqrt{1-|\lambda_j|^2} \left(\frac{\eta(1-|p|^2)\overline{h_{ij}(\lambda_{j-1})}}{(\eta\overline{\lambda_j} - \overline{p})(1 - p\eta\overline{\lambda_j})} - \frac{\eta\overline{h_{ij}(0)}}{\eta\overline{\lambda_j} - \overline{p}} \right),$$

where $h_{ij} = \frac{k_{\lambda_i}}{\|k_{\lambda_i}\|}\varphi_{n-j+2} \cdots \varphi_{n-i+2}$, and the last equality follows from the $j \geq 3$ case of Lemma 5. Note that $h_{ij}(\lambda_{j-1})$ has $\varphi_{n-j+2}(\lambda_{j-1}) = \lambda_{n+1} = 0$ as a factor. In addition, $h_{ij}(0) = \sqrt{1-|\lambda_i|^2}\lambda_{n-j+3} \cdots \lambda_{n-i+3}$. Therefore, when $i \geq j \geq 3$, we conclude that

$$\langle C_\varphi g_j, g_i \rangle = -\frac{\eta\overline{\lambda_{n-j+3}} \cdots \overline{\lambda_{n-i+3}}\sqrt{1-|\lambda_j|^2}\sqrt{1-|\lambda_i|^2}}{\eta\overline{\lambda_j} - \overline{p}}.$$

Therefore, in all cases, $\langle C_\varphi g_j, g_i \rangle$ has the value given in the theorem. □

Lemma 7. *If the assumptions in (12) hold and $1 \leq i, j \leq n$, then, for $n \geq 3$,*

$$\langle C_\varphi g_j, Bg_i \rangle = \begin{cases} 0 & \text{if } i = j = 1, \\ 0 & \text{if } i > 1,\ 1 \leq j \leq n, \\ \dfrac{-\overline{p}\overline{\lambda_3} \cdots \overline{\lambda_n}}{\sqrt{1-|p|^2}} & \text{if } i = 1,\ j = 2, \\ \dfrac{-\overline{p}\,\overline{\lambda_3} \cdots \overline{\lambda_{n-j+3}}\sqrt{1-|\lambda_j|^2}}{\eta\overline{\lambda_j} - \overline{p}} & \text{if } i = 1,\ 3 \leq j \leq n. \end{cases}$$

When $n = 2$, the relevant (i, j) values are the same except that $\langle C_\varphi g_2, Bg_1 \rangle = -\frac{\overline{p}}{\sqrt{1-|p|^2}}$.

Proof. If $j = 1$, then $\langle C_\varphi g_j, Bg_i\rangle = \langle 1, Bg_i\rangle = 0$ for all i, since $B(0) = 0$. If $i > 1$ and $j > 1$, then

$$\begin{aligned}
\langle C_\varphi g_j, Bg_i\rangle &= \left\langle C_\varphi\left(\frac{k_{\lambda_j}}{\|k_{\lambda_j}\|}\varphi_n\cdots\varphi_{n-j+2}\right), B\frac{k_{\lambda_i}}{\|k_{\lambda_i}\|}\varphi_n\cdots\varphi_{n-i+2}\right\rangle\\
&= \left\langle C_\varphi\left(\frac{k_{\lambda_j}}{\|k_{\lambda_j}\|}\right)\varphi_1\varphi_n\cdots\varphi_{n-j+3}, B\frac{k_{\lambda_i}}{\|k_{\lambda_i}\|}\varphi_n\cdots\varphi_{n-i+2}\right\rangle\\
&= \left\langle C_\varphi\left(\frac{k_{\lambda_j}}{\|k_{\lambda_j}\|}\right), \varphi_2\cdots\varphi_{n-j+2}\frac{k_{\lambda_i}}{\|k_{\lambda_i}\|}\varphi_n\cdots\varphi_{n-i+2}\right\rangle\\
&= \left\langle \frac{k_{\lambda_j}}{\|k_{\lambda_j}\|}, C_\varphi^*\left[\varphi_2\cdots\varphi_{n-j+2}\frac{k_{\lambda_i}}{\|k_{\lambda_i}\|}\varphi_n\cdots\varphi_{n-i+2}\right]\right\rangle.
\end{aligned}$$

When $j \neq 2$, the function in the square brackets to which C_φ^* is applied has a zero at $\lambda_1 = 0$ and at λ_{j-1} due to the factors φ_n and φ_{n-j+2}, respectively. When $j = 2$, the factor φ_n appears twice in this function, which consequently has a zero of order two at $\lambda_1 = 0$. Therefore Corollary 4 shows that $\langle C_\varphi g_j, Bg_i\rangle = 0$ whenever $i > 1$.

If $i = 1$ and $2 \le j \le n$, then

$$\begin{aligned}
\langle C_\varphi g_j, Bg_i\rangle &= \langle C_\varphi g_j, B\rangle\\
&= \left\langle C_\varphi\left(\frac{k_{\lambda_j}}{\|k_{\lambda_j}\|}\varphi_n\cdots\varphi_{n-j+2}\right), B\right\rangle\\
&= \left\langle C_\varphi\left(\frac{k_{\lambda_j}}{\|k_{\lambda_j}\|}\right), \varphi_2\cdots\varphi_{n-j+2}\right\rangle\\
&= \sqrt{1-|\lambda_j|^2}\left\langle k_{\lambda_j}, C_\varphi^*\left[\varphi_2\cdots\varphi_{n-j+2}\right]\right\rangle.
\end{aligned}$$

Corollary 4 provides the values to the expression above as follows. Define $f_j = \varphi_2\cdots\varphi_{n-j+2}$. When $n \ge 3$ and $j = 2$, $f_2(z) = \varphi_2(z)\cdots\varphi_{n-1}(z)z$, so $f_2(0) = 0$ and

$$f_2'(0) = \varphi_2(0)\cdots\varphi_{n-1}(0) = \lambda_3\cdots\lambda_n.$$

Therefore

$$\langle C_\varphi g_2, B\rangle = \frac{-\overline{p}}{\sqrt{1-|p|^2}}\overline{\lambda_3}\cdots\overline{\lambda_n}.$$

When $n = j = 2$, $f_2(z) = z$, so $f_2'(0) = 1$, and therefore $\langle C_\varphi g_2, B\rangle = -\frac{\overline{p}}{\sqrt{1-|p|^2}}$.

When $j \ge 3$, the value $f_j(\lambda_{j-1})$ has $\varphi_{n-j+2}(\lambda_{j-1}) = 0$ as a factor, so Corollary 4 yields

$$\sqrt{1-|\lambda_j|^2}\langle k_{\lambda_j}, C_\varphi^*[\varphi_2\cdots\varphi_{n-j+2}]\rangle = \frac{-\overline{p}\overline{f_j(0)}\sqrt{1-|\lambda_j|^2}}{\eta\overline{\lambda_j}-\overline{p}} = \frac{-\overline{p}\overline{\lambda_3}\cdots\overline{\lambda_{n-j+3}}\sqrt{1-|\lambda_j|^2}}{\eta\overline{\lambda_j}-\overline{p}}.$$

This concludes the proof. □

3. Block Toeplitz form of C_φ

We next review the material (available in [3] or [6]) about block Toeplitz operators that is used in what follows. Let $\mathcal{H}$ be a Hilbert space. Define $\ell^2(\mathcal{H})$ to be the Hilbert space of square-summable sequences with entries in $\mathcal{H}$. That is, $\ell^2(\mathcal{H})$ consists of the set of sequences

$$C = \left\{(C_0, C_1, C_2, \ldots) \mid C_j \in \mathcal{H} \text{ for } j = 0, 1, 2, \ldots\right\}$$

that satisfy $\sum_{j=0}^\infty \|C_j\|_{\mathcal{H}}^2 < \infty$, where $\|\cdot\|_{\mathcal{H}}$ denotes the norm in $\mathcal{H}$ and the expression $\sum_{j=0}^\infty \|C_j\|_{\mathcal{H}}^2$ defines $\|C\|^2_{\ell^2(\mathcal{H})}$. There is a natural Hilbert space isomorphism between $\ell^2(\mathcal{H})$ and the space $H^2(\mathcal{H})$ of $\mathcal{H}$-valued analytic functions in the unit disk. That is, a function $f \in H^2(\mathcal{H})$ has a power series expansion valid in $\mathbb{D}$ of the form $f(z) = \sum_{n=0}^\infty C_n z^n$ with square-summable coefficients.

The space $\mathcal{L}^\infty(\mathcal{B}(\mathcal{H}))$ is the set of essentially bounded, weakly measurable functions defined on the unit circle $\partial\mathbb{D}$ with values in $\mathcal{B}(\mathcal{H})$. For $F \in \mathcal{L}^\infty(\mathcal{B}(\mathcal{H}))$, the norm is defined by

$$\|F\|_\infty = \text{ess sup}\left\{\|F(z)\|_{\mathcal{B}(\mathcal{H})} \mid z \in \partial\mathbb{D}\right\}.$$

The functions in $\mathcal{L}^\infty(\mathcal{B}(\mathcal{H}))$ have Fourier expansions of the form $F(z) = \sum_{n=-\infty}^{\infty} A_n z^n$, where $A_n \in \mathcal{B}(\mathcal{H})$ for all integers n.

If T is a bounded linear operator on $H^2(\mathcal{H})$ (or equivalently $\ell^2(\mathcal{H})$), then T has a block matrix representation of the form $(B_{ij})_{i,j=0}^{\infty}$, where each B_{ij} is in $\mathcal{B}(\mathcal{H})$. If there exists a sequence $\{A_n\}_{n=-\infty}^{\infty}$ in $\mathcal{B}(\mathcal{H})$ such that the (i,j) block B_{ij} satisfies $B_{ij} = A_{i-j}$, then T is called a block Toeplitz operator. The symbol of T is defined to be the following $\mathcal{B}(\mathcal{H})$-valued function defined on the unit circle $\partial\mathbb{D}$:

$$F(z) = \sum_{n=-\infty}^{\infty} A_n z^n.$$

As in the scalar-valued case, if T is a block Toeplitz operator on $H^2(\mathcal{H})$ with symbol F, then T will be denoted T_F. The operator T_F is the projection onto $H^2(\mathcal{H})$ of multiplication by F. Consequently,

$$\|T_F\| = \|F\|_\infty = \text{ess sup}\left\{\|F(z)\|_{\mathcal{B}(H)} \mid z \in \partial\mathbb{D}\right\}. \tag{14}$$

If $F(z) = \sum_{n=-M}^{M} A_n z^n$ with A_M or A_{-M} nonzero, then F is called a trigonometric polynomial of degree M, and F is continuous on $\partial\mathbb{D}$. In this case, the supremum in (14) is attained at some $z_0 \in \partial\mathbb{D}$.

The main result of this section is that a composition operator on $H^2(\mathbb{D})$ with minimal polynomial equal to $z^n - 1$ has a block Toeplitz matrix with respect to the orthogonal decomposition $H^2 = \oplus_{N=0}^{\infty} B^N G$. In particular, the matrix with respect to this decomposition is

$$\mathcal{M}(C_\varphi) = \begin{pmatrix} A_0 & 0 & 0 & 0 & 0 & \cdots \\ A_1 & A_0 & 0 & 0 & 0 & \cdots \\ 0 & A_1 & A_0 & 0 & 0 & \cdots \\ 0 & 0 & A_1 & A_0 & 0 & \\ & & & & & \ddots \\ 0 & 0 & 0 & A_1 & A_0 & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{15}$$

where A_0 and A_1 will be defined in the theorem below.

Theorem 8. *Assume the hypotheses in (12). The matrix $\mathcal{M}(C_\varphi)$ for C_φ with respect to the basis $\mathcal{E}$ in (7) is block Toeplitz of the form (15), where A_0 and A_1 are $n \times n$ matrices whose entries depend on η and p. For $1 \le i,j \le n$, the (i,j) entries of A_0 are given by $\langle C_\varphi g_j, g_i\rangle$ in Lemma 6, and the (i,j) entries of A_1 are given by $\langle C_\varphi g_j, Bg_i\rangle$ in Lemma 7.*

Remark. The form of A_1 is particularly simple, since Lemma 7 implies that

$$A_1 = \begin{pmatrix} 0 & \langle C_\varphi g_2, B\rangle & \langle C_\varphi g_3, B\rangle & \cdots & \langle C_\varphi g_n, B\rangle \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Proof of Theorem 8. Let $n \ge 2$ be an integer. Let φ be a disk automorphism of the form (1). The corresponding composition operator has minimal polynomial $z^n - 1$ if and only if φ satisfies both conditions in Proposition 1. Let ℓ and m be non-negative integers, and let $1 \le i,j \le n$. We will compute the (i,j) entry of the (ℓ, m) block of the matrix $\mathcal{M}(C_\varphi)$ with respect to the basis $\mathcal{E}$; this entry is given by $\langle C_\varphi B^m g_j, B^\ell g_i\rangle$, so the remainder of this proof will consist of computing this expression for different values of $\ell - m$.

If $\ell - m > 1$, then (6) shows that for $1 \le i,j \le n$,

$$\langle C_\varphi B^m g_j, B^\ell g_i\rangle = \langle B^m(C_\varphi g_j), B^\ell g_i\rangle = \langle C_\varphi g_j, B^{\ell-m} g_i\rangle = \langle g_j, C_\varphi^*(B^{\ell-m} g_i)\rangle = 0.$$

The final equality follows from Corollary 4, because each g_j is a linear combination of the kernels $k_{\lambda_1}, \ldots, k_{\lambda_j}$, and the function $B^{\ell-m} g_i$ has zeros of order at least $\ell - m > 1$ at $\lambda_1, \ldots, \lambda_n$.

If $\ell - m < 0$, then again (6) shows that for $1 \le i,j \le n$,

$$\langle C_\varphi B^m g_j, B^\ell g_i\rangle = \langle B^m(C_\varphi g_j), B^\ell g_i\rangle = \langle B^{m-\ell} C_\varphi g_j, g_i\rangle.$$

If either $i = 1$ or $j = 1$, then orthonormality of the basis $\mathcal{E}$ and the fact that $B(0) = 0$ show that the above expression is zero. If $2 \le i, j \le n$, then there is at least one factor in each of g_i and g_j that is an iterate of φ, and we obtain

$$\begin{aligned}\langle C_\varphi B^m g_j, B^\ell g_i\rangle &= \left\langle B^{m-\ell} C_\varphi\left(\frac{k_{\lambda_j}}{\|k_{\lambda_j}\|}\varphi_n\cdots\varphi_{n-j+2}\right), \frac{k_{\lambda_i}}{\|k_{\lambda_i}\|}\varphi_n\cdots\varphi_{n-i+2}\right\rangle \\ &= \left\langle B^{m-\ell-1}\varphi_{n-i+1}\cdots\varphi_1 C_\varphi\left(\frac{k_{\lambda_j}}{\|k_{\lambda_j}\|}\varphi_n\cdots\varphi_{n-j+2}\right), \frac{k_{\lambda_i}}{\|k_{\lambda_i}\|}\right\rangle \\ &= \sqrt{1-|\lambda_i|^2} f(\lambda_i) = 0,\end{aligned}$$

where

$$f = B^{m-\ell-1}\varphi_{n-i+1}\cdots\varphi_1 C_\varphi\left(\frac{k_{\lambda_j}}{\|k_{\lambda_j}\|}\varphi_n\cdots\varphi_{n-j+2}\right)$$

has a zero at λ_i because $\varphi_{n-i+1}(\lambda_i) = \lambda_{n+1} = 0$.

Hence the only possible nonzero blocks in the matrix for C_φ occur when $\ell = m$ and when $\ell = m + 1$. Furthermore, the identity (6) shows that for each (i, j), the entry $\langle C_\varphi B^m g_j, B^\ell g_i\rangle$ depends only on $\ell - m$ and not on the separate values of ℓ and m. Therefore C_φ is block Toeplitz, with the form given by (15). To complete the proof, we will compute the entries of A_0 and A_1.

To compute the entries of A_0, set $\ell = m$, and let $1 \le i, j \le n$. The (i, j) entry is

$$\langle C_\varphi B^m g_j, B^\ell g_i\rangle = \langle B^m C_\varphi g_j, B^m g_i\rangle = \langle C_\varphi g_j, g_i\rangle,$$

and these values are given in Lemma 6.

Next, we will compute the entries of the matrix A_1, so set $\ell = m + 1$ and $1 \le i, j \le n$. The same reasoning as before yields

$$\langle C_\varphi B^m g_j, B^\ell g_i\rangle = \langle C_\varphi g_j, B g_i\rangle.$$

Therefore the entries of A_1 are given by Lemma 7, which results in the form for A_1 stated in the theorem. □

When $n = 1$, Theorem 8 trivially holds, because both the composition operator with minimal polynomial $z - 1$ and the associated Toeplitz operator are the identity operator. In fact, the only composition operator that is also scalar Toeplitz is the identity operator. See [19,23] for more general connections between composition operators and scalar Toeplitz operators.

The corollary below follows directly from the theorem and basic properties of block Toeplitz matrices.

Corollary 9. *Assume the hypotheses in (12). The symbol of the block Toeplitz form of the matrix for C_φ with respect to the basis (7) is the $n \times n$ matrix-valued function defined for $z \in \partial\mathbb{D}$ by*

$$A(z) = A_0 + A_1 z, \tag{16}$$

where A_0 and A_1 are defined as in Theorem 8. That is, C_φ is unitarily equivalent to T_A.

It is well known (e.g., see [8]) that the norm of the composition operator C_φ on $H^2(\mathbb{D})$ with automorphic symbol φ is given by $\|C_\varphi\| = \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\frac{1}{2}} = \left(\frac{1+|p|}{1-|p|}\right)^{\frac{1}{2}}$. Therefore, the norm identity (14) shows that, for all $n \ge 2$, the L^∞ norm of the symbol A given in (16) is equal to the composition operator norm:

$$\sup_{z\in\partial\mathbb{D}} \|A(z)\|_{\mathcal{B}(\mathbb{C}^n)} = \left(\frac{1+|p|}{1-|p|}\right)^{\frac{1}{2}}.$$

Our original goal was to study the numerical ranges of composition operators with minimal polynomial $z^n - 1$. Recently, Bebiano and Spitkovsky [2] described the numerical range of Toeplitz operators with matrix-valued symbol in terms of the numerical ranges of the values of the symbol. Let $\mathcal{R}(a)$ denote the essential range of the matrix-valued function a defined on $\partial\mathbb{D}$; for a continuous function a, the essential range is simply the range. The convex hull of a set S in the complex plane will be denoted conv S, and the closure of S is denoted clos S. One of the results from [2] that is relevant to this paper is stated below.

Theorem 10 (*Bebiano–Spitkovsky*). *If $a \in \mathcal{L}^\infty(\mathcal{B}(\mathbb{C}^n))$ and T_a is the Toeplitz operator with symbol a on $H^2(\mathcal{B}(\mathbb{C}^n))$, then*

$$\operatorname{clos} W(T_a) = \operatorname{conv}\{W(A) : A \in \mathcal{R}(a)\}.$$

Bebiano and Spitovsky also showed that the closure of the numerical range of the multiplication operator with symbol a is the same as the set given above. The $n = 1$ version of their result was proved by Klein [14].

Since the numerical range of an operator is invariant under unitary equivalence, an immediate consequence of the combination of Theorems 8 and 10 is the following.

Theorem 11. *Assume the hypotheses in (12). The closure of the numerical range of C_φ is given by*

$$\operatorname{clos} W(C_\varphi) = \operatorname{conv}\{W(A(z)) : z \in \partial\mathbb{D}\},$$

where $A(z)$ is defined in Corollary 9.

As an application of Theorem 11, we will use the $n = 2$ case to find the closure of the numerical range of the conformal automorphism $\varphi = \frac{p-z}{1-\bar{p}z}$. The original proof of this result appeared in [1].

Recall that the numerical range of a 2×2 matrix of the form

$$\begin{pmatrix} 1 & v \\ 0 & -1 \end{pmatrix} \tag{17}$$

is an elliptic disk with foci at 1 and -1, and it is also straightforward to show that the major axis of the ellipse has length $\sqrt{4+|v|^2}$ while the minor axis has length $|v|$. Thus, any collection of matrices of the form (17) has numerical ranges that are increasing with $|v|$ with respect to set containment. If the Toeplitz symbol A is computed in the $n = 2$ case, then we get that $A(z)$ has form (17) with

$$v = \frac{p}{\sqrt{1-|p|^2}} - z\frac{\bar{p}}{\sqrt{1-|p|^2}}.$$

The value of $|v|$ is clearly maximized on $\partial\mathbb{D}$ when $z = z_0 = -\frac{p^2}{|p|^2}$, in which case $|v| = \frac{2|p|}{\sqrt{1-|p|^2}}$. Therefore, for any $z \in \partial\mathbb{D}$, the set containment

$$W(A(z)) \subseteq W(A(z_0))$$

holds, and consequently

$$\operatorname{conv}\{W(A(z)) : z \in \partial\mathbb{D}\} = W(A(z_0)).$$

Thus the closure of the numerical range of $W(C_\varphi)$ equals the elliptic disk with foci at ± 1 and with major axis of length $\frac{2}{\sqrt{1-|p|^2}}$, which is exactly the result that appears in [1].

Using a similar argument with increasing nested numerical ranges of certain 3×3 matrices, Theorem 11 will be used in the next section to answer the question of Bourdon and Shapiro affirmatively in the $n = 3$ case. That is, we will show that the numerical range of a composition operator on $H^2(\mathbb{D})$ with minimal polynomial $z^3 - 1$ is not a disk.

4. The case where $n = 3$

We next review some properties of the support function of a convex set. We will use some of these concepts to prove the conjecture of Bourdon and Shapiro in the $n = 3$ case. Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$. Since $W(T)$ is a convex set in $\mathbb{C}$, every boundary point of $W(T)$ intersects a line (called a support line) such that the interior of $W(T)$ lies entirely on one side of the line. The support function of $W(T)$, which will be denoted p_T, can be defined as follows. For each $\theta \in [0, 2\pi)$,

$$p_T(\theta) = \sup\{\operatorname{Re}(e^{-i\theta}\langle Tv\, v\rangle) \mid v \in \mathcal{H}, \|v\| = 1\}.$$

The value $p_T(\theta)$ is the maximum scalar projection of $W(T)$ in the direction of θ. For every operator T discussed in this paper, the numerical range $W(T)$ contains the origin. Therefore, for each value $\theta \in [0, 2\pi)$, the ray in the direction of the vector $(\cos\theta, \sin\theta)$ hits a support line L_θ of $W(T)$ that is orthogonal to $(\cos\theta, \sin\theta)$. The value $p_T(\theta)$ is the distance from the origin to the line L_θ. Properties of convex sets show that the closure of $W(T)$ is completely determined by its support lines. Consequently, if T and R are bounded linear operators such that $p_T(\theta) = p_R(\theta)$ for all $\theta \in [0, 2\pi)$, then the closure of $W(T)$ equals the closure of $W(R)$.

Furthermore, the following proposition immediately follows from the definition of p_T.

Proposition 12. *If T and R are bounded linear operators on a Hilbert space H such that $p_T(\theta) \le p_R(\theta)$ for all $\theta \in [0, 2\pi)$, then*

$$\operatorname{clos} W(T) \subseteq \operatorname{clos} W(R).$$

A simple calculation shows that $\operatorname{Re}\left(e^{-i\theta}\langle Tv\, v\rangle\right) = \langle \operatorname{Re}(e^{-i\theta}T)v, v\rangle$. Therefore, it is convenient to define the operator $H_\theta = \operatorname{Re}\left(e^{-i\theta}T\right) = \frac{e^{-i\theta}}{2}T + \frac{e^{i\theta}}{2}T^*$. Clearly H_θ is self-adjoint. When T is an operator on a finite-dimensional Hilbert space, properties of self-adjoint operators imply that $p_T(\theta)$ is the maximum eigenvalue of H_θ.

Define a subset S of the complex plane to have three-fold symmetry about the origin if $z \in S$ implies that $e^{i\frac{2\pi}{3}} z \in S$. If φ is an elliptic automorphism with rotation parameter equal to a cube root of unity, that is, if φ satisfies (3), then the numerical range of C_φ has three-fold symmetry about the origin, as noted in [4]. The values of the Toeplitz symbol in the $n = 3$ case of Theorem 8 have numerical range with three-fold symmetry about the origin. Such matrices were classified in [11], where the following result appeared.

Theorem 13. *Let M be any 3×3 matrix. Assume that $W(M)$ is not a disk. Then the following are equivalent.*
(i) *$W(M)$ has three-fold symmetry about the origin.*
(ii) *$\mathrm{Tr}(M^2M^*) = 0$ and the spectrum $\sigma(M)$ has three-fold symmetry about the origin.*
(iii) *There exist $p, q, r \in \mathbb{C}$ such that M is unitarily equivalent to the matrix V of the form*

$$\begin{pmatrix} 0 & 0 & p \\ q & 0 & 0 \\ 0 & r & 0 \end{pmatrix}.$$

As mentioned in the introduction, the numerical range of an $n \times n$ matrix with no repeated eigenvalues cannot be a circular disk, so if the minimal polynomial of a 3×3 matrix M is $z^3 - 1$, then $W(M)$ is not a disk. The support function for a 3×3 matrix M with minimal polynomial equal to $z^3 - 1$, derived in [11], is given by

$$p_M(\theta) = \frac{2}{\sqrt{3}}\sqrt{s}\cos\left(\frac{1}{3}\arccos\left(\frac{t(\theta)}{2}\sqrt{\frac{27}{s^3}}\right)\right),$$

where $s = \mathrm{Tr}(MM^*)/4$ and $t(\theta) = \frac{1}{4}\left[\cos(3\theta) + \mathrm{Re}\,\mathrm{Tr}(M^2M^*)\cos(\theta) + \mathrm{Im}\,\mathrm{Tr}(M^2M^*)\sin(\theta)\right]$.

Hence, if the minimal polynomial of the 3×3 matrix M is $z^3 - 1$ and $W(M)$ has three-fold symmetry about the origin, then

$$\frac{t(\theta)}{2}\sqrt{\frac{27}{s^3}} = \frac{\sqrt{27}\cos(3\theta)}{(\mathrm{Tr}(MM^*))^{\frac{3}{2}}}.$$

In this case, the support function simplifies to

$$p_M(\theta) = \frac{1}{\sqrt{3}}\sqrt{\mathrm{Tr}(MM^*)}\cos\left(\frac{1}{3}\arccos\left(\frac{\sqrt{27}\cos(3\theta)}{(\mathrm{Tr}(MM^*))^{\frac{3}{2}}}\right)\right),$$

a function which achieves its absolute maximum only at $\theta = 0, \frac{2\pi}{3}$, and $\frac{4\pi}{3}$.

When $n = 3$, the calculations in the previous sections can be done explicitly. Definition (4) results in $\lambda_1 = 0$, $\lambda_2 = \eta p$, and $\lambda_3 = p$. By Theorem 11, we know that if φ is an automorphism of form (1) that satisfies (3), then

$$\mathrm{clos}\, W(C_\varphi) = \mathrm{conv}\,\{W(A(z)) : z \in \partial\mathbb{D}\},$$

where $A(z) = A_0 + A_1 z$ with A_0 and A_1 equal to the 3×3 matrices defined in Theorem 8. Substituting into the values given by Lemmas 6, 7, and (3) yields

$$A_0 = \begin{pmatrix} 1 & \dfrac{\eta p}{\sqrt{1-|p|^2}} & 0 \\ 0 & -\eta & p \\ 0 & -\eta\overline{p} & -1+\eta \end{pmatrix} \tag{18}$$

and

$$A_1 = \begin{pmatrix} 0 & -\dfrac{\overline{p}^2}{\sqrt{1-|p|^2}} & -\dfrac{\overline{p}(\overline{\eta}-1)}{\sqrt{1-|p|^2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{19}$$

We are now prepared to show that the conjecture of Bourdon and Shapiro is correct in the $n = 3$ case.

Theorem 14. *If φ is a disk automorphism of form (1) such that the associated composition operator C_φ has minimal polynomial equal to $z^3 - 1$, then the closure of the numerical range of the composition operator C_φ is equal to the numerical range of the 3×3 matrix $A_0 + A_1 z_0$, where A_0 and A_1 are given by (18) and (19), respectively, and $z_0 = -\frac{\eta p^3}{|p|^3}$. The support function of C_φ is*

$$p_{C_\varphi}(\theta) = \frac{1}{\sqrt{3}}\sqrt{\frac{3-|p|^4+2|p|^3}{1-|p|^2}}\cos\left(\frac{1}{3}\arccos\left(\frac{\sqrt{1-|p|^2}^{\,3}\sqrt{27}\cos(3\theta)}{(3-|p|^4+2|p|^3)^{\frac{3}{2}}}\right)\right). \tag{20}$$

In particular, $W(C_\varphi)$ is not a disk.

Proof. Let C_φ be a composition operator with symbol φ satisfying (1). The operator C_φ has minimal polynomial $z^3 - 1$ if and only if the parameters of φ satisfy (3). We will show that the support function for C_φ is given by (20), from which it will follow that $W(C_\varphi)$ is not a disk.

By Theorem 11, the closure of $W(C_\varphi)$ is given by

$$\operatorname{clos} W(C_\varphi) = \operatorname{conv}\{A(z) : z \in \mathbb{D}\},$$

where $A(z) = A_0 + A_1 z$ and the 3×3 matrices A_0 and A_1 were computed above so that

$$A(z) = \begin{pmatrix} 1 & \frac{\eta p}{\sqrt{1-|p|^2}} - z\frac{\overline{p}^2}{\sqrt{1-|p|^2}} & -z\frac{\overline{p}(\overline{\eta}-1)}{\sqrt{1-|p|^2}} \\ 0 & -\eta & p \\ 0 & -\eta\overline{p} & -1+\eta \end{pmatrix}.$$

For each z on the unit circle, $A(z)$ has distinct eigenvalues 1, $e^{i\frac{2\pi}{3}}$, and $e^{i\frac{4\pi}{3}}$, and therefore the numerical range of $A(z)$ is not a disk. Furthermore, the eigenvalues of $A(z)$ have three-fold symmetry about the origin. A straightforward but lengthy computation using identity (3) shows that

$$A(z)A(z)^* = \begin{pmatrix} \frac{1+|p|^2-\eta p^3\overline{z}-\overline{\eta}\,\overline{p}^3 z}{1-|p|^2} & \frac{-p+\overline{p}^2 z}{\sqrt{1-|p|^2}} & \frac{-p^2+\overline{\eta}\overline{p}z}{\sqrt{1-|p|^2}} \\ \frac{-\overline{p}+p^2\overline{z}}{\sqrt{1-|p|^2}} & 1+|p|^2 & \overline{\eta}p \\ \frac{-\overline{p}^2+\eta p\overline{z}}{\sqrt{1-|p|^2}} & \eta\overline{p} & 1 \end{pmatrix}.$$

From this calculation, we obtain

$$\operatorname{Tr} A(z)A(z)^* = \frac{3-|p|^4-\overline{\eta}\,\overline{p}^3 z-\eta p^3\overline{z}}{1-|p|^2},$$

and another matrix multiplication and trace computation show that $\operatorname{Tr} A(z)^2 A(z)^* = 0$. Therefore, Theorem 13 shows that $W(A(z))$ has three-fold symmetry about the origin. The remarks following the statement of Theorem 13 show that the support function for $A(z)$ is

$$p_{A(z)}(\theta) = \frac{1}{\sqrt{3}}\sqrt{\frac{3-|p|^4-\overline{\eta}\,\overline{p}^3 z-\eta p^3\overline{z}}{1-|p|^2}}\cos\left(\frac{1}{3}\arccos\left(\frac{\sqrt{1-|p|^2}^{\,3}\sqrt{27}\cos(3\theta)}{(3-|p|^4-\overline{\eta}\,\overline{p}^3 z-\eta p^3\overline{z})^{\frac{3}{2}}}\right)\right). \tag{21}$$

For any real value of k, the function $x\cos(\frac{1}{3}\arccos(\frac{k}{x^3}))$ is an increasing function of x at any positive x in its domain. Therefore, for each value of $\theta \in [0, 2\pi)$, $p_{A(z)}(\theta)$ attains its maximum at the value of z on the unit circle that maximizes $\operatorname{Tr} A(z)A(z)^*$. This occurs when $z_0 = -\frac{\eta p^3}{|p|^3}$. Hence

$$p_{A(z)}(\theta) \le p_{A(z_0)}(\theta)$$

for all $\theta \in [0, 2\pi)$. Proposition 12 thus shows that

$$W(A(z)) \subseteq W(A(z_0))$$

for all $z \in \partial\mathbb{D}$. Consequently,

$$\operatorname{clos} W(C_\varphi) = \operatorname{conv}\{A(z) : z \in \mathbb{D}\} = W(A(z_0)).$$

Since $W(A(z_0))$ is not a disk, it follows that $W(C_\varphi)$ is not a disk.

In addition, the support function for C_φ is equal to the support function for $A(z_0)$, namely, the function given in (21) with $z = z_0$. Hence,

$$p_{C_\varphi}(\theta) = \frac{1}{\sqrt{3}}\sqrt{\frac{3-|p|^4+2|p|^3}{1-|p|^2}}\cos\left(\frac{1}{3}\arccos\left(\frac{\sqrt{1-|p|^2}^{\,3}\sqrt{27}\cos(3\theta)}{(3-|p|^4+2|p|^3)^{\frac{3}{2}}}\right)\right). \quad \square$$

The question of whether the numerical range of a composition operator on $H^2(\mathbb{D})$ with minimal polynomial equal to $z^n - 1$ can be a disk is still open for $n \ge 4$. It seems possible that in these cases, the closure of the numerical range of C_φ will also be equal to the numerical range of a fixed $n \times n$ matrix with distinct eigenvalues, and thus will not be a disk.

Acknowledgment

The author thanks Estelle Basor and Ilya Spitkovsky for helpful discussions and references.

References

[1] A. Abdollahi, The numerical range of a composition operator with conformal automorphism symbol, Linear Algebra Appl. 408 (2005) 177–188.
[2] N. Bebiano, I.M. Spitkovsky, Numerical ranges of Toeplitz operators with matrix symbols, Linear Algebra Appl. 436 (2012) 1721–1726.
[3] H. Bercovici, Operator Theory and Arithmetic in H^∞, American Mathematical Society, 1988.
[4] P.S. Bourdon, J.H. Shapiro, The numerical ranges of automorphic composition operators, J. Math. Anal. Appl. 251 (2000) 839–854.
[5] P.S. Bourdon, J.H. Shapiro, When is zero in the numerical range of a composition operator? Integral Equations Operator Theory 44 (2002) 401–441.
[6] A. Böttcher, B. Silbermann, Analysis of Toeplitz Operators, second ed., Springer, Berlin, 2006.
[7] C.C. Cowen, Linear fractional composition operators on $H^2(\mathbb{D})$, Integral Equations Operator Theory 11 (1988) 151–160.
[8] C.C. Cowen, B.D. MacCluer, Composition Operators on Spaces of Analytic Functions, CRC Press, Boca Raton, 1995.
[9] K.E. Gustafson, D.K.M. Rao, Numerical Range: The Field of Values of Linear Operators and Matrices, Springer-Verlag, New York, 1996.
[10] J. Guyker, On reducing subspaces of composition operators, Acta Sci. Math. (Szeged) 53 (3–4) (1989) 369–376.
[11] T.R. Harris, M. Mazzella, L.J. Patton, D. Renfrew, I.M. Spitkovsky, Numerical ranges of cube roots of the identity, Linear Algebra Appl. 435 (2011) 2639–2657.
[12] F. Hausdorff, Der Wertvorrat einer Bilinearform, Math. Z. 3 (1919) 314–316.
[13] R.A. Horn, C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, Cambridge, 1991.
[14] E.M. Klein, The numerical range of a Toeplitz operator, Proc. Amer. Math. Soc. 35 (1972) 101–103.
[15] F. Malmquist, Sur la détermination d'une classe de fonctions analytiques par leurs valeurs dans un ensemble donné de points, in: Comptes Rendus du Sixième Congrès (1925) des Mathématiciens Scandinaves, Kopenhagen, 1926, pp. 253–259.
[16] J. Maroulas, P.J. Psarrokos, M.J. Tsatsomeros, Perron–Frobenius type results on the numerical range, Linear Algebra Appl. 348 (2002) 49–62.
[17] V. Matache, Numerical ranges of composition operators, Linear Algebra Appl. 331 (2001) 61–74.
[18] V. Matache, Numerical ranges of composition operators with inner symbols, Rocky Mountain J. Math. 42 (1) (2012) 235–249.
[19] F. Nazarov, J.H. Shapiro, On the Toeplitzness of composition operators, Complex Var. Elliptic Equ. 52 (2007) 193–210.
[20] J.R. Partington, Good bases, in: Harmonic Analysis and Rational Approximation, in: Lecture Notes in Control and Inform. Science, vol. 327, Springer, Berlin, 2006, pp. 95–102.
[21] L. Rodman, I. Spitkovsky, On generalized numerical ranges of quadratic operators, in: Recent Advances in Matrix and Operator Theory, in: Oper. Theory Adv. Appl., vol. 179, Birkhäuser, Basel, 2008, pp. 241–256.
[22] J.H. Shapiro, Composition Operators and Classical Function Theory, Springer-Verlag, New York, 1993.
[23] J.H. Shapiro, Composition operators (heart) Toeplitz operators, in: Five Lectures in Complex Analysis, in: Contemp. Math., vol. 525, American Mathematical Society, 2010, pp. 117–139.
[24] S. Takenaka, On the orthogonal functions and a new formula of interpolation, Jpn. J. Math. 2 (1925) 129–145.
[25] O. Toeplitz, Das algebraische Analogon zu einem Satz von Fejér, Math. Z. 2 (1918) 187–197.
[26] S.-H. Tso, P.Y. Wu, Matricial ranges of quadratic operators, Rocky Mountain J. Math. 29 (3) (1999).
[27] J.L. Walsh, Interpolation and Approximation by Rational Functions in the Complex Domain, American Math. Soc., 1935.
[28] P.Y. Wu, Numerical ranges as circular discs, Appl. Math. Lett. 24 (2011) 2115–2117.