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Osgood type condition for the Volterra integral equations with bounded and nonincreasing kernels

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ABSTRACT

This paper provides the necessary and sufficient Osgood type condition for the existence of blow-up solutions of Volterra equation with kernels being nonincreasing and bounded functions. Examples of such equations related to models of anomalous diffusion as well as some integral estimates of blow-up time are also presented.

1. Introduction

Many papers (see for instance [1,2], the survey paper [9] and references therein) are devoted to the blow-up solutions u of the Volterra integral equations of the convolution type

$$u(t) = \int_0^t k(t-s)g\big(u(s)\big)\,ds, \quad t \geqslant 0, \tag{1}$$

where $g, k \geqslant 0$ satisfy some additional conditions (such as g is increasing and k is locally integrable). Such equations appear in various applications. For instance, in recent years [5–7,10] that type of equation appeared in mathematical models of the classical diffusion as well as anomalous one (sub- and superdiffusion). It turns out that most of the kernels in these models are nonincreasing and bounded functions. Because, in addition, in the aforementioned papers authors did examine the blow-up of Eq. (1) only in the case $g(0) > 0$, motivated by this fact we fill the gap and give the necessary and the sufficient condition of the existence of the blow-up solutions of (1) with nonincreasing and bounded kernels and $g(0) = 0$. Furthermore, that condition is expressed in terms of the convergence of some integral which has exactly the same form as the integral in the famous Osgood condition [8] in ODE theory. Our method used in the proof of that condition allows us also to link that integral with the estimation of the blow-up time (see [3] for some series estimates).

2. Background information

We consider Volterra integral equation (1) with the following assumptions about nonlinearity g and kernel k: $g : [0,\infty) \to [0,\infty)$ – strictly increasing absolutely continuous function which satisfies the following conditions:

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$$g(0) = 0, \tag{2}$$

$$x/g(x) \to 0 \quad \text{as } x \to 0^+, \tag{3}$$

$$x/g(x) \to 0 \quad \text{as } x \to \infty, \tag{4}$$

k – nonincreasing and bounded positive function defined at least on $(0,\infty)$ (what implies in particular that k is locally integrable on $[0,\infty)$).

We say that u is a nontrivial solution of (1) if it is a continuous solution of (1) with the maximal interval of its existence $[0,T)$ such that $u(0)=0$ and $u>0$ in $(0,T)$. It is known [4] that under our assumptions about g and k Eq. (1) has at most one nontrivial solution u and, moreover, u is then a strictly increasing absolutely continuous function. If nontrivial solution u additionally satisfies $u(t)\to\infty$ as $t\to T^-$, $T<\infty$, then we call u a blow-up solution of (1) with a blow-up time T.

Throughout this paper we impose an extra condition on the kernel k, i.e. the condition of the form

$$\lim_{t\to\infty} K(t) \geqslant \gamma \max_{t\in(0,\infty)} \frac{t}{g(t)}, \tag{5}$$

for some $\gamma>1$, where $K(t) := \int_0^t k(s)\,ds$. Obviously in our case K is a strictly increasing continuous function, thus the inverse function K^{-1} to it exists. Finally, let us formulate the following sufficient condition for blow-up solutions, an easy consequence of Theorems 5.1. and 5.3. from [2] (for aforementioned γ just take $w(t) := g(\frac{t}{\gamma})$ in these two theorems):

Theorem 2.1. *Let strictly increasing absolutely continuous function g satisfy conditions (2)–(4), k be a locally integrable function positive a.e. in $[0,\infty)$ which satisfies (5) and let the mapping $t\to\frac{t}{g(t)}$ be strictly increasing in some right neighbourhood of 0 and strictly decreasing in some neighbourhood of ∞. If*

$$\int_0^\infty K^{-1}\left(\frac{\gamma s}{g(s)}\right)\frac{ds}{s} < \infty,$$

then a blow-up solution to Eq. (1) exists.

3. Auxiliary result

Now we prove a result which shows, under our assumptions about g and k, that in fact condition (5) is the necessary condition of the existence of the blow-up solutions of (1).

Theorem 3.1. *If u is the blow-up solution of (1), then*

$$\lim_{t\to\infty} K(t) > \max_{t\in(0,\infty)} \frac{t}{g(t)}. \tag{6}$$

Proof. From the monotonicity of functions u and g we obtain

$$u(t) \leqslant g\big(u(t)\big)\int_0^t k(t-s)\,ds = g\big(u(t)\big)K(t), \quad t\in(0,T),$$

hence

$$\frac{u(t)}{g(u(t))} \leqslant K(t), \quad t\in(0,T). \tag{7}$$

Our assumptions about g imply that the mapping $t\to\frac{t}{g(t)}$ has the global maximum achieved, let us say, at $t=t^*$. On the other hand, obviously, there also exists $\tau\in(0,T)$ such that $u(\tau)=t^*$. Then, using (7), we finally have

$$\max_{t\in(0,\infty)}\frac{t}{g(t)} = \frac{t^*}{g(t^*)} = \frac{u(\tau)}{g(u(\tau))} \leqslant K(\tau) < \lim_{t\to\infty} K(t). \qquad \square$$

Remark 3.2. Our assumptions about g and k imply that inequalities (5) and (6) are equivalent.

4. Main result

Theorem 4.1. *Let the mapping $t \to \frac{t}{g(t)}$ be strictly increasing in some right neighbourhood of 0 and strictly decreasing in some neighbourhood of ∞. Eq. (1) with the nonlinearity g satisfying (2)–(4) and with nonincreasing bounded kernel $k > 0$ satisfying (5) has a blow-up solution if and only if when*

$$\int_0^\infty \frac{ds}{g(s)} < \infty. \tag{8}$$

Proof. The necessity part of theorem. From absolutely continuity of blow-up solution u we have

$$u'(t) = \int_0^t k(t-s)g'\big(u(s)\big)u'(s)\,ds = \int_0^{u(t)} k\big(t-u^{-1}(s)\big)g'(s)\,ds \quad \text{a.e.} \tag{9}$$

Making the substitution $u^{-1}(t)$ for t in (9) and using the formula for the derivative of the inverse function we obtain

$$\big(u^{-1}\big)'(t)\int_0^t k\big(u^{-1}(t)-u^{-1}(s)\big)g'(s)\,ds = 1 \quad \text{a.e.}$$

Hence

$$\big(u^{-1}\big)'(t) = \left(\int_0^t k\big(u^{-1}(t)-u^{-1}(s)\big)g'(s)\,ds\right)^{-1} \quad \text{a.e.} \tag{10}$$

The values of mapping $s \to k(u^{-1}(t)-u^{-1}(s))$ for $s \in [0,t]$ are bounded from above by k_0, where

$$k_0 = \begin{cases} k(0), & \text{if } k \text{ is defined at } 0, \\ \lim_{t\to 0^+} k(t), & \text{otherwise,} \end{cases} \tag{11}$$

so

$$\int_0^t k\big(u^{-1}(t)-u^{-1}(s)\big)g'(s)\,ds \leqslant k_0\int_0^t g'(s)\,ds \leqslant k_0 g(t).$$

Using (10) we get the inequality

$$\big(u^{-1}\big)'(t) \geqslant \frac{1}{k_0 g(t)} \quad \text{a.e.}$$

which implies that

$$u^{-1}(t) \geqslant \frac{1}{k_0}\int_0^t \frac{ds}{g(s)}$$

for all $t \in (0,\infty)$. Then finally

$$\lim_{t\to\infty} u^{-1}(t) = T \geqslant \frac{1}{k_0}\int_0^\infty \frac{ds}{g(s)}.$$

The sufficient part of theorem. Let $\gamma > 1$ be a real number from (5). Condition (8) implies then that

$$\int_0^\infty \frac{\gamma\,ds}{g(s)} < \infty. \tag{12}$$

Because k is nonincreasing, K^{-1} is convex. Moreover, the mapping $s \to \frac{\gamma s}{g(s)}$ has the global maximum in $(0,\infty)$ what means that there exists $M > 0$ such that

$$K^{-1}\left(\frac{\gamma s}{g(s)}\right) \leqslant \frac{M\gamma s}{g(s)}, \quad s \in (0,\infty). \tag{13}$$

Hence from (12) and (13) we obtain

$$\int_0^\infty K^{-1}\left(\frac{\gamma s}{g(s)}\right)\frac{ds}{s} \leqslant M\int_0^\infty \frac{\gamma\, ds}{g(s)} < \infty,$$

and now the use of Theorem 2.1 ends the proof. □

5. Estimations of the blow-up time

A technique we use in the proof of Theorem 4.1 could be slightly modified in order to obtain the implicit estimations of the blow-up time of blow-up solution of (1).

Theorem 5.1. *Let the mapping $t \to \frac{t}{g(t)}$ be strictly increasing in some right neighbourhood of 0 and strictly decreasing in some neighbourhood of ∞. If Eq. (1) with the nonlinearity g satisfying (2)–(4) and with nonincreasing bounded kernel $k > 0$ satisfying (5) has a blow-up solution with a blow-up time T, then*

$$\frac{1}{k_0}\int_0^\infty \frac{ds}{g(s)} \leqslant T \leqslant \frac{1}{k(T)}\int_0^\infty \frac{ds}{g(s)}, \tag{14}$$

where k_0 is defined by (11).

Proof. The first part of inequality (14) was shown in the proof of Theorem 4.1. To show that

$$T \leqslant \frac{1}{k(T)}\int_0^\infty \frac{ds}{g(s)},$$

we notice that the minimum of mapping $s \to k(u^{-1}(t) - u^{-1}(s))$ for $s \in [0,t]$ is achieved for $s = 0$ and it is equal to $k(u^{-1}(t))$. In such a case (10) can be modified in the following way:

$$\left(u^{-1}\right)'(t) \leqslant \frac{1}{\int_0^t k(u^{-1}(t))g'(s)\,ds} = \frac{1}{k(u^{-1}(t))\int_0^t g'(s)\,ds} = \frac{1}{k(u^{-1}(t))g(t)} \quad \text{a.e.}$$

From last inequality it follows that

$$u^{-1}(t) \leqslant \int_0^t \frac{ds}{k(u^{-1}(s))g(s)}, \quad t \in (0,\infty),$$

and hence

$$\lim_{t\to\infty} u^{-1}(t) = T \leqslant \int_0^\infty \frac{ds}{k(u^{-1}(s))g(s)} \leqslant \frac{1}{k(T)}\int_0^\infty \frac{ds}{g(s)}. \qquad \square$$

6. Applications

Now we show how the results of previous sections can be applied to examine the existence of the blow-up solutions of some multidimensional models of anomalous diffusion.

Example 6.1. As it was shown in [6], the superdiffusion in the unbounded spatial domain of dimension N, $N = 1, 2, 3$, can be modelled by the fractional diffusion equation

$$\frac{\partial}{\partial t}T(x,t) = \sum_{n=1}^N \frac{\partial^\mu}{\partial |x_n|^\mu}T(x,t) + \lambda D(x|0)g\big(T(0,t)\big), \quad x \in \mathbb{R}^N,\ t > 0, \tag{15}$$

subject to the constraints

$$T(x,0) = 0, \quad x \in \mathbb{R}^N, \tag{16}$$

$$\lim_{|x|\to\infty} T(x,t) = 0, \quad t > 0. \tag{17}$$

The operator $\frac{\partial^\mu}{\partial|x_n|^\mu}$, where $1<\mu<2$, is the so-called Riesz fractional derivative operator, the parameter of superdiffusion $\lambda>0$ and the localization function $D(x|0)$ is defined as follows:

$$D(x|0)=\begin{cases}1, & x\in\Omega,\\ 0, & x\notin\Omega,\end{cases} \tag{18}$$

where $\Omega=\{x\in\mathbb{R}^N\colon -a<x_n<a\}$, $n=1,2,\dots,N$, $0<a\ll 1$.

Remark 6.2. The following equality is valid for all $x\in\mathbb{R}^N$ and $t\geqslant 0$:

$$T(x,t)=\lambda\left(\frac{2}{\pi}\right)^N\int_0^t\left(\prod_{n=1}^N\int_0^\infty\frac{\sin az\cos zx_n}{z}\exp\bigl(-z^\mu(t-s)\bigr)\,dz\right)g\bigl(T(0,s)\bigr)\,ds.$$

The conversion of (15)–(17) to an integral equation, accomplished through the use of Green's function, leads to Eq. (1) with

$$u(t)\equiv T(0,t)$$

and

$$k(t)=:k_N(t)=\lambda\left(\frac{2}{\pi}\int_0^\infty\frac{\sin az}{z}\exp\bigl(-z^\mu t\bigr)\,dz\right)^N. \tag{19}$$

The kernels k_N can be expressed in terms of Fox H functions

$$k_N(t)=\lambda\left(2\int_0^{at^{-1/\mu}}\frac{1}{\mu z}H^{1,1}_{2,2}\left[z\,\middle|\,\begin{matrix}(1,\mu^{-1}),(1,\frac{1}{2})\\ (1,1),\quad(1,\frac{1}{2})\end{matrix}\right]dz\right)^N.$$

Using this form, one can show that kernels k_N are nonnegative and nonincreasing. We have also $k_N(0)=\lambda$, so k_N is bounded. Moreover, further analysis of k_N leads to the following asymptotic relation:

$$k_N(t)\sim\lambda\left(\frac{2a}{\pi\mu}\Gamma\left(\frac{1}{\mu}\right)\right)^N t^{-\frac{N}{\mu}}\quad\text{as } t\to\infty,$$

what implies that k_N are in fact positive,

$$K_1(t)\sim\frac{2\lambda a}{\pi(\mu-1)}\Gamma\left(\frac{1}{\mu}\right)t^{1-\frac{1}{\mu}}\quad\text{as } t\to\infty$$

and

$$\lim_{t\to\infty}K_N(t)=:\mathcal{K}(N,\lambda,a,\mu)<\infty,\qquad N=2,3.$$

Hence in one-dimensional case ($N=1$) the condition (5) holds for an arbitrary λ but when $N=2,3$ the condition (5) does not need to be satisfied. Let us note that in the latter case the magnitude of λ is crucial for the occurrence of blow-up, i.e. if only λ is sufficiently large, then

$$\mathcal{K}(N,\lambda,a,\mu)>\max_{t\in(0,\infty)}\frac{t}{g(t)}. \tag{20}$$

Now an application of Theorems 3.1 and 4.1 allows us to formulate that dimensional influence on blow-up behaviour of Eq. (1) as the following result:

Theorem 6.3. *Let in Eq. (1) with kernel k_N given by (19) the nonlinearity g satisfies (2)–(4) and let the mapping $t\to\frac{t}{g(t)}$ be strictly increasing in some right neighbourhood of 0 and strictly decreasing in some neighbourhood of ∞.*

1. *For $N=1$ Eq. (1) has a blow-up solution if and only if when*

$$\int_0^\infty\frac{ds}{g(s)}<\infty.$$

2. *For $N = 2, 3$ Eq. (1) has a blow-up solution if and only if when*

$$\int_0^\infty \frac{ds}{g(s)} < \infty$$

provided that condition (20) is satisfied.

3. *For $N = 2, 3$ if condition (20) does not hold, then a blow-up solution of Eq. (1) does not exist.*

Moreover, in cases when blow-up solution exists, the blow-up time T could be estimated by (14) with $k_0 = \lambda$.

Example 6.4. In our second example we consider the equation

$$\frac{\partial}{\partial t}T(x,t) = v\frac{\partial}{\partial x_1}T(x,t) + \sum_{n=1}^{N}\frac{\partial^2}{\partial x_n^2}D_t^{1-\alpha}\big[T(x,t)\big] + \lambda D(x|0)g\big(T(0,t)\big), \tag{21}$$

where $x \in \mathbb{R}^N$, $N = 1, 2, 3$, $t > 0$, the parameter of subdiffusion $\lambda > 0$ and $v > 0$ is the constant advection speed associated with the x_1-direction, with the initial condition

$$T(x,0) = 0, \quad x \in \mathbb{R}^N, \tag{22}$$

and the boundary conditions

$$\lim_{|x|\to\infty} T(x,t) = 0, \quad t > 0. \tag{23}$$

The fractional derivative operator $D_t^{1-\alpha}$ in (21) is given by

$$D_t^{1-\alpha}\big[T(x,t)\big] = \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t (t-\tau)^{\alpha-1}T(x,\tau)\,d\tau, \tag{24}$$

$0 < \alpha < 1$, and the localization function $D(x|0)$ is defined in the same way as in the previous example. The problem (21)–(23) can serve [5] as a model of the subdiffusion with advection in the unbounded spatial domain of dimension N. In this case the given PDE problem (21)–(23) can be connected, via Green's function, with the integral equation of type (1) with $u(t) \equiv T(0,t)$ and $k(t) = k_N(t)$, where

$$k_N(t) = \frac{\lambda}{2\sqrt{\pi}}\int_0^\infty \frac{f_\alpha(z)}{\sqrt{t^\alpha z}}\left(\int_{-a}^{a}\exp\left(-\frac{(s-vt^\alpha z)^2}{4t^\alpha z}\right)ds\right)\left(\operatorname{erf}\left(\frac{a}{2\sqrt{t^\alpha z}}\right)\right)^{N-1}dz. \tag{25}$$

In (25) the function f_α is defined as follows:

$$f_\alpha(z) = \sum_{j=0}^{\infty}\frac{(-1)^j z^j}{j!\,\Gamma(1-\alpha-\alpha j)}, \quad z \geqslant 0. \tag{26}$$

It can be shown that kernel k_N above is the nonincreasing and positive function (nonnegativity of f_α (see [11]) plays a crucial role in showing these properties for that kernel) with $k_N(t) \sim \lambda$ as $t \to 0^+$, so this kernel belongs to the class of kernels considered in this paper. Hence we only need to check if condition (5) is valid to know when the blow-up solution to (1) exists and because any blow-up solution of (21)–(23) is associated with the blow-up solution of Eq. (1), we would know then also when the subdiffusion with advection PDE problem possesses the blow-up solutions. In order to do that we use the asymptotic behaviour of kernel k_N:

$$k_N(t) \sim \frac{\lambda\mathcal{C}(N,a,v)}{\Gamma(1-\alpha)}t^{-\alpha} \quad \text{as } t \to \infty,$$

where

$$0 < \mathcal{C}(N,a,v) \leqslant \frac{2a}{v},$$

what implies that

$$K_N(t) \sim \frac{\lambda\mathcal{C}(N,a,v)}{\Gamma(2-\alpha)}t^{1-\alpha} \quad \text{as } t \to \infty. \tag{27}$$

From (27) it follows immediately that $\lim_{t\to\infty} K_N(t) = \infty$, so on the basis of Theorem 4.1 we just proved the following result:

Theorem 6.5. *Let in Eq. (1) with kernel k_N given by (25) the nonlinearity g satisfies (2)–(4) and let the mapping $t \to \frac{t}{g(t)}$ be strictly increasing in some right neighbourhood of 0 and strictly decreasing in some neighbourhood of ∞. Then Eq. (1) has a blow-up solution if and only if when*

$$\int_0^\infty \frac{ds}{g(s)} < \infty. \tag{28}$$

Moreover, the blow-up time T could be estimated by (14) with $k_0 = \lambda$.

Remark 6.6. It is very interesting that one can show that for the classical diffusion with advection problem [5] an analogue of Theorem 6.5 is not valid, i.e. it can happen that blow-up solution does not exist even if condition (28) is satisfied. This is due to fact that kernel

$$k_N(t) = \frac{\lambda}{2\sqrt{\pi t}} \left(\int_{-a}^{a} \exp\left(-\frac{(s - vt)^2}{4t} \right) ds \right) \left(\operatorname{erf}\left(\frac{a}{2\sqrt{t}} \right) \right)^{N-1} \tag{29}$$

of respective Volterra integral equation in this case does not necessarily satisfy condition (5).

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