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Bishop–Phelps–Bollobás property for certain spaces of operators ☆

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ABSTRACT

We characterize the Banach spaces Y for which certain subspaces of operators from $L_1(\mu)$ into Y have the Bishop–Phelps–Bollobás property in terms of a geometric property of Y, namely AHSP. This characterization applies to the spaces of compact and weakly compact operators. New examples of Banach spaces Y with AHSP are provided. We also obtain that certain ideals of Asplund operators satisfy the Bishop–Phelps–Bollobás property.

1. Introduction

E. Bishop and R. Phelps in [6] proved that every continuous linear functional x^* on a Banach space X can be uniformly approximated on the closed unit ball of X by a continuous linear functional y^* that attains its norm. This result is called the Bishop–Phelps Theorem. Shortly thereafter, B. Bollobás [7] showed that this approximation can be obtained with the additional property that the point at which x^* almost attains its norm is close in norm to a point at which y^* attains its norm. This is a "quantitative version" of the Bishop–Phelps Theorem, known as the Bishop–Phelps–Bollobás Theorem. Throughout the paper, X and Y will be Banach spaces over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). As usual, S_X, B_X and X^* will denote the unit sphere, the closed unit ball, and the (topological) dual of X, respectively.

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Theorem 1.1 *(Bishop–Phelps–Bollobás Theorem). (See [8, Theorem 16.1].) Let X be a Banach space and $0 < \varepsilon < 1$. Given $x \in B_X$ and $x^* \in S_{X^*}$ with $|1 - x^*(x)| < \frac{\varepsilon^2}{4}$, there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y) = 1$, $\|y - x\| < \varepsilon$ and $\|y^* - x^*\| < \varepsilon$.*

In what follows K will be a compact Hausdorff space and μ will be a σ-finite measure. Different versions of the Bishop–Phelps–Bollobás Theorem for operators were proved in [1]. Amongst them it is shown a characterization of the Banach spaces Y satisfying an analogous result to the Bishop–Phelps–Bollobás Theorem for operators from ℓ_1 into Y. There are also positive results for operators from $L_1(\mu)$ into $L_\infty[0,1]$ [4,10] and for operators from an Asplund space into $\mathcal{C}(K)$ [3]. For some more results on the subject see also [9,19,20].

Our aim in this paper is to provide classes of spaces satisfying a version of the Bishop–Phelps–Bollobás Theorem for operators. By $\mathcal{L}(X,Y)$ we denote the Banach space of bounded linear operators from X into Y. Before going on, we need the following definitions.

The next property was introduced in [1].

Definition 1.2. Let X and Y be both real or complex Banach spaces. The pair (X,Y) satisfies the *Bishop–Phelps–Bollobás property for operators* if given $\varepsilon > 0$, there are $\eta(\varepsilon) > 0$ and $\beta(\varepsilon) > 0$ with $\lim_{t\to 0}\beta(t) = 0$ such that for any $T \in S_{\mathcal{L}(X,Y)}$, if $x_0 \in S_X$ is such that $\|Tx_0\| > 1 - \eta(\varepsilon)$, then there exist a point $u_0 \in S_X$ and an operator $S \in S_{\mathcal{L}(X,Y)}$ that satisfy the following conditions:

$$\|Su_0\| = 1, \qquad \|u_0 - x_0\| < \beta(\varepsilon) \quad \text{and} \quad \|S - T\| < \varepsilon.$$

In this case, we also say that the space $\mathcal{L}(X,Y)$ has the Bishop–Phelps–Bollobás property.

When the operator T (in the definition above) belongs to a certain class, we expect that S also belongs to the same class. Therefore we introduce the following notion.

Definition 1.3. Let X and Y be both real or complex Banach spaces and M a subspace of $\mathcal{L}(X,Y)$. We say that M satisfies the *Bishop–Phelps–Bollobás property* if given $\varepsilon > 0$, there is $\eta(\varepsilon) > 0$ such that for any $T \in S_M$, if $x_0 \in S_X$ satisfies that $\|Tx_0\| > 1 - \eta(\varepsilon)$, then there exist a point $u_0 \in S_X$ and an operator $S \in S_M$ satisfying the following conditions:

$$\|Su_0\| = 1, \qquad \|u_0 - x_0\| < \varepsilon \quad \text{and} \quad \|S - T\| < \varepsilon.$$

Remark 1.4. The above definition can be reformulated as follows. Given $\varepsilon > 0$, there are $\eta(\varepsilon) > 0$ and $\beta(\varepsilon) > 0$ with $\lim_{t\to 0}\beta(t) = 0$ such that for any $T \in S_M$, if $x_0 \in S_X$ satisfies that $\|Tx_0\| > 1 - \eta(\varepsilon)$, then there exist a point $u_0 \in S_X$ and an operator $S \in S_M$ satisfying the following conditions:

$$\|Su_0\| = 1, \qquad \|u_0 - x_0\| < \beta(\varepsilon) \quad \text{and} \quad \|S - T\| < \beta(\varepsilon).$$

Notice that if $M = \mathcal{L}(X,Y)$, Definitions 1.2 and 1.3 are equivalent.

To study the Bishop–Phelps–Bollobás property for operators on ℓ_1, the following geometric property was introduced in [1, Definition 3.1].

Definition 1.5. A Banach space X has the *approximate hyperplane series property* (AHSP) if for every $\varepsilon > 0$ there exist $\gamma(\varepsilon) > 0$ and $\eta(\varepsilon) > 0$ with $\lim_{t\to 0^+}\gamma(t) = 0$ such that for every sequence $(x_k) \subset S_X$ (or $(x_k) \subset B_X$) and every convex series $\sum_{k\geqslant 1}\alpha_k$ satisfying

$$\left\|\sum_{k=1}^{\infty}\alpha_k x_k\right\| > 1 - \eta(\varepsilon),$$

there exist a subset $D \subset \mathbb{N}$, $\{z_k\colon\ k \in D\} \subset S_X$ and $x^* \in S_{X^*}$ such that

(i) $\sum_{k\in D} \alpha_k > 1 - \gamma(\varepsilon)$,
(ii) $\|z_k - x_k\| < \varepsilon$ for all $k \in D$,
(iii) $x^*(z_k) = 1$ for all $k \in D$.

Note that X has AHSP if whenever we have a convex series of vectors in B_X whose norm is very close to 1, then a preponderance of these vectors are uniformly close to unit vectors that lie in the same affine hyperplane. For instance, finite-dimensional spaces, uniformly convex spaces, $\mathcal{C}(K)$ and $L_1(\mu)$ have AHSP [1, §3].

The outline of the paper is as follows. In Section 2, we characterize the Banach spaces Y such that certain subspaces of operators from $L_1(\mu)$ into Y satisfy the Bishop–Phelps–Bollobás property. As a consequence, we show that the following conditions are equivalent:

(1) Y satisfies AHSP.
(2) $\mathcal{F}(L_1(\mu), Y)$ (finite-rank operators) has the Bishop–Phelps–Bollobás property.
(3) $\mathcal{K}(L_1(\mu), Y)$ (compact operators) has the Bishop–Phelps–Bollobás property.
(4) $\mathcal{W}(L_1(\mu), Y)$ (weakly compact operators) has the Bishop–Phelps–Bollobás property.
(5) $\mathcal{RN}(L_1(\mu), Y)$ (Radon–Nikodým operators) has the Bishop–Phelps–Bollobás property.

We also deal with the Bishop–Phelps–Bollobás property for Asplund operators. In Section 3, we extend Theorem 2.4 and Corollary 2.5 of [3] to some spaces of vector valued continuous functions. As a consequence, we obtain new spaces of operators satisfying the Bishop–Phelps–Bollobás property. We prove that the pairs $(X, \mathcal{K}(Y, \mathcal{C}(K)))$, $(X, \mathcal{W}(Y, \mathcal{C}(K)))$, and $(X, \mathcal{L}(Y, \mathcal{C}(K)))$ satisfy the Bishop–Phelps–Bollobás property if X is an Asplund space and Y has property α of Schachermayer [23] (for instance $Y = \ell_1$). Finally, new examples of spaces having AHSP are provided in Section 4, for instance $\mathcal{K}(X, \mathcal{C}(K))$ and $\mathcal{L}(X, \mathcal{C}(K))$ whenever X is uniformly smooth.

2. Bishop–Phelps–Bollobás property for the space of Radon–Nikodým operators

It will be convenient to begin by recalling a few definitions and results related to Radon–Nikodým operators. Let (Ω, Σ, μ) be a finite measure space. A bounded linear operator $T : L_1(\mu) \to Y$ is said to be *representable* if there exists $g \in S_{L_\infty(\mu,Y)}$ such that

$$T(f) = \int_\Omega g f \, d\mu \quad \text{for all } f \in L_1(\mu)$$

(see [14, p. 61] or [16, Definition 5.5.15]).

We recall that a *Radon–Nikodým operator* is an operator $T : X \to Y$ such that TS is representable for every operator $S : L_1(\mu) \to X$ (see [16, Definition 5.5.12 and Theorem 5.5.19]). A bounded operator $T : L_1(\mu) \to Y$ is representable if and only if T is a Radon–Nikodým operator (see [16, Proposition 5.5.18]). Also, a Banach space Y has the Radon–Nikodým property if and only if every operator $T : L_1(\mu) \to Y$ is a Radon–Nikodým operator (see [16, Proposition 5.5.16]).

Following [13, Definition 9.1], an *operator ideal* $\mathcal{I}$ is a subclass of the class $\mathcal{L}$ such that for any pair of Banach spaces (X, Y), $\mathcal{I}(X, Y)$ is a subspace of $\mathcal{L}(X, Y)$ which contains the finite rank operators and satisfies the so-called "ideal property". That is, given arbitrary Banach spaces X_0, Y_0, we have $R \circ S \circ T \in \mathcal{I}(X, Y)$ for any S in $\mathcal{I}(X_0, Y_0)$, T in $\mathcal{L}(X, X_0)$, and R in $\mathcal{L}(Y_0, Y)$, and for every Banach spaces X and Y. The operator ideal $\mathcal{I}$ is said to be closed if the subspace $\mathcal{I}(X,Y)$ is closed in $\mathcal{L}(X,Y)$ for all Banach spaces X and Y.

As mentioned above, we denote by $\mathcal{RN}$ the closed operator ideal of all *Radon–Nikodým operators*. Also we have $\mathcal{F} \subseteq \mathcal{K} \subseteq \mathcal{W} \subseteq \mathcal{RN}$ (see [16, Proposition 5.5.20]).

The elementary result below will be useful in the sequel.

Lemma 2.1. *(See [1, Lemma 3.3].) Let (c_n) be a sequence of complex numbers with $|c_n| \leqslant 1$ for every n, and let $\eta > 0$ be such that for some convex series $\sum_n \alpha_n$, $\operatorname{Re}\sum_{n=1}^{\infty} \alpha_n c_n > 1-\eta$. Then for every $0<r<1$, the set $D := \{i \in \mathbb{N}\colon \operatorname{Re} c_i > r\}$, satisfies the estimate*

$$\sum_{i\in D} \alpha_i \geqslant 1 - \frac{\eta}{1-r}.$$

The following result is a refinement of [10, Theorem 2.1].

Proposition 2.2. *Let (Ω,Σ,μ) be a measure space such that $L_1(\mu)$ is infinite-dimensional, Y a Banach space, and M a subspace of $\mathcal{L}(L_1(\mu),Y)$ containing all finite-rank operators. If M has the Bishop–Phelps–Bollobás property, then Y has AHSP.*

Proof. For every $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ satisfying Definition 1.3.

Now, given $0 < \varepsilon < \frac{1}{9}$, we will prove that Y satisfies AHSP for the functions $\bar{\eta}(\varepsilon) = \min\{\eta(\varepsilon^3), \varepsilon\}$ and γ given by

$$\gamma(\varepsilon) := 8\varepsilon(1-\varepsilon) + \varepsilon + \varepsilon^3(1-\varepsilon). \tag{2.1}$$

It is clear that $\gamma(\varepsilon) > 0$ and $\lim_{\varepsilon\to 0}\gamma(\varepsilon) = 0$ as it is required in Definition 1.5.

Let (y_n) be a sequence in S_Y and a convex series $\sum_n \alpha_n$ satisfying

$$\left\| \sum_{n=1}^{\infty} \alpha_n y_n \right\| > 1 - \bar{\eta}(\varepsilon).$$

Fix N such that

$$\left\| \sum_{n=1}^{N} \alpha_n y_n \right\| > 1 - \bar{\eta}(\varepsilon) \geqslant 1-\varepsilon > 0. \tag{2.2}$$

If we write $\tilde{\alpha}_n = \frac{\alpha_n}{\sum_{k=1}^N \alpha_k}$ then

$$\left\| \sum_{k=1}^{N} \tilde{\alpha}_k y_k \right\| \geqslant \left\| \sum_{k=1}^{N} \alpha_k y_k \right\| > 1 - \eta\bigl(\varepsilon^3\bigr) \quad \text{and} \quad \sum_{k=1}^{N} \tilde{\alpha}_k = 1. \tag{2.3}$$

By assumption, there is a sequence (E_n) of pairwise disjoint subsets in Σ satisfying $0 < \mu(E_n) < \infty$ for each n. For every positive integer n, let x_n^* be the functional on $L_1(\mu)$ associated to χ_{E_n}, that is,

$$x_n^*(f) := \int_{E_n} f\, d\mu \quad \bigl(f \in L_1(\mu)\bigr).$$

Now we define the finite-rank operator (therefore in M) $T : L_1(\mu)\to Y$ by

$$T(f) = \sum_{k=1}^{N} x_k^*(f) y_k \quad \bigl(f \in L_1(\mu)\bigr).$$

Note that $\|T\| \leqslant 1$ and $\|T(\chi_{E_k})\| = \|\chi_{E_k}\|_1$ for all $k \leqslant N$, then $T \in S_M$.

Define $f_0 := \sum_{k=1}^{N} \tilde{\alpha}_k \frac{\chi_{E_k}}{\mu(E_k)}$. By (2.3), $\|f_0\|_1 = 1$ and $\|T(f_0)\| = \|\sum_{k=1}^{N} \tilde{\alpha}_k y_k\| > 1 - \eta(\varepsilon^3)$.

Since M has the Bishop–Phelps–Bollobás property, there exist $g_0 \in S_{L_1(\mu)}$ and $S \in S_M$ satisfying

$$\|Sg_0\| = 1, \qquad \|g_0 - f_0\|_1 < \varepsilon^3 \quad \text{and} \quad \|S - T\| < \varepsilon^3. \tag{2.4}$$

Proceeding as in [10, Theorem 2.1] we obtain

$$\sum_{k=1}^{N} \operatorname{Re} x_k^*(g_0) > 1 - \varepsilon^3. \tag{2.5}$$

Let $s = 1 - \frac{\varepsilon^2}{8}$ and $D := \{k \in \mathbb{N}\colon\ k \leqslant N,\ \operatorname{Re} x_k^*(g_0) > s x_k^*(|g_0|)\}$. By (2.5) and following the proof of [10, Theorem 2.1] we obtain

$$\sum_{k\in D} \operatorname{Re} x_k^*(g_0) > 1 - \frac{\varepsilon^3}{1-s} = 1 - 8\varepsilon > 0. \tag{2.6}$$

Thus $D \neq \emptyset$.

Combining (2.6) and (2.4) and using $\varepsilon < \frac{1}{9}$ we deduce that

$$\sum_{k\in D} \tilde{\alpha}_k \geqslant \sum_{k\in D} \operatorname{Re} x_k^*(g_0) - \|g_0 - f_0\|_1 > 1 - 8\varepsilon - \|g_0 - f_0\|_1 > 1 - 8\varepsilon - \varepsilon^3 > 0.$$

By (2.2) and the previous inequality

$$\sum_{k\in D} \alpha_k = \Bigl(\sum_{k\in D} \tilde{\alpha}_k\Bigr)\Bigl(\sum_{k=1}^{N} \alpha_k\Bigr) > \bigl(1 - 8\varepsilon - \varepsilon^3\bigr)\bigl(1 - \bar{\eta}(\varepsilon)\bigr) \geqslant \bigl(1 - 8\varepsilon - \varepsilon^3\bigr)(1 - \varepsilon) = 1 - \gamma(\varepsilon).$$

Therefore, condition (i) of Definition 1.5 is satisfied. Now, note that for a complex number w with $|w| \leqslant 1$ and $\operatorname{Re} w > r > 0$ it is satisfied $|1 - w|^2 = 1 + |w|^2 - 2\operatorname{Re} w < 2(1 - r)$. So for every $k \in D$ we have

$$\left|1 - \frac{x_k^*(g_0)}{x_k^*(|g_0|)}\right|^2 < 2(1 - s) = \frac{\varepsilon^2}{4}. \tag{2.7}$$

For $k \in \mathbb{N}$ we define $z_k = S(\frac{g_0 \chi_{E_k}}{x_k^*(|g_0|)})$ if $x_k^*(|g_0|) \neq 0$ and 0 otherwise. In particular, $\|z_k\| \leqslant 1$ for every k. We write $\Omega_1 = \Omega \setminus \bigcup_{k=1}^{\infty} E_k$. Let us notice that $g_0 = \sum_{k=1}^{\infty} g_0 \chi_{E_k} + g_0 \chi_{\Omega_1}$ and the series is norm convergent. Then

$$S(g_0) = \sum_{k=1}^{\infty} S(g_0 \chi_{E_k}) + S(g_0 \chi_{\Omega_1}) = \sum_{k=1}^{\infty} x_k^*\bigl(|g_0|\bigr) z_k + S(g_0 \chi_{\Omega_1}).$$

By the Hahn–Banach Theorem, there is a functional $y^* \in S_{Y^*}$ attaining its norm at $S(g_0)$. Then

$$1 = y^*\bigl(S(g_0)\bigr) = \sum_{k=1}^{\infty} x_k^*\bigl(|g_0|\bigr) y^*(z_k) + y^*\bigl(S(g_0 \chi_{\Omega_1})\bigr) \leqslant \sum_{k=1}^{\infty} \Bigl(\int_{E_k} |g_0|\, d\mu\Bigr) + \|g_0 \chi_{\Omega_1}\| = \|g_0\|_1 = 1.$$

Therefore

$$y^*(z_k) = 1 \quad \text{for all } k \in \mathbb{N} \text{ with } x_k^*\big(|g_0|\big) \neq 0.$$

In particular, $z_k \in S_Y$ for $k \in D$ and condition (iii) of Definition 1.5 is also satisfied.

Now for every $k \in D$ we have that $x_k^*(g_0) \neq 0$ and $T(\frac{g_0 \chi_{E_k}}{x_k^*(g_0)}) = y_k$. Hence by (2.4) for every $k \in D$ we deduce that

$$\left\| z_k - \frac{x_k^*(g_0)}{x_k^*(|g_0|)} y_k \right\| = \left\| S\left(\frac{g_0 \chi_{E_k}}{x_k^*(|g_0|)}\right) - T\left(\frac{g_0 \chi_{E_k}}{x_k^*(|g_0|)}\right) \right\| \leqslant \|S - T\| < \varepsilon^3.$$

Finally, by (2.7), for every $k \in D$ we obtain

$$\|z_k - y_k\| \leqslant \left\| z_k - \frac{x_k^*(g_0)}{x_k^*(|g_0|)} y_k \right\| + \left\| \left(\frac{x_k^*(g_0)}{x_k^*(|g_0|)} - 1\right) y_k \right\| \leqslant \varepsilon^3 + \frac{\varepsilon}{2} < \varepsilon,$$

and Y has AHSP. □

Improving [10, Theorem 2.2], we give a partial converse of Proposition 2.2.

Theorem 2.3. *Let (Ω, Σ, μ) be a finite measure space, Y a Banach space with AHSP and M a subspace of $\mathcal{L}(L_1(\mu), Y)$ such that contains all finite-rank operators and it is contained in the subspace of all representable operators. Also, assume that the operator $S_A(f) = S(f\chi_A)$ belongs to M whenever $S \in M$ and A is any measurable subset of Ω. Then M has the Bishop–Phelps–Bollobás property for operators.*

Proof. By assumption Y has AHSP; let γ and η be the functions satisfying Definition 1.5. Given $0 < \varepsilon < 1$, we choose $0 < \delta < \frac{\varepsilon}{6}$ such that $0 < \gamma(\delta) < \frac{\varepsilon}{6}$ and $0 < \delta' < \min\{\frac{\varepsilon}{6}, \frac{\eta(\delta)}{4}\}$. Define $\rho(\varepsilon) := \frac{\eta(\delta)}{2}$ and assume that $T \in S_M$ and $f_0 \in S_{L_1(\mu)}$ satisfy that $\|Tf_0\| > 1 - \rho(\varepsilon)$. There is a function $h \in L_\infty(\mu)$ such that $|h(t)| = 1$ for every $t \in \Omega$ and satisfying also that $h(t)f_0(t) = |f_0(t)|$ for every $t \in \Omega$. Now we define a surjective linear isometry $\psi : L_1(\mu) \to L_1(\mu)$ given by

$$\psi(f) = hf \quad \big(f \in L_1(\mu)\big),$$

that satisfies $\psi(f_0)(t) \in \mathbb{R}_0^+$ for every $t \in \Omega$.

We write $R = T\psi^{-1}$ and $u_0 = \psi(f_0)$. Clearly, we have $\|R(u_0)\| = \|T(f_0)\| > 1 - \rho(\varepsilon)$, with $u_0 \in S_{L_1(\mu)}$ nonnegative and $R \in S_{\mathcal{L}(L_1(\mu), Y)}$.

Since T is a representable operator, R is also representable. So there is $g \in L_\infty(\mu, Y)$ such that

$$R(f) = \int\limits_\Omega gf \, d\mu \quad \text{for all } f \in L_1(\mu).$$

By [14, Lemma III.1.4], g also satisfies that $\|g\|_\infty = \|R\| = 1$. By [14, Corollary II.1.3], there exist a measurable function $h : \Omega \to Y$, whose range is countable, and a μ-null subset E of Ω such that $\|(g-h)\chi_{\Omega \setminus E}\|_\infty < \frac{\varepsilon}{4}$. Write $h = \sum_{n=1}^\infty \chi_{B_n} w_n$ (pointwise convergence) with $(w_n) \subset Y$ and (B_n) a sequence of pairwise disjoint measurable sets of Ω with $\bigcup_n B_n = \Omega$. Hence, fixed $n \in \mathbb{N}$ and $s, t \in B_n \setminus E$ we have

$$\big\|g(s) - g(t)\big\| \leqslant \big\|g(s) - h(s)\big\| + \big\|h(s) - h(t)\big\| + \big\|h(t) - g(t)\big\| < \frac{\varepsilon}{2}.$$

Both functions g and $g\chi_{\Omega \setminus E}$ represent R, then we may assume that

$$\big\|g(s) - g(t)\big\| < \frac{\varepsilon}{2} \quad \text{for all } s, t \in B_n \text{ and } n \in \mathbb{N}. \tag{2.8}$$

By the Monotone Convergence Theorem the sequence $(u_0\chi_{\bigcup_{k=1}^n B_k})$ converges to u_0 in $L_1(\mu)$. Since $1-\rho(\varepsilon)<\|R(u_0)\|$, for some m large enough we have

$$1-\rho(\varepsilon)<\big\|R(u_0\chi_{\bigcup_{k=1}^m B_k})\big\| \quad \text{and} \quad \|u_0-u_0\chi_{\bigcup_{k=1}^m B_k}\|<\delta'. \tag{2.9}$$

We write $B=\bigcup_{i=1}^m B_k$. Since u_0 is a non-negative function in $S_{L_1(\mu)}$, there is a non-negative simple function v_0 in $B_{L_1(\mu)}$ with support contained in B satisfying $\|v_0-u_0\chi_B\|<\delta'$ and $\|v_0\|=\|u_0\chi_B\|$ and so $0<1-\delta'\leqslant\|v_0\|\leqslant 1$. The element $s_0=\frac{v_0}{\|v_0\|}$ belongs to $S_{L_1(\mu)}$. Its support is contained in B and also satisfies that

$$\begin{aligned}\|s_0-u_0\chi_B\|&\leqslant\|s_0-v_0\|+\|v_0-u_0\chi_B\|=1-\|v_0\|+\|v_0-u_0\chi_B\|<2\delta'\\&<\min\left\{\frac{\varepsilon}{3},\frac{\eta(\delta)}{2}\right\}.\end{aligned} \tag{2.10}$$

Hence, there is a finite number of pairwise disjoint measurable sets in B, $\{A_1,\dots,A_N\}$, such that s_0 belongs to the space generated by $\{\chi_{A_i}\colon 1\leqslant i\leqslant N\}$.

Let $\{C_i\colon 1\leqslant i\leqslant p\}$ be the family of pairwise disjoint measurable subsets obtained by indexing the set $\{A_i\cap B_j\colon 1\leqslant i\leqslant N,\ 1\leqslant j\leqslant m,\ \mu(A_i\cap B_j)>0\}$. Write $s_0=\sum_{k=1}^p\beta_k\chi_{C_k}$ with $\beta_k\geqslant 0$ and $\sum_{k=1}^p\beta_k\mu(C_k)=\|s_0\|=1$.

From (2.9) and (2.10) we obtain that

$$1-\eta(\delta)=1-\rho(\varepsilon)-\frac{\eta(\delta)}{2}<\big\|R(u_0\chi_B)\big\|-\frac{\eta(\delta)}{2}<\big\|R(s_0)\big\|=\left\|\sum_{k=1}^p\beta_k\mu(C_k)R\left(\frac{\chi_{C_k}}{\mu(C_k)}\right)\right\|.$$

Since $R\in S_{\mathcal{L}(L_1(\mu),Y)}$, $y_k=R(\frac{\chi_{C_k}}{\mu(C_k)})\in B_Y$ for $1\leqslant k\leqslant p$ and

$$1-\eta(\delta)<\left\|\sum_{k=1}^p\beta_k\mu(C_k)y_k\right\|. \tag{2.11}$$

Observe that by (2.8), for every $k\leqslant p$ and $t\in C_k$ we have that

$$\big\|g(t)-y_k\chi_{C_k}(t)\big\|=\left\|\int_{C_k}\frac{g(t)}{\mu(C_k)}\,d\mu(u)-\int_{C_k}\frac{g(u)}{\mu(C_k)}\,d\mu(u)\right\|\leqslant\int_{C_k}\frac{\|g(t)-g(u)\|}{\mu(C_k)}\,d\mu(u)\leqslant\frac{\varepsilon}{2}. \tag{2.12}$$

Since Y has AHSP and $\sum_{k=1}^p\beta_k\mu(C_k)=1$, by (2.11), there are sets $D\subset\{1,\dots,p\}$, $\{z_k\colon k\in D\}\subset S_Y$ and $y^*\in S_{Y^*}$ satisfying

$$y^*(z_k)=1,\qquad \|z_k-y_k\|<\delta\quad\text{for all }k\in D\quad\text{and}\quad\sum_{k\in D}\beta_k\mu(C_k)>1-\gamma(\delta)>0. \tag{2.13}$$

Now define the function $g_1:\Omega\to Y$ given by $g_1=g\chi_{\Omega\setminus C}+\sum_{k\in D}z_k\chi_{C_k}$, where $C=\bigcup_{k\in D}C_k$. It is clear that $g_1\in B_{L_\infty(\mu,Y)}$. By (2.12) and (2.13), we have

$$\|g_1-g\|_\infty=\big\|(g_1-g)\chi_C\big\|_\infty<\delta+\frac{\varepsilon}{2}<\varepsilon.$$

Let R_1 be the element in $\mathcal{L}(L_1(\mu),Y)$ associated to g_1. Then $\|R_1\|\leqslant 1$ and

$$\|R_1-R\|=\|g_1-g\|_\infty<\varepsilon. \tag{2.14}$$

Let $s_1 = \sum_{k\in D} \beta_k \chi_{C_k}$, which by (2.13) is nonzero and satisfies

$$\|s_1\| = \sum_{k\in D} \beta_k \mu(C_k) = y^*\left(\sum_{k\in D} \beta_k \mu(C_k) z_k\right) = y^*\big(R_1(s_1)\big) \leqslant \|y^*\|\|R_1\|\|s_1\| = \|s_1\|.$$

Then, $\|R_1\| \leqslant 1$ and R_1 attains its norm at $s_2 = \frac{s_1}{\|s_1\|}$. By (2.9), (2.10) and (2.13) we have

$$\begin{aligned}\|s_2 - u_0\| &\leqslant \left\|\frac{s_1}{\|s_1\|} - s_1\right\| + \|s_1 - s_0\| + \|s_0 - u_0\| = \big|1 - \|s_1\|\big| + \sum_{k\leqslant p,\, k\notin D} \beta_k \mu(C_k) + \|s_0 - u_0\| \\ &\leqslant 2 \sum_{k\leqslant p,\, k\notin D} \beta_k \mu(C_k) + \|s_0 - u_0\chi_B\| + \|u_0\chi_B - u_0\| \leqslant 2\gamma(\delta) + \frac{\varepsilon}{3} + \delta' < \varepsilon.\end{aligned}$$

Now, define $T_1 = R_1\psi$ and $f_2 = \psi^{-1}s_2$. Since ψ is an isometry, $T_1 \in S_{\mathcal{L}(L_1(\mu),Y)}$, $f_2 \in S_{L_1(\mu)}$ and T_1 attains its norm at f_2. By (2.14), $\|T_1 - T\| = \|R_1 - R\| < \varepsilon$, also $\|f_2 - f_0\| < \varepsilon$.

Let us notice that $R_1 - R$ is the operator associated to the function $g_1 - g$. Hence, for every $f \in L_1(\mu)$ we have

$$(R_1 - R)(f) = (R_1 - R)(f\chi_C) = \sum_{k\in D}\left(\int_{C_k} f\, d\mu\right) z_k - R(f\chi_C) = S(f) - R_C(f),$$

where $R_C(f) = R(f\chi_C)$ and S is the finite-rank operator given by $S(f) = \sum_{k\in D}(\int_{C_k} f\, d\mu) z_k$. Hence

$$T_1 - T = (R_1 - R)\psi = (S - R_C)\psi. \tag{2.15}$$

To show that $T_1 \in M$ note that

$$R_C\big(\psi(f)\big) = R_C(hf) = R(hf\chi_C) = T\big(\psi^{-1}(hf\chi_C)\big) = T(\bar{h}hf\chi_C) = T_C(f),$$

where $\bar{h}$ stands for the conjugate of h.

Now, the hypothesis on M implies that $R_C \circ \psi$ also belongs to M. On the other hand, M contains all finite-rank operators, thus (2.15) gives that T_1 is in M. Therefore M has the Bishop–Phelps–Bollobás property. □

As a consequence of Theorem 2.3, if $\mathcal{I}$ is an operator ideal such that $\mathcal{I}(L_1(\mu), Y) \subset \mathcal{RN}(L_1(\mu), Y)$, Y has AHSP and μ is any finite measure, then the space $\mathcal{I}(L_1(\mu), Y)$ satisfies the Bishop–Phelps–Bollobás property. By Proposition 2.2, we deduce the following:

Corollary 2.4. *Let Y be a Banach space and (Ω, Σ, μ) a finite measure space such that $L_1(\mu)$ is infinite-dimensional. The following conditions are equivalent*:

(1) *Y satisfies AHSP.*
(2) *$\mathcal{F}(L_1(\mu), Y)$ has the Bishop–Phelps–Bollobás property.*
(3) *$\mathcal{K}(L_1(\mu), Y)$ has the Bishop–Phelps–Bollobás property.*
(4) *$\mathcal{W}(L_1(\mu), Y)$ has the Bishop–Phelps–Bollobás property.*
(5) *$\mathcal{RN}(L_1(\mu), Y)$ has the Bishop–Phelps–Bollobás property.*

There are very different Banach spaces having AHSP. For instance, finite-dimensional spaces, uniformly convex spaces, $\mathcal{C}(K)$, $L_1(\mu)$ (μ σ-finite) and $\mathcal{K}(H)^*$ ($\mathcal{K}(H)$ = compact operators on a Hilbert space) satisfy this property (see [1, §3] and [2, Proposition 4.7]). Also every lush space has AHSP [11] (see also [9]). We will provide later some examples of spaces of operators satisfying AHSP.

Remark 2.5. Theorem 2.3 (and hence Corollary 2.4) actually holds whenever μ is a σ-finite measure. To obtain this let us notice that every element g in $L_\infty(\mu, Y)$ is the almost μ-everywhere pointwise limit of a sequence of μ-measurable and countably valued functions. So the range of g is essentially separable and almost μ-everywhere coincides with a Borel measurable function. Hence the classical Lebesgue's horizontal approximation method can be applied to show that every element in $L_\infty(\mu, Y)$ is the μ-almost everywhere uniform limit of countably valued measurable functions that are bounded. The proof of Theorem 2.3 goes exactly along the same lines by using this fact and standard techniques in case that μ is σ-finite.

Theorem 2.3 applies not only to operator ideals, as the next remark shows.

Remark 2.6. There is a closed subspace $M \subset \mathcal{L}(L_1(\mu), Y)$ satisfying the hypothesis of Theorem 2.3 which is not an operator ideal. Let Y be any infinite dimensional Banach space and $L_1(\mu) = L_1[-1, 1]$. Let $M \subset \mathcal{L}(L_1(\mu), Y)$ be the linear space given by

$$M = \{K + R\colon\ K \in \mathcal{K}\big(L_1(\mu), Y\big),\ R \in \mathcal{RN}\big(L_1(\mu), Y\big) \text{ and } R = R_{[0,1]}\},$$

where $R_C(f) = R(f\chi_C)$ for any $f \in L_1(\mu)$ and any measurable subset $C \subset [-1, 1]$. Note that $T_{[-1,0]}$ is a compact operator for any T in M. Let $\phi(t) = -t$ and $C_\phi\colon L_1(\mu) \to L_1(\mu)$ be the surjective linear isometry defined by $C_\phi(f) = f \circ \phi$. Take $T \in \mathcal{RN}(L_1(\mu), Y) \setminus \mathcal{K}(L_1(\mu), Y)$. Since $(T \circ C_\phi)_{[-1,0]} = T_{[0,1]}$ is noncompact and, therefore, M satisfies all the requirements.

3. Bishop–Phelps–Bollobás property for the space of Asplund operators

We recall that an operator $T \in \mathcal{L}(X, Y)$ is said to be an *Asplund operator* if T^* is a Radon–Nikodým operator (see [16, Definition 5.5.22]). We denote by $\mathcal{A}$ the closed operator ideal of all Asplund operators.

A Banach space Y is said to have *property* β (of Lindenstrauss [21]) if there are two sets $\{y_\alpha\colon\ \alpha \in \Lambda\} \subset S_Y$, $\{y^*_\alpha\colon\ \alpha \in \Lambda\} \subset S_{Y^*}$ and $0 \leqslant \rho < 1$ such that the following conditions hold

(1) $y^*_\alpha(y_\alpha) = 1$,
(2) $|y^*_\alpha(y_\gamma)| \leqslant \rho < 1$ if $\alpha \neq \gamma$,
(3) $\|y\| = \sup\{|y^*_\alpha(y)|\colon\ \alpha \in \Lambda\}$, for all $y \in Y$.

Aron, Cascales and Kozhushkina in [3, Theorem 2.4 and Corollary 2.5] proved that $\mathcal{A}(X, C(K))$ has the Bishop–Phelps–Bollobás property. In this section we extend this result to some spaces of vector-valued continuous functions $\mathcal{C}(K, Y)$ (Theorem 3.1).

In general, it is known that not every operator into a $\mathcal{C}(K)$ space can be approximated by norm attaining operators (see [22, Theorem A] or [18, Corollary 2]). Moreover, in view of [5, Example 4.2], we have to introduce some restrictions on Y in order to get a positive result of Bishop–Phelps–Bollobás property for operators into $\mathcal{C}(K, Y)$.

We recall that a subspace Z of Y^* is said to be norming for Y, if for every $y \in Y$, we have $\|y\| = \sup\{|\phi(y)|\colon\ \phi \in B_Z\}$ for any $y \in Y$. We also say that a subset C of Y^* is 1-norming, if $\|y\| = \sup\{|\phi(y)|\colon\ \phi \in C\}$ for every $y \in Y$. We denote by $\sigma(Y, Z)$ the topology on Y of pointwise convergence on Z. If Z is any norming subspace for Y and τ is any linear topology on Y with $\sigma(Y, Z) \subset \tau \subset n$ where n is the norm topology then $C(K, (Y, \tau))$ is a Banach space with the norm induced by $\ell_\infty(K, Y)$. Also $C(K, (Y, \tau))$ is stable under products by elements of $C(K)$.

Theorem 3.1. *Let Y be a Banach space satisfying property β for the subset of functionals $\Delta = \{y_\alpha^*\colon\ \alpha \in \Lambda\}$ and Z the closed subspace of Y^* generated by Δ. Let τ be a linear topology on Y with $\sigma(Y,Z) \subseteq \tau \subseteq n$. Then for every closed operator ideal $\mathcal{I}$ such that $\mathcal{I} \subseteq \mathcal{A}$, we have that $\mathcal{I}(X, \mathcal{C}(K,(Y,\tau)))$ has the Bishop–Phelps–Bollobás property for every Banach space X and every compact Hausdorff topological space K.*

Proof. Let us fix T in the unit sphere of $\mathcal{I}(X, \mathcal{C}(K,(Y,\tau)))$, $0 < \varepsilon < 1$ and $x_0 \in S_X$ such that

$$\big\|T(x_0)\big\| > 1 - \frac{\varepsilon^2}{4}.$$

We will prove that there exist $u_0 \in S_X$ and R in the unit sphere of $\mathcal{I}(X, \mathcal{C}(K,(Y,\tau)))$ such that

$$\big\|R(u_0)\big\| = 1, \qquad \|u_0 - x_0\| < \varepsilon \quad \text{and} \quad \|R - T\| \leqslant \varepsilon\left(3 + \frac{8\rho}{1-\rho}\right),$$

where ρ is the constant appearing in the definition of property β.

Since Y has property β, the set

$$B := \big\{\delta_t \otimes y_\alpha^*\colon\ t \in K,\ \alpha \in \Lambda\big\}$$

is a 1-norming subset of $B_{\mathcal{C}(K,(Y,\tau))^*}$. By [3, Lemma 2.3] one can find a w^*-open subset U of X^* so that $U \cap T^*(B) \neq \emptyset$ and two elements $u_0 \in S_X$, $u_0^* \in S_{X^*}$ such that

$$u_0^*(u_0) = 1, \qquad \|u_0 - x_0\| < \varepsilon \quad \text{and} \quad \big\|x^* - u_0^*\big\| < 3\varepsilon \quad \text{for all } x^* \in U \cap T^*(B). \tag{3.1}$$

Since $U \cap T^*(B)$ is nonempty, we can find some $t_0 \in K$ and $\alpha_0 \in \Lambda$ such that $T^*(\delta_{t_0} \otimes y_{\alpha_0}^*) \in U$. Consider the set

$$W := \big\{t \in K\colon\ T^*\big(\delta_t \otimes y_{\alpha_0}^*\big) \in U\big\}$$

which is open and contains t_0.

By Urysohn's Lemma, there is a continuous function $f : K \to [0,1]$ whose support is contained in W such that $f(t_0) = 1$. Define the operator $S : X \to \mathcal{C}(K,(Y,\tau))$ by

$$S(x)(t) = T(x)(t) + \big((1+\eta)u_0^*(x) - T^*\big(\delta_t \otimes y_{\alpha_0}^*\big)(x)\big) f(t) y_{\alpha_0} \quad (x \in X,\ t \in K),$$

where $\eta = \frac{4\varepsilon\rho}{1-\rho}$. The operator S is clearly bounded and linear.

Our aim now is to show that S belongs to $\mathcal{I}(X, \mathcal{C}(K,(X,\tau)))$. In order to do that, we consider the bounded linear operators $R\ :\ X \to \mathcal{C}(K,(Y,\tau))$ and $F, M_f\ :\ \mathcal{C}(K,(Y,\tau)) \to \mathcal{C}(K,(Y,\tau))$ given by

$$R(x)(t) = (1+\eta)u_0^*(x) f(t) y_{\alpha_0} \quad (x \in X,\ t \in K),$$
$$M_f(g)(t) = f(t)g(t) \quad \text{and} \quad F(g)(t) = y_{\alpha_0}^*\big(g(t)\big) y_{\alpha_0} \quad \big(g \in \mathcal{C}\big(K,(Y,\tau)\big),\ t \in K\big).$$

It is clearly satisfied that $S = T + R - F \circ M_f \circ T$. Since $\mathcal{I}$ is an operator ideal we have that the rank-one operators R, $F \circ M_f \circ T$ and so S belong to $\mathcal{I}(X, \mathcal{C}(K,(Y,\tau)))$.

We will check that $\|S\| = \|S(u_0)\| = 1 + \eta$. Indeed, we have that

$$y_{\alpha_0}^*\big(S(u_0)(t_0)\big) = (1+\eta)u_0^*(u_0) = 1 + \eta. \tag{3.2}$$

On the one hand, for $t \in K \setminus W$ we know that $f(t) = 0$, so $S(x)(t) = T(x)(t)$, hence

$$\|S(x)(t)\| \leqslant 1 \quad \text{for all } x \in B_X. \tag{3.3}$$

On the other hand, if $t \in W$ we distinguish two cases to estimate $|y_\alpha^*(S(x)(t))|$.

For $\alpha = \alpha_0$ we obtain that

$$\begin{aligned} \left|y_{\alpha_0}^*\big(S(x)(t)\big)\right| &= \left|y_{\alpha_0}^*\big(T(x)(t)\big) + \big((1+\eta)u_0^*(x) - T^*\big(\delta_t \otimes y_{\alpha_0}^*\big)(x)\big)f(t)\right| \\ &= \left|\big(1 - f(t)\big)y_{\alpha_0}^*\big(T(x)(t)\big) + (1+\eta)u_0^*(x)f(t)\right| \\ &\leqslant \left|\big(1 - f(t)\big)y_{\alpha_0}^*\big(T(x)(t)\big) + f(t)u_0^*(x)\right| + \eta\left|u_0^*(x)\right| \leqslant 1 + \eta, \end{aligned} \tag{3.4}$$

since $(1 - f(t))y_{\alpha_0}^*(T(x)(t)) + f(t)u_0^*(x)$ is a convex combination of $y_{\alpha_0}^*(T(x)(t))$ and $u_0^*(x)$.

For $\alpha \in \Lambda \setminus \{\alpha_0\}$, since t is in W, by (3.1) we know that $\|u_0^* - T^*(\delta_t \otimes y_{\alpha_0}^*)\| < 3\varepsilon$. Thus,

$$\begin{aligned} \left|y_\alpha^*\big(S(x)(t)\big)\right| &\leqslant \left|y_\alpha^*\big(T(x)(t)\big)\right| + \left|\big(u_0^* - T^*\big(\delta_t \otimes y_{\alpha_0}^*\big)\big)(x) + \eta u_0^*(x)\right|\left|y_\alpha^*(y_{\alpha_0})\right|f(t) \\ &\leqslant 1 + (3\varepsilon + \eta)\rho < 1 + \eta. \end{aligned} \tag{3.5}$$

By (3.3), (3.4) and (3.5), we have that $\|S\| \leqslant 1+\eta$ and by (3.2) we obtain $\|S\| = 1+\eta$ and $\|S(u_0)\| = 1+\eta$. We will check that $\|S - T\| \leqslant \varepsilon(3 + \frac{4\rho}{1-\rho})$. If $t \in K \setminus W$ then $S(x)(t) = T(x)(t)$. If $t \in W$ then by (3.1)

$$\left\|S(x)(t) - T(x)(t)\right\| = \left\|\big((1+\eta)u_0^*(x) - T^*\big(\delta_t \otimes y_{\alpha_0}^*(x)\big)\big)f(t)y_{\alpha_0}\right\| \leqslant 3\varepsilon + \eta = \varepsilon\left(3 + \frac{4\rho}{1-\rho}\right).$$

Finally, taking $R = \frac{S}{\|S\|}$ we get

$$\|R - T\| \leqslant \left\|\frac{S}{\|S\|} - S\right\| + \|S - T\| = \big(1 - \|S\|\big) + \|S - T\| \leqslant \eta + \varepsilon\left(3 + \frac{4\rho}{1-\rho}\right) = \varepsilon\left(3 + \frac{8\rho}{1-\rho}\right),$$

which completes the proof. □

Our aim now is to provide examples of pairs of Banach spaces with the Bishop–Phelps–Bollobás property for operators. Recall that the spaces $\mathcal{C}(K, Y^*)$, $\mathcal{C}(K, (Y^*, w))$ and $\mathcal{C}(K, (Y^*, w^*))$ can be isometrically identified with $\mathcal{K}(Y, \mathcal{C}(K))$, $\mathcal{W}(Y, \mathcal{C}(K))$ and $\mathcal{L}(Y, \mathcal{C}(K))$, respectively (see [15, Theorem VI.7.1, p. 490]). It is also known that $\mathcal{L}(X, Y) = \mathcal{A}(X, Y)$ whenever X is an Asplund space. The following property will be required.

A Banach space Y is said to have *property* α (of Schachermayer) if there are two sets $\{y_\alpha\colon\ \alpha \in \Lambda\} \subset S_Y$, $\{y_\alpha^*\colon\ \alpha \in \Lambda\} \subset S_{Y^*}$ and $0 \leqslant \rho < 1$ such that the following conditions hold

(1) $y_\alpha^*(y_\alpha) = 1$ for all $\alpha \in \Lambda$,
(2) $|y_\alpha^*(y_\gamma)| \leqslant \rho < 1$ for $\alpha, \gamma \in \Lambda$, $\alpha \neq \gamma$,
(3) the unit ball of Y is the closed, circled convex hull of $\{y_\alpha\colon\ \alpha \in \Lambda\}$.

For every set Λ the space $\ell_1(\Lambda)$ has property α. Property α is quite general if we admit equivalent norms (see [23, Theorem 4.4] and [17]). It is clear that Y^* has property β whenever Y has property α. Hence, we obtain the following corollary:

Corollary 3.2. *Let X be an Asplund space and Y a Banach space satisfying property α. Then $(X, \mathcal{K}(Y, \mathcal{C}(K)))$, $(X, \mathcal{W}(Y, \mathcal{C}(K)))$, and $(X, \mathcal{L}(Y, \mathcal{C}(K)))$ have the Bishop–Phelps–Bollobás property for operators for every compact Hausdorff topological space K.*

4. New examples of spaces with the approximate hyperplane series property

It is known that uniformly convex spaces have AHSP (see [1, Proposition 3.8]). Hence X^* has AHSP whenever X is uniformly smooth. We will generalize this fact by providing some spaces of operators satisfying the same property.

We recall that a Banach space X is *uniformly convex* if for every $\varepsilon > 0$ there is $0 < \delta < 1$ such that

$$u, v \in B_X, \quad \frac{\|u+v\|}{2} > 1 - \delta \quad \Rightarrow \quad \|u - v\| < \varepsilon.$$

In such a case, *the modulus of convexity* of X is given by

$$\delta(\varepsilon) := \inf\left\{1 - \frac{\|u+v\|}{2}\colon\ u, v \in B_X,\ \|u - v\| \geqslant \varepsilon\right\}.$$

Given a (non-empty) bounded subset A of X, an element $x^* \in X^*$ and $\alpha > 0$, the *slice* $S(A, x^*, \alpha)$ is the subset of A given by

$$S\bigl(A, x^*, \alpha\bigr) := \Bigl\{z \in A\colon\ \operatorname{Re} x^*(z) > \sup_{x \in A} \operatorname{Re} x^*(x) - \alpha\Bigr\}.$$

The following elementary fact will be useful below.

Lemma 4.1. *(See [2, Lemma 2.1].) If X is uniformly convex, then for every $\varepsilon > 0$,*

$$\operatorname{diam} S\bigl(B_X, x^*, \delta(\varepsilon)\bigr) \leqslant \varepsilon \quad \textit{for all } x^* \in S_{X^*}.$$

Theorem 4.2. *Let X be a uniformly convex Banach space and τ be a linear topology on X satisfying $w \subseteq \tau \subseteq n$. Then the space $\mathcal{C}(K, (X, \tau))$ has AHSP for any compact Hausdorff topological space K.*

Proof. We write $Y = \mathcal{C}(K, (X, \tau))$ and denote by δ the modulus of convexity of X. Take $(f_i)_{i=1}^n \subset B_Y$ and a finite convex series $\sum_{i=1}^n \alpha_i$ satisfying

$$\left\|\sum_{i=1}^n \alpha_i f_i\right\| > 1 - \varepsilon\delta(\varepsilon).$$

Choose $x_0^* \in S_{X^*}$ and $t_0 \in K$ so that

$$x_0^*\left(\sum_{i=1}^n \alpha_i f_i(t_0)\right) > 1 - \varepsilon\delta(\varepsilon).$$

By Lemma 2.1, the set $D := \{1 \leqslant i \leqslant n\colon\ \operatorname{Re} x_0^*(f_i(t_0)) > 1 - \delta(\varepsilon)\}$ satisfies that $\sum_{k \in D} \alpha_k > 1 - \varepsilon$. Consider the subset U of K given by

$$U = \bigcap_{i \in D} f_i^{-1}\bigl(S\bigl(B_X, x_0^*, \delta(\varepsilon)\bigr)\bigr).$$

Since $w \subseteq \tau$, U is open and it clearly contains t_0. By Urysohn's Lemma, there exists a continuous function $\phi : K \to [0, 1]$ with $\operatorname{supp}(\phi) \subset U$ and $\phi(t_0) = 1$.

By assumption X is reflexive, so there is $x_0 \in S_X$ so that $x_0^*(x_0) = 1$. For each $i \in D$, define $g_i \in B_Y$ by

$$g_i = \phi x_0 + (\mathbf{1} - \phi) f_i.$$

For $i \in D$, we have that $g_i(t_0) = x_0$ and by Lemma 4.1 we obtain

$$\|g_i - f_i\| = \|\phi(x_0 \cdot \mathbf{1} - f_i)\| \leqslant \sup_{t \in U} \|x_0 - f_i(t)\| \leqslant \operatorname{diam} S(B_X, x_0^*, \delta(\varepsilon)) \leqslant \varepsilon.$$

On the other hand, the element $x_0^* \circ \delta_{t_0}$ belongs to S_{Y^*} and $(x_0^* \circ \delta_{t_0})(g_i) = x_0^*(g_i(t_0)) = 1$ for every $i \in D$. □

$\mathcal{C}(K, X)$ has AHSP whenever X also satisfies AHSP [12, Theorem 2.15]. As we already noticed in the previous section, sometimes vector-valued spaces of continuous functions can be identified with spaces of operators. Hence, we deduce the following result.

Corollary 4.3. *Let X be a Banach space whose dual has AHSP. Then the space $\mathcal{K}(X, \mathcal{C}(K))$ has AHSP for every compact Hausdorff topological space K.*

The above corollary implies that $\mathcal{L}(X, \mathcal{C}(K))$ has AHSP for any finite-dimensional space X. It is a natural question whether or not there are infinite-dimensional spaces with the previous property. The answer is positive since it is not difficult to show that for every set I, the space $(\bigoplus_{i \in I} Y)_{\ell_\infty}$ has AHSP whenever Y satisfies AHSP. Hence the space $\mathcal{L}(\ell_1, Y) = (\bigoplus_{n \in \mathbb{N}} Y)_{\ell_\infty}$ has also AHSP. We will provide another example that follows from the main result of this section.

Corollary 4.4. *The spaces $\mathcal{L}(X, \mathcal{C}(K))$ and $\mathcal{K}(X, \mathcal{C}(K))$ have AHSP for every uniformly smooth Banach space X and every compact Hausdorff topological space K.*

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