J. Math. Anal. Appl. 415 (2014) 825–845

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Maps which preserve norms of non-symmetrical quotients between groups of exponentials of Lipschitz functions ☆

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ARTICLE INFO

Article history:
Received 14 October 2013
Available online 10 February 2014
Submitted by K. Jarosz

Keywords:
Lipschitz algebra
Peaking function
Algebra isomorphism
Isometric isomorphism

ABSTRACT

Let $\Phi : \exp \mathrm{Lip}(X_1) \to \exp \mathrm{Lip}(X_2)$ be a surjective mapping where X_1 and X_2 are compact metric spaces. We prove that if Φ satisfies the non-symmetric-quotient norm condition for the uniform norm:

$$\left\| \frac{g}{f} - \mathbf{1} \right\|_\infty = \left\| \frac{\Phi(g)}{\Phi(f)} - \mathbf{1} \right\|_\infty \quad \big(f, g \in \exp \mathrm{Lip}(X_1)\big),$$

then Φ is of the form

$$\Phi(f)(y) = \begin{cases} \Phi(\mathbf{1})(y) f(\phi(y)) & \text{if } y \in K, \\ \Phi(\mathbf{1})(y) \overline{f(\phi(y))} & \text{if } y \in X_2 \backslash K \end{cases} \quad \big(f \in \exp \mathrm{Lip}(X_1)\big),$$

where $\phi : X_2 \to X_1$ is a homeomorphism and K is a closed open subset of X_2. On the other hand, if Φ satisfies the non-symmetric-quotient norm condition for the Lipschitz algebra norm:

$$\left\| \frac{g}{f} - \mathbf{1} \right\|_\infty + \left\| \frac{g}{f} - \mathbf{1} \right\|_L = \left\| \frac{\Phi(g)}{\Phi(f)} - \mathbf{1} \right\|_\infty + \left\| \frac{\Phi(g)}{\Phi(f)} - \mathbf{1} \right\|_L \quad \big(f, g \in \exp \mathrm{Lip}(X_1)\big),$$

we show that Φ is of the form

$$\Phi(f)(y) = \Phi(\mathbf{1})(y) f\big(\phi(y)\big) \quad \big(y \in X_2,\ f \in \exp \mathrm{Lip}(X_1)\big),$$

or

$$\Phi(f)(y) = \Phi(\mathbf{1})(y) \overline{f\big(\phi(y)\big)} \quad \big(y \in X_2,\ f \in \exp \mathrm{Lip}(X_1)\big),$$

where $\phi : X_2 \to X_1$ is a surjective isometry.

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☆ The second and third authors were partially supported by MICINN under project MTM 2010-17687.
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http://dx.doi.org/10.1016/j.jmaa.2014.01.088
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1. Introduction

Non-symmetrically norm preserving maps were initially studied in [5] motivated by the seminal paper of Molnár [13] on the multiplicatively spectrum preserving surjections on certain Banach algebras. It was proved that multiplicatively non-symmetrically spectral-radius preserving maps on commutative Banach algebras are closely related to the isomorphisms on these algebras, and it turns several authors' attention to the subject [9,2,7,4,12]. Miura, Honma and Shindo [12] considered the non-symmetrically quotient spectral-radius preserving maps on semisimple unital commutative Banach algebras. They showed that such maps are real algebra isomorphisms followed by multiplications. It is interesting to study such maps for the *original norms* of the given Banach algebras, but it seems that there has not yet been a literature on the non-symmetrically original norm preserving maps other than uniform norms. In this paper we give a result for maps preserving (Banach algebra) norms of non-symmetrical quotients between groups of exponentials of Lipschitz functions.

Throughout the paper, (X,d) denotes a compact metric space and let $\mathrm{Lip}(X)$ be the algebra of all complex-valued Lipschitz functions f on X with the norm $\|\cdot\| = \|\cdot\|_\infty + \|\cdot\|_L$, where

$$\|f\|_\infty = \sup\{|f(x)|:\ x\in X\}$$

and

$$\|f\|_L = \inf\{K>0:\ |f(x)-f(y)| \leqslant Kd(x,y),\ \forall x,y\in X\}.$$

It is known (see [16]) that $\mathrm{Lip}(X)$ is a semisimple unital commutative Banach algebra. The unity of $\mathrm{Lip}(X)$, denoted by $\mathbf{1}$, is the function constantly equal to 1 on X, and the maximal ideal space of $\mathrm{Lip}(X)$ is homeomorphic to X. Hence the spectral radius coincides with the uniform norm on X for every function in $\mathrm{Lip}(X)$. The group of all invertible elements in $\mathrm{Lip}(X)$ is denoted by $\mathrm{Lip}(X)^{-1}$ and $\exp\mathrm{Lip}(X) = \{\exp(f):\ f\in \mathrm{Lip}(X)\}$. Note that $\exp\mathrm{Lip}(X)$ is the principal component (the connected component of $\mathrm{Lip}(X)^{-1}$ which contains the function $\mathbf{1}$) of $\mathrm{Lip}(X)^{-1}$.

From [12, Theorem 3.2] we infer that a surjection $\Phi:\mathrm{Lip}(X_1)^{-1}\to\mathrm{Lip}(X_2)^{-1}$ satisfies the equality

$$\left\|\frac{g}{f}-1\right\|_\infty = \left\|\frac{\Phi(g)}{\Phi(f)}-1\right\|_\infty$$

for every $f,g\in\mathrm{Lip}(X_1)^{-1}$ if and only if there exists a homeomorphism $\phi:X_2\to X_1$ and a closed open subset K of X_2 such that

$$\Phi(f)(y) = \begin{cases} \Phi(\mathbf{1})(y)f(\phi(y)) & \text{if } y\in K,\\ \Phi(\mathbf{1})(y)\overline{f(\phi(y))} & \text{if } y\in X_2\backslash K,\end{cases}$$

for every $f\in\mathrm{Lip}(X_1)^{-1}$. In Theorem 1, we show that this result also holds for surjective mappings $\Phi:\exp\mathrm{Lip}(X_1)\to\exp\mathrm{Lip}(X_2)$. Then we give in Corollary 2 some sufficient conditions for Φ to be extendible to an algebra isomorphism. Our method of proof of Theorem 1 is an adaptation of the reasoning used in [2,9].

On the other hand, surjective isometries with respect to the Lipschitz Banach norm $\|\cdot\|_\infty + \|\cdot\|_L$ between groups $\exp\mathrm{Lip}(X)$ are of a much restrictive form. Namely, we show in the main result of this paper, Theorem 8, that Φ satisfies the non-symmetric-quotient norm condition for the Lipschitz algebra norm:

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$$\left\|\frac{g}{f}-\mathbf{1}\right\|_\infty+\left\|\frac{g}{f}-\mathbf{1}\right\|_L=\left\|\frac{\Phi(g)}{\Phi(f)}-\mathbf{1}\right\|_\infty+\left\|\frac{\Phi(g)}{\Phi(f)}-\mathbf{1}\right\|_L \quad \bigl(f,g\in\exp\operatorname{Lip}(X_1)\bigr),$$

if and only if there exists a surjective isometry $\phi: X_2\to X_1$ such that

$$\Phi(f)(y)=\Phi(\mathbf{1})(y)f\bigl(\phi(y)\bigr)$$

for all $y\in X_2$ and $f\in\exp\operatorname{Lip}(X_1)$, or

$$\Phi(f)(y)=\Phi(\mathbf{1})(y)\overline{f\bigl(\phi(y)\bigr)}$$

for all $y\in X_2$ and $f\in\exp\operatorname{Lip}(X_1)$. Note that if, in addition, $\Phi(\mathbf{1})=\mathbf{1}$, then Φ is extendible to either an isometric complex-linear algebra isomorphism or an isometric conjugate-linear algebra isomorphism.

For the proof of Theorem 8, we first show by adapting the proof of Jarosz's theorem on isometries in semisimple commutative Banach algebras [8] that every real-linear isometry with respect to the Lipschitz Banach norm T from $\operatorname{Lip}(X_1)$ onto $\operatorname{Lip}(X_2)$ such that $T(\mathbf{1})=\mathbf{1}$ and either $T(i\mathbf{1})=i\mathbf{1}$ or $T(i\mathbf{1})=-i\mathbf{1}$, is an isometry from $\operatorname{Lip}(X_1)$ onto $\operatorname{Lip}(X_2)$ for the uniform norm. Apart from this fact, our approach for proving Theorem 8 requires the use of tools concerning d-preserving maps between groups [3], continuous one-parameter groups of functions [14], the famous theorems of Mazur–Ulam and Stone–Weierstrass and real-linear isometries between function algebras [11]. We remark that the proof of Theorem 8 has been motivated by the proof of Theorem 1 in [6].

We point out in a final remark that similar results to those above are valid for surjections Φ between groups $\exp\operatorname{lip}_\alpha(X)$ of spaces of little Lipschitz complex-valued functions on compact metric spaces (X,d^α) with $\alpha\in(0,1)$.

2. Case: Uniform norm

Our purpose in this section is to obtain the following result.

Theorem 1. *Let X_1 and X_2 be compact metric spaces and let Φ be a surjective mapping from $\exp\operatorname{Lip}(X_1)$ to $\exp\operatorname{Lip}(X_2)$. Then Φ satisfies the non-symmetric-quotient norm condition for the uniform norm:*

$$\left\|\frac{g}{f}-\mathbf{1}\right\|_\infty=\left\|\frac{\Phi(g)}{\Phi(f)}-\mathbf{1}\right\|_\infty,\quad \forall f,g\in\exp\operatorname{Lip}(X_1),$$

if and only if there exists a homeomorphism $\phi: X_2\to X_1$ and a closed open subset $K\subset X_2$ such that

$$\Phi(f)(y)=\begin{cases}\Phi(\mathbf{1})(y)f(\phi(y)) & \text{if } y\in K,\\ \Phi(\mathbf{1})(y)\overline{f(\phi(y))} & \text{if } y\in X_2\backslash K,\end{cases}$$

for all $f\in\exp\operatorname{Lip}(X_1)$.

From the description given for Φ, we give sufficient conditions for Φ to be extendable to be an algebra isomorphism.

Corollary 2. *Let X_1 and X_2 be compact metric spaces and let Φ be a surjective mapping from $\exp\operatorname{Lip}(X_1)$ to $\exp\operatorname{Lip}(X_2)$ satisfying the non-symmetric-quotient norm condition for the uniform norm. Then the following assertions are satisfied:*

(1) *If* $\Phi(\mathbf{1}) = \mathbf{1}$*, then* Φ *is extendible to a real-linear algebra isomorphism.*
(2) *If* $\Phi(\mathbf{1}) = \mathbf{1}$ *and* $\Phi(\mathbf{1}i) = \mathbf{1}i$*, then* Φ *is extendible to a complex-linear algebra isomorphism.*
(3) *If* $\Phi(\mathbf{1}) = \mathbf{1}$ *and* $\Phi(\mathbf{1}i) = -\mathbf{1}i$*, then* Φ *is extendible to a conjugate-linear algebra isomorphism.*

Given a compact metric space X and $x \in X$, denote

$$F_x(X) = \{f \in \exp \operatorname{Lip}(X): \ |f(x)| = \|f\|_\infty = 1\}.$$

We prepare the proof of Theorem 1 proving first the following lemma.

Lemma 3. *Let* X *be a compact metric space and* $f, g \in \operatorname{Lip}(X)$*.*

i) *If* $x \in X$ *and* $f(x) \neq 0$*, then there exists* $h_{f,x} \in \exp \operatorname{Lip}(X)$ *such that* $h_{f,x}(X) \subset (0,1]$*,* $h_{f,x}(x) = 1$ *and, for all* $z \in X$ *with* $z \neq x$*,* $h_{f,x}(z) < 1$ *and* $|h_{f,x}(z)f(z)| < |f(x)|$*.*
ii) *If* $x, z \in X$ *and* $F_x(X) \subset F_z(X)$*, then* $z = x$*.*
iii) $|f| \leqslant |g|$ *if and only if* $\|fh\|_\infty \leqslant \|gh\|_\infty$ *for all* $h \in \exp \operatorname{Lip}(X)$*.*

Proof. i) Let $x \in X$ with $f(x) \neq 0$, $g_1, g_2 : X \to (-\infty, 0]$ be defined by

$$g_1(z) = \min\left\{0, 1 - \frac{|f(z)|}{|f(x)|}\right\},$$
$$g_2(z) = -d(x,z),$$

and let $h_{f,x} = \exp(g_1 + g_2)$. Clearly $g_1, g_2 \in \operatorname{Lip}(X)$ and, taking into account that $e^{1-t} \leqslant 1/t$ for all $t \geqslant 1$, it is easy to prove that $h_{f,x}$ satisfies the conditions given in the statement i).

ii) Given $x, z \in X$ with $F_x(X) \subset F_z(X)$, just consider $h_{\mathbf{1},x} \in F_x(X)$ to see that $z = x$.

iii) If $|f| \leqslant |g|$, it is clear that $\|fh\|_\infty \leqslant \|gh\|_\infty$ for all $h \in \exp \operatorname{Lip}(X)$. Reciprocally, assume that $\|fh\|_\infty \leqslant \|gh\|_\infty$ for all $h \in \exp \operatorname{Lip}(X)$. Let $x \in X$. Suppose $|g(x)| < |f(x)|$ and let ε be a real number such that $|g(x)| < \varepsilon < |f(x)|$. By the continuity of g at x, there exists $\delta > 0$ such that $|g(z)| < \varepsilon$ for all $z \in X$ with $d(x,z) < \delta$. Let h be in $\exp \operatorname{Lip}(X)$ defined by

$$h(z) = \exp\left(-\frac{d(x,z)}{\delta} \ln\left(\frac{\varepsilon + \|g\|_\infty}{\varepsilon}\right)\right), \quad \forall z \in X.$$

An easy calculation shows that $\|gh\|_\infty < \varepsilon$. Therefore

$$\varepsilon < |f(x)| = |f(x)h(x)| \leqslant \|fh\|_\infty \leqslant \|gh\|_\infty < \varepsilon,$$

which yields a contradiction. This proves that $|f| \leqslant |g|$. $\square$

Our next purpose is to show that each surjection $\Phi : \exp \operatorname{Lip}(X_1) \to \exp \operatorname{Lip}(X_2)$ that satisfies the non-symmetric-quotient norm condition for the uniform norm gives rise to a homeomorphism $\phi : X_2 \to X_1$ in such a way that $|\Phi(f)(y)| = |f(\phi(y))|$ for all $y \in X_2$ and $f \in \exp \operatorname{Lip}(X_1)$.

Proposition 4. *Let* X_1 *and* X_2 *be compact metric spaces and let* Φ *be a surjective mapping from* $\exp \operatorname{Lip}(X_1)$ *to* $\exp \operatorname{Lip}(X_2)$ *such that* $\Phi(\mathbf{1}) = \mathbf{1}$ *and*

$$\left\|\frac{g}{f} - \mathbf{1}\right\|_\infty = \left\|\frac{\Phi(g)}{\Phi(f)} - \mathbf{1}\right\|_\infty, \quad \forall f, g \in \exp \operatorname{Lip}(X_1).$$

Then the following assertions hold:

i) *Φ is injective.*
ii) $\|g/f\|_\infty = \|\Phi(g)/\Phi(f)\|_\infty$ *for all* $f, g \in \exp \mathrm{Lip}(X_1)$.
iii) $\|g\|_\infty = \|\Phi(g)\|_\infty$ *for all* $g \in \exp \mathrm{Lip}(X_1)$.
iv) *Given* $f, g \in \exp \mathrm{Lip}(X_1)$, $|f| \leqslant |g|$ *if and only if* $|\Phi(f)| \leqslant |\Phi(g)|$.
v) *For each* $x \in X_1$ *there is a unique* $y \in X_2$ *such that* $\Phi(F_x(X_1)) \subset F_y(X_2)$.
vi) *There exists a homeomorphism* $\phi: X_2 \to X_1$ *such that* $|\Phi(f)(y)| = |f(\phi(y))|$ *for all* $y \in X_2$ *and* $f \in \exp \mathrm{Lip}(X_1)$.

Proof. i) If $f, g \in \exp \mathrm{Lip}(X_1)$ satisfy $\Phi(f) = \Phi(g)$, then $\|g/f - \mathbf{1}\|_\infty = \|\Phi(g)/\Phi(f) - \mathbf{1}\|_\infty = 0$, thereupon $f = g$.

ii) Let $f, g \in \exp \mathrm{Lip}(X_1)$ and $\varepsilon > 0$. It is clear that

$$\left\|\Phi\left(\frac{2}{\varepsilon}g\right)\frac{\mathbf{1}}{\Phi(g)}\right\|_\infty \leqslant \left\|\Phi\left(\frac{2}{\varepsilon}g\right)\frac{\mathbf{1}}{\Phi(g)} - \mathbf{1}\right\|_\infty + 1 = \left|\frac{2}{\varepsilon} - \mathbf{1}\right| + 1 \leqslant \frac{2}{\varepsilon} + 2.$$

Hence

$$\begin{aligned}\frac{2}{\varepsilon}\left\|\frac{g}{f}\right\|_\infty - 1 &\leqslant \left\|\frac{2}{\varepsilon}\frac{g}{f} - \mathbf{1}\right\|_\infty = \left\|\Phi\left(\frac{2}{\varepsilon}g\right)\frac{\mathbf{1}}{\Phi(f)} - \mathbf{1}\right\|_\infty \leqslant \left\|\Phi\left(\frac{2}{\varepsilon}g\right)\frac{\mathbf{1}}{\Phi(f)}\right\|_\infty + 1 \\ &= \left\|\Phi\left(\frac{2}{\varepsilon}g\right)\frac{\mathbf{1}}{\Phi(g)}\frac{\Phi(g)}{\Phi(f)}\right\|_\infty + 1 \leqslant \left(\frac{2}{\varepsilon} + 2\right)\left\|\frac{\Phi(g)}{\Phi(f)}\right\|_\infty + 1,\end{aligned}$$

that is, $\|g/f\|_\infty \leqslant (1 + \varepsilon)\|\Phi(g)/\Phi(f)\|_\infty + \varepsilon$. By the arbitrariness of ε, we deduce that $\|g/f\|_\infty \leqslant \|\Phi(g)/\Phi(f)\|_\infty$. As Φ is bijective by the assumption on Φ and i), Φ^{-1} is well defined and the opposite inequality results from the fact that Φ^{-1} has the same properties as Φ.

iii) follows immediately from ii) taking into account that $\Phi(\mathbf{1}) = \mathbf{1}$.

iv) Fix $f, g \in \exp \mathrm{Lip}(X_1)$ and suppose that $|f| \leqslant |g|$. Then $\|f/h\|_\infty \leqslant \|g/h\|_\infty$ for all $h \in \exp \mathrm{Lip}(X_1)$. By ii), it follows that $\|\Phi(f)/\Phi(h)\|_\infty \leqslant \|\Phi(g)/\Phi(h)\|_\infty$ for all $h \in \exp \mathrm{Lip}(X_1)$. Given $k \in \exp \mathrm{Lip}(X_2)$, as Φ is surjective, there is $h \in \exp \mathrm{Lip}(X_1)$ such that $\Phi(h) = 1/k$. Therefore $\|\Phi(f)k\|_\infty \leqslant \|\Phi(g)k\|_\infty$ for all $k \in \exp \mathrm{Lip}(X_2)$. Thus, by Lemma 3, $|\Phi(f)| \leqslant |\Phi(g)|$. Conversely, assume that $|\Phi(f)| \leqslant |\Phi(g)|$. Since Φ^{-1} has the same properties as Φ, we infer that $|f| = |\Phi^{-1}(\Phi(f))| \leqslant |\Phi^{-1}(\Phi(g))| = |g|$.

v) We follow here the method of proof used in [15]. Let $x \in X_1$. For every $f \in F_x(X_1)$, define

$$P(f) = \{y \in X_2\colon\ |\Phi(f)(y)| = 1\}.$$

Since X_2 is compact, we deduce from iii) that $P(f)$ is nonempty. Furthermore, it is easy to prove that the family $\{P(f)\colon f \in F_x(X_1)\}$ has the finite intersection property simply by considering $f_1, \ldots, f_n \in F_x(X_1)$ and taking $g = f_1 \cdots f_n$. Consequently, $\bigcap_{f \in F_x(X_1)} P(f)$ is nonempty, and picking $y \in \bigcap_{f \in F_x(X_1)} P(f)$, it is clear that $\Phi(F_x(X_1)) \subset F_y(X_2)$.

To prove the uniqueness of y, pick $z \in X_2$ with $\Phi(F_x(X_1)) \subset F_z(X_2)$. Let $g \in \exp \mathrm{Lip}(X_1)$ and $h \in \exp \mathrm{Lip}(X_2)$ be the functions defined by

$$g(w) = e^{-d_1(w,x)}, \quad \forall w \in X_1; \qquad h(w) = e^{-d_2(w,y)}, \quad \forall w \in X_2.$$

Since Φ is surjective, $\Phi(f) = \Phi(g)h$ for some $f \in \exp \mathrm{Lip}(X_1)$. Obviously, $|\Phi(f)| = |\Phi(g)|h \leqslant |\Phi(g)|$. Then, by iv), it follows that $|f| \leqslant |g|$. Moreover, as $g \in F_x(X_1)$, it holds that $\Phi(g) \in F_y(X_2) \cap F_z(X_2)$. Thus

$$\|f\|_\infty = \|\Phi(f)\|_\infty = \|\Phi(g)h\|_\infty = |\Phi(g)(y)h(y)| = 1.$$

Now an easy calculation shows that $f \in F_x(X_1)$. By assumption, $\Phi(f) \in F_z(X_2)$, whereupon

$$1 = \left|\Phi(f)(z)\right| = \left|\Phi(g)(z)\right| h(z) = e^{-d_2(z,y)},$$

and this implies that $z = y$.

vi) Let $\psi : X_1 \to X_2$ be the map that takes every point $x \in X_1$ to the unique point $\psi(x) \in X_2$ satisfying $\Phi(F_x(X_1)) \subset F_{\psi(x)}(X_2)$. Analogously, we can define a map $\phi : X_2 \to X_1$ such that $\Phi^{-1}(F_y(X_2)) \subset F_{\phi(y)}(X_1)$ for all $y \in X_2$. From Lemma 3, it follows that ϕ is bijective and $\phi^{-1} = \psi$. Moreover, given $f \in \exp \mathrm{Lip}(X_1)$ and $x \in X_1$, it is obvious that the function $h_{\mathbf{1}/f,x}$ obtained in Lemma 3 belongs to $F_x(X_1)$. Thus $\Phi(h_{\mathbf{1}/f,x}) \in F_{\psi(x)}(X_2)$ and we have

$$\frac{1}{|\Phi(f)(\psi(x))|} = \left|\frac{\Phi(h_{\mathbf{1}/f,x})(\psi(x))}{\Phi(f)(\psi(x))}\right| \leqslant \left\|\frac{\Phi(h_{\mathbf{1}/f,x})}{\Phi(f)}\right\|_\infty = \left\|\frac{h_{\mathbf{1}/f,x}}{f}\right\|_\infty = \frac{1}{|f(x)|}.$$

Hence $|f(x)| \leqslant |\Phi(f)(\psi(x))|$. Similarly, $|g(y)| \leqslant |\Phi^{-1}(g)(\phi(y))|$ for all $y \in X_2$ and $g \in \exp \mathrm{Lip}(X_2)$. Therefore $|f(\phi(y))| = |\Phi(f)(y)|$ for all $y \in X_2$ and $f \in \exp \mathrm{Lip}(X_1)$.

Now, we prove that ϕ is continuous. Let $y_0 \in X_2$ and $\varepsilon > 0$. Consider $h \in \exp \mathrm{Lip}(X_1)$ defined by

$$h(x) = \exp\left(-\frac{d_1(x, \phi(y_0))}{\varepsilon}\right), \quad \forall x \in X_1,$$

and fix $U = \{y \in X_2 : |\Phi(h)(y)| > 1/e\}$. Notice that U is an open neighborhood of y_0 in X_2. Furthermore, given $y \in U$, we have $1/e < |\Phi(h)(y)| = |h(\phi(y))|$, and thus $d_1(\phi(y), \phi(y_0)) < \varepsilon$. Hence ϕ is continuous at y_0. As ϕ is bijective and continuous, X_2 is compact and X_1 is Hausdorff, then ϕ is a homeomorphism. □

The following straightforward lemma will facilitate the reading of the subsequent proofs.

Lemma 5. *Let $\alpha, \beta \in \mathbb{C}$.*

i) *If $|\alpha - 1| = |\beta| + 1$ and $|\alpha| = |\beta|$, then $\alpha = -|\beta|$.*
ii) *If $|\beta| = |\alpha|$, $|\beta - 1| \leqslant |\alpha - 1|$ and $|\beta + 1| \leqslant |\alpha + 1|$, then $\beta = \alpha$ or $\beta = \bar{\alpha}$.*

Next we study the homogeneity of the mapping Φ on constant functions.

Lemma 6. *Let X_1 and X_2 be compact metric spaces, $\Phi : \exp \mathrm{Lip}(X_1) \to \exp \mathrm{Lip}(X_2)$ be a surjective mapping such that $\Phi(\mathbf{1}) = \mathbf{1}$ and*

$$\left\|\frac{g}{f} - \mathbf{1}\right\|_\infty = \left\|\frac{\Phi(g)}{\Phi(f)} - \mathbf{1}\right\|_\infty, \quad \forall f, g \in \exp \mathrm{Lip}(X_1);$$

and let $\phi : X_2 \to X_1$ be the homeomorphism obtained in Proposition 4. Then:

i) *$\Phi(\alpha h)(y) = \Phi(\alpha \mathbf{1})(y)$ for all $\alpha \in \mathbb{C}\backslash\{0\}$, $y \in X_2$ and $h \in F_{\phi(y)}(X_1)$ with $h(\phi(y)) = 1$.*
ii) *$\Phi(-\alpha \mathbf{1}) = -\Phi(\alpha \mathbf{1})$ for all $\alpha \in \mathbb{C}\backslash\{0\}$.*
iii) *Given $y \in X_2$, either $\Phi(i\mathbf{1})(y) = i$ or $\Phi(i\mathbf{1})(y) = -i$.*
iv) *If $y \in X_2$ and $\Phi(i\mathbf{1})(y) = i$, then $\Phi(\alpha \mathbf{1})(y) = \alpha$ for all $\alpha \in \mathbb{C}\backslash\{0\}$.*
v) *If $y \in X_2$ and $\Phi(i\mathbf{1})(y) = -i$, then $\Phi(\alpha \mathbf{1})(y) = \bar{\alpha}$ for all $\alpha \in \mathbb{C}\backslash\{0\}$.*

Proof. i)–ii) Let $y \in X_2$, $\alpha \in \mathbb{C}\backslash\{0\}$, $h \in F_{\phi(y)}(X_1)$ with $h(\phi(y)) = 1$, and let $g \in F_{\phi(y)}(X_1)$ be defined by $g(x) = \exp(-d_1(x, \phi(y)))$ for all $x \in X_1$. Since $\|\Phi(\alpha g)/\Phi(-\alpha/h) - \mathbf{1}\|_\infty = \|-gh - \mathbf{1}\|_\infty = 2$

and X_2 is compact, we can find $z \in X_2$ such that $|\Phi(\alpha g)(z)/\Phi(-\alpha/h)(z) - 1| = 2$. Proposition 4 iv) yields

$$2 \leqslant \left|\frac{\Phi(\alpha g)(z)}{\Phi(-\alpha/h)(z)}\right| + 1 = \left|g\big(\phi(z)\big)\right|\left|h\big(\phi(z)\big)\right| + 1 \leqslant g\big(\phi(z)\big) + 1 = e^{-d_1(\phi(z),\phi(y))} + 1.$$

This clearly forces $z = y$. Consequently, we have

$$\left|\frac{\Phi(\alpha g)(y)}{\Phi(-\alpha/h)(y)} - 1\right| = 2, \qquad \left|\frac{\Phi(\alpha g)(y)}{\Phi(-\alpha/h)(y)}\right| = 1.$$

By Lemma 5 i), it follows that $\Phi(\alpha g)(y) = -\Phi(-\alpha/h)(y)$. Analogously, $\Phi(\alpha h)(y) = -\Phi(-\alpha/g)(y)$. Since h is arbitrary, in particular,

$$\Phi(\alpha g)(y) = -\Phi(-\alpha\mathbf{1})(y), \qquad \Phi(\alpha g)(y) = -\Phi(-\alpha/g)(y) = \Phi(\alpha\mathbf{1})(y),$$

and thus

$$-\Phi(-\alpha\mathbf{1})(y) = \Phi(\alpha g)(y) = \Phi(\alpha\mathbf{1})(y).$$

iii) Let $y \in X_2$ and $\alpha \in \mathbb{C}\backslash\{0\}$. From Proposition 4 iv) we can deduce that $|\Phi(\alpha\mathbf{1})(y)| = |\alpha|$. By using ii), it follows that

$$\left|\Phi(\alpha\mathbf{1})(y) + 1\right| \leqslant \left\|\Phi(\alpha\mathbf{1}) + \mathbf{1}\right\|_\infty = \left\|\frac{\Phi(-\alpha\mathbf{1})}{\Phi(\mathbf{1})} - \mathbf{1}\right\|_\infty = |\alpha + 1|.$$

Moreover

$$\left|\Phi(\alpha\mathbf{1})(y) - 1\right| \leqslant \left\|\frac{\Phi(\alpha\mathbf{1})}{\Phi(\mathbf{1})} - \mathbf{1}\right\|_\infty = |\alpha - 1|.$$

Now Lemma 5 ii) gives

$$\Phi(\alpha\mathbf{1})(y) = \alpha \quad \text{or} \quad \Phi(\alpha\mathbf{1})(y) = \bar{\alpha}. \tag{2.1}$$

In particular, for $\alpha = i$, it holds $\Phi(i\mathbf{1})(y) = i$ or $\Phi(i\mathbf{1})(y) = -i$.

We next show that iv) and v) follow analogously. So, fix $y \in X_2$ and assume $\Phi(i\mathbf{1})(y) = i$. Let $\alpha \in \mathbb{C}\backslash\{0\}$. Then assertion ii) gives

$$\left|i\Phi(\alpha\mathbf{1})(y) - 1\right| = \left|\frac{\Phi(\alpha\mathbf{1})(y)}{\Phi(-i\mathbf{1})(y)} - 1\right| \leqslant \left\|\frac{\Phi(\alpha\mathbf{1})}{\Phi(-i\mathbf{1})} - \mathbf{1}\right\|_\infty = \left|\frac{\alpha}{-i} - 1\right| = |i\alpha - 1|$$

and, similarly, $|i\Phi(\alpha\mathbf{1})(y) + 1| \leqslant |i\alpha + 1|$. Moreover, by Proposition 4 vi), it is clear that $|i\Phi(\alpha\mathbf{1})(y)| = |i\alpha|$. Thus, taking into account Lemma 5 ii), it follows that $\mathrm{Re}(i\Phi(\alpha\mathbf{1})(y)) = \mathrm{Re}(i\alpha)$, or equivalently $\mathrm{Im}(\Phi(\alpha\mathbf{1})(y)) = \mathrm{Im}(\alpha)$. From (2.1), we deduce that $\Phi(\alpha\mathbf{1})(y) = \alpha$. $\square$

We now are ready to prove Theorem 1.

Proof of Theorem 1. It is straightforward to check that every surjective mapping Φ of the form given in the statement of Theorem 1 verifies

$$\left\|\frac{g}{f} - \mathbf{1}\right\|_\infty = \left\|\frac{\Phi(g)}{\Phi(f)} - \mathbf{1}\right\|_\infty, \quad \forall f, g \in \exp \mathrm{Lip}(X_1). \tag{2.2}$$

Let us prove the contrary implication. Suppose first that Φ satisfies $\Phi(\mathbf{1}) = \mathbf{1}$ and (2.2), and let $\phi: X_2 \to X_1$ be the homeomorphism obtained in Proposition 4. Let $f \in \exp \mathrm{Lip}(X_1)$, $y \in X_2$ and $h_{\mathbf{1}/f,\phi(y)} \in F_{\phi(y)}(X_1)$ be the function given in Lemma 3 i). Set

$$\alpha = \frac{-\Phi(f)(y)}{|f(\phi(y))|}, \qquad \lambda = \mathrm{Re}(\alpha) + \Phi(i\mathbf{1})(y)\,\mathrm{Im}(\alpha).$$

By applying Lemma 6, we obtain

$$\left|\frac{\Phi(\lambda h_{\mathbf{1}/f,\phi(y)})(y)}{\Phi(f)(y)} - 1\right| = \left|\frac{\Phi(\lambda\mathbf{1})(y)}{\Phi(f)(y)} - 1\right| = \left|\frac{\alpha}{\Phi(f)(y)} - 1\right| = \left|\frac{-1}{|f(\phi(y))|} - 1\right| = \frac{1}{|f(\phi(y))|} + 1,$$

hence

$$\frac{1}{|f(\phi(y))|} + 1 \leqslant \left\|\frac{\Phi(\lambda h_{\mathbf{1}/f,\phi(y)})}{\Phi(f)} - \mathbf{1}\right\|_\infty = \left\|\frac{\lambda h_{\mathbf{1}/f,\phi(y)}}{f} - 1\right\|_\infty.$$

From Proposition 4 vi) and Lemma 6 iv), v) we have $|\lambda| = |\alpha| = 1$, hence

$$\left|\frac{\lambda h_{\mathbf{1}/f,\phi(y)}(x)}{f(x)} - 1\right| \leqslant \left|\frac{\lambda h_{\mathbf{1}/f,\phi(y)}(x)}{f(x)}\right| + 1 < \frac{1}{|f(\phi(y))|} + 1$$

for all $x \in X_1$ with $x \neq \phi(y)$. Now the compactness of X_1 gives

$$\left|\frac{\lambda}{f(\phi(y))} - 1\right| = \left|\frac{\lambda h_{\mathbf{1}/f,\phi(y)}(\phi(y))}{f(\phi(y))} - 1\right| = \left\|\frac{\lambda h_{\mathbf{1}/f,\phi(y)}}{f} - \mathbf{1}\right\|_\infty = \frac{1}{|f(\phi(y))|} + 1.$$

In view of Lemma 5 i), this shows that $\lambda/f(\phi(y)) = -1/|f(\phi(y))|$. As a consequence,

$$f(\phi(y)) = \begin{cases} \Phi(f)(y) & \text{if } \Phi(i\mathbf{1})(y) = i, \\ \overline{\Phi(f)(y)} & \text{if } \Phi(i\mathbf{1})(y) = -i, \end{cases}$$

that is,

$$\Phi(f)(y) = \begin{cases} f(\phi(y)) & \text{if } \Phi(i\mathbf{1})(y) = i, \\ \overline{f(\phi(y))} & \text{if } \Phi(i\mathbf{1})(y) = -i. \end{cases}$$

Now, if $\Phi(\mathbf{1}) \neq \mathbf{1}$, we can take $\Phi_0 = \Phi/\Phi(\mathbf{1})$. Then Φ_0 is surjective, $\Phi_0(\mathbf{1}) = \mathbf{1}$ and $\|g/f - \mathbf{1}\|_\infty = \|\Phi_0(g)/\Phi_0(f) - \mathbf{1}\|_\infty$ for all $f, g \in \exp \mathrm{Lip}(X_1)$. By above-proved there is a homeomorphism $\phi: X_2 \to X_1$ such that

$$\Phi(f)(y) = \begin{cases} \Phi(\mathbf{1})(y) f(\phi(y)) & \text{if } \Phi(i\mathbf{1})(y) = i\Phi(\mathbf{1})(y), \\ \Phi(\mathbf{1})(y)\overline{f(\phi(y))} & \text{if } \Phi(i\mathbf{1})(y) = -i\Phi(\mathbf{1})(y), \end{cases}$$

for every $f \in \exp \mathrm{Lip}(X_1)$. Finally, just take

$$K = \{y \in X_2\colon\ \Phi_0(i\mathbf{1})(y) = i\} = \left\{y \in X_2\colon\ \frac{\Phi(i\mathbf{1})(y)}{\Phi(\mathbf{1})(y)} = i\right\}$$

which is a closed open subset by Lemma 6 iii). $\square$

3. Case: Lipschitz algebra norm

Let $C(Y)$ be the algebra of all continuous complex-valued functions on a compact Hausdorff space Y. The following proposition is a weaker version of the main theorem of Jarosz in [8] on surjective complex-linear isometries T with $T(\mathbf{1}) = \mathbf{1}$ between complex-linear subspaces of $C(Y)$ that contain constant functions equipped with certain natural norms. Instead of these assumptions on T, we will assume here that T is a surjective real-linear isometry with $T(\mathbf{1}) = \mathbf{1}$ and $T(i\mathbf{1}) = i\mathbf{1}$ or $-i\mathbf{1}$ between spaces $\operatorname{Lip}(X)$. We will apply this proposition to prove the main theorem of this paper.

We first need the following terminology and notation introduced in [8]. Let A be a complex-linear subspace of $C(Y)$ that contains the function $\mathbf{1}$. By $\operatorname{Ch} A$ we denote the Choquet boundary of A, that is, the subset of all points $x \in Y$ such that the evaluation functional at x, from A to $\mathbb{C}$, is an extreme point of the unit ball of $(A, \|\cdot\|_\infty)^*$. Recall that A is said to be regular if for any $\varepsilon > 0$, any $x_0 \in \operatorname{Ch} A$ and any open neighborhood U of x_0, there is an $f \in A$ with $\|f\|_\infty \leqslant 1 + \varepsilon$, $f(x_0) = 1$, and $|f(x)| < \varepsilon$ for $x \in Y \setminus U$.

It is known (see [16]) that $(\operatorname{Lip}(X), \|\cdot\|_\infty + \|\cdot\|_L, \mathbf{1})$ is a semisimple commutative Banach algebra with unit and the maximal ideal space of $\operatorname{Lip}(X)$ is homeomorphic to X. Then $\operatorname{Lip}(X)$ is a regular subspace of $C(X)$ by [8, Proposition 2].

If K and H are subsets of $\mathbb{C}$, we represent by $\operatorname{co}(K)$ the convex hull of K and

$$K + H = \{w + z\colon\ w \in K,\ z \in H\}.$$

If $f \in \operatorname{Lip}(X)$, we put $\widetilde{\sigma}(f) = \operatorname{co}(f(X))$. For $z_0 \in \mathbb{C}$ and $r \geqslant 0$, we write

$$K(z_0, r) = \bigl\{z \in \mathbb{C}\colon\ |z - z_0| \leqslant r\bigr\}, \qquad K(r) = K(0, r),$$

and, for $K \subset \mathbb{C}$ and $z_0 \in K$, we denote

$$\rho(K, z_0) = \sup\bigl\{r \geqslant 0\colon\ \exists z \in K,\ z_0 \in K(z, r) \subset K\bigr\},$$
$$\rho(K) = \inf\bigl\{\rho(K, z)\colon\ z \in K\bigr\}.$$

Proposition 7. *Let X_1 and X_2 be compact metric spaces and let T be a real-linear isometry from $(\operatorname{Lip}(X_1), \|\cdot\|_1)$ onto $(\operatorname{Lip}(X_2), \|\cdot\|_2)$, where $\|\cdot\|_j = \|\cdot\|_\infty + \|\cdot\|_L$ for $j = 1, 2$, with $T(\mathbf{1}) = \mathbf{1}$ and either $T(i\mathbf{1}) = i\mathbf{1}$ or $T(i\mathbf{1}) = -i\mathbf{1}$. Then T is an isometry from $(\operatorname{Lip}(X_1), \|\cdot\|_\infty)$ onto $(\operatorname{Lip}(X_2), \|\cdot\|_\infty)$.*

Proof. We only give a proof when $T(i\mathbf{1}) = i\mathbf{1}$. The case $T(i\mathbf{1}) = -i\mathbf{1}$ can be deduced from the case $T(i\mathbf{1}) = i\mathbf{1}$ considering the mapping $\overline{T}$ from $\operatorname{Lip}(X_1)$ onto $\operatorname{Lip}(X_2)$ defined by $\overline{T}(f) = \overline{T(f)}$ for every $f \in \operatorname{Lip}(X_1)$.

We follow essentially the proof of [8, Theorem] although some parts have to be revised to fit for our T. For any nonempty bounded convex subset $K \subset \mathbb{C}$ and any $\varphi \in [0, 2\pi)$, define

$$c(K, \varphi) = \sup\bigl\{a \in \mathbb{R}\colon\ \text{there is a } b \in \mathbb{R} \text{ with } (a + ib)e^{i\varphi} \in K\bigr\}.$$

For $j = 1, 2$, define the functions

$$c_j : \operatorname{Lip}(X_j) \times [0, 2\pi) \to \mathbb{R}, \qquad c_j(f, \varphi) = c\bigl(\widetilde{\sigma}(f), \varphi\bigr),$$

and

$$r_j : \operatorname{Lip}(X_j) \times \mathbb{R}^+ \times [0, 2\pi) \to \mathbb{R}^+, \qquad r_j(f, t, \varphi) = \bigl\|f + e^{i\varphi} t\mathbf{1}\bigr\|_\infty.$$

For every $\varphi \in [0, 2\pi)$, $f \in \mathrm{Lip}(X_j)$ and $t \in \mathbb{R}^+$, we have

$$t + c_j(f, \varphi) \leqslant r_j(f, t, \varphi) \leqslant \sqrt{\big(t + c_j(f, \varphi)\big)^2 + \|f\|_\infty^2},$$

and therefore

$$\lim_{t \to +\infty} \big(r_j(f, t, \varphi) - t\big) = c_j(f, \varphi). \tag{3.1}$$

Fix $f \in \mathrm{Lip}(X_1)$. Using that T is a real-linear isometry, $T(\mathbf{1}) = \mathbf{1}$ and $T(i\mathbf{1}) = i\mathbf{1}$, a simple calculation yields

$$r_1(f, t, \varphi) + \|f\|_L = r_2\big(T(f), t, \varphi\big) + \big\|T(f)\big\|_L$$

for any $t \in \mathbb{R}^+$ and $\varphi \in [0, 2\pi)$. Using (3.1), it follows that

$$c_2\big(T(f), \varphi\big) - c_1(f, \varphi) = \|f\|_L - \big\|T(f)\big\|_L \tag{3.2}$$

for all $f \in \mathrm{Lip}(X_1)$ and $\varphi \in [0, 2\pi)$.

For every $f \in \mathrm{Lip}(X_1)$, set $\Delta f = \|f\|_L - \|T(f)\|_L$. Since T is an isometry from $(\mathrm{Lip}(X_1), \|\cdot\|_1)$ onto $(\mathrm{Lip}(X_2), \|\cdot\|_2)$, we get that

$$\Delta f = \big\|T(f)\big\|_\infty - \|f\|_\infty. \tag{3.3}$$

For any $r \geqslant 0$ and any nonempty compact convex subset $K \subset \mathbb{C}$, we have that

$$c\big(K + K(r), \varphi\big) = c(K, \varphi) + r \tag{3.4}$$

for all $\varphi \in [0, 2\pi)$. By (3.2) and [8, Lemma 1], we have

$$\begin{aligned} \Delta f \geqslant 0 \quad &\Rightarrow \quad \widetilde{\sigma}\big(T(f)\big) = \widetilde{\sigma}(f) + K(\Delta f), \\ \Delta f \leqslant 0 \quad &\Rightarrow \quad \widetilde{\sigma}(f) = \widetilde{\sigma}\big(T(f)\big) + K(-\Delta f). \end{aligned} \tag{3.5}$$

Since T^{-1} satisfies the same conditions as T, the proof will be finished if we show that

$$\big\|T(f)\big\|_\infty - \|f\|_\infty = \Delta f \geqslant 0 \tag{3.6}$$

for all $f \in \mathrm{Lip}(X_1)$. For every $\varepsilon > 0$, denote

$$\mathcal{A}_\varepsilon = \big\{f \in \mathrm{Lip}(X_1)\colon\ \rho\big(\widetilde{\sigma}(f)\big) \leqslant \varepsilon\big\}.$$

The inequality in (3.6) follows from the following assertions:

(1) T is a continuous mapping from $(\mathrm{Lip}(X_1), \|\cdot\|_\infty)$ onto $(\mathrm{Lip}(X_2), \|\cdot\|_\infty)$.
(2) For each $\varepsilon > 0$, the set $\mathcal{A}_\varepsilon$ is dense in $(\mathrm{Lip}(X_1), \|\cdot\|_\infty)$.
(3) For each $\varepsilon > 0$ and each $f \in \mathcal{A}_\varepsilon$, we have that $\|T(f)\|_\infty \geqslant \|f\|_\infty - \varepsilon$.

The proof of the second and third assertions is the same as in the proof of [8, Theorem]. The proof of the first one is slightly different from the corresponding in [8, p. 69]. This change is rather ambitious. We also point out that the terms $-\pi/2$ and $\pi/2$ which appear in the formulae (7) and (8) in [8] seem not be appropriate; they read, for example, as $3\pi/4$ and $\pi/4$, respectively.

We now proceed to prove the first statement. Aiming for a contradiction, suppose that T is not continuous from $(\mathrm{Lip}(X_1), \|\cdot\|_\infty)$ to $(\mathrm{Lip}(X_2), \|\cdot\|_\infty)$. Let ε be a positive real number less than $1/100$. Then there is a function $f_0 \in \mathrm{Lip}(X_1)$ such that $\|f_0\|_\infty \leqslant \varepsilon$ and $\|T(f_0)\|_\infty = 1$. Then there exist $y_0 \in X_2$ and $\varphi_0 \in [0, 2\pi)$ such that $T(f_0)(y_0) = e^{i\varphi_0}$. Note that if T is complex-linear, we may assume without loss of generality that $\varphi_0 = 0$ as in [8], but we cannot assume this here for our T.

From (3.3) and (3.5), we deduce that $\Delta f_0 = \|T(f_0)\|_\infty - \|f_0\|_\infty \geqslant 1-\varepsilon$ and $\widetilde{\sigma}(T(f_0)) = \widetilde{\sigma}(f_0) + K(\Delta f_0)$. Thus we have

$$K(1-2\varepsilon) \subset \widetilde{\sigma}\bigl(T(f_0)\bigr) \subset K(1). \tag{3.7}$$

Consider the open neighborhood U_0 of y_0 in X_2 given by

$$U_0 = \bigl\{y \in X_2\colon\ \bigl|T(f_0)(y) - e^{i\varphi_0}\bigr| < \varepsilon\bigr\}.$$

We infer that U_0 is a proper subset of X_2 by (3.7). Then, by [8, Lemma 2], there exists $g \in \mathrm{Lip}(X_2)$ such that $\|g\|_\infty \leqslant 1+\varepsilon$, $g(y_0) = 1$, $|g(y)+1| < \varepsilon$ for every $y \in X_2 \backslash U_0$ and $|\operatorname{Im} g(y)| < \varepsilon$ for all $y \in X_2$. If H denotes the closed rectangle whose vertices are the four points $\pm(1+\varepsilon) \pm \varepsilon i$, we have

$$\widetilde{\sigma}(g) \subset H. \tag{3.8}$$

Consider now the set

$$L = \bigl\{e^{i(3\pi/4+\varphi_0)}z\colon\ |z| \leqslant 1,\ \operatorname{Re} z \geqslant 1-2\varepsilon\bigr\}.$$

We claim that $T(f_0)(X_2) \cap L \neq \emptyset$. Suppose that $T(f_0)(X_2) \cap L = \emptyset$. Then (3.7) gives $T(f_0)(X_2) \subset K(1)\backslash L$. Hence $\widetilde{\sigma}(T(f_0))$ is contained in the convex set $K(1)\backslash L$. On the other hand, $(1-2\varepsilon)e^{i(3\pi/4+\varphi_0)} \in K(1-2\varepsilon) \subset \widetilde{\sigma}(T(f_0))$ by (3.7). As $(1-2\varepsilon)e^{i(3\pi/4+\varphi_0)} \in L$, this contradicts to $\widetilde{\sigma}(T(f_0)) \subset K(1)\backslash L$, and this proves our claim. Hence there is $y \in X_2$ with $T(f_0)(y) \in L$. As $\varepsilon \leqslant 1/100$, it follows that $|T(f_0)(y) - e^{i\varphi_0}| \geqslant \varepsilon$ and so $y \in X_2 \backslash U_0$. Hence

$$\bigl|\bigl(T(f_0)(y) - e^{i\varphi_0}\bigr) - \bigl(e^{i\varphi_0}g(y) + T(f_0)(y)\bigr)\bigr| = \bigl|g(y)+1\bigr| < \varepsilon,$$

and this says us that $e^{i\varphi_0}g(y) + T(f_0)(y)$ is in $K(T(f_0)(y) - e^{i\varphi_0}, \varepsilon)$. Then $e^{i\varphi_0}g(y) + T(f_0)(y)$ is in $L - e^{i\varphi_0} + K(\varepsilon)$. Thus we have

$$1 + \frac{\sqrt{2}}{2} - 3\varepsilon \leqslant c_2\biggl(e^{i\varphi_0}g + T(f_0), \frac{3\pi}{4} + \varphi_0\biggr). \tag{3.9}$$

We claim that

$$\widetilde{\sigma}\bigl(e^{i\varphi_0}g + T(f_0)\bigr) \subset \operatorname{co}\bigl(K\bigl(-e^{i\varphi_0}, 1\bigr) \cup \bigl\{2e^{i\varphi_0}\bigr\}\bigr) + K(3\varepsilon).$$

Let $x \in X_2$. We distinguish two cases. Suppose first that $|T(f_0)(x) - e^{i\varphi_0}| < \varepsilon$. Since $e^{i\varphi_0}g(X_2) \subset e^{i\varphi_0}H$ by (3.8), we have

$$T(f_0)(x) + e^{i\varphi_0}g(x) \in K\bigl(e^{i\varphi_0}, \varepsilon\bigr) + e^{i\varphi_0}H = e^{i\varphi_0}(H+1) + K(\varepsilon). \tag{3.10}$$

Secondly suppose that $|T(f_0)(x)-e^{i\varphi_0}| \geqslant \varepsilon$. Then $x \in X_2 \setminus U_0$ and so $|e^{i\varphi_0}g(x)+e^{i\varphi_0}| < \varepsilon$. Hence $|e^{i\varphi_0}g(x)+T(f_0)(x) - (T(f_0)(x) - e^{i\varphi_0})| < \varepsilon$ and thus $e^{i\varphi_0}g(x) + T(f_0)(x)$ is in $K(T(f_0)(x) - e^{i\varphi_0}, \varepsilon)$. Moreover, $|T(f_0)(x)| \leqslant 1$. Therefore we have

$$e^{i\varphi_0}g(x) + T(f_0)(x) \in K(1) - e^{i\varphi_0} + K(\varepsilon) = K\bigl(-e^{i\varphi_0}, 1\bigr) + K(\varepsilon). \tag{3.11}$$

It follows from (3.10) and (3.11) that

$$\bigl(e^{i\varphi_0}g + T(f_0)\bigr)(X_2) \subset \bigl(K\bigl(-e^{i\varphi_0}, 1\bigr) \cup e^{i\varphi_0}(H+1)\bigr) + K(\varepsilon).$$

Furthermore, it is easy to see that $H \subset \mathrm{co}(K(-2,1) \cup \{1\}) + K(2\varepsilon)$, whereupon

$$K\bigl(-e^{i\varphi_0}, 1\bigr) \cup e^{i\varphi_0}(H+1) \subset \mathrm{co}\bigl(K\bigl(-e^{i\varphi_0}, 1\bigr) \cup \bigl\{2e^{i\varphi_0}\bigr\}\bigr) + K(2\varepsilon).$$

Hence

$$\widetilde{\sigma}\bigl(e^{i\varphi_0}g + T(f_0)\bigr) \subset \mathrm{co}\bigl(K\bigl(-e^{i\varphi_0}, 1\bigr) \cup \bigl\{2e^{i\varphi_0}\bigr\}\bigr) + K(3\varepsilon)$$

as is claimed. Therefore we have

$$c_2\left(e^{i\varphi_0}g + T(f_0), \frac{\pi}{4} + \varphi_0\right) \leqslant \sqrt{2} + 3\varepsilon. \tag{3.12}$$

Put $f_1 = T^{-1}(e^{i\varphi_0}g)$. We claim that $\Delta f_1 \leqslant \varepsilon$. If $\Delta f_1 < 0$, there is nothing to prove. Suppose that $\Delta f_1 \geqslant 0$. Then, by (3.5), we have

$$\widetilde{\sigma}\bigl(e^{i\varphi_0}g\bigr) = \widetilde{\sigma}(f_1) + K(\Delta f_1). \tag{3.13}$$

Since $\widetilde{\sigma}(e^{i\varphi_0}g) \subset e^{i\varphi_0}H$ by (3.8), it follows that $e^{i\varphi_0}H \supset \widetilde{\sigma}(f_1) + K(\Delta f_1)$. As $e^{i\varphi_0}H$ does not include a closed disk with the radius greater than ε, we conclude that $\Delta f_1 \leqslant \varepsilon$.

In the following we will consider two cases: $0 \leqslant \Delta f_1 \leqslant \varepsilon$ and $\Delta f_1 < 0$. Suppose first that $0 \leqslant \Delta f_1 \leqslant \varepsilon$. Then (3.8) and (3.13) yield

$$e^{i\varphi_0}H \supset \widetilde{\sigma}\bigl(e^{i\varphi_0}g\bigr) = \widetilde{\sigma}(f_1) + K(\Delta f_1) \supset \widetilde{\sigma}(f_1).$$

From $\|f_0\|_\infty \leqslant \varepsilon$ we deduce that $\widetilde{\sigma}(f_0) \subset K(\varepsilon)$. From (3.4) we infer that

$$\begin{aligned} c_1\left(f_1 + f_0, \frac{3\pi}{4} + \varphi_0\right) &\leqslant c\left(e^{i\varphi_0}H + K(\varepsilon), \frac{3\pi}{4} + \varphi_0\right) \\ &= c\left(e^{i\varphi_0}H, \frac{3\pi}{4} + \varphi_0\right) + \varepsilon \\ &= \frac{\sqrt{2}}{2} + (1+\sqrt{2})\varepsilon. \end{aligned} \tag{3.14}$$

By (3.13) and $e^{i\varphi_0} = e^{i\varphi_0}g(y_0)$, we deduce that $e^{i\varphi_0} \in \widetilde{\sigma}(f_1) + K(\Delta f_1)$. Thus there is $z \in \widetilde{\sigma}(f_1)$ such that $|z - e^{i\varphi_0}| \leqslant \Delta f_1$. It follows that $\sqrt{2}/2 - \Delta f_1 \leqslant c_1(f_1, \pi/4 + \varphi_0)$, hence we have

$$\frac{\sqrt{2}}{2} - 2\varepsilon \leqslant c_1\left(f_1 + f_0, \frac{\pi}{4} + \varphi_0\right) \tag{3.15}$$

as $\|f_0\|_\infty \leqslant \varepsilon$ and $0 \leqslant \Delta f_1 \leqslant \varepsilon$.

Since $T(f_1+f_0)=e^{i\varphi_0}g+T(f_0)$, from (3.9) and (3.14) we obtain that

$$1-(4+\sqrt{2})\varepsilon \leqslant c_2\left(T(f_1+f_0),\frac{3\pi}{4}+\varphi_0\right)-c_1\left(f_1+f_0,\frac{3\pi}{4}+\varphi_0\right). \tag{3.16}$$

We also get by (3.12) and (3.15) that

$$c_2\left(T(f_1+f_0),\frac{\pi}{4}+\varphi_0\right)-c_1\left(f_1+f_0,\frac{\pi}{4}+\varphi_0\right)\leqslant \frac{\sqrt{2}}{2}+5\varepsilon. \tag{3.17}$$

On the other hand, $c_2(T(f_1+f_0),\varphi)-c_1(f_1+f_0,\varphi)$ does not depend on φ by (3.2). From (3.16) and (3.17) we deduce that $\varepsilon \geqslant (2-\sqrt{2})/2(9+\sqrt{2})$ and this contradicts that $\varepsilon \leqslant 1/100$.

For the second case, suppose next that $\Delta f_1<0$. Then, by (3.5), we have

$$\widetilde{\sigma}(f_1)=\widetilde{\sigma}\big(e^{i\varphi_0}g\big)+K(-\Delta f_1), \tag{3.18}$$

and, by (3.8), it follows that $\widetilde{\sigma}(f_1)\subset e^{i\varphi_0}H+K(-\Delta f_1)$. Moreover, $\widetilde{\sigma}(f_0)\subset K(\varepsilon)$ since $\|f_0\|_\infty \leqslant \varepsilon$. Using (3.4), we infer that

$$\begin{aligned} c_1\left(f_1+f_0,\frac{3\pi}{4}+\varphi_0\right) &\leqslant c\left(e^{i\varphi_0}H+K(-\Delta f_1)+K(\varepsilon),\frac{3\pi}{4}+\varphi_0\right)\\ &= c\left(e^{i\varphi_0}H,\frac{3\pi}{4}+\varphi_0\right)+(-\Delta f_1)+\varepsilon\\ &= \frac{\sqrt{2}}{2}+(1+\sqrt{2})\varepsilon+(-\Delta f_1). \end{aligned} \tag{3.19}$$

By (3.18), we obtain that $\widetilde{\sigma}(f_1)\supset e^{i\varphi_0}g(X_2)+K(-\Delta f_1)$, and as $e^{i\varphi_0}g(y_0)=e^{i\varphi_0}$, we infer that $\widetilde{\sigma}(f_1)\supset e^{i\varphi_0}+K(-\Delta f_1)$. Hence $\sqrt{2}/2+(-\Delta f_1)\leqslant c_1(f_1,\pi/4+\varphi_0)$, so that

$$\frac{\sqrt{2}}{2}+(-\Delta f_1)-\varepsilon \leqslant c_1\left(f_1+f_0,\frac{\pi}{4}+\varphi_0\right) \tag{3.20}$$

as $\|f_0\|_\infty\leqslant\varepsilon$. Since $T(f_1+f_0)=e^{i\varphi_0}g+T(f_0)$, we obtain by (3.9) and (3.19) that

$$1-(4+\sqrt{2})\varepsilon-(-\Delta f_1)\leqslant c_2\left(T(f_1+f_0),\frac{3\pi}{4}+\varphi_0\right)-c_1\left(f_1+f_0,\frac{3\pi}{4}+\varphi_0\right). \tag{3.21}$$

We also obtain by (3.12) and (3.20) that

$$c_2\left(T(f_1+f_0),\frac{\pi}{4}+\varphi_0\right)-c_1\left(f_1+f_0,\frac{\pi}{4}+\varphi_0\right)\leqslant\frac{\sqrt{2}}{2}+4\varepsilon-(-\Delta f_1). \tag{3.22}$$

Since $c_2(T(f_1+f_0),\varphi)-c_1(f_1+f_0,\varphi)$ does not depend on φ by (3.2), from (3.21) and (3.22) we deduce that $\varepsilon\geqslant(2-\sqrt{2})/2(8+\sqrt{2})$ and this is impossible since $\varepsilon\leqslant 1/100$. This completes the proof of the proposition. □

The following is the main result in this paper.

Theorem 8. *Let X_1 and X_2 be compact metric spaces, let Φ be a mapping from* $\exp\mathrm{Lip}(X_1)$ *into* $\exp\mathrm{Lip}(X_2)$ *and let $\|\cdot\|_j=\|\cdot\|_\infty+\|\cdot\|_L$ for $j=1,2$. Then Φ is a surjective mapping that satisfies the non-symmetric-quotient norm condition for the Lipschitz algebra norm:*

$$\left\|\frac{g}{f}-\mathbf{1}\right\|_1=\left\|\frac{\Phi(g)}{\Phi(f)}-\mathbf{1}\right\|_2,\quad \forall f,g\in\exp\mathrm{Lip}(X_1),$$

if and only if there exists a surjective isometry $\phi: X_2\to X_1$ *such that*

$$\Phi(f)(y)=\Phi(\mathbf{1})(y)f\bigl(\phi(y)\bigr)$$

for all $y\in X_2$ *and* $f\in\exp\mathrm{Lip}(X_1)$, *or*

$$\Phi(f)(y)=\Phi(\mathbf{1})(y)\overline{f\bigl(\phi(y)\bigr)}$$

for all $y\in X_2$ *and* $f\in\exp\mathrm{Lip}(X_1)$. *If, in addition,* $\Phi(\mathbf{1})=\mathbf{1}$, *then* Φ *is extendable to either an isometric complex-linear algebra isomorphism or an isometric conjugate-linear algebra isomorphism.*

From the description given for Φ, we give sufficient conditions for Φ is extendable to be an isometrical algebra isomorphism.

Corollary 9. *Let* X_1 *and* X_2 *be compact metric spaces and let* Φ *be a surjective mapping from* $\exp\mathrm{Lip}(X_1)$ *to* $\exp\mathrm{Lip}(X_2)$ *satisfying the non-symmetric-quotient norm condition for the Lipschitz algebra norm. Then the following assertions are satisfied*:

(1) *If* $\Phi(\mathbf{1})=\mathbf{1}$ *and* $\Phi(\mathbf{1}i)=\mathbf{1}i$, *then* Φ *is extendable to an isometrical complex-linear algebra isomorphism.*
(2) *If* $\Phi(\mathbf{1})=\mathbf{1}$ *and* $\Phi(\mathbf{1}i)=-\mathbf{1}i$, *then* Φ *is extendable to an isometrical conjugate-linear algebra isomorphism.*

Proof of Theorem 8. Suppose that Φ has the form of a weighted composition operator as in the statement of Theorem 8. Using that ϕ is bi-Lipschitz, we infer that Φ is surjective. A simple calculation shows that Φ satisfies the non-symmetric-quotient norm condition.

Suppose conversely that Φ is surjective and obeys the non-symmetric-quotient norm condition. For $j=1,2$, define

$$d_j(f,g)=\left\|\frac{g}{f}-\mathbf{1}\right\|_j+\left\|\frac{f}{g}-\mathbf{1}\right\|_j\quad\bigl(f,g\in\exp\mathrm{Lip}(X_j)\bigr).$$

Clearly, $d_j(f,g)\geqslant 0$, and $d_j(f,g)=0$ holds only if $f=g$. Now define

$$\Phi_0(f)=\frac{\Phi(f)}{\Phi(\mathbf{1})}\quad\bigl(f\in\exp\mathrm{Lip}(X_1)\bigr).$$

By an easy verification we deduce that $\Phi_0:\exp\mathrm{Lip}(X_1)\to\exp\mathrm{Lip}(X_2)$ is bijective and satisfies the equality $d_2(\Phi_0(f),\Phi_0(g))=d_1(f,g)$ for all $f,g\in\exp\mathrm{Lip}(X_1)$. We claim that

$$\Phi_0(gfg)=\Phi_0(g)\Phi_0(f)\Phi_0(g)$$

for every pair $f,g\in\exp\mathrm{Lip}(X_1)$. We will use [3, Corollary 3.9] to prove this equality. Let $f=\exp(u)$ and $g=\exp(v)$ be in $\exp\mathrm{Lip}(X_1)$ for $u,v\in\mathrm{Lip}(X_1)$. Let ε be a positive real number with $\varepsilon(3\varepsilon/2+5)<1/4$. We infer there is a positive integer n with

$$\left\|\exp\left(\frac{\pm(u-v)}{2^{n-1}}\right)-\mathbf{1}\right\|_\infty<\frac{\varepsilon}{4},\qquad\left\|\exp\left(\frac{\pm(u-v)}{2^{n-1}}\right)-\mathbf{1}\right\|_L<\frac{\varepsilon}{4}.\tag{3.23}$$

For $0 \leqslant k \leqslant 2^n$, put

$$f_k = \exp\left(u - \frac{k(u-v)}{2^{n-1}}\right).$$

Then $f_0 = f$, $f_{2^{n-1}} = g$ and $f_{2^n} = gf^{-1}g$. We also have $f_{k+2} = f_{k+1}f_k^{-1}f_{k+1}$ for $0 \leqslant k \leqslant 2^n - 2$. For $0 \leqslant k \leqslant 2^n - 2$, set

$$L_{f_k, f_{k+1}} = \bigl\{h \in \exp \mathrm{Lip}(X_1)\colon\ d_1(f_k, h) = d_1(f_{k+2}, h) = d_1(f_k, f_{k+1})\bigr\}.$$

Note that $d_1(f_{k+2}, f_{k+1}) = d(f_k, f_{k+1})$, hence $f_{k+1} \in L_{f_k, f_{k+1}}$. Note also that $d_1(f_k, f_{k+1}) < \varepsilon$ by (3.23) since $f_{k+1}/f_k = \exp(-(u-v)/2^{n-1})$. We first observe that $d_1(h, f_{k+1}) < 1/4$ for every $h \in L_{f_k, f_{k+1}}$. To prove this, let $h \in L_{f_k, f_{k+1}}$. Since

$$\max\left\{\left\|\frac{f_k}{h}\right\|_L, \left\|\frac{f_k}{h} - \mathbf{1}\right\|_\infty\right\} \leqslant d_1(f_k, h) = d_1(f_k, f_{k+1}) < \varepsilon,$$

we have

$$\begin{aligned}
\left\|\frac{f_{k+1}}{h} - \mathbf{1}\right\|_L &\leqslant \left\|\frac{f_k}{h}\right\|_L \left\|\exp\left(\frac{-(u-v)}{2^{n-1}}\right)\right\|_\infty + \left\|\frac{f_k}{h}\right\|_\infty \left\|\exp\left(\frac{-(u-v)}{2^{n-1}}\right)\right\|_L \\
&\leqslant d_1(f_k, f_{k+1})\left(\left\|\exp\left(\frac{-(u-v)}{2^{n-1}}\right) - \mathbf{1}\right\|_\infty + 1\right) \\
&\quad + \left(\left\|\frac{f_k}{h} - \mathbf{1}\right\|_\infty + 1\right)\left\|\exp\left(\frac{-(u-v)}{2^{n-1}}\right)\right\|_L \\
&\leqslant d_1(f_k, f_{k+1})\left(\frac{\varepsilon}{4} + 1\right) + \bigl(d_1(f_k, f_{k+1}) + 1\bigr)\frac{\varepsilon}{4} \leqslant \varepsilon\left(\frac{5}{4} + \frac{\varepsilon}{2}\right).
\end{aligned}$$

In a similar way we obtain

$$\left\|\frac{h}{f_{k+1}} - \mathbf{1}\right\|_L \leqslant \varepsilon\left(\frac{5}{4} + \frac{\varepsilon}{2}\right).$$

On the other hand, we check that

$$\begin{aligned}
\left\|\frac{f_{k+1}}{h} - \mathbf{1}\right\|_\infty &= \left\|\frac{f_k \exp(\frac{-(u-v)}{2^{n-1}})}{h} - \mathbf{1}\right\|_\infty \\
&\leqslant \left\|\frac{f_k}{h} - \mathbf{1}\right\|_\infty \left(\left\|\exp\left(\frac{-(u-v)}{2^{n-1}}\right) - \mathbf{1}\right\|_\infty + 1\right) + \left\|\exp\left(\frac{-(u-v)}{2^{n-1}}\right) - \mathbf{1}\right\|_\infty \\
&\leqslant d_1(f_k, h)\left(\frac{\varepsilon}{4} + 1\right) + \frac{\varepsilon}{4} \leqslant \varepsilon\left(\frac{\varepsilon}{4} + \frac{5}{4}\right).
\end{aligned}$$

Similarly, we get

$$\left\|\frac{h}{f_{k+1}} - \mathbf{1}\right\|_\infty \leqslant \varepsilon\left(\frac{\varepsilon}{4} + \frac{5}{4}\right).$$

Finally, we obtain the desired inequality

$$d_1(h, f_{k+1}) \leqslant 2\varepsilon\left(\frac{5}{4} + \frac{\varepsilon}{2}\right) + 2\varepsilon\left(\frac{\varepsilon}{4} + \frac{5}{4}\right) = \varepsilon\left(\frac{3\varepsilon}{2} + 5\right) < \frac{1}{4} \tag{3.24}$$

for every $h \in L_{f_k,f_{k+1}}$. We also have

$$\begin{aligned}\left\|\left(\frac{f_{k+1}}{h}\right)^2 - \mathbf{1}\right\|_L &= \left\|\left(\frac{f_{k+1}}{h} - \mathbf{1}\right)\left(\frac{f_{k+1}}{h} - \mathbf{1} + 2\mathbf{1}\right)\right\|_L \\ &\geqslant 2\left\|\frac{f_{k+1}}{h} - \mathbf{1}\right\|_L - \left\|\left(\frac{f_{k+1}}{h} - \mathbf{1}\right)^2\right\|_L \\ &\geqslant 2\left\|\frac{f_{k+1}}{h} - \mathbf{1}\right\|_L - 2\left\|\frac{f_{k+1}}{h} - \mathbf{1}\right\|_\infty\left\|\frac{f_{k+1}}{h} - \mathbf{1}\right\|_L \\ &\geqslant 2\big(1 - d_1(h, f_{k+1})\big)\left\|\frac{f_{k+1}}{h} - \mathbf{1}\right\|_L,\end{aligned}$$

and

$$\left\|\left(\frac{h}{f_{k+1}}\right)^2 - \mathbf{1}\right\|_L \geqslant 2\big(1 - d_1(h, f_{k+1})\big)\left\|\frac{h}{f_{k+1}} - \mathbf{1}\right\|_L.$$

On the other hand, we get

$$\begin{aligned}\left\|\left(\frac{f_{k+1}}{h}\right)^2 - \mathbf{1}\right\|_\infty &\geqslant 2\left\|\frac{f_{k+1}}{h} - \mathbf{1}\right\|_\infty - \left\|\frac{f_{k+1}}{h} - \mathbf{1}\right\|_\infty^2 \\ &\geqslant 2\big(1 - d_1(h, f_{k+1})\big)\left\|\frac{f_{k+1}}{h} - \mathbf{1}\right\|_\infty,\end{aligned}$$

and

$$\left\|\left(\frac{h}{f_{k+1}}\right)^2 - \mathbf{1}\right\|_\infty \geqslant 2\big(1 - d_1(h, f_{k+1})\big)\left\|\frac{h}{f_{k+1}} - \mathbf{1}\right\|_\infty.$$

It follows that

$$\begin{aligned}d_1\big(f_{k+1}h^{-1}f_{k+1}, h\big) &\geqslant 2\big(1 - d_1(h, f_{k+1})\big)d_1(f_{k+1}, h) \\ &\geqslant 2\left(1 - \varepsilon\left(\frac{3\varepsilon}{2} + 5\right)\right)d_1(f_{k+1}, h) \\ &\geqslant \frac{3}{2}d_1(f_{k+1}, h)\end{aligned} \tag{3.25}$$

for every $h \in L_{f_k,f_{k+1}}$. By a simple calculation we have

$$d_1\big(f_{k+1}F^{-1}f_{k+1}, f_{k+1}G^{-1}f_{k+1}\big) = d_1(F, G) \tag{3.26}$$

for every $F, G \in \exp \mathrm{Lip}(X_1)$. By [3, Definition 3.2], we have proved that the pair $(\exp \mathrm{Lip}(X_1), d_1)$ satisfies the condition $\mathrm{B}(f_k, f_{k+1})$ for every $0 \leqslant k \leqslant 2^n - 2$ by (3.24), (3.25) and (3.26). Moreover, it is easy to see that the pair $(\exp \mathrm{Lip}(X_2), d_2)$ satisfies the condition $\mathrm{C}_1(\Phi_0(f_k), \Phi_0(f_{k+1}f_k^{-1}f_{k+1}))$ (cf. [3, Definition 3.3]). Then, by [3, Corollary 3.9], the equation

$$\Phi_0\big(f_{k+1}f_k^{-1}f_{k+1}\big) = \Phi_0(f_{k+1})\Phi_0(f_k)^{-1}\Phi_0(f_{k+1})$$

holds for every $0 \leqslant k \leqslant 2^n - 2$. Applying [3, Lemma 4.2], we deduce that

$$\Phi_0\big(f_{2^{n-1}}f_0^{-1}f_{2^{n-1}}\big) = \Phi_0(f_{2^{n-1}})\Phi_0(f_0)^{-1}\Phi_0(f_{2^{n-1}}).$$

Since $f_0 = f$ and $f_{2^{n-1}} = g$, we have

$$\Phi_0\bigl(gf^{-1}g\bigr) = \Phi_0(g)\Phi_0(f)^{-1}\Phi_0(g) \tag{3.27}$$

for every pair $f, g \in \exp \mathrm{Lip}(X_1)$. Letting $g = \mathbf{1}$ in (3.27) yields

$$\Phi_0\bigl(f^{-1}\bigr) = \Phi_0(f)^{-1} \tag{3.28}$$

for every $f \in \exp \mathrm{Lip}(X_1)$. Then, by (3.27), we conclude that

$$\Phi_0(gfg) = \Phi_0(g)\Phi_0(f)\Phi_0(g) \tag{3.29}$$

for every pair $f, g \in \exp \mathrm{Lip}(X_1)$, and this proves our claim.

Then, it is easy to deduce from (3.28) and (3.29) that

$$\Phi_0\bigl(f^n\bigr) = \Phi_0(f)^n \tag{3.30}$$

for every $f \in \exp \mathrm{Lip}(X_1)$ and $n \in \mathbb{Z}$.

Pick $u \in \mathrm{Lip}(X_1)$ and define $S_u : \mathbb{R} \to \exp \mathrm{Lip}(X_2)$ by

$$S_u(t) = \Phi_0\bigl(\exp(tu)\bigr).$$

We assert that S_u is a continuous one-parameter group with the values in $\exp \mathrm{Lip}(X_2)$. Suppose that $t_0 \in \mathbb{R}$ and $t \to t_0$. Then we check that

$$\left\| \frac{\exp(tu)}{\exp(t_0u)} - \mathbf{1} \right\|_\infty \to 0, \qquad \left\| \frac{\exp(tu)}{\exp(t_0u)} - \mathbf{1} \right\|_L \to 0$$

and

$$\left\| \frac{\exp(t_0u)}{\exp(tu)} - \mathbf{1} \right\|_\infty \to 0, \qquad \left\| \frac{\exp(t_0u)}{\exp(tu)} - \mathbf{1} \right\|_L \to 0,$$

hence

$$d_2\bigl(\Phi_0\bigl(\exp(tu)\bigr), \Phi_0\bigl(\exp(t_0u)\bigr)\bigr) = d_1\bigl(\exp(tu), \exp(t_0u)\bigr) \to 0$$

as $t \to t_0$. Hence S_u is continuous with respect to $\|\cdot\|_2$. Notice that $S_u(0) = \Phi_0(\mathbf{1}) = \mathbf{1}$. We now prove that $S_u(t + t') = S_u(t)S_u(t')$ for every $t, t' \in \mathbb{R}$. First select rational numbers n/m and n'/m' with integers m, m', n, n'. We compute

$$\begin{aligned}
S_u\left(\frac{n}{m} + \frac{n'}{m'}\right) &= \Phi_0\left(\exp\left(\frac{nm' + n'm}{mm'}u\right)\right) \\
&= \Phi_0\left(\exp\left(\frac{1}{mm'}u\right)\right)^{nm'+n'm} \\
&= \Phi_0\left(\exp\left(\frac{nm'}{mm'}u\right)\right)\Phi_0\left(\exp\left(\frac{n'm}{mm'}u\right)\right) \\
&= S_u\left(\frac{n}{m}\right)S_u\left(\frac{n'}{m'}\right).
\end{aligned}$$

Since S_u is continuous, we obtain that $S_u(t+t') = S_u(t)S_u(t')$ for all $t, t' \in \mathbb{R}$. Hence S_u is a continuous one-parameter group. Then, by [14, Proposition 6.4.6], there exists a unique $u' \in \mathrm{Lip}(X_2)$ such that $S_u(t) = \exp(tu')$ holds for every $t \in \mathbb{R}$.

Define a mapping $T : \mathrm{Lip}(X_1) \to \mathrm{Lip}(X_2)$ for which

$$\Phi_0\bigl(\exp(tu)\bigr) = S_u(t) = \exp\bigl(t\bigl(T(u)\bigr)\bigr) \quad \bigl(t \in \mathbb{R},\ u \in \mathrm{Lip}(X_1)\bigr).$$

Considering Φ_0^{-1} in the place of Φ_0, we infer that there is a mapping $T' : \mathrm{Lip}(X_2) \to \mathrm{Lip}(X_1)$ such that $\Phi_0^{-1}(\exp(tw)) = \exp(tT'(w))$ holds for every $w \in \mathrm{Lip}(X_2)$ and $t \in \mathbb{R}$. This easily implies that $w = T(T'(w))$ for all $w \in \mathrm{Lip}(X_2)$. Hence T is a surjection from $\mathrm{Lip}(X_1)$ onto $\mathrm{Lip}(X_2)$.

We next prove that T is an isometry from $(\mathrm{Lip}(X_1), \|\cdot\|_1)$ onto $(\mathrm{Lip}(X_2), \|\cdot\|_2)$. Since

$$\left\|\frac{\exp(tT(u))}{\exp(tT(v))} - \mathbf{1}\right\|_2 = \left\|\frac{\exp(tu)}{\exp(tv)} - \mathbf{1}\right\|_1$$

for all $t \in \mathbb{R}$ and $u, v \in \mathrm{Lip}(X_1)$, we obtain

$$\left\|\frac{\exp(t(T(u) - T(v))) - \mathbf{1}}{t}\right\|_2 = \left\|\frac{\exp(t(u - v)) - \mathbf{1}}{t}\right\|_1 \tag{3.31}$$

for $t \neq 0$. Given $j \in \{1, 2\}$ and $w \in \mathrm{Lip}(X_j)$, it is known that the function $t \mapsto \exp(tw)$ from $\mathbb{R}$ to $\mathrm{Lip}(X_j)$ is derivable and its derivative function is $t \mapsto w\exp(tw)$. In particular, the derivative of this function at 0 is w, that is, $\lim_{t\to 0}(\exp(tw) - \mathbf{1})/t = w$. Then $\lim_{t\to 0} \|(\exp(tw) - \mathbf{1})/t\|_j = \|w\|_j$. Letting $t \to 0$ for the both sides of Eq. (3.31), we obtain that $\|T(u) - T(v)\|_2 = \|u - v\|_1$ for every $u, v \in \mathrm{Lip}(X_1)$. Hence T is a surjective isometry from $(\mathrm{Lip}(X_1), \|\cdot\|_1)$ onto $(\mathrm{Lip}(X_2), \|\cdot\|_2)$. We denote by $\mathbf{0}$ the function constantly equal to 0. By the definition of T, $T(\mathbf{0}) = \mathbf{0}$ is easily to be deduced. Then the celebrated Mazur–Ulam theorem asserts that T is real-linear.

We claim that $T(\mathbf{1}) = \mathbf{1}$. In order to prove it, we first show that $\Phi_0(e^{1/n}\mathbf{1}) = e^{1/n}\mathbf{1}$ for all $n \in \mathbb{N}$. Suppose that $\|\Phi_0(e^{1/n}\mathbf{1})/e^{1/n}\mathbf{1}\|_\infty < 1$ for some $n \in \mathbb{N}$. Then we have

$$\left\|\left(\frac{\Phi_0(e^{1/n}\mathbf{1})}{e^{1/n}}\right)^m\right\|_L \leqslant m\left\|\frac{\Phi_0(e^{1/n}\mathbf{1})}{e^{1/n}}\right\|_\infty^{m-1}\left\|\frac{\Phi_0(e^{1/n}\mathbf{1})}{e^{1/n}}\right\|_L \to 0$$

as $m \to \infty$. Since $\Phi_0(\mathbf{1}) = \mathbf{1}$ and $\Phi_0(f^m) = \Phi_0(f)^m$ for any $f \in \exp \mathrm{Lip}(X_1)$ and $m \in \mathbb{N}$, we obtain that

$$\begin{aligned}
1 - e^{-m/n} &= \left\|\frac{e^{m/n}\mathbf{1} - \mathbf{1}}{e^{m/n}}\right\|_1 \\
&= \left\|\frac{\Phi_0(e^{m/n}\mathbf{1}) - \mathbf{1}}{e^{m/n}}\right\|_2 \\
&\leqslant \left\|\frac{\Phi_0(e^{1/n}\mathbf{1})}{e^{1/n}}\right\|_\infty^m + e^{-m/n} + \left\|\frac{\Phi_0(e^{m/n}\mathbf{1})}{e^{m/n}}\right\|_L \to 0
\end{aligned}$$

as $m \to \infty$, which is a contradiction. Thus

$$\bigl\|\Phi_0\bigl(e^{1/n}\mathbf{1}\bigr)\bigr\|_\infty \geqslant e^{1/n} \tag{3.32}$$

for all $n \in \mathbb{N}$. We compute

$$\begin{aligned} e^{1/n}-1 &= \big\|e^{1/n}\mathbf{1}-\mathbf{1}\big\|_1 \\ &= \big\|\Phi_0\big(e^{1/n}\mathbf{1}\big)-\mathbf{1}\big\|_2 \\ &\geqslant \big\|\Phi_0\big(e^{1/n}\mathbf{1}\big)\big\|_\infty - 1 + \big\|\Phi_0\big(e^{1/n}\mathbf{1}\big)-\mathbf{1}\big\|_L \\ &\geqslant e^{1/n}-1+\big\|\Phi_0\big(e^{1/n}\mathbf{1}\big)-\mathbf{1}\big\|_L. \end{aligned}$$

Hence we infer that $\|\Phi_0(e^{1/n}\mathbf{1})-\mathbf{1}\|_L=0$, and thus $\Phi_0(e^{1/n}\mathbf{1})$ is a constant function. By

$$e^{1/n}-1=\big\|e^{1/n}\mathbf{1}-\mathbf{1}\big\|_1=\big\|\Phi_0\big(e^{1/n}\mathbf{1}\big)-\mathbf{1}\big\|_2=\big\|\Phi_0\big(e^{1/n}\mathbf{1}\big)-\mathbf{1}\big\|_\infty$$

and (3.32), we obtain that $\Phi_0(e^{1/n}\mathbf{1})=e^{1/n}\mathbf{1}$. Thus, by the definition of T, it follows that

$$\exp\big(1/nT(\mathbf{1})\big)=\Phi_0\big(e^{1/n}\mathbf{1}\big)=e^{1/n}\mathbf{1}$$

for all $n\in\mathbb{N}$. Hence $n(e^{T(1)/n}-\mathbf{1})=n(e^{1/n}\mathbf{1}-\mathbf{1})$ for all $n\in\mathbb{N}$, and letting $n\to\infty$ we infer that $T(\mathbf{1})=\mathbf{1}$.

We claim that $T(i\mathbf{1})=i\mathbf{1}$ or $T(i\mathbf{1})=-i\mathbf{1}$. By the definition of T and (3.30), we obtain that $\Phi_0(-\mathbf{1})=\Phi_0(\exp(i\pi)\mathbf{1})=\exp(\pi T(i\mathbf{1}))$ and $\Phi_0(-\mathbf{1})^2=\mathbf{1}$. As Φ_0 is injective and $\Phi_0(\mathbf{1})=\mathbf{1}$, we deduce that the function $\Phi_0(-\mathbf{1})$ takes the value -1. On the other hand, we compute

$$2=\|-\mathbf{1}-\mathbf{1}\|_1=\big\|\Phi_0(-\mathbf{1})-\mathbf{1}\big\|_2=\big\|\Phi_0(-\mathbf{1})-\mathbf{1}\big\|_\infty+\big\|\Phi_0(-\mathbf{1})-\mathbf{1}\big\|_L.$$

As $\Phi_0(-\mathbf{1})$ takes the value -1, we obtain $\|\Phi_0(-\mathbf{1})-\mathbf{1}\|_\infty=2$, hence $\|\Phi_0(-\mathbf{1})-\mathbf{1}\|_L=0$, so that $\Phi_0(-\mathbf{1})$ is a constant function. As $\Phi_0(-\mathbf{1})$ takes the value -1, we conclude that $\Phi_0(-\mathbf{1})=-\mathbf{1}$. Thus

$$-\mathbf{1}=\Phi_0(-\mathbf{1})=\Phi_0\big(\exp(i\pi)\mathbf{1}\big)=\exp\big(\pi T(i\mathbf{1})\big).$$

Hence, for every $x\in X_2$, there is an integer l_x such that $T(i\mathbf{1})(x)=(2l_x+1)i$. Since T is an isometry, we compute

$$\sqrt{2}=\|i\mathbf{1}-\mathbf{1}\|_1=\big\|T(i\mathbf{1})-T(\mathbf{1})\big\|_2=\big\|T(i\mathbf{1})-\mathbf{1}\big\|_\infty+\big\|T(i\mathbf{1})-\mathbf{1}\big\|_L.$$

Since $T(i\mathbf{1})(x)=(2l_x+1)i$, we obtain $l_x=0$ or $l_x=-1$, and $\|T(i\mathbf{1})-\mathbf{1}\|_L=0$. Hence we infer that $T(i\mathbf{1})=i\mathbf{1}$ or $T(i\mathbf{1})=-i\mathbf{1}$.

By Proposition 7, we see that T is a surjective real-linear isometry from $(\mathrm{Lip}(X_1),\|\cdot\|_\infty)$ onto $(\mathrm{Lip}(X_2),\|\cdot\|_\infty)$. Hence T can be extended to a surjective real-linear isometry $\widetilde{T}$ from the uniform closure of $\mathrm{Lip}(X_1)$ onto the uniform closure of $\mathrm{Lip}(X_2)$. By the Stone–Weierstrass theorem, the uniform closure of $\mathrm{Lip}(X_j)$ is $C(X_j)$, the algebra of all complex-valued continuous functions on X_j, $j=1,2$. Thus $\widetilde{T}$ is a surjective real-linear isometry from $C(X_1)$ onto $C(X_2)$. Applying a theorem of Miura [11, Theorem 1.1], for example, there exists a homeomorphism ϕ from X_2 onto X_1 such that

$$\widetilde{T}(u)=u\circ\phi,\quad \forall u\in C(X_1)$$

if $T(i\mathbf{1})=i\mathbf{1}$, or

$$\widetilde{T}(u)=\bar{u}\circ\phi,\quad \forall u\in C(X_1)$$

if $T(i\mathbf{1})=-i\mathbf{1}$. By the definition of T, we obtain that

$$\Phi(f)=\Phi(\mathbf{1})\cdot(f\circ\phi),\quad \forall f\in\exp\mathrm{Lip}(X_1),$$

or

$$\Phi(f) = \Phi(\mathbf{1}) \cdot (\overline{f} \circ \phi), \quad \forall f \in \exp \mathrm{Lip}(X_1).$$

The rest of the proof is to observe that ϕ is an isometry. We can prove it in the same way as in the proof of [1, Theorem 2.1]. Since $T(\mathbf{0}) = \mathbf{0}$, T is an isometry from $(\mathrm{Lip}(X_1), \|\cdot\|_1)$ onto $(\mathrm{Lip}(X_2), \|\cdot\|_2)$, and also an isometry from $(\mathrm{Lip}(X_1), \|\cdot\|_\infty)$ onto $(\mathrm{Lip}(X_2), \|\cdot\|_\infty)$, it follows that $\|T(f)\|_L = \|f\|_L$ for every $f \in \mathrm{Lip}(X_1)$. Let $x, y \in X_2$. Consider the function $h_y : X_1 \to \mathbb{R}$ defined by

$$h_y(z) = d_1\bigl(z, \phi(y)\bigr) \quad (z \in X_1).$$

For all $z, w \in X_1$, we have

$$\bigl|h_y(z) - h_y(w)\bigr| = \bigl|d_1\bigl(z, \phi(y)\bigr) - d_1\bigl(w, \phi(y)\bigr)\bigr| \leqslant d_1(z, w).$$

Hence $h_y \in \mathrm{Lip}(X_1)$ and $\|h_y\|_L \leqslant 1$, so that $\|T(h_y)\|_L \leqslant 1$. Then

$$\begin{aligned} d_1\bigl(\phi(x), \phi(y)\bigr) &= \bigl|h_y\bigl(\phi(x)\bigr) - h_y\bigl(\phi(y)\bigr)\bigr| \\ &= \bigl|T(h_y)(x) - T(h_y)(y)\bigr| \\ &\leqslant d_2(x, y). \end{aligned}$$

Considering T^{-1} instead of T, we see in a similar way as above that $d_2(\phi^{-1}(z), \phi^{-1}(w)) \leqslant d_1(z, w)$ for every $z, w \in X_1$. It follows that $d_1(\phi(x), \phi(y)) = d_2(x, y)$ for all $x, y \in X_2$. $\square$

Remark 10. Given a real number $\alpha \in (0, 1)$ and a compact metric space (X, d), let $\mathrm{Lip}_\alpha(X)$ be the Banach algebra of all complex-valued functions f on X such that

$$\|f\|_{L_\alpha} = \sup\left\{\frac{|f(x) - f(y)|}{d(x, y)^\alpha} \colon\ x, y \in X,\ x \neq y\right\} < \infty,$$

endowed with the norm $\|f\|_\alpha = \|f\|_{L_\alpha} + \|f\|_\infty$. Define $\mathrm{lip}_\alpha(X)$ as the subset of $\mathrm{Lip}_\alpha(X)$ formed by all those functions f for which

$$\lim_{d(x,y) \to 0} \frac{|f(x) - f(y)|}{d(x, y)^\alpha} = 0.$$

Then $\mathrm{lip}_\alpha(X)$ is a closed subalgebra of $\mathrm{Lip}_\alpha(X)$ with maximal ideal space X and unity $\mathbf{1}$ (see [17]).

Let X_1 and X_2 be compact metric spaces and α in $(0, 1)$. An analogous result to Theorem 1 can be stated with a similar proof for surjections $\Phi : \exp \mathrm{lip}_\alpha(X_1) \to \exp \mathrm{lip}_\alpha(X_2)$ satisfying the non-symmetric-quotient norm condition for the uniform norm.

Analogously to Proposition 7, we may also show that if T is a real-linear isometry from $(\mathrm{lip}_\alpha(X_1), \|\cdot\|_\alpha)$ onto $(\mathrm{lip}_\alpha(X_2), \|\cdot\|_\alpha)$ with $T(\mathbf{1}) = \mathbf{1}$ and either $T(i\mathbf{1}) = i\mathbf{1}$ or $T(i\mathbf{1}) = -i\mathbf{1}$, then T is an isometry from $(\mathrm{lip}_\alpha(X_1), \|\cdot\|_\infty)$ onto $(\mathrm{lip}_\alpha(X_2), \|\cdot\|_\infty)$. Using this result and following steps analogous to those of the proof of Theorem 8 above, we obtain that if Φ is a mapping from $\exp \mathrm{lip}_\alpha(X_1)$ to $\exp \mathrm{lip}_\alpha(X_2)$, then Φ is a surjection satisfying the equality

$$\left\|\frac{g}{f} - \mathbf{1}\right\|_\alpha = \left\|\frac{\Phi(g)}{\Phi(f)} - \mathbf{1}\right\|_\alpha, \quad \forall f, g \in \exp \mathrm{lip}_\alpha(X_1),$$

if and only if Φ is of the form either $\Phi(f) = \Phi(\mathbf{1}) \cdot (f \circ \phi)$ for all $f \in \exp \mathrm{lip}_\alpha(X_1)$, or $\Phi(f) = \Phi(\mathbf{1}) \cdot (\overline{f} \circ \phi)$ for all $f \in \exp \mathrm{lip}_\alpha(X_1)$, where $\phi : X_2 \to X_1$ is a surjective isometry. We only need to make a modification in the final part of the proof of Theorem 8 and substitute the functions h_y by the following functions h_{xy} (cf. the proof of [1, Theorem 2.1]). Fix $x, y \in X_2$, $x \neq y$, choose $\beta \in (\alpha, 1)$ and define $h_{xy} : X_1 \to \mathbb{R}$ by

$$h_{xy}(z) = \frac{d(z, \phi(y))^\beta - d(z, \phi(x))^\beta}{2d(\phi(x), \phi(y))^{\beta - \alpha}}.$$

Then $h_{xy} \in \mathrm{lip}_\alpha(X_1)$ and $\|h_{xy}\|_{L_\alpha} = 1$ (see [10, p. 62]). An easy verification gives

$$\begin{aligned} d\big(\phi(x), \phi(y)\big)^\alpha &= \big|h_{xy}\big(\phi(x)\big) - h_{xy}\big(\phi(y)\big)\big| \\ &= \big|T(h_{xy})(x) - T(h_{xy})(y)\big| \\ &\leqslant \big\|T(h_{xy})\big\|_{L_\alpha} d(x, y)^\alpha \\ &= d(x, y)^\alpha. \end{aligned}$$

Hence we have $d(\phi(x), \phi(y)) \leqslant d(x, y)$ for all $x, y \in X_2$.

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