



Existence of a lower bound for the distance between point masses of relative equilibria in spaces of constant curvature



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ARTICLE INFO

Article history:

Received 14 January 2014

Available online 22 February 2014

Submitted by P. Yao

Keywords:

n-body problems in spaces of constant curvature
Dynamical systems
Mathematics
n-body problems

ABSTRACT

We prove that if for the curved *n*-body problem the masses are given, the minimum distance between the point masses of a specific type of relative equilibrium solution to that problem has a universal lower bound that is not equal to zero. We furthermore prove that the set of all such relative equilibria is compact. This class of relative equilibria includes all relative equilibria of the curved *n*-body problem in \mathbb{H}^2 and a significant subset of the relative equilibria for \mathbb{S}^2 , \mathbb{S}^3 and \mathbb{H}^3 .

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1. Introduction

By *n*-body problems, we mean problems where we want to find the dynamics of *n* point particles. By relative equilibria, we mean solutions to such problems where the point particles represent rotating configurations of fixed size and shape. The *n*-body problem in spaces of constant curvature, or curved *n*-body problem is an extension of the Newtonian *n*-body problem (in Euclidean space) into spaces of nonzero, constant Gaussian curvature, which means that the space is either spherical (if the curvature is positive), or hyperbolic (if the curvature is negative) (see [11,9,10]). It was noted in [5] and [7] that it suffices to consider the case that the curvature is equal to either +1, or −1. More precisely, following [11,9,10,6,7], if we define the space

$$\mathbb{M}_\sigma^k = \{(x_1, \dots, x_{k+1}) \in \mathbb{R}^{k+1} \mid x_1^2 + \dots + x_k^2 + \sigma x_{k+1}^2 = \sigma\},$$

where σ equals either +1, or −1 and for $x, y \in \mathbb{M}_\sigma^k$ define the inner product

$$x \odot_k y = x_1 y_1 + \dots + x_k y_k + \sigma x_{k+1} y_{k+1},$$

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we mean the problem of finding the dynamics of n point particles with respective masses m_1, \dots, m_n and coordinates $q_1, \dots, q_n \in \mathbb{M}_\sigma^k$, $k \geq 2$, as described by the system of differential equations

$$\ddot{q}_i = \sum_{j=1, j \neq i}^n \frac{m_j(q_j - \sigma(q_i \odot_k q_j)q_i)}{(\sigma - \sigma(q_i \odot_k q_j)^2)^{\frac{3}{2}}} - \sigma(\dot{q}_i \odot_k \dot{q}_i)q_i, \quad i \in \{1, \dots, n\}. \quad (1.1)$$

The first to investigate n -body problems for spaces of constant curvature were Bolyai [1] and Lobachevsky [19], who independently proposed a curved 2-body problem in hyperbolic space \mathbb{H}^3 in the 1830s. Since then, n -body problems for spaces of constant curvature have been studied by mathematicians such as Dirichlet, Schering [21,20], Killing [12–14], Liebmann [16–18] and more recently Kozlov and Harin [15] and Cariñena, Rañada and Santander [2]. However, the study of n -body problems in spaces of constant curvature for the case that $n \geq 2$ started with [11,9,10] by Diacu, Pérez-Chavela and Santoprete. After this breakthrough, additional results for the $n \geq 2$ case were then obtained by Diacu [3,4,6], Diacu, Kordlou [7], Diacu, Pérez-Chavela [8]. For a more detailed historical overview, please see [4,6,5,7], or [11].

M. Shub proved for the Newtonian n -body problem that if we fix the masses and the angular velocity (i.e. the speed with which the angle of the rotation changes), the set of possible relative equilibria is compact and as a direct consequence that there exists a universal nonzero lower bound for the distance between the point particles of the relative equilibria in such a set (see [22]). Shub's results were a potential first step in what may lead to a proof of the famous sixth Smale problem (see [23]) which states that such sets are not only compact, but, in fact, finite.

In this paper, following Shub's line of thought, we will make a first attempt at investigating to which extent we can extend his results to the constant curvature case. More specifically, let

$$T(t) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$$

be a 2×2 rotation matrix, $A > 0$, $Q_1, \dots, Q_n \in \mathbb{R}^2$ and $Z \in \mathbb{R}^{k-1}$ constant. Then we will call any solution q_1, \dots, q_n of (1.1) of the form

$$q_i(t) = \begin{pmatrix} T(At)Q_i \\ Z \end{pmatrix}, \quad i \in \{1, \dots, n\}, \quad (1.2)$$

a *relative equilibrium* and A its *angular velocity*. Let $\|\cdot\|_p$ be the Euclidean norm on \mathbb{R}^p . Let $\epsilon > 0$ and let R_ϵ be the set of all relative equilibria in \mathbb{S}^k for which $\|Z\|_{k-1} > \epsilon$ together with all relative equilibria in \mathbb{H}^k . We will prove that

Theorem 1.1. *There exists a universal constant $C > 0$ such that for any relative equilibrium solution in R_ϵ of (1.1) $\|q_i - q_j\|_k > C$ for all $i, j \in \{1, \dots, n\}$, $i \neq j$ if the masses m_1, \dots, m_n are given.*

and

Theorem 1.2. *If we write any set of vectors $Q_1, \dots, Q_n \in \mathbb{R}^2$ of a relative equilibrium solution q_1, \dots, q_n in R_ϵ as a $2n$ -dimensional vector*

$$\begin{pmatrix} Q_1 \\ \vdots \\ Q_n \end{pmatrix},$$

then the set of all such $2n$ -dimensional vectors, for fixed masses m_1, \dots, m_n and angular velocity A , is compact in \mathbb{R}^{2n} .

Remark 1.3. Note that the definition of a relative equilibrium used in [Theorem 1.1](#) and [Theorem 1.2](#) includes all relative equilibria of the n -body problem in \mathbb{H}^2 , a subclass of the relative equilibria of the n -body problem in \mathbb{S}^2 , a subclass of the positive elliptic relative equilibria in \mathbb{S}^3 as defined in [\[6\]](#) and a subclass of the negative elliptic relative equilibria in \mathbb{H}^3 as defined in [\[6\]](#), which are two out of all four possible classes of relative equilibria in \mathbb{M}_σ^3 (see [\[6\]](#)). The restriction that the relative equilibria lie in \mathbb{R}_ϵ is needed to restrict relative equilibria in \mathbb{S}^k for which $Z = 0$, which is a special case that requires different techniques than used in this paper.

We will first formulate two lemmas, which will be done in [Section 2](#), that are related to Criterion 1 in [\[4\]](#) and then use those lemmas to prove [Theorem 1.1](#) in [Section 3](#) and [Theorem 1.2](#) in [Section 4](#).

2. Background theory

In order to formulate the aforementioned lemmas we need for the proofs of [Theorem 1.1](#) and [Theorem 1.2](#), we need to introduce some notation:

Let $m \in \mathbb{N}$. Let $\langle \cdot, \cdot \rangle_m$ be the Euclidean inner product on \mathbb{R}^m . Let $i, j \in \{1, \dots, n\}$. Let

$$q_1(t) = \begin{pmatrix} T(At)Q_1 \\ Z \end{pmatrix}, \quad \dots, \quad q_n(t) = \begin{pmatrix} T(At)Q_n \\ Z \end{pmatrix}$$

be a relative equilibrium, define $r := \|Q_i\|$ for all $i \in \{1, \dots, n\}$ and let α_i be the angle between Q_i and the first coordinate axis. Then the first lemma we will need is:

Lemma 2.1. *Let q_1, \dots, q_n be a relative equilibrium solution as in [\(1.2\)](#). Then*

$$0 = \sum_{j=1, j \neq i}^n \frac{m_j \sin(\alpha_i - \alpha_j)}{(1 - \cos(\alpha_i - \alpha_j))^{\frac{3}{2}} (2 - \sigma r^2 (1 - \cos(\alpha_i - \alpha_j)))^{\frac{3}{2}}}. \quad (2.1)$$

Proof. This lemma is a direct consequence of Criterion 1 in [\[24\]](#), but the proof for our case is very short, which is why it has been added here regardless:

Inserting our expressions for q_1, \dots, q_n into [\(1.1\)](#), using that $(T(At))'' = -A^2 T(At)$ and that $(T(At))' = AT(At) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, gives

$$\begin{aligned} \begin{pmatrix} -A^2 T(At)Q_i \\ \vec{0} \end{pmatrix} &= \sum_{j=1, j \neq i}^n \frac{m_j \left(\begin{pmatrix} T(At)Q_j \\ Z \end{pmatrix} - \sigma(q_i \odot_k q_j) \begin{pmatrix} T(At)Q_i \\ Z \end{pmatrix} \right)}{(\sigma - \sigma(q_i \odot_k q_j)^2)^{\frac{3}{2}}} \\ &\quad - \sigma(\dot{q}_i \odot_k \dot{q}_i) \begin{pmatrix} T(At)Q_i \\ Z \end{pmatrix}, \quad i \in \{1, \dots, n\}, \end{aligned} \quad (2.2)$$

where $\vec{0} \in \mathbb{R}^{k-2}$. Writing out the identities for the first two coordinates of the vectors of [\(2.2\)](#) gives

$$\begin{aligned} -A^2 T(At)Q_i &= \sum_{j=1, j \neq i}^n \frac{m_j (T(At)Q_j - \sigma(q_i \odot_k q_j) T(At)Q_i)}{(\sigma - \sigma(q_i \odot_k q_j)^2)^{\frac{3}{2}}} \\ &\quad - \sigma(\dot{q}_i \odot_k \dot{q}_i) T(At)Q_i, \quad i \in \{1, \dots, n\}. \end{aligned} \quad (2.3)$$

Multiplying both sides of [\(2.3\)](#) with $(T(At))^{-1}$ and consequently taking inner products at both sides with

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Q_i$$

gives

$$0 = \sum_{j=1, j \neq i}^n \frac{m_j \langle Q_j, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Q_i \rangle_2}{(\sigma - \sigma(q_i \odot_k q_j)^2)^{\frac{3}{2}}}, \quad i \in \{1, \dots, n\},$$

which can be rewritten as

$$0 = \sum_{j=1, j \neq i}^n \frac{m_j \|Q_j\| \|Q_i\| \sin(\alpha_j - \alpha_i)}{(\sigma - \sigma(\|Q_j\| \|Q_i\| \cos(\alpha_j - \alpha_i) + Z \odot_{k-2} Z)^2)^{\frac{3}{2}}}, \quad i \in \{1, \dots, n\}. \quad (2.4)$$

Using that $\sigma = q_i \odot_k q_i = \|Q_i\|_2^2 + Z \odot_{k-2} Z$ and that $\|Q_i\| = \|Q_j\| = r$ allows us to rewrite (2.4) as

$$0 = \sum_{j=1, j \neq i}^n \frac{m_j r^2 \sin(\alpha_j - \alpha_i)}{(\sigma - \sigma(r^2 \cos(\alpha_j - \alpha_i) + \sigma - r^2)^2)^{\frac{3}{2}}}, \quad i \in \{1, \dots, n\},$$

which means that

$$0 = \sum_{j=1, j \neq i}^n \frac{m_j \sin(\alpha_j - \alpha_i)}{(1 - \cos(\alpha_j - \alpha_i))^{\frac{3}{2}} (2 - \sigma r^2 (1 - \cos(\alpha_j - \alpha_i))^2)^{\frac{3}{2}}}, \quad i \in \{1, \dots, n\},$$

which completes the proof. \square

Lemma 2.2. For any relative equilibrium solution to (1.1)

$$\begin{pmatrix} T(At)Q_i \\ Z \end{pmatrix}, \quad i \in \{1, \dots, n\},$$

we have that if $Z \neq 0$, then

$$\sigma A^2 r^2 = \sum_{j=1, j \neq i}^n \frac{m_j (1 - \sigma(q_i \odot_k q_j))}{(\sigma - \sigma(q_i \odot_k q_j)^2)^{\frac{3}{2}}}, \quad i \in \{1, \dots, n\}.$$

Proof. Because of (2.2), we have that

$$\vec{0} = \sum_{j=1, j \neq i}^n \frac{m_j (Z - \sigma(q_i \odot_k q_j) Z)}{(\sigma - \sigma(q_i \odot_k q_j)^2)^{\frac{3}{2}}} - \sigma(\dot{q}_i \odot_k \dot{q}_i) Z, \quad i \in \{1, \dots, n\},$$

which can be rewritten as

$$\sigma(\dot{q}_i \odot_k \dot{q}_i) Z = \sum_{j=1, j \neq i}^n \frac{m_j (Z - \sigma(q_i \odot_k q_j) Z)}{(\sigma - \sigma(q_i \odot_k q_j)^2)^{\frac{3}{2}}}, \quad i \in \{1, \dots, n\}. \quad (2.5)$$

Because $Z \neq 0$, there has to be at least one nonzero entry of Z , so if we divide the identity in (2.5) for that entry by that entry, we get

$$\sigma(\dot{q}_i \odot_k \dot{q}_i) = \sum_{j=1, j \neq i}^n \frac{m_j (1 - \sigma(q_i \odot_k q_j))}{(\sigma - \sigma(q_i \odot_k q_j)^2)^{\frac{3}{2}}}, \quad i \in \{1, \dots, n\}. \quad (2.6)$$

Because $\dot{q}_i \odot_k \dot{q}_i = A^2 r^2$, this proves the lemma. \square

3. Proof of Theorem 1.1

Proof. Assume that the contrary is true. Then there exist sequences $\{Q_{ip}\}_{p=1}^{\infty} \subset \mathbb{R}^2$, $i = 1, \dots, n$, with respective sequences of relative equilibria

$$\left\{ \begin{pmatrix} T(A_p t) Q_{ip} \\ Z_p \end{pmatrix} \right\}_{p=1}^{\infty}, \quad i \in \{1, \dots, n\},$$

for which, after renumbering the

$$\begin{pmatrix} T(A_p t) Q_{ip} \\ Z_p \end{pmatrix}$$

in terms of i if necessary, there exists an $l \in \{1, \dots, n\}$, such that

$$\begin{pmatrix} T(A_p t) Q_{1p} \\ Z_p \end{pmatrix}, \dots, \begin{pmatrix} T(A_p t) Q_{lp} \\ Z_p \end{pmatrix}$$

go to the same limit for p going to infinity.

For each of those p , we have because of Lemma 2.1 that

$$0 = \sum_{j=1, j \neq i}^n \frac{m_j \sin(\alpha_{ip} - \alpha_{jp})}{(1 - \cos(\alpha_{ip} - \alpha_{jp}))^{\frac{3}{2}} (2 - \sigma r_p^2 (1 - \cos(\alpha_{ip} - \alpha_{jp})))^{\frac{3}{2}}}, \quad (3.1)$$

where α_{ip} and α_{jp} are the angles between the first coordinate axis and Q_{ip} and the angle between the first coordinate axis and Q_{jp} respectively and $r_p = \|Q_{ip}\|$.

Because of (3.1), we thus get that

$$\begin{aligned} 0 &= \sum_{j=2}^l \frac{m_j \sin(\alpha_{1p} - \alpha_{jp})}{(1 - \cos(\alpha_{1p} - \alpha_{jp}))^{\frac{3}{2}} (2 - \sigma r_p^2 (1 - \cos(\alpha_{1p} - \alpha_{jp})))^{\frac{3}{2}}} \\ &\quad + \sum_{j=l+1}^n \frac{m_j \sin(\alpha_{1p} - \alpha_{jp})}{(1 - \cos(\alpha_{1p} - \alpha_{jp}))^{\frac{3}{2}} (2 - \sigma r_p^2 (1 - \cos(\alpha_{1p} - \alpha_{jp})))^{\frac{3}{2}}}. \end{aligned} \quad (3.2)$$

There are two possibilities:

1. $\alpha_{1p} - \alpha_{jp}$ goes to zero for $j \in \{1, \dots, l\}$ and r_p is bounded for p going to infinity.
2. $\alpha_{1p} - \alpha_{jp}$ goes to zero for $j \in \{1, \dots, l\}$ and r_p is not bounded for p going to infinity.

For the first case, note that by l'Hôpital and by renumbering the α_{ip} in terms of i and taking subsequences if necessary such that $\alpha_{1p} - \alpha_{jp}$ decreases to zero for all $j \in \{1, \dots, l\}$ that

$$\lim_{(\alpha_{1p} - \alpha_{jp}) \downarrow 0} \frac{m_j \sin(\alpha_{1p} - \alpha_{jp})}{1 - \cos(\alpha_{1p} - \alpha_{jp})} = \lim_{(\alpha_{1p} - \alpha_{jp}) \downarrow 0} \frac{m_j \cos(\alpha_{1p} - \alpha_{jp})}{\sin(\alpha_{1p} - \alpha_{jp})} = +\infty, \quad (3.3)$$

which means that if we take the limit where p goes to infinity on both sides of (3.2), we get that $0 = \infty$, which is a contradiction.

For the second case, the n -body problem is defined on \mathbb{H}^k and thus $\sigma = -1$. Then multiplying both sides of (3.2) with r_p^3 and noting that for p going to infinity

$$\frac{r_p^3}{(2 - \sigma r_p^2(1 - \cos(\alpha_{1p} - \alpha_{jp})))^{\frac{3}{2}}} = \frac{r_p^3}{(2 + r_p^2(1 - \cos(\alpha_{1p} - \alpha_{jp})))^{\frac{3}{2}}}$$

does not go to zero, leads, combined with (3.3), to the desired contradiction we got for the first case. This completes the proof. \square

4. Proof of Theorem 1.2

Assume that the contrary is true. Then there exist sequences $\{Q_{ip}\}_{p=1}^\infty$, $i \in \{1, \dots, n\}$, and corresponding relative equilibria

$$q_{ip} = \begin{pmatrix} T(A)Q_{ip} \\ Z_p \end{pmatrix}, \quad i \in \{1, \dots, n\},$$

where q_{1p}, \dots, q_{np} solve (1.1), such that $r_p := \|Q_{ip}\|$ goes to infinity for p going to infinity.

As consequently, for p large enough, taking subsequences if necessary, $Z_p \neq 0$, we have by Lemma 2.2 that

$$\sigma A^2 r_p^2 = \sum_{j=1, j \neq i}^n \frac{m_j(1 - \sigma(q_{ip} \odot_k q_{jp}))}{(\sigma - \sigma(q_{ip} \odot_k q_{jp})^2)^{\frac{3}{2}}}, \quad i \in \{1, \dots, n\}. \quad (4.1)$$

Letting p go to infinity on both sides of (4.1) means that the left-hand side of (4.1) goes to infinity, which is only possible if the right-hand side of (4.1) does the same. The right-hand side of (4.1) can only become infinitely large if for at least one term

$$\frac{m_j(1 - \sigma(q_{ip} \odot_k q_{jp}))}{(\sigma - \sigma(q_{ip} \odot_k q_{jp})^2)^{\frac{3}{2}}}$$

the denominator goes to zero, which means that $\lim_{p \rightarrow \infty} q_{ip} \odot_k q_{jp} = -1$, which means that q_{ip} and q_{jp} have the same limit. This contradicts Theorem 1.1.

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