



Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Some remarks on the fifth-order KdV equations

C.T. Lee

Department of Mathematics, Shantou University, Shantou, Guangdong, 515063, PR China

ARTICLE INFO

Article history:

Received 13 May 2010

Available online xxxx

Submitted by R. Curto

Keywords:

Hamiltonian system

Fifth-order KdV equation

Kaup–Kupershmidt equation

Skew-adjoint operator

Jacobi identity

Prolongation

ABSTRACT

In this paper, we present the differential operators for the generalized fifth-order KdV equation. We give formal proofs on the Hamiltonian properties including the skew-adjointness and Jacobi identity by using the prolongation method. Our results show that there are three third-order Hamiltonian operators which can be used to construct the Hamiltonians. However, no fifth-order operators are shown to pass the Hamiltonian test, although there are an infinite number of them, and they are skew-adjoint.

© 2014 Published by Elsevier Inc.

1. Introduction

The study of the generalized fifth-order Korteweg–de Vries (fifth-order KdV) equation

$$u_t + \alpha uu_{xxx} + \beta u_x u_{xx} + \gamma u^2 u_x + u_{xxxxx} = 0, \quad (1)$$

where α , β and γ are arbitrary real parameters, has always been an important topic in nonlinear physical phenomena. This equation not only describes the motion of long waves in shallow water under gravity and in a one-dimensional nonlinear lattice, but is also an important mathematical model for a chain of coupled nonlinear oscillators [14] and magneto-sound propagation in plasmas [7]. A great deal of research work has been conducted during the past decades for exact solutions, such as soliton solutions [17,15,16,20,21]. Several different approaches, such as Bäcklund transformation, a bilinear form, and Lax pairs, have been used independently for various constants α , β and γ to obtain multisoliton solutions [18]. Interesting and deeply examined examples of the fifth-order KdV equation (1) are the

E-mail address: chuntelee2000@googlemail.com.

<http://dx.doi.org/10.1016/j.jmaa.2014.10.021>

0022-247X/© 2014 Published by Elsevier Inc.

- Sawada–Kotera equation (SK equation) [19]

$$u_t + 5uu_{xxx} + 5u_xu_{xx} + 5u^2u_x + u_{xxxxx} = 0. \quad (2)$$

- Lax equation [9]

$$u_t + 10uu_{xxx} + 20u_xu_{xx} + 30u^2u_x + u_{xxxxx} = 0. \quad (3)$$

- Kaup–Kupersmidt equation (KK equation, [2])

$$u_t + 10uu_{xxx} + 25u_xu_{xx} + 20u^2u_x + u_{xxxxx} = 0. \quad (4)$$

- Ito equation [4]

$$u_t + 3uu_{xxx} + 6u_xu_{xx} + 2u^2u_x + u_{xxxxx} = 0. \quad (5)$$

As the constants α , β and γ in (1) vary, the properties of the whole equation change drastically. For instance, the SK equation (2) and Lax equation (3) are completely integrable and have N -soliton solutions [19]. Notice that Caudrey et al. [1] have found new hierarchies of Korteweg–de Vries and Boussinesq equations by studying a fifth-order equation as

$$u_t + 30uu_{xxx} + 30u_xu_{xx} + 180u^2u_x + u_{xxxxx} = 0, \quad (6)$$

which is the same as (2) under the transformation $u \mapsto \frac{1}{6}u$. Moreover, the KK equation (4) is known to be integrable and possesses bilinear representations [15,16]. In addition, Kaup and Kupersmidt [6,8] have studied the following equation

$$u_t - 15uu_{xxx} - \frac{75}{2}u_xu_{xx} + 45u^2u_x + u_{xxxxx} = 0, \quad (7)$$

for multisoliton solutions and integrability. However, it is straightforward to check that (4) is just the same as (7) under $u \mapsto -\frac{3}{2}u$. On the other hand, the Ito equation (5) has been shown to be non-integrable, but it is confirmed to have a limited number of special conserved densities [5].

The question of Hamiltonian structures for the fifth-order KdV equation is very important and deserves serious considerations, since soliton equations often come with some surprising Hamiltonian structures and these are now being recognized as an important aspect of soliton theory [3,12,11].

In this paper, we rummage through all possible differential operators for the generalized fifth-order KdV equation (1), and give full validation tests of the Hamiltonian structure with regard to the skew-adjoint property and Jacobi identity. These two important properties will be verified in a systematic way, carefully and directly using the prolongation method. Importantly, we will successfully present all possible Hamiltonian operators for Eq. (1).

2. Preliminaries

Most integrable systems are known to have Hamiltonian structures. In fact, most of them possess a bi-Hamiltonian structure, i.e., they can be formulated in two distinct ways as a Hamiltonian system. An infinite-dimensional Hamiltonian system takes the general form

$$\frac{\partial u}{\partial t} + \mathcal{D} \frac{\delta \mathcal{H}}{\delta u} = 0, \quad (8)$$

where \mathcal{D} is a skew-adjoint differential operator and \mathcal{H} is a functional

$$\mathcal{H}[u, u^{(n)}] := \int_{-\infty}^{\infty} H(u, u_x, \dots, u_{nx}) \, dx, \quad (9)$$

with the density function H coming from the space of differentiable functions. Then one can define the Poisson bracket as

$$\{\mathcal{F}, \mathcal{G}\} = \int_{-\infty}^{\infty} \frac{\delta \mathcal{F}}{\delta u} \mathcal{D} \frac{\delta \mathcal{G}}{\delta u} \, dx, \quad (10)$$

for any two smooth functionals \mathcal{F}, \mathcal{G} . For instance, it is well known that the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0,$$

admits a Hamiltonian system as

$$\frac{\partial u}{\partial t} = \mathcal{D} \frac{\delta \mathcal{H}}{\delta u}, \quad \mathcal{H} = \int_{-\infty}^{\infty} \left(\frac{1}{2} u_x^2 + u^3 \right) \, dx, \quad (11)$$

where $\mathcal{D} = \partial_x$, \mathcal{H} is called the Hamiltonian and $\delta/\delta u$ is the variational derivative. Its Poisson bracket is

$$\{\mathcal{F}, \mathcal{G}\} = \int_{-\infty}^{\infty} \frac{\delta \mathcal{F}}{\delta u} \partial_x \frac{\delta \mathcal{G}}{\delta u} \, dx. \quad (12)$$

Throughout this paper, the functionals such as \mathcal{F}, \mathcal{G} , and \mathcal{H} above are all specified with the vanishing boundary conditions, e.g., $\mathcal{F} \rightarrow 0$ as $|x| \rightarrow \infty$.

The general differential operator \mathcal{D} in (8) is of the form

$$\mathcal{D} := \sum_{i=0}^n P_i(u, u_x, \dots, u_{nx}) \frac{\partial^i}{\partial x^i}, \quad (13)$$

for some smooth differentiable functions P_i , where n is a finite, natural number. For simplicity, one can write \mathcal{D} as

$$\mathcal{D} = \sum_j P_j[u] \partial_j, \quad (14)$$

where $j = 0, 1, 2, \dots, n$, for some finite positive integers, and $\partial_j := \partial^j / \partial x^j$. A differential operator \mathcal{D} is of order n if its leading coefficient is not zero, i.e. $P_n \neq 0$, and of order 0 if it is a single differentiable function.

Definition 1. Let \mathcal{A} be the space of differentiable functions and suppose

$$\mathcal{D} = \sum_j P_j \partial_j, \quad P_j \in \mathcal{A}. \quad (15)$$

Then its *adjoint* \mathcal{D}^* is a differential operator which satisfies

$$\int P \mathcal{D} Q \, dx = \int Q \mathcal{D}^* P \, dx, \quad \forall P, Q \in \mathcal{A}.$$

An easy integration by parts shows that

$$\mathcal{D}^* = \sum_j (-\partial)^j \cdot P_j = \sum_{j=0}^n (-1)^j \frac{\partial^j}{\partial x^j} (P_j \cdot),$$

meaning that for any $Q \in \mathcal{A}$,

$$\mathcal{D}^* Q = \sum_j (-\partial)^j [P_j Q] = \sum_{j=0}^n (-1)^j \frac{\partial^j}{\partial x^j} (P_j Q).$$

Here we note that, in order to get the above results, P_j , Q are smooth (differentiable) functions depending on x , u and derivatives of u up to some finite order n with $P_j[u(x)] \rightarrow 0$ as $|x| \rightarrow \infty$.

Definition 2. A differential operator \mathcal{D} is called *Hamiltonian* if its Poisson bracket (10) satisfies the “*skew-symmetry*” property

$$\{\mathcal{P}, \mathcal{Q}\} = -\{\mathcal{Q}, \mathcal{P}\}, \quad (16)$$

and the “*Jacobi identity*”

$$\{\{\mathcal{P}, \mathcal{Q}\}, \mathcal{R}\} + \{\{\mathcal{R}, \mathcal{P}\}, \mathcal{Q}\} + \{\{\mathcal{Q}, \mathcal{R}\}, \mathcal{P}\} = 0, \quad (17)$$

for any smooth functionals \mathcal{P} , \mathcal{Q} , and \mathcal{R} .

To explain the terms in (17), we first note that for a given functional \mathcal{Q} , there is an evolutionary vector field $\hat{\mathbf{v}}_{\mathcal{Q}}$ associated with \mathcal{Q} which satisfies [13]

$$\{\mathcal{P}, \mathcal{Q}\} = \text{pr } \hat{\mathbf{v}}_{\mathcal{Q}}(\mathcal{P}),$$

for any functional \mathcal{P} . Note that $\hat{\mathbf{v}}_{\mathcal{Q}}$ has characteristic $\mathcal{D} \frac{\delta \mathcal{Q}}{\delta u} = \mathcal{D} E(Q)$ with

$$E = \sum_{j=0}^{\infty} \left(-\frac{\partial}{\partial x} \right)^j \frac{\partial}{\partial u_j} = \frac{\partial}{\partial u} - \frac{\partial}{\partial x} \frac{\partial}{\partial u_x} + \frac{\partial^2}{\partial x^2} \frac{\partial}{\partial u_{xx}} - \cdots,$$

being called the Euler operator.

For functionals \mathcal{P} , \mathcal{Q} , \mathcal{R} with variational derivatives $\frac{\delta \mathcal{P}}{\delta u} = P$, $\frac{\delta \mathcal{Q}}{\delta u} = Q$, $\frac{\delta \mathcal{R}}{\delta u} = R$ in \mathcal{A}^q , we have

$$\{\{\mathcal{P}, \mathcal{Q}\}, \mathcal{R}\} = \text{pr } \hat{\mathbf{v}}_{\mathcal{R}} \left(\int_{-\infty}^{\infty} P \cdot \mathcal{D}Q \, dx \right) = \int_{-\infty}^{\infty} \text{pr } \mathbf{v}_{\mathcal{D}R} (P \cdot \mathcal{D}Q) \, dx.$$

Using Leibniz’ rule and Lie derivative formula [13]

$$\text{pr } \mathbf{v}_Q(\mathcal{D}P) = \text{pr } \mathbf{v}_Q(\mathcal{D}) \cdot P + \mathcal{D}[\text{pr } \mathbf{v}_Q(P)],$$

we have

$$\begin{aligned} & \{\{\mathcal{P}, \mathcal{Q}\}, \mathcal{R}\} \\ &= \int_{-\infty}^{\infty} \{ \text{pr } \mathbf{v}_{\mathcal{D}R}(P) \cdot \mathcal{D}Q + P \cdot \text{pr } \mathbf{v}_{\mathcal{D}R}(\mathcal{D})Q + P \cdot \mathcal{D}[\text{pr } \mathbf{v}_{\mathcal{D}R}(Q)] \} \, dx. \end{aligned} \quad (18)$$

Introducing the Frechet derivatives

$$D_p(Q) = \text{pr } \mathbf{v}_Q(P),$$

the first term in (18) becomes

$$\int_{-\infty}^{\infty} \text{pr } \mathbf{v}_{\mathcal{D}R}(P) \cdot \mathcal{D}Q \, dx = \int_{-\infty}^{\infty} D_p(\mathcal{D}R) \cdot \mathcal{D}Q \, dx.$$

Similarly, if we use the fact that \mathcal{D} is skew-adjoint, the third term in (18) has an analogous form

$$\int_{-\infty}^{\infty} P \cdot \mathcal{D}[\text{pr } \mathbf{v}_{\mathcal{D}R}(Q)] \, dx = - \int_{-\infty}^{\infty} \mathcal{D}P \cdot D_Q(\mathcal{D}R) \, dx.$$

The second and third components in the Jacobi identity (17) contribute similar expressions; for example, $\{\mathcal{Q}, \mathcal{R}, \mathcal{P}\}$ contains the terms

$$\int_{-\infty}^{\infty} D_Q(\mathcal{D}P) \cdot \mathcal{D}R \, dx, \quad - \int_{-\infty}^{\infty} \mathcal{D}Q \cdot D_R(\mathcal{D}P) \, dx.$$

Thus one can expand the Jacobi identity in this way, six of terms cancel and we are left with the equivalent form

$$\int_{-\infty}^{\infty} [P \cdot \text{pr } \mathbf{v}_{\mathcal{D}R}(\mathcal{D})Q + R \cdot \text{pr } \mathbf{v}_{\mathcal{D}Q}(\mathcal{D})P + Q \cdot \text{pr } \mathbf{v}_{\mathcal{D}P}(\mathcal{D})R] \, dx,$$

which must vanish for all P, Q, R which are variational derivatives of functionals.

Notice that if \mathcal{D} is a skew-adjoint differential operator whose coefficients do not depend on u or its derivatives, then \mathcal{D} is automatically a Hamiltonian operator.

Although we should check the Jacobi identity in (17) by direct verifications, it still requires quite a lot of calculations even in such a relatively simple example like the KdV equation. A radical simplification is offered by Olver [13] using the theory of multi-vectors and prolongation of the vector field. In this section we will apply such method to justify a Hamiltonian operator.

Let \mathcal{A}^q be the space of general q -tuple differentiable functions. One can define the general multi k -vector as follows

$$\Theta = \int \left(\sum_{\alpha, J} R_J^\alpha(u(x), u^{(n)}(x)) \theta_{J_1}^{\alpha_1} \wedge \cdots \wedge \theta_{J_k}^{\alpha_k} \right) dx, \quad (19)$$

where $R_J^\alpha \in \mathcal{A}^q$ are differentiable functions depending on u and derivatives of u up to some finite number. Here $x = (x^1, \dots, x^p)$ are independent variables, $u = (u^1, \dots, u^q)$ are dependent variables and $J = (J_1, J_2, \dots, J_k)$ is a k -th order multi-index with $0 \leq J_k \leq p$ indicating which derivatives are being taken. Here $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is a multi-index with $1 \leq \alpha_i \leq q$ indicating which variables are being using. The variables $\theta_{J_i}^{\alpha_i}$ are unit vectors of u corresponding to the derivatives $\partial/\partial u_{J_i}^{\alpha_i}$ and \wedge is the wedge product. Olver [13] further indicated that any skew-adjoint differential operator \mathcal{D} can be written as a canonical form of functional *bi-vector*

$$\Theta_{\mathcal{D}} = \frac{1}{2} \int \{\theta \wedge \mathcal{D}\theta\} dx = \frac{1}{2} \int \left(\sum_{\alpha, \beta=1}^q \theta^\alpha \wedge \mathcal{D}_{\alpha\beta} \theta^\beta \right) dx, \quad (20)$$

where $\mathcal{D} = (\mathcal{D}_{\alpha\beta})$ is a $(q \times q)$ -dimensional differential operator. Thus, studying the Hamiltonian operator of the differential equation is equally important as studying the bi-vector of the equation.

Theorem 1. *Let \mathcal{D} be a skew-adjoint operator and let θ be a unit vector of \mathcal{D} . Suppose*

$$\Theta_{\mathcal{D}} = \frac{1}{2} \int (\theta \wedge \mathcal{D}(\theta)) dx, \quad (21)$$

is the corresponding functional bi-vector. Then \mathcal{D} is Hamiltonian if and only if

$$\text{pr } \mathbf{v}_{\mathcal{D}\theta}(\Theta_{\mathcal{D}}) = 0, \quad (22)$$

where “pr” stands for prolongation calculations.

3. Third-order operator

We start with the general third-order differential operator of the form

$$\mathcal{D} = \partial_x^3 + \alpha_1 u \partial_x + \beta_1 u_x, \quad (23)$$

where α_1 and β_1 are real parameters to be determined. In order to generate the fifth-order KdV equation (1) from (8), let us assume a Hamiltonian of the form

$$\mathcal{H} = \int \left(\frac{1}{2} u u_{xx} + \frac{\gamma_1}{3} u^3 \right) dx, \quad (24)$$

where γ_1 is a real parameter to be determined. Then one substitutes (23) and (24) into (8) to obtain

$$u_t + (\partial_x^3 + \alpha_1 u \partial_x + \beta_1 u_x)(u_{xx} + \gamma_1 u^2) = 0, \quad (25)$$

which leads to a nonlinear system of algebraic equations for α_1 , β_1 and γ_1 :

(1) SK equation:

$$\begin{cases} 2\gamma_1 + \alpha_1 = 5, \\ 6\gamma_1 + \beta_1 = 5, \\ 2\alpha_1\gamma_1 + \beta_1\gamma_1 = 5. \end{cases}$$

Then we have solutions $\alpha_1 = 3$, $\beta_1 = -1$ and $\gamma_1 = 1$, and the “potential” Hamiltonian operators as well as Hamiltonians are

$$\mathcal{D}_{sk-3} = \partial_x^3 + 3u \partial_x - u_x, \quad (26)$$

$$\mathcal{H}_{sk-3} = \int \left(\frac{1}{2} u u_{xx} + \frac{1}{3} u^3 \right) dx. \quad (27)$$

If we set $\alpha_1 = 2\beta_1$ for the simplest form of skew-adjoint operator, then the algebraic equations for α_1 , β_1 and γ_1 turn out to be

$$\begin{cases} 2\gamma_1 + 2\beta_1 = 5, \\ 6\gamma_1 + \beta_1 = 5, \\ 5\beta_1\gamma_1 = 5. \end{cases}$$

Solving the above system of equations, we obtain $\alpha_1 = 4$, $\beta_1 = 2$ and $\gamma_1 = 1/2$, and

$$\mathcal{D}_{sk-3-1} = \partial_x^3 + 4u\partial_x + 2u_x, \quad (28)$$

$$\mathcal{H}_{sk-3-1} = \int \left(\frac{1}{2}uu_{xx} + \frac{1}{6}u^3 \right) dx. \quad (29)$$

(2) Lax equation:

$$\begin{cases} 2\gamma_1 + \alpha_1 = 10, \\ 6\gamma_1 + \beta_1 = 20, \\ 2\alpha_1\gamma_1 + \beta_1\gamma_1 = 30. \end{cases}$$

Then we have $\alpha_1 = 4$, $\beta_1 = 2$ and $\gamma_1 = 3$, and the “potential” Hamiltonian operators as well as Hamiltonians are

$$\mathcal{D}_{Lax-3} = \partial_x^3 + 4u\partial_x + 2u_x, \quad (30)$$

$$\mathcal{H}_{Lax-3} = \int \left(\frac{1}{2}uu_{xx} + u^3 \right) dx. \quad (31)$$

If we set $\alpha_1 = 2\beta_1$ for the simplest form of skew-adjoint operator, then the algebraic equations for α_1 , β_1 and γ_1 turn out to be

$$\begin{cases} \gamma_1 + \beta_1 = 5, \\ 6\gamma_1 + \beta_1 = 20, \\ \beta_1\gamma_1 = 6. \end{cases}$$

Solving the above system of equations, we obtain $\alpha_1 = 4$, $\beta_1 = 2$ and $\gamma_1 = 3$, which is the same as above results.

(3) KK equation:

$$\begin{cases} 2\gamma_1 + \alpha_1 = 10, \\ 6\gamma_1 + \beta_1 = 25, \\ 2\alpha_1\gamma_1 + \beta_1\gamma_1 = 20. \end{cases}$$

Then we have $\alpha_1 = 2$, $\beta_1 = 1$ and $\gamma_1 = 4$, and the “potential” Hamiltonian operators as well as Hamiltonians are

$$\mathcal{D}_{KK-3} = \partial_x^3 + 2u\partial_x + u_x, \quad (32)$$

$$\mathcal{H}_{KK-3} = \int \left(\frac{1}{2}uu_{xx} + \frac{4}{3}u^3 \right) dx. \quad (33)$$

If we set $\alpha_1 = 2\beta_1$ for the simplest form of skew-adjoint operator, then the algebraic equations for α_1 , β_1 and γ_1 turn out to be

$$\begin{cases} \gamma_1 + \beta_1 = 5, \\ 6\gamma_1 + \beta_1 = 25, \\ \beta_1\gamma_1 = 4. \end{cases}$$

Solving the above system of equations, we obtain the same results as in (32) and (33).

(4) Ito equation:

$$\begin{cases} 2\gamma_1 + \alpha_1 = 3, \\ 6\gamma_1 + \beta_1 = 6, \\ 2\alpha_1\gamma_1 + \beta_1\gamma_1 = 2. \end{cases}$$

Then we have $\alpha_1 = 1$, $\beta_1 = 0$ and $\gamma_1 = 1$, and the “potential” Hamiltonian operators as well as Hamiltonians are

$$\mathcal{D}_{Ito-3} = \partial_x^3 + u\partial_x, \quad (34)$$

$$\mathcal{H}_{Ito-3} = \int \left(\frac{1}{2}uu_{xx} + \frac{1}{3}u^3 \right) dx. \quad (35)$$

If we set $\alpha_1 = 2\beta_1$ for the simplest form of skew-adjoint operator, then the algebraic equations for α_1 , β_1 and γ_1 turn out to be

$$\begin{cases} 2\gamma_1 + 2\beta_1 = 3, \\ 6\gamma_1 + \beta_1 = 6, \\ 5\beta_1\gamma_1 = 2. \end{cases}$$

Solving the above system of equations, we found that no solutions for α_1 , β_1 and γ_1 can be obtained.

4. Fifth-order operator

In this section take the fifth-order, skew-adjoint operator of the form

$$\mathcal{D} = \partial_x^5 + a\partial_x^3 + \partial_x^3a + b\partial_x + \partial_xb, \quad (36)$$

where $a = a(x, t)$, $b = b(x, t)$ are to be determined. The reason for choosing such forms of fifth-order operator is that (36) is skew-adjoint, and is likely to lead to the fifth-order KdV equation (1) with suitable parameters.

We suppose that

$$a = \alpha_3u, \quad b = \beta_3u_{xx} + \gamma_3u^2, \quad (37)$$

where α_3 , β_3 and γ_3 are real parameters to be determined. A quick judgment of suitable Hamiltonian form from (8) and (36) is

$$\mathcal{H} = \int \frac{1}{2}u^2 dx. \quad (38)$$

Substituting (36) and (38) into (8), we obtain

$$u_t + \left[\begin{aligned} &u_{xxxxx} + 2\alpha_3uu_{xxx} + \alpha_3u_{xxx}u + 3\alpha_3u_{xx}u_x + 3\alpha_3u_xu_{xx} \\ &+ 2(\beta_3u_{xx} + \gamma_3u^2)u_x + (\beta_3u_{xxx} + 2\gamma_3uu_x)u \end{aligned} \right] = 0, \quad (39)$$

which can be simplified to

$$u_t + 4\gamma_3u^2u_x + 2(3\alpha_3 + \beta_3)u_xu_{xx} + (3\alpha_3 + \beta_3)uu_{xxx} + u_{xxxxx} = 0. \quad (40)$$

By comparing to the coefficients of the SK equation (2), Lax equation (3), KK equation (4), and Ito equation (5), one can see the possibilities of deducing the Hamiltonian operators for SK and KK equations have been ruled out, while the others yield a linear system of algebraic equations for α_3 , β_3 and γ_3 as follows:

(1) Lax equation:

$$\begin{cases} 3\alpha_3 + \beta_3 = 10, \\ 6\alpha_3 + 2\beta_3 = 20, \\ 4\gamma_3 = 30. \end{cases}$$

This system gives infinite solutions under the condition $3\alpha_3 + \beta_3 = 10$ and $\gamma_3 = 15/2$. It gives infinite numbers of skew-adjoint operators as

$$\mathcal{D}_{Lax-5} = \begin{pmatrix} \partial_x^5 + 2\alpha_3 u \partial_{xxx} + (\alpha_3 + \beta_3) u_{xxx} \\ + (3\alpha_3 + 2\beta_3) u_{xx} \partial_x + 3\alpha_3 u_x \partial_{xx} \\ + 15u^2 \partial_x + 15uu_x \end{pmatrix}, \quad (41)$$

$$\mathcal{H}_{Lax-5} = \int \frac{1}{2} u^2 dx. \quad (42)$$

(2) Ito equation:

$$\begin{cases} 3\alpha_3 + \beta_3 = 3, \\ 6\alpha_3 + 2\beta_3 = 6, \\ 4\gamma_3 = 2. \end{cases}$$

This system gives solutions when $3\alpha_3 + \beta_3 = 3$ and $\gamma_3 = 1/2$, so that we have

$$\mathcal{D}_{Ito-5} = \begin{pmatrix} \partial_x^5 + 2\alpha_3 u \partial_{xxx} + (\alpha_3 + \beta_3) u_{xxx} \\ + (3\alpha_3 + 2\beta_3) u_{xx} \partial_x + 3\alpha_3 u_x \partial_{xx} \\ + u^2 \partial_x + uu_x \end{pmatrix}, \quad (43)$$

$$\mathcal{H}_{Ito-5} = \int \frac{1}{2} u^2 dx. \quad (44)$$

5. Hamiltonian operator

In previous sections, we show some “potential” Hamiltonian operators of third-order and fifth-order for Eq. (1). These operators are all nontrivial, and need to be further verified for the Hamiltonian nature. In this section we will use the method of prolongation [13] to verify the skew-adjoint property and Jacobi identity in order to justify the Hamiltonian structure for Eq. (1).

The skew adjoint property of differential operators is easily checked. The Jacobi identity is normally easier to check by examining the closure of the corresponding symplectic form. However, most of the operators are highly nontrivial, making it extremely difficult to invert. Thus we will turn to the use of the method of prolongation. We refer the interested readers to [13,10] for details on this method and simply note that if we define a bi-vector as $\Theta_{\mathcal{D}} = \frac{1}{2} \int \theta \wedge \mathcal{D}(\theta) dx$, then \mathcal{D} would satisfy the Jacobi identity provided

$$\text{pr } \mathbf{v}_{\mathcal{D}\theta}(\Theta_{\mathcal{D}}) = 0.$$

Here the assumption is that

$$\theta \neq \theta[u],$$

and by definition, prolongation acts only on coefficients functionally dependent on u .

First we examine \mathcal{D}_{Lax-3} in (30) by writing its bi-vector form as

$$\Theta_{\mathcal{D}_{Lax-3}} = \int \left(\frac{1}{2} \theta \wedge \theta_{xxx} + 2u\theta \wedge \theta_x + u_x \theta \wedge \theta \right) dx.$$

Notice that

$$\text{pr } \mathbf{v}_{\mathcal{D}_{Lax-3}\theta}(u) = \theta_{xxx} + 4u\theta_x + 2u_x,$$

which leads to

$$\begin{aligned} & \text{pr } \mathbf{v}_{\mathcal{D}_{Lax-3}\theta}(\Theta_{\mathcal{D}_{Lax-3}}) \\ &= \text{pr } \mathbf{v}_{\mathcal{D}_{Lax-3}\theta} \int \left(\frac{1}{2} \theta \wedge \theta_{xxx} + 2u\theta \wedge \theta_x + u_x \theta \wedge \theta \right) dx \\ &= 2 \int (\theta_{xxx} \wedge \theta \wedge \theta_x + 4u\theta_x \wedge \theta \wedge \theta_x + 2u_x \theta \wedge \theta \wedge \theta_x) dx \\ &= 0. \end{aligned}$$

This shows that \mathcal{D}_{Lax-3} is a Hamiltonian operator, and

$$\mathcal{H}_{Lax-3} = \int \left(\frac{1}{2} uu_{xx} + u^3 \right) dx$$

is its corresponding Hamiltonian. Here we note that Lax equation (3) can be written in Hamiltonian form in two distinct ways as

$$u_t = -\mathcal{D}_{Lax} \frac{\delta \mathcal{H}_{Lax}}{\delta u} = -\mathcal{D}_{Lax-3} \frac{\delta \mathcal{H}_{Lax-3}}{\delta u},$$

where $\mathcal{D}_{Lax} = \partial_x$, $\mathcal{H}_{Lax} = \int_{-\infty}^{\infty} (\frac{1}{2} uu_{xxx} - 5uu_x^2 - \frac{5}{2}u^4) dx$, \mathcal{D}_{Lax-3} in (30) and \mathcal{H}_{Lax-3} in (31).

Following similar procedures, \mathcal{D}_{KK-3} , \mathcal{D}_{sk-3-1} are all Hamiltonian operators.

We are interested in seeing if the Lax operator (41) and Ito operator (43) are qualified as Hamiltonian operators. If by chance the answer is positive, then we will obtain extra Hamiltonian results.

To verify \mathcal{D}_{Lax-5} in (41), we first write its bi-vector form as

$$\begin{aligned} \Theta_{\mathcal{D}_{Lax-5}} &= \frac{1}{2} \int_{-\infty}^{\infty} \theta \wedge \mathcal{D}_{Lax-5}(\theta) dx \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \left(\begin{aligned} & \theta \wedge \theta_{xxxxx} + 2\alpha_3 u \theta \wedge \theta_{xxx} + (\alpha_3 + \beta_3) \theta \wedge u_{xxx} \theta \\ & + (3\alpha_3 + 2\beta_3) \theta \wedge u_{xx} \theta_x + 3\alpha_3 \theta \wedge u_x \theta_{xx} \\ & + 15u^2 \theta \wedge \theta_x + 15\theta \wedge uu_x \theta \end{aligned} \right) dx. \end{aligned}$$

Noticing that

$$\text{pr } \mathbf{v}_{\mathcal{D}_{Lax-5}\theta}(u) = \begin{pmatrix} \theta_{xxxxx} + 2\alpha_3 u \theta_{xxx} + (\alpha_3 + \beta_3) u_{xxx} \theta \\ + (3\alpha_3 + 2\beta_3) u_{xx} \theta_x + 3\alpha_3 u_x \theta_{xx} \\ + 15u^2 \theta_x + 15uu_x \theta \end{pmatrix},$$

we have

$$\begin{aligned}
 & \text{pr } \mathbf{v}_{\mathcal{D}_{Lax-5}} \theta (\Theta_{\mathcal{D}_{Lax-5}}) \\
 &= \frac{1}{2} \int_{-\infty}^{\infty} \left[\begin{aligned} & (2\alpha_3 + 2\beta_3) \left(\begin{aligned} & \theta_{xxxxx} + 2\alpha_3 u \theta_{xxx} \\ & + (\alpha_3 + \beta_3) u_{xxx} \theta \\ & + (3\alpha_3 + 2\beta_3) u_{xx} \theta_x \\ & + 3\alpha_3 u_x \theta_{xx} \\ & + 15u^2 \theta_x + 15u u_x \theta \end{aligned} \right) \wedge \theta \wedge \theta_{xxx} \\ & - 2\beta_3 \left(\begin{aligned} & \theta_{xxxxx} + 2\alpha_3 u \theta_{xxx} \\ & + (\alpha_3 + \beta_3) u_{xxx} \theta \\ & + (3\alpha_3 + 2\beta_3) u_{xx} \theta_x + 3\alpha_3 u_x \theta_{xx} \\ & + 15u^2 \theta_x + 15u u_x \theta \end{aligned} \right) \wedge \theta_{xx} \wedge \theta_x \\ & + 30u \left(\begin{aligned} & \theta_{xxxxx} + 2\alpha_3 u \theta_{xxx} \\ & + (\alpha_3 + \beta_3) u_{xxx} \theta \\ & + (3\alpha_3 + 2\beta_3) u_{xx} \theta_x + 3\alpha_3 u_x \theta_{xx} \\ & + 15u^2 \theta_x + 15u u_x \theta \end{aligned} \right) \wedge \theta \wedge \theta_x \end{aligned} \right] dx \\
 &= \frac{1}{2} \int_{-\infty}^{\infty} \left[\begin{aligned} & -(2\alpha_3 + 4\beta_3) \theta_x \wedge \theta_{xxxx} \wedge \theta_{xxx} \\ & + 3\alpha_3 u_x (2\alpha_3 + 2\beta_3) \theta_{xx} \wedge \theta \wedge \theta_{xxx} \\ & + (-4\alpha_3 \beta_3 u - 30u) \theta_{xxx} \wedge \theta_{xx} \wedge \theta_x \\ & - 2\beta_3 \left(\begin{aligned} & (\alpha_3 + \beta_3) u_{xxx} \\ & + 15u u_x - 90\alpha_3 u u_x \end{aligned} \right) \theta \wedge \theta_{xx} \wedge \theta_x \end{aligned} \right] dx,
 \end{aligned}$$

by using the integration by parts and vanishing boundary conditions.

When $3\alpha_3 + \beta_3 = 10$, we have

$$\begin{aligned}
 & \text{pr } \mathbf{v}_{\mathcal{D}_{Lax-5}} \theta (\Theta_{\mathcal{D}_{Lax-5}}) \\
 &= \frac{1}{2} \int \left[\begin{aligned} & -(40 - 10\alpha_3) \theta_x \wedge \theta_{xxxx} \wedge \theta_{xxx} \\ & + 3\alpha_3 u_x (20 - 4\alpha_3) \theta_{xx} \wedge \theta \wedge \theta_{xxx} \\ & + (-4\alpha_3 (10 - 3\alpha_3) - 30) u \theta_{xxx} \wedge \theta_{xx} \wedge \theta_x \\ & - 2\beta_3 \left(\begin{aligned} & (10 - 2\alpha_3) u_{xxx} \\ & + (15 - 90\alpha_3) u u_x \end{aligned} \right) \theta \wedge \theta_{xx} \wedge \theta_x \end{aligned} \right] dx.
 \end{aligned}$$

Therefore

$$\text{pr } \mathbf{v}_{\mathcal{D}_{Lax-5}} \theta (\Theta_{\mathcal{D}_{Lax-5}}) \neq 0,$$

which shows that \mathcal{D}_{Lax-5} is not Hamiltonian in the form of (36).

We make one more effort by writing the Ito operator in (43) into its bi-vector form as

$$\begin{aligned}
 \Theta_{\mathcal{D}_{Ito-5}} &= \frac{1}{2} \int_{-\infty}^{\infty} \theta \wedge \mathcal{D}_{Ito-5}(\theta) dx \\
 &= \frac{1}{2} \int_{-\infty}^{\infty} \left[\begin{aligned} & \theta \wedge \theta_{xxxxx} + 2\alpha_3 u \theta \wedge \theta_{xxx} + \theta \wedge (\alpha_3 + \beta_3) u_{xxx} \theta \\ & + \theta \wedge (3\alpha_3 + 2\beta_3) u_{xx} \theta_x + \theta \wedge 3\alpha_3 u_x \theta_{xx} \\ & + \theta \wedge u^2 \theta_x + \theta \wedge u u_x \theta \end{aligned} \right] dx.
 \end{aligned}$$

Since

$$\text{pr } \mathbf{v}_{\mathcal{D}_{It0-5}\theta}(u) = \begin{pmatrix} \theta_{xxxxx} + 2\alpha_3 u \theta_{xxx} + (\alpha_3 + \beta_3) u_{xxx} \theta \\ + (3\alpha_3 + 2\beta_3) u_{xx} \theta_x + 3\alpha_3 u_x \theta_{xx} + u^2 \theta_x + u u_x \theta \end{pmatrix},$$

it is tedious but straightforward to show that

$$\begin{aligned} & \text{pr } \mathbf{v}_{\mathcal{D}_{It0-5}\theta}(\Theta_{\mathcal{D}_{It0-5}}) \\ &= \frac{1}{2} \int \left[\begin{aligned} & (2\alpha_3 + 2\beta_3) \theta_{xxxxx} \wedge \theta \wedge \theta_{xxx} \\ & + 3\alpha_3 (2\alpha_3 + 2\beta_3) u_x \theta_{xx} \wedge \theta \wedge \theta_{xxx} \\ & - 2\beta_3 \begin{pmatrix} \theta_{xxxxx} + 2\alpha_3 u \theta_{xxx} \\ + (\alpha_3 + \beta_3) u_{xxx} \theta \\ + u u_x \theta \end{pmatrix} \theta_{xx} \wedge \theta_x \\ & + 2u \begin{pmatrix} \theta_{xxxxx} + 2\alpha_3 u \theta_{xxx} \\ + 3\alpha_3 u_x \theta_{xx} \end{pmatrix} \theta \wedge \theta_x \end{aligned} \right] dx \\ &= \frac{1}{2} \int \left[\begin{aligned} & -(2\alpha_3 + 2\beta_3) \theta_{xx} \wedge \theta_{xxxx} \wedge \theta_{xx} \\ & + (2\alpha_3 + 4\beta_3) \theta_{xxxxx} \wedge \theta_x \wedge \theta_{xx} \\ & + (3\alpha_3 (2\alpha_3 + 2\beta_3) u_x) \theta_{xx} \wedge \theta \wedge \theta_{xxx} \\ & + (-4\alpha_3 \beta_3 u - 2u) \theta_{xxx} \wedge \theta_{xx} \wedge \theta_x \\ & + \begin{pmatrix} 6\alpha_3 u u_x + 2\beta_3 (\alpha_3 + \beta_3) u_{xxx} \\ + 2\beta_3 u u_x \end{pmatrix} \theta_{xx} \wedge \theta \wedge \theta_x \end{aligned} \right] dx, \end{aligned}$$

by using the integration by parts and boundary conditions.

Notice that with the condition $3\alpha_3 + \beta_3 = 3$, the above expression turns out to be

$$\begin{aligned} & \text{pr } \mathbf{v}_{\mathcal{D}_{It0-5}\theta}(\Theta_{\mathcal{D}_{It0-5}}) \\ &= \frac{1}{2} \int \left[\begin{aligned} & -(6 - 4\alpha_3) \theta_{xx} \wedge \theta_{xxxx} \wedge \theta_{xx} \\ & + (12 - 10\alpha_3) \theta_{xxxxx} \wedge \theta_x \wedge \theta_{xx} \\ & + (3\alpha_3 (6 - 4\alpha_3) u_x) \theta_{xx} \wedge \theta \wedge \theta_{xxx} \\ & + (-4\alpha_3 (3 - 3\alpha_3) u - 2u) \theta_{xxx} \wedge \theta_{xx} \wedge \theta_x \\ & + (6u u_x + 2(3 - 3\alpha_3)(3 - 2\alpha_3) u_{xxx}) \theta_{xx} \wedge \theta \wedge \theta_x \end{aligned} \right] dx \\ &\neq 0, \end{aligned}$$

for any α_3 . This shows that the Ito equation has no fifth-order Hamiltonian operator of the form (36).

6. Summary

In this paper, we have presented all of the differential operators for the fifth-order KdV equation (1), including the third-order and fifth-order. All the skew-adjoint and Hamiltonian operators have been presented and identified. We have shown that there are three Hamiltonian operators of order-3. Also, there are skew-adjoint operators of order-5, but these are not Hamiltonian although they are perfectly skew-adjoint and there are an infinite number of them.

In addition, some remarks have to be highlighted. If a general finite-dimensional, say second-order, Hamiltonian system possesses n first integrals $P_1(x), \dots, P_n(x)$ which are *in involution*:

$$\{P_i, P_j\} = 0, \quad \forall i, j, \quad (45)$$

then it is called a *complete integrable* Hamiltonian system, since in principle, its solutions can be determined by quadrature alone. Actually, much more can be said about such completely integrable systems and topic forms a significant chapter in the classical theory of Hamiltonian dynamics. In this paper, we have shown that for the infinite-dimensional Hamiltonian-like system (8), it is crucial for the differential operator \mathcal{D} to satisfy the Jacobi identity and that the prolongation approach provides a relatively straightforward means of checking this. If one just takes a Hamiltonian-like form (8) and the skew-adjoint property of \mathcal{D} , it is still not enough for the system to be completely integrable. Otherwise Ito equation would be integrable since it has a Hamiltonian-like form (40) and skew-adjoint operator (43).

We introduce a remarkable property of the form (8) which, like the Korteweg–de Vries equation, can be written in Hamiltonian form in two different ways

$$\frac{\partial u}{\partial t} = \mathcal{D}\delta\mathcal{H}_1 = \mathcal{E}\delta\mathcal{H}_0, \quad (46)$$

in which both \mathcal{D} and \mathcal{E} are Hamiltonian operators, and $\mathcal{H}_0[u]$ and $\mathcal{H}_1[u]$ are appropriate Hamiltonian functionals. However, this is still not enough to guarantee the complete integrability for (8). Accordingly, one often refers to the *bi-Hamiltonian* structure in which by subjecting to a compatibility condition between the two Poisson structures determined by \mathcal{D} and \mathcal{E} , one will be able to recursively construct an infinite hierarchy of symmetries and conservation laws for the system. The basic theorem on bi-Hamiltonian system is due to Magri [12] who was also the first to publish the second Hamiltonian structure for the Korteweg–de Vries and other equations. Bi-Hamiltonian system appears to be a very special situation, yet, occurs in numerous situations as a model equation for more complicated physical systems. This paper is aimed to give all possible Hamiltonian structures for the fifth-order KdV equation (8). Those interested in bi-Hamiltonian system can refer to Olver [13] for more details.

References

- [1] P.J. Caudrey, R.K. Dood, J.D. Gibbon, A new hierarchy of Korteweg–de Vries equations, *Proc. R. Soc. Lond. Ser. A* 351 (1976) 407–422.
- [2] A.P. Fordy, D.J. Kaup, Some remarkable nonlinear transformations, *Phys. Lett. A* 75 (1980) 325–327.
- [3] I.M. Gelfand, L.A. Dikii, The resolvents and Hamiltonian systems, *Funktsional. Anal. i Prilozhen.* 11 (1977) 11–27.
- [4] M. Ito, An extension of nonlinear evolution equations of the K–dV (mK–dV) type to higher orders, *J. Phys. Soc. Jpn.* 49 (1980) 771–778.
- [5] M. Ito, A reduce program for finding symmetries of nonlinear evolution equations with uniform rank, *Comput. Phys. Comm.* 42 (1986) 351–357.
- [6] D.J. Kaup, On the inverse scattering problem for the cubic eigenvalue problems of the class $\Psi_{3x} + 6Q\Psi_x + 6R\Psi = \lambda\Psi$, *Stud. Appl. Math.* 62 (1980) 189–216.
- [7] T. Kawahara, Oscillatory solitary waves in dispersive media, *J. Phys. Soc. Jpn.* 33 (1972) 260–264.
- [8] B.A. Kupershmidt, A super Korteweg–de Vries equation: an integrable system, *Phys. Lett. A* 102 (1984) 213–215.
- [9] P.D. Lax, Integrals of nonlinear equations of evolution and solitary waves, *Comm. Pure Appl. Math.* 62 (1968) 467–490.
- [10] C.T. Lee, Multi-soliton solution of the two-mode KdV equations, PhD thesis, Oxford University, UK, 2007.
- [11] C.T. Lee, J.L. Liu, C.C. Lee, Y.H. Kang, The second-order KdV equation and its soliton-like solution, *Modern Phys. Lett. B* 23 (2009) 1771–1780.
- [12] F. Magri, A simple model of the integrable Hamiltonian system, *J. Math. Phys.* 19 (1978) 1156–1162.
- [13] P.J. Olver, *Applications of Lie Groups to Differential Equations*, 2nd edition, Springer-Verlag, New York, 2000.
- [14] L.A. Ostrovsky, K.A. Gorshkov, V.V. Papko, On the existence of stationary multisolitons, *Phys. Lett. A* 74 (1979) 177–179.
- [15] A. Parker, On soliton solutions of the Kaup–Kupershmidt equation. I. Direct bilinearisation and solitary wave, *Phys. D* 137 (2000) 25–48.
- [16] A. Parker, On soliton solutions of the Kaup–Kupershmidt equation. II. ‘Anomalous’ N-soliton solutions, *Phys. D* 137 (2000) 34–48.
- [17] A.H. Salas, Exact solutions for the general fifth KdV equation by the exp function method, *Appl. Math. Comput.* 205 (2008) 291–297.
- [18] J. Satsuma, D.J. Kaup, A Backlund transformation for a higher order Korteweg–de Vries equation, *J. Phys. Soc. Jpn.* 43 (1977) 692–697.

- [19] K. Sawada, T. Kortera, A method finding N-soliton solutions of the KdV equation and KdV-like equation, *Progr. Theoret. Phys.* 51 (1974) 1355–1367.
- [20] A.M. Wazwaz, Abundant solitons solutions for several forms of the fifth-order KdV equation by using the tanh method, *Appl. Math. Comput.* 182 (2006) 283–300.
- [21] A.M. Wazwaz, N-soliton solutions for the combined KdV–CDG equation and the KdV–Lax equation, *Appl. Math. Comput.* 203 (2008) 402–407.