J. Math. Anal. Appl. 430 (2015) 144–151

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Characterizations of automorphisms of operator algebras on Banach spaces

Lizhong Huang [a,b], Lin Chen [a,c], Fangyan Lu [a,*]

[a] Department of Mathematics, Soochow University, Suzhou 215006, People's Republic of China
[b] Department of Mathematics, Shanxi Datong University, Datong 037009, People's Republic of China
[c] Department of Mathematics and Physics, Anshun University, Anshun 561000, People's Republic of China

A R T I C L E I N F O

Article history:
Received 26 May 2014
Available online 29 April 2015
Submitted by D. Blecher

Keywords:
Automorphisms
Projection constant
Projections

A B S T R A C T

Let X be a complex Banach space of dimension greater than one, and denote by $B(X)$ the algebra of all the bounded linear operators on X. It is shown that if $\phi : B(X) \to B(X)$ is a multiplicative map (not assumed linear) and if ϕ is sufficiently close to a linear automorphism of $B(X)$ in some uniform sense, then it is actually an automorphism.

1. Introduction

It is a classical result [3] that every algebra automorphism ϕ of $B(X)$, the algebra of all bounded linear operators on a Banach space X, is spatial, i.e., there exists an invertible operator T in $B(X)$ such that $\phi(A) = TAT^{-1}$ for all $A \in B(X)$. There are two main directions in characterizing automorphisms of $B(X)$. One is to study when a linear map is an automorphism [2,4,8,11,12,16–18]. Among other results, Jafarian and Sourour [8] showed that a surjective linear map of $B(X)$ preserves spectrum if and only if it is either an automorphism or an anti-automorphism. Using this, Larson and Sourour [12] proved that each surjective local automorphism of $B(X)$ is actually an automorphism when X is infinite-dimensional, which implies that the automorphism space of $B(X)$ is reflexive.

Another direction of characterizing automorphisms of $B(X)$ is to investigate when a multiplicative map (not assumed linear) is an automorphism [1,7,13–15]. Semrl [15] considered a bijection φ of $B(X)$ satisfying $\|\varphi(AB) - \varphi(A)\varphi(B)\| < \epsilon$ for all $A, B \in B(X)$, and showed that φ is either an automorphism or an anti-automorphism in the case X is infinite-dimensional. Let H be a Hilbert space. Molnar [14] showed that a continuous multiplicative map of $B(H)$ which preserves co-rank is a linear automorphism or a

* Corresponding author.
E-mail addresses: huanglizhongsx@126.com (L. Huang), linchen198112@163.com (L. Chen), fylu@suda.edu.cn (F. Lu).

http://dx.doi.org/10.1016/j.jmaa.2015.04.086
0022-247X/

conjugate-linear automorphism. Recently, Marcoux, Radjavi and Sourour [13] proved that a multiplicative map $\varphi : B(H) \to B(H)$ (not assumed linear and bijective) which is sufficiently close to a linear automorphism ψ, i.e. there exists a $0 < \delta < \frac{1}{4}$ such that $\|\varphi(A) - \psi(A)\| < \delta\|\psi(A)\|$ for all $A \in B(H)$, then φ is an automorphism.

The aim of the present paper is to establish the result of [13] mentioned above in the Banach space setting. Moreover, the upper bound of δ is expanded, namely, the map can be farther away from the (fixed) automorphism. Because of the obvious difference between the Hilbert space and the Banach space, our approach is very different from that in [13]. In fact, our result is closely linked with projection constants.

For a closed subspace E of X, we let

$$\lambda(X, E) = \inf\{\|P\| : P \in B(X) \text{ is a projection with range } E\},$$

and $\lambda(X, E) = \infty$ if there are no projections with range E. Further, for $n \in \mathbb{N}$, let

$$\lambda_n(X) = \sup\{\lambda(X, E) : E \text{ is an } n\text{-dimensional subspace of } X\},$$

and call it the n-dimensional projection constant of X. Obviously, $\lambda_n(H) = 1$ for any Hilbert space H. Due to Kadec and Snobar [9], $\lambda_n(X) \leq \sqrt{n}$ for any Banach space X. For more information, see, for example, [5,6,10].

Let's now introduce some terminologies. Throughout, X is a complex Banach space with topological dual X^*. Denote by $B(X)$ the algebra of all bounded linear operators on X. For $A \in B(X)$, by A^* denote the adjoint of A, by $\operatorname{im} A$ the range of A, by $\operatorname{rank} A$ the dimension of $\operatorname{im} A$. For $x \in X$, $f \in X^*$, the rank at most one operator $x \otimes f$ is defined by $x \otimes f(z) = f(z)x$. A projection P is an operator in $B(X)$ satisfying $P^2 = P$.

The following lemma can be found in any textbook of functional analysis and we omit the proof here.

Lemma 1.1 *(Riesz lemma). For a non-dense subspace E of X, given $0 < r < 1$, there is $x \in X$ with $\|x\| = 1$ but $\operatorname{dist}(x, E) = \inf_{y \in E}\|x - y\| > r$.*

We close this section with a result about the rank of projections. This is surely known, but we include a proof for completeness.

Lemma 1.2. *Suppose P and Q are projections in $B(X)$. If $\|P - Q\| < 1$ then $\operatorname{rank} P = \operatorname{rank} Q$.*

Proof. Let $M = \operatorname{im} P = \{Px : x \in X\}$. For $y \in M$, we have $Py = y$ and hence $\|Qy\| \geq \|Py\| - \|(P - Q)y\| \geq (1 - \|P - Q\|)\|y\|$. This shows that the restriction $Q|_M$ of Q to M is injective, which further implies that $\operatorname{rank} Q \geq \dim M = \operatorname{rank} P$. Similarly, $\operatorname{rank} P \geq \operatorname{rank} Q$. The desired equality follows. □

2. Auxiliary lemmas

Throughout this section, X is a complex Banach space of dimension greater than one. By $\mathcal{P}_0(X)$ and $\mathcal{P}_1(X)$, we denote, respectively, the set of all projections of rank one in $B(X)$ and the set of all projections of rank one with norm 1 in $B(X)$. We always suppose that ϕ is a multiplicative map from $B(X)$ into itself and satisfies $\|\phi(A) - A\| \leq \delta\|A\|$ for all $0 \neq A \in B(X)$, where $\delta \in (0, 1)$ is fixed.

Lemma 2.1. *ϕ preserves rank-1 operators.*

Proof. For non-zero vectors $x \in X$ and $f \in X^*$, choose $g \in X^*$ such that $g(x) = 1$ and $\|g\| = \frac{1}{\|x\|}$. Let $P = x \otimes g$. Then $P \in \mathcal{P}_1(X)$. Therefore, $\phi(P)$ is a projection and $\|\phi(P) - P\| < \|P\| = 1$. It follows from

Lemma 1.2 that $\phi(P)$ is of rank one. Now since $\phi(x \otimes f) = \phi(Px \otimes f) = \phi(P)\phi(x \otimes f)$, we know that the rank of $\phi(x \otimes f)$ is at most one. If $\phi(x \otimes f) = 0$, then $\|x \otimes f\| = \|\phi(x \otimes f) - x \otimes f\| \leq \delta\|x \otimes f\| < \|x \otimes f\|$, a contradiction. So $\phi(x \otimes f)$ is of rank one. $\square$

Lemma 2.2. *For $A \in B(X)$, $\phi(A) = 0$ if and only if $A = 0$.*

Proof. The necessity is clear, and we only need to verify the sufficiency. For $P \in \mathcal{P}_0(X)$, since $\phi(P)$ is of rank one by Lemma 2.1, we have some scalar α_P such that

$$\phi(0) = \phi(P0P) = \phi(P)\phi(0)\phi(P) = \alpha_P\phi(P).$$

Since $\phi(0)$ and $\phi(P)$ are projections, we can get that $\alpha_P = 0$ or $\alpha_P = 1$. If $\alpha_P = 0$ for some $P \in \mathcal{P}_0(X)$, it is trivial to see that $\phi(0) = 0$.

Now suppose that $\alpha_P = 1$ for all $P \in \mathcal{P}_0(X)$. Then we have $\phi(P) = \phi(0)$ for all $P \in \mathcal{P}_0(X)$. Take vectors $x_1 \in X$ and $f_1 \in X^*$ satisfying $\|x_1\| = \|f_1\| = f_1(x_1) = 1$. Since the dimension of X is greater than one, we can take x from $\ker f_1$ with $\|x\| = 3$. Let $x_2 = \frac{1}{\|x_1 - x\|}(x_1 - x)$. Then $\|x_2\| = 1$ and $|f_1(x_2)| = \frac{1}{\|x_1 - x\|} \leq \frac{1}{\|x\| - \|x_1\|} = \frac{1}{2}$. Choose $f \in X^*$ such that $\|f\| = f(x_2) = 1$. If $f(x_1) \neq 0$, we let $f_2 = f$; if $f(x_1) = 0$, we let $f_2 = \frac{\|x_1 - x\|}{1 + \|x_1 - x\|}(f_1 + f)$. Then $f_2(x_2) = 1$ and $0 < |f_2(x_1)| \leq 1$. Let $P_i = x_i \otimes f_i$, $i = 1, 2$. Then $P_1, P_2 \in \mathcal{P}_0(X)$. Thus, we have $\phi(P_1P_2P_1) = \phi(P_1)\phi(P_2)\phi(P_1) = \phi(0)^3 = \phi(0)$ and hence $\phi((P_1P_2P_1)^n) = \phi(P_1P_2P_1)^n = \phi(0)$ for all $n \in \mathbb{N}$. Since $(P_1P_2P_1)^n = (f_1(x_2)f_2(x_1))^nP_1 \neq 0$, we have

$$\begin{aligned}\|\phi(0) - (P_1P_2P_1)^n\| &= \|\phi((P_1P_2P_1)^n) - (P_1P_2P_1)^n\| \\ &\leq \|(P_1P_2P_1)^n\| \leq \|(P_1P_2P_1)\|^n.\end{aligned}$$

Letting $n \to \infty$ in the above equation and noting $\|P_1P_2P_1\| = |f_1(x_2)||f_2(x_1)| \leq \frac{1}{2}$, we get that $\phi(0) = 0$. $\square$

Lemma 2.3. *Let A and B be in $B(X)$ and suppose that $\phi(A) = \phi(B)$. Then A and B are linearly dependent.*

Proof. If one of A and B is zero, then another is also zero by Lemma 2.2. Now suppose that A and B are of rank one, namely, $A = x_1 \otimes f_1$ and $B = x_2 \otimes f_2$ for some non-zero vectors $x_1, x_2 \in X$, $f_1, f_2 \in X^*$. Then for any $y \in X$, we have $\phi(f_1(y)x_1 \otimes f_1) = \phi((x_1 \otimes f_1)(y \otimes f_1)) = \phi((x_2 \otimes f_2)(y \otimes f_1)) = \phi(f_2(y)x_2 \otimes f_1)$. By Lemma 2.2, $f_1(y) = 0$ if and only if $f_2(y) = 0$. By the arbitrariness of $y \in X$, we conclude that f_1 and f_2 are linearly dependent. Similarly, for $h \in X^*$ we have that $\phi(h(x_1)x_1 \otimes f_1) = \phi(h(x_2)x_1 \otimes f_2)$. This implies that $h(x_1) = 0$ if and only if $h(x_2) = 0$. So x_1 and x_2 are linearly dependent. Consequently, A and B are linearly dependent.

Finally, assume that one of A and B is not of rank one. For any $x \in X$, $f \in X^*$, we have $\phi(Ax \otimes f) = \phi(Bx \otimes f)$. By the preceding result, Ax and Bx are linearly dependent for all $x \in X$. This together with the assumption shows that A and B are linearly dependent. $\square$

Lemma 2.4. $\operatorname{span}\{\operatorname{im}\phi(P) : P \in \mathcal{P}_1(X)\}$ *is norm dense in X. Furthermore, for each non-zero $f \in X^*$,* $\operatorname{span}\{\operatorname{im}\phi(x \otimes f) : x \in X\}$ *is norm dense in X.*

Proof. Let X_1 be the (norm) closure of $\operatorname{span}\{\operatorname{im}\phi(P),\ P \in \mathcal{P}_1(X)\}$. Suppose on the contrary that $X_1 \neq X$. Then by the Riesz lemma, there exists a unit vector $x \in X$ such that $\alpha := \operatorname{dist}(x,\ X_1) > \frac{1}{2}(1 + \delta)$. By the Hahn–Banach theorem, there exists $f \in X^*$ such that $\|f\| = 1$, $f(X_1) = 0$ and $f(x) = \alpha$. Let $P = x \otimes f$. Then $\|P\| = 1$ and $\|x - Px\| = 1 - \alpha$. Choose $g \in X^*$ such that $\|g\| = g(x) = 1$. Then $x \otimes g \in \mathcal{P}_1(X)$. Since $\phi(P) = \phi((x \otimes g)(x \otimes f)) = \phi(x \otimes g)\phi(P)$, we have that $\operatorname{im}\phi(P) \subseteq \operatorname{im}\phi(x \otimes g) \subseteq X_1$. Thus

$$\begin{aligned}\alpha = f(x) = f(x - \phi(P)x) &\leq \|x - \phi(P)x\| \\ &\leq \|x - Px\| + \|\phi(P)x - Px\| \leq 1 - \alpha + \delta.\end{aligned}$$

From this we get $\alpha \leq \frac{1+\delta}{2}$, a contradiction. So $X_1 = X$.

Now let $0 \neq f \in X^*$ and choose $y \in X$ such that $f(y) = 1$. For $P = x \otimes g \in \mathcal{P}_1(X)$, we have $\phi(P) = \phi((x \otimes f)(y \otimes g)) = \phi(x \otimes f)\phi(y \otimes g)$. So the range of $\phi(P)$ is contained in the range of $\phi(x \otimes f)$. By the preceding result, $\operatorname{span}\{\operatorname{im} \phi(x \otimes f) : x \in X\}$ is norm dense in X. □

Lemma 2.5. *ϕ is homogeneous.*

Proof. We first show a claim.

Claim. *There holds $\phi(zP) = z\phi(P)$ for all $P \in \mathcal{P}_1(X)$ and $z \in \mathbb{C}$.*

Let $P \in \mathcal{P}_1(X)$. By Lemma 2.1, $\phi(P)$ is a projection of rank one. Therefore, for $z \in \mathbb{C}$ there corresponds a scalar $\theta(z)$ such that $\phi(P)\phi(zP)\phi(P) = \theta(z)\phi(P)$. Thus, $\phi(zP) = \theta(z)\phi(P)$. Since P is of rank one, there exists a scalar α such that $P\phi(P)P = \alpha P$. Hence

$$\begin{aligned}|\alpha\theta(z) - z| &= \|\theta(z)P\phi(P)P - zP\| \leq \|P\|\|\theta(z)\phi(P) - zP\|\|P\| \\ &= \|\theta(z)\phi(P) - zP\| = \|\phi(zP) - zP\| \\ &\leq \delta|z|.\end{aligned}$$

By [13, Lemma 3.4], we obtain $\theta(z) = z$ for all $z \in \mathbb{C}$. This establish the claim.

Now let $z \in \mathbb{C}$. Then for any $Q \in \mathcal{P}_1(X)$, we have $\phi(zI)\phi(Q) = \phi(zQ) = z\phi(Q)$, i.e. $(\phi(zI) - zI)\phi(Q) = 0$. By Lemma 2.4, we can get $\phi(zI) = zI$. Hence, $\phi(zA) = \phi(zI)\phi(A) = z\phi(A)$ for all $A \in B(X)$. □

Lemma 2.6. *ϕ is injective.*

Proof. Suppose that $\phi(A) = \phi(B)$, $A, B \in B(X)$. By Lemma 2.3, A and B are linearly dependent, say $B = \alpha A$ for some scalar α. By Lemma 2.5, we have $\phi(A) = \phi(B) = \phi(\alpha A) = \alpha\phi(A)$. Unless $\phi(A) = 0$, this implies $\alpha = 1$ and then $A = B$. If $\phi(A) = \phi(B) = 0$, by Lemma 2.2, $A = B = 0$. So ϕ is injective. □

Lemma 2.7. *If ϕ is additive, then there exists an invertible operator $S \in B(X)$ such that $\phi(A) = SAS^{-1}$, for all $A \in B(X)$.*

Proof. First we notice that ϕ is linear by Lemma 2.5. Fix unit vectors $x_0 \in X$ and $f_0 \in X^*$ with $f_0(x_0) = 1$. By Lemma 2.1, we can suppose that $\phi(x_0 \otimes f_0) = y_0 \otimes g_0$ for $y_0 \in X$ and $g_0 \in X^*$ with $g_0(y_0) = 1$ and $\|g_0\| = 1$. Define a map $S : X \to X$ by

$$Sx = \phi(x \otimes f_0)y_0, \ x \in X.$$

Then S is linear. For $0 \neq x \in X$, we have

$$\begin{aligned}\|Sx\| = \|\phi(x \otimes f_0)y_0\| &\leq \|\phi(x \otimes f_0)\|\|y_0\| \\ &\leq (\|x \otimes f_0\| + \|\phi(x \otimes f_0) - x \otimes f_0\|)\|y_0\| \\ &\leq (1 + \delta)\|x\|\|f_0\|\|y_0\| = (1 + \delta)\|y_0\|\|x\|,\end{aligned}$$

which obviously holds for $x = 0$ by Lemma 2.2. So S is bounded. Moreover, for $A \in B(X)$, there holds

$$SAx = \phi(Ax \otimes f_0)y_0 = \phi(A)\phi(x \otimes f_0)y_0 = \phi(A)Sx$$

for all $x \in X$. So $\phi(A)S = SA$. It now remains to show that S is invertible, which can be deduced from the following two claims.

Claim 1. *S is bounded below.*

For $x \in X$, since

$$\phi(x \otimes f_0) = \phi(x \otimes f_0)\phi(x_0 \otimes f_0) = \phi(x \otimes f_0)y_0 \otimes g_0 = Sx \otimes g_0,$$

we have

$$\begin{aligned}\|Sx\| = \|\phi(x \otimes f_0)\| &\geq \|x \otimes f_0\| - \|\phi(x \otimes f_0) - x \otimes f_0\| \\ &\geq (1-\delta)\|x \otimes f_0\| = (1-\delta)\|x\|.\end{aligned}$$

So S is bounded below.

Claim 2. *S has dense range.*

For $P = x \otimes f \in \mathcal{P}_1(X)$, from

$$\begin{aligned}\phi(P) &= \phi((x \otimes f_0)(x_0 \otimes f_0)(x_0 \otimes f)) = \phi(x \otimes f_0)\phi(x_0 \otimes f_0)\phi(x_0 \otimes f) \\ &= \phi(x \otimes f_0)y_0 \otimes g_0\phi(x_0 \otimes f) = Sx \otimes \phi(x_0 \otimes f)^* g_0,\end{aligned}$$

we see that the range of $\phi(P)$ is contained in the range of S. Hence the range of S is dense by Lemma 2.4. □

3. Main results

Recall that the n-dimensional projection constant $\lambda_n(X)$ of a Banach space X has the following property: whenever E is an n-dimensional subspace of X and $\lambda > \lambda_n(X)$, there is a projection $P : X \to E$ with $\|P\| < \lambda$.

Theorem 3.1. *Let X be a Banach space of dimension greater than one and $0 < \delta < \frac{1}{\lambda_2(X)}$. Let $\phi : B(X) \to B(X)$ be a multiplicative map and suppose that $\|\phi(A) - A\| \leq \delta\|A\|$ for all $0 \neq A \in B(X)$. Then there exists an invertible operator $S \in B(X)$ such that $\phi(A) = S^{-1}AS$, for all $A \in B(X)$.*

Proof. By Lemma 2.7, it suffices to show that ϕ is additive. For this, we take some steps.

Step 1. Let P_1 and P_2 be in $\mathcal{P}_0(X)$ satisfying $P_1P_2 = P_2P_1 = 0$. Then $\phi(P_1 + P_2) = \phi(P_1) + \phi(P_2)$.

Since $\phi(P_1)\phi(P_2) = \phi(P_2)\phi(P_1) = 0$, by Lemma 2.1, $\phi(P_1) + \phi(P_2)$ is a projection of rank 2. Let $P = P_1 + P_2$. Then P is a projection of rank 2. Let Q be the projection of rank 2 onto the range of P with $\|Q\| < \frac{1}{\delta}$. Then $\|\phi(Q) - Q\| \leq \delta\|Q\| < 1$. It follows from Lemma 1.2 that $\phi(Q)$ is of rank 2. Since $QP = P$ and $PQ = Q$, we have that $\phi(Q)\phi(P) = \phi(P)$ and $\phi(P)\phi(Q) = \phi(Q)$, which implies that $\operatorname{im}\phi(P) = \operatorname{im}\phi(Q)$. So $\phi(P)$ is of rank 2. An easy computation gives

$$\phi(P)(\phi(P_1) + \phi(P_2)) = (\phi(P_1) + \phi(P_2))\phi(P) = \phi(P_1) + \phi(P_2).$$

Together with the result shown just that both $\phi(P)$ and $\phi(P_1)+\phi(P_2)$ are projections of rank 2, this yields $\phi(P) = \phi(P_1) + \phi(P_2)$.

Step 2. For $x_1, x_2 \in X$ and $f \in X^*$, there holds $\phi(x_1 \otimes f + x_2 \otimes f) = \phi(x_1 \otimes f) + \phi(x_2 \otimes f)$.

If x_1 and x_2 are linearly dependent, say $x_2 = \mu x_1$ for some scalar μ, then by the homogeneity of ϕ,

$$\phi(x_1 \otimes f + x_2 \otimes f) = \phi((1+\mu)x_1 \otimes f) = (1+\mu)\phi(x_1 \otimes f) = \phi(x_1 \otimes f) + \phi(x_2 \otimes f).$$

Now suppose that x_1 and x_2 are linearly independent. Then we have functionals $f_1, f_2 \in X^*$ satisfying $f_i(x_j) = \delta_{ij}$. By Step 1,

$$\begin{aligned}
&\phi(x_1 \otimes f + x_2 \otimes f)\\
&\quad = \phi((x_1 \otimes f_1 + x_2 \otimes f_2)(x_1 \otimes f + x_2 \otimes f))\\
&\quad = \phi(x_1 \otimes f_1 + x_2 \otimes f_2)\phi(x_1 \otimes f + x_2 \otimes f)\\
&\quad = (\phi(x_1 \otimes f_1) + \phi(x_2 \otimes f_2))\phi(x_1 \otimes f + x_2 \otimes f)\\
&\quad = \phi((x_1 \otimes f_1)(x_1 \otimes f + x_2 \otimes f)) + \phi((x_2 \otimes f_2)(x_1 \otimes f + x_2 \otimes f))\\
&\quad = \phi(x_1 \otimes f) + \phi(x_2 \otimes f).
\end{aligned}$$

Step 3. ϕ is additive.

Let A and B be in $B(X)$. Then for each $P \in \mathcal{P}_1(X)$, by Step 2, we have

$$\begin{aligned}
\phi(A+B)\phi(P) &= \phi((A+B)P) = \phi(AP+BP)\\
&= \phi(AP) + \phi(BP) = (\phi(A)+\phi(B))\phi(P).
\end{aligned}$$

It follows from Lemma 2.4 that $\phi(A+B) = \phi(A) + \phi(B)$. □

By a result of Kadec and Snobar [9], $\lambda_2(X) \le \sqrt{2}$, so $\frac{\sqrt{2}}{2}$ is the common bound of δ in Theorem 3.1. The following theorem shows that when the space is reflexive, the bound of δ can be expanded to 1.

Theorem 3.2. *Let X be a reflexive Banach space of dimension greater than one and $0 < \delta < 1$. Let $\phi : B(X) \to B(X)$ be a multiplicative map and suppose that $\|\phi(A) - A\| \le \delta\|A\|$ for all $0 \ne A \in B(X)$. Then there exists an invertible operator $S \in B(X)$ such that $\phi(A) = S^{-1}AS$, for all $A \in B(X)$.*

Proof. We only need to verify the additivity of ϕ by Lemma 2.7. Firstly, we note that using the reflexivity of X the following claim can be proven in a similar way to that in Lemma 2.4. For the convenience of the reader, we include the proof.

Claim. $\operatorname{span}\{\operatorname{im}\phi(P)^* : P \in \mathcal{P}_1(X)\}$ *is norm dense in X^*. Furthermore, for each non-zero $x \in X$,* $\operatorname{span}\{\operatorname{im}\phi(x \otimes f)^* : f \in X^*\}$ *is norm dense in X^*.*

Let Y be the norm closure of $\operatorname{span}\{\operatorname{im}\phi(P)^*,\ P \in \mathcal{P}_1(X)\}$. Suppose on the contrary that $Y \ne X^*$. Then by the Riesz lemma, there exists a vector $g \in X^*$ with $\|g\| = 1$ such that $\alpha := \operatorname{dist}(g,\ Y) > \frac{1}{2}(1+\delta)$. By the Hahn–Banach theorem, there exists $F \in X^{**}$ such that $\|F\| = 1$, $F(Y) = 0$ and $F(g) = \alpha$. Since X is reflexive, there exists a vector $x \in X$ such that $F = x^{**}$. Let $P = x \otimes g$. Then $\|P\| = 1$ and $\|g - P^*g\| = 1-\alpha$. By the reflexivity of X again, we can choose $y \in X$ such that $\|y\| = g(y) = 1$. Then $y \otimes g \in \mathcal{P}_1(X)$. Since

$\phi(P) = \phi((x \otimes g)(y \otimes g)) = \phi(x \otimes g)\phi(y \otimes g)$, we have that $\phi(P)^* = \phi(y \otimes g)^*\phi(x \otimes g)^*$, which shows $\operatorname{im}\phi(P)^* \subseteq \operatorname{im}\phi(y \otimes g)^* \subseteq Y$. Thus

$$\begin{aligned}\alpha = F(g) = F(g - \phi(P)^*g) &\leq \|g - \phi(P)^*g\| \\ &\leq \|g - P^*g\| + \|\phi(P)^*g - P^*g\| \\ &\leq 1 - \alpha + \|\phi(P)^* - P^*\| \\ &= 1 - \alpha + \|\phi(P) - P\| \leq 1 - \alpha + \delta.\end{aligned}$$

From this we get $\alpha \leq \frac{1+\delta}{2}$, a contradiction. So $Y = X^*$.

Now let $0 \neq x \in X$ and choose $g \in X^*$ such that $g(x) = 1$. For $P = y \otimes f \in \mathcal{P}_1(X)$, since $\phi(P) = \phi((y \otimes g)(x \otimes f)) = \phi(y \otimes g)\phi(x \otimes f)$, we have $\phi(P)^* = \phi(x \otimes f)^*\phi(y \otimes g)^*$. So the range of $\phi(P)^*$ is contained in the range of $\phi(x \otimes f)^*$. By the preceding result, $\operatorname{span}\{\operatorname{im}\phi(x \otimes f)^* : f \in X^*\}$ is norm dense in X^*. The claim is established.

For $x \otimes f \in B(X)$, by Lemma 2.1, we can suppose $\phi(x \otimes f) = y \otimes g$. Squaring it and using the homogeneity of ϕ, we get

$$\phi(f(x)x \otimes f) = \phi(x \otimes f)\phi(x \otimes f) = (y \otimes g)(y \otimes g) = g(y)y \otimes g = \phi(g(y)x \otimes f).$$

By Lemma 2.5, we have $f(x) = g(y)$. In other words, the trace of $\phi(x \otimes f)$ is equal to the trace of $x \otimes f$. Consequently, ϕ preserves the trace of rank one operators.

Now let A and B be in $B(X)$. Then for each operator F of rank one, we have

$$\begin{aligned}&tr(\phi(A + B)\phi(F)) = tr(\phi((A + B)F)) \\ &= tr((A + B)F) = tr(AF) + tr(BF) \\ &= tr(\phi(AF)) + tr(\phi(BF)) = tr(\phi(A)\phi(F)) + tr(\phi(B)\phi(F)) \\ &= tr((\phi(A) + \phi(B))\phi(F)).\end{aligned}$$

Fix $x_0 \otimes f_0 \in \mathcal{P}_1(X)$ and suppose $\phi(x_0 \otimes f_0) = y_0 \otimes g_0$ with $g_0(y_0) = 1$. Then for $x \in X$ and $f \in X^*$, we can get $\phi(x \otimes f_0) = y \otimes g_0$ and $\phi(x_0 \otimes f) = y_0 \otimes g$ for some $y \in X$ and $g \in X^*$. Putting $F = x \otimes f$ in the above displayed equation and noting $\phi(x \otimes f) = y \otimes g$, we have $g(\phi(A + B)y) = g((\phi(A) + \phi(B))y)$. This implies that $\phi(x_0 \otimes f)\phi(A + B)\phi(x \otimes f_0) = \phi(x_0 \otimes f)(\phi(A) + \phi(B))\phi(x \otimes f_0)$ for all $x \in X$ and $f \in X^*$. From Lemma 2.4 and Claim, we can conclude that $\phi(A + B) = \phi(A) + \phi(B)$, completing the proof. □

Corollary 3.3. *Let X be a Banach space of dimension greater than one and $0 < \delta < \frac{1}{\lambda_2(X)}$ (if X is reflexive, $0 < \delta < 1$). Let $\varphi : B(X) \to B(X)$ be a linear automorphism and $\phi : B(X) \to B(X)$ be a multiplicative map satisfying*

$$\|\phi(A) - \varphi(A)\| \leq \delta\|\varphi(A)\|$$

for all $0 \neq A \in B(X)$. Then there exists an invertible operator $T \in B(X)$ such that $\phi(A) = T^{-1}AT$, for all $A \in B(X)$.

Proof. Set $\tau = \phi \circ \varphi^{-1}$, so that τ is a multiplicative map on $B(X)$. Then

$$\|\tau(\varphi(A)) - \varphi(A)\| \leq \delta\|\varphi(A)\|$$

for all $0 \neq A \in B(X)$. By the above two theorems, there exists an invertible $R \in B(X)$ such that $\tau(B) = R^{-1}BR$ for all $B \in B(X)$. Hence $\phi = \tau \circ \varphi$ is again an automorphism of $B(X)$. But every automorphism of $B(X)$ is spatial, and so we can find invertible $T \in B(X)$ so that $\phi(A) = T^{-1}AT$ for all $A \in B(X)$. $\square$

Finally we investigate some properties of operators S obtained in the above theorems in the Hilbert space setting.

Proposition 3.4. *Let H be a complex Hilbert space of dimension greater than one and S be an invertible operator in $B(H)$ with $\|S\| = 1$. If $\|S^{-1}AS - A\| \leq \delta\|A\|$ for all $A \in B(H)$, then $\mathrm{dist}(S, \mathbb{C}I) \leq \frac{\delta}{2}$. Moreover, if S is unitary, then the condition is also sufficient.*

Proof. Replacing A by SA in $\|S^{-1}AS - A\| \leq \delta\|A\|$, we get $\|AS - SA\| \leq \delta\|SA\| \leq \delta\|A\|$ for all $A \in B(H)$. This implies that $\|\Delta_S\| \leq \delta$, where Δ_S is the map (inner derivation) sending A to $AS - SA$. By [19, Theorem 4], $\mathrm{dist}(S, \mathbb{C}I) = \frac{1}{2}\|\Delta_S\| \leq \frac{\delta}{2}$.

Now suppose that S is unitary and $\mathrm{dist}(S, \mathbb{C}I) \leq \frac{\delta}{2}$. Then by [19, Theorem 4] again, $\|\Delta_S\| = 2\mathrm{dist}(S, \mathbb{C}I) \leq \delta$. Thus, for all $A \in B(H)$, we have $\|AS - SA\| \leq \delta\|A\|$, and hence $\|S^{-1}AS - A\| \leq \delta\|S^{-1}A\| = \delta\|A\|$. $\square$

Acknowledgments

The research is supported by the National Natural Science Foundation of China (Grant No. 11171244). The first author is supported by the Foundation of Shanxi Datong University (Grant No. 2010Q2). The second author is supported by the joint project from the commission of Science & technology of Guizhou Province, Anshun government and Anshun University (Grant No. LKA[2013]04). The authors thank the anonymous referee for the very thorough reading of the paper and valuable comments.

References

[1] N. Boudi, P. Semrl, Semigroup automorphisms of nest algebras, Houston J. Math. 38 (2012) 1197–1206.
[2] M. Bresar, P. Semrl, Linear maps preserving the spectral radius, J. Funct. Anal. 142 (1996) 360–368.
[3] P.R. Chernoff, Representations, automorphisms, and derivations of some operator algebras, J. Funct. Anal. 12 (1973) 275–289.
[4] L. Fang, G. Ji, Y. Pang, Maps preserving the idempotency of products of operators, Linear Algebra Appl. 426 (2007) 40–52.
[5] B. Grunbaum, Projection constants, Trans. Amer. Math. Soc. 95 (1960) 451–465.
[6] K. Hermann, T.J. Nicole, Norms of minimal projection, J. Funct. Anal. 119 (1994) 253–280.
[7] S.H. Hochwald, Multiplicative maps on matrices that preserve spectrum, Linear Algebra Appl. 212–213 (1994) 339–351.
[8] A.A. Jafarian, A.R. Sourour, Spectrum-preserving linear maps, J. Funct. Anal. 66 (1986) 439–458.
[9] I.M. Kadec, M.G. Snobar, Certain functionals on Minkowsli compactum, Math. Notes 10 (1971) 694–696.
[10] H. Konig, D.R. Lewis, P.K. Lin, Finite dimensional projection constants, Studia Math. 75 (1983) 341–358.
[11] B. Kuzma, Additive idempotent preservers, Linear Algebra Appl. 355 (2002) 103–117.
[12] D.R. Larson, A.R. Sourour, Local derivations and local automorphisms of $B(X)$, Proc. Sympos. Pure Math. 51 (1990) 187–194.
[13] L.W. Marcoux, H. Radjavi, A.R. Sourour, Multiplicative maps that are close to an automorphism on algebras of linear transformations, Studia Math. 214 (2013) 279–296.
[14] L. Molnar, Some multiplicative preservers on $B(H)$, Linear Algebra Appl. 301 (1999) 1–13.
[15] P. Semrl, Isomorphisms of standard operator algebras, Proc. Amer. Math. Soc. 123 (1995) 1851–1855.
[16] P. Semrl, Linear maps that preserve the nilpotent operators, Acta Sci. Math. (Szeged) 61 (1995) 523–534.
[17] P. Semrl, Two characterizations of automorphisms on B(X), Studia Math. 105 (1993) 143–149.
[18] A.R. Sourour, Invertibility preserving linear maps on L(X), Trans. Amer. Math. Soc. 348 (1996) 13–30.
[19] J.G. Stampfli, The norm of a derivation, Pacific J. Math. 33 (1970) 737–744.