J. Math. Anal. Appl. 431 (2015) 365–405

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Large solutions of elliptic semilinear equations in the borderline case. An exhaustive and intrinsic point of view

S. Alarcón [a], G. Díaz [b,*], J.M. Rey [b]

[a] *Departamento de Matemática, Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile*
[b] *Instituto de Matemática Interdisciplinar, Departamento de Matemática Aplicada, Universidad Complutense de Madrid, 28040 Madrid, Spain*

ARTICLE INFO

Article history:
Received 23 February 2015
Available online 29 May 2015
Submitted by A. Cianchi

Keywords:
Large solutions
Asymptotic behavior
Boundary blow-up

ABSTRACT

We revisit the problem $\Delta u = f(u)$ in Ω, $u(x) \to \infty$ as $x \to \partial\Omega$, where $\Omega \subset \mathbb{R}^N$, $N > 1$, is a bounded smooth domain and f is an increasing and continuous function in $\mathbb{R}_+$ with $f(0^+) = 0$ for which the Keller–Osserman condition holds. We study uniqueness of solutions, extending known results about the boundary blow-up behavior of solutions. Furthermore, we obtain explicit representations for the second order terms in the explosive boundary expansion of solutions under intrinsic and direct assumptions. Our study is exhaustive including both ordinary and borderline cases providing new and sharp results.

1. Introduction

This paper deals with the uniqueness and asymptotic behavior near the boundary of the solutions of the problem

$$\begin{cases} -\Delta u + f(u) = 0 & \text{in } \Omega, \\ u(x) \to \infty \quad \text{as} & x \to \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N > 1$, is a bounded domain, with $\partial\Omega$ bounded, and f is a continuous and increasing function in $\mathbb{R}_+$ with $f(0^+) = 0$. Both assumptions on Ω and f will be assumed every time in the whole paper, without necessity to mention them. In some results we will require extra assumptions to be specified. This kind of solutions are usually called *large solutions* due to the boundary blow-up behavior.

* Corresponding author.
E-mail addresses: salomon.alarcon@usm.cl (S. Alarcón), gdiaz@ucm.es (G. Díaz), jrey@ucm.es (J.M. Rey).

http://dx.doi.org/10.1016/j.jmaa.2015.05.068

A first study known of (1.1) is due to Bieberbach [12] who considered the two-dimensional case for the particular choice $f(t) = e^t$. Existence of solutions of (1.1) was established by Keller [25] and Osserman [38] under the necessary and sufficient condition

$$\int^{\infty} \frac{ds}{\sqrt{F(s)}} < \infty, \quad \text{where } F(t) = \int_0^t f(s)\, ds. \tag{1.2}$$

From then the property (1.2) is known as the *Keller–Osserman condition*. Clearly, (1.2) implies the behavior $\lim_{t\to\infty} f(t) = \infty$.

Our goal here is to study uniqueness and asymptotic behavior near the boundary of the solutions of (1.1). In this paper, we present an exhaustive point of view by a simple alternative in the unique available framework where (1.2) holds (see Remark A.2 in Appendix A for other equivalent ones).

About uniqueness, it is known that the question may be reduced to find appropriate conditions on f for which any couple of nonnegative solutions u and v have the same main explosive boundary behavior as

$$\lim_{x\to\partial\Omega} \frac{u(x)}{v(x)} = 1. \tag{1.3}$$

A way to obtain a result as (1.3) is through the study of the blow-up rate of solutions (see [20] or [22] for other methods). Then, we obtain uniqueness in the explosive boundary behavior of the solutions whence we confer the uniqueness on the whole domain by a kind of Maximum Principle. Another aspect of our goal is the study of the explosive boundary expansion of solutions; by simplicity we only focus on the two first terms, where the second one leads to the eventual influence of the mean curvature. So, we find representations connecting our results with the question about the difference between a solution of (1.1) and its blow-up rate at the boundary near $\partial\Omega$. Here, the reasonings follow a simple approach already used by the authors for the power like choice $f(t) = t^p$, $p > 1$ (see [1,2]). It consists in appropriate balances on the terms involving the rest of the approximation immediately preceding. Advantages of our approach will be clarified later through some concrete examples.

In order to introduce our results, it is worth noting that condition (1.2) is directly related to the solution of a one-dimensional problem. Indeed, if we define

$$\psi(t) = \int_t^{\infty} \frac{ds}{\sqrt{2F(s)}}, \quad t > 0, \tag{1.4}$$

then, by construction, the function

$$\phi(\delta) = \psi^{-1}(\delta), \quad 0 < \delta < b = \lim_{t\searrow 0} \psi(t) \le \infty, \tag{1.5}$$

solves the singular problem

$$\begin{cases} -\phi'' + f(\phi) = 0 & \text{in } (0,b), \\ \lim_{\delta\to 0^+} \phi(\delta) = \infty. \end{cases}$$

Moreover, by construction ψ is decreasing and then for $\eta > 1$ one satisfies $\psi(\eta t) < \psi(t)$ for large t, whence

$$\limsup_{t\to\infty} \frac{\psi(\eta t)}{\psi(t)} \le 1. \tag{1.6}$$

Many results on the blow-up rate at the boundary of solutions of (1.1) are known in the literature. The first one was due to Loewner and Nirenberg [31] for the special choice $f(t) = t^{(N+2)/(N-2)}$, $N > 2$, and later extended by Kondrat'ev and Nikishkin [26] to $f(t) = t^p$, $p > 1$ (see also [19]). For general nonlinearities Bandle and Essén [8,9] proved the blow-up rate of solutions of (1.1)

$$\lim_{x \to \partial\Omega} \frac{\psi(u(x))}{\operatorname{dist}(x, \partial\Omega)} = 1$$

(see also [36]). In order to obtain an explicit characterization for the solutions an additional assumption on f was considered by Bandle and Essén [9]

$$\limsup_{t \to \infty} \frac{\psi(\eta t)}{\psi(t)} < 1 \quad \text{for } \eta > 1 \tag{1.7}$$

or, equivalently,

$$\liminf_{t \to \infty} \frac{\psi(\eta t)}{\psi(t)} > 1 \quad \text{for } \eta \in (0, 1)$$

(see also [10] and [36]).

In order to simplify, from now on we call *nonlinearities in the ordinary case* to those nonlinearities, as the power like choices, for which (1.7) holds. Our first result is related to the blow-up rate in this ordinary case. In Section 3 the main behavior will be proved

Theorem 1.1 *(Blow-up rate in the ordinary case). Assume $\partial\Omega \in \mathcal{C}^2$ and f such that (1.2) and (1.7) hold. Then every solution u of (1.1) verifies*

$$\lim_{\mathrm{d}(x) \to 0} \frac{u(x)}{\phi(\mathrm{d}(x))} = 1, \tag{1.8}$$

where $\mathrm{d}(x) = \operatorname{dist}(x, \partial\Omega)$.

Now, we can ask

What can we say about functions f for which (1.2) holds but (1.7) fails?

as it happens for choices as $f(t) = t(\log t)^p$, $p > 2$, for large t. According to (1.6), this situation corresponds to functions f for which

$$\limsup_{t \to \infty} \frac{\psi(\eta_0 t)}{\psi(t)} = 1 \quad \text{for some } \eta_0 > 1 \tag{1.9}$$

or, equivalently,

$$\liminf_{t \to \infty} \frac{\psi(\eta_0^{-1} t)}{\psi(t)} = 1 \quad \text{for some } \eta_0 > 1$$

holds. We note that the alternative (1.7) or (1.9) is obviously exhaustive, provided (1.2).

As it is proved in Lemma A.2 below, condition (1.9) is equivalent to

$$\limsup_{t \to \infty} \frac{\eta f(t)}{f(\eta t)} = 1 \quad \text{for } \eta \in [1, \eta_0]. \tag{1.10}$$

Again for clarity, hereafter on we call *nonlinearities in the borderline case* to those nonlinearities for which (1.9) or (1.10) holds. Assumptions with *regularly varying* at infinity as (1.10) involve the Karamata regular variation theory (see [13,15] or [34]).

Following [2], we study in [3] the boundary behavior of the large solutions of quasilinear elliptic equations in the borderline case.

Several works including results on the borderline case under different hypothesis on f are known (see, for instance, [6,14,24,30,37,39]). We emphasize two of them. In [14], Cîrstea and Du regarded the problem for the choice $f(t) = c_1 t(\log t)^p + c_2(\log t)^{p-1}$ if $t > 1$, $p > 2$, where $c_1 > 0$ and $c_2 \in \mathbb{R}$. On the other hand, Zhang [39] considered choices as $f(t) = t(\log t)^2(\log(\log t))^p$ if $t > 1$, $p > 2$. In both [14] and [39], the results may be applied to more general equations including a weight $b \in \mathcal{C}^\alpha(\overline{\Omega})$ as in the equation

$$\Delta u = b(x)f(u) \quad \text{in } \Omega,$$

being b nonnegative in Ω, positive near $\partial\Omega$ and such that can be vanishing on the boundary.

In this paper, we focus on functions f verifying

$$\liminf_{t\to\infty} \frac{tf'(t)}{f(t)} = \alpha$$

provided that f is differentiable for large t. The Keller–Osserman condition (1.2) leads to $\alpha \geq 1$ (see (A.3)). We prove in Lemma A.3 below that the condition of the borderline case (1.9) implies

$$\lim_{t\to\infty} \frac{tf'(t)}{f(t)} = 1 \tag{1.11}$$

(see (A.4)). In fact, in Remark A.2 we show that, under the Keller–Osserman condition, properties (1.9) and (1.11) are equivalent and property (A.6) below is equivalent to (1.7). We also obtain other equivalent alternatives. On the other hand, in Lemma A.4 we prove that $\alpha > 1$ (related to the ordinary case) implies the Keller–Osserman condition (1.2) (see also [35]). It does not happen whenever $\alpha = 1$, as it is pointed out in Remark A.2.

The assumption (1.11) on f is more simple and generic than the assumption one used in [39]. Moreover, the reasonings use similar arguments to those of the proof of Theorem 1.1, now squeezing as much as possible the condition (1.9). As it is proved in Section 3 one has

Theorem 1.2 *(Blow-up rate in the borderline case). Let f such that (1.2) and (1.9) hold, as well as f is differentiable at the infinity. Then for every solution u of (1.1) the property (1.8)*

$$\lim_{\mathrm{d}(x)\to 0} \frac{u(x)}{\phi(\mathrm{d}(x))} = 1$$

also holds.

So that, the explosive behavior (1.8) of the solutions of (1.1) is fully obtained by the alternative: the ordinary case given by (1.7) or equivalently by

$$\liminf_{t\to\infty} \frac{tf'(t)}{f(t)} > 1, \quad \liminf_{t\to\infty} \frac{tf(t)}{2F(t)} > 1 \quad \text{or} \quad \liminf_{t\to\infty} \frac{2f'(t)F(t)}{\big(f(t)\big)^2} > 1,$$

or the borderline case given by (1.9) or equivalently by

$$\lim_{t\to\infty}\frac{tf'(t)}{f(t)}=1,\quad \lim_{t\to\infty}\frac{tf(t)}{2F(t)}=1\quad\text{or}\quad \lim_{t\to\infty}\frac{2f'(t)F(t)}{\big(f(t)\big)^2}=1$$

(see Remark A.2). We refer to the alternative by the couple (1.7)–(1.9), but we may refer to the alternative by other useful equivalent couple.

As it was pointed out, there is not a unique way to study uniqueness. The uniqueness of solutions of (1.1) is usually proved by means of the exact main term of the boundary blow-up expansion (see for example [1,2,9,10,14,19,22,26,31]). In particular, in [6,14,39] uniqueness was proved for special borderline cases. On the other hand, Marcus and Véron [32,33] and Dong, Kim and Safonov [20] obtained uniqueness without precise estimates of the boundary blow-up rate of solutions by means of reasoning requiring less regularity on $\partial\Omega$, but under additional conditions on f. García-Melián [22] obtained also uniqueness of large solutions by estimating the boundary blow-up rate under the assumption

$$\frac{f(t)}{t^\sigma}\ \text{is increasing for large } t,$$

for some $\sigma>1$, that implies (1.2) and (1.7). At this point on the uniqueness of solutions of (1.1), it should be noted a recent result by Costin, Dupaigne and Goubet [17] whose proof is direct, without using blow-up rates, provided that $\partial\Omega\in\mathcal{C}^3$ has nonnegative mean curvature and $\sqrt{F(t)}$ is convex for large t.

Since Theorems 1.1 and 1.2 prove (1.3) we have the key in our reasoning in obtaining uniqueness.

Theorem 1.3 *(Uniqueness). Let us suppose Ω is smooth and let f such that (1.2) holds. Assume also (1.7) (the ordinary case) or (1.9) as well as that f is differentiable for large t (the borderline case). In both cases, if*

$$\frac{f(t)}{t}\ \textit{is increasing for large } t, \tag{1.12}$$

the problem (1.1) admits a unique nonnegative solution that verifies (1.8).

From

$$\left(\frac{f(t)}{t}\right)'=\frac{f(t)}{t^2}\left(\frac{tf'(t)}{f(t)}-1\right),$$

it follows that

$$\begin{cases}\dfrac{tf'(t)}{f(t)}\to\alpha>1\\ \text{or}\\ \dfrac{tf'(t)}{f(t)}\searrow 1\end{cases}\qquad \text{as } t\to\infty, \tag{1.13}$$

implies (1.12), provided that f is differentiable for large t (see Lemma A.4). Obviously uniqueness says that the unique main explosive term $\phi\big(\mathrm{d}(x)\big)$ (see (1.8)) of the boundary expansion of solutions implies that in fact the whole explosive boundary expansion is unique. Theorem 1.3 is near to the relative results on the ordinary case studied in [22].

The second goal of this paper is the study of lower terms of the asymptotic explosive boundary expansion of the solutions of (1.1). It is known its dependence on the mean curvature of $\partial\Omega$ when (1.7) is assumed. It has been studied in several works, first for the power like choice $f(t)=t^p$, $1<p<3$, in [18] (see also [1,2]) and then for a wide class of functions which verify (1.7) (see, for instance, [5–7,11]). In particular, the more general result that we know under assumption (1.7) is due to Bandle and Marcus [11]. In all works it is shown that the eventual influence of the mean curvature from the explosive boundary expansion of the solutions of (1.1) appears from the second order term.

Since our interest is different to [11], the contributions use another point of view to study the second order terms in the asymptotic expansion near the boundary of the solutions of (1.1).

Is it possible to find an explicit formula for the second order term if (1.7) holds?

In order to answer this question, for large t, we introduce the function

$$\varphi(t)=\begin{cases} -\displaystyle\int_t^\infty \frac{s}{\sqrt{2F(s)}}ds & \text{provided } \displaystyle\int^\infty \frac{s}{\sqrt{2F(s)}}ds<\infty \text{ and } \displaystyle\int_{0+} \frac{s}{\sqrt{2F(s)}}ds=\infty,\\ \displaystyle\int_0^t \frac{s}{\sqrt{2F(s)}}ds & \text{provided } \displaystyle\int^\infty \frac{s}{\sqrt{2F(s)}}ds<\infty \text{ and } \displaystyle\int_{0+} \frac{s}{\sqrt{2F(s)}}ds<\infty,\\ \displaystyle\int_0^t \frac{s}{\sqrt{2F(s)}}ds & \text{provided } \displaystyle\int^\infty \frac{s}{\sqrt{2F(s)}}ds=\infty \text{ and } \displaystyle\int_{0+} \frac{s}{\sqrt{2F(s)}}ds<\infty,\\ \displaystyle\int_{t_0}^t \frac{s}{\sqrt{2F(s)}}ds & \text{provided } \displaystyle\int^\infty \frac{s}{\sqrt{2F(s)}}ds=\infty \text{ and } \displaystyle\int_{0+} \frac{s}{\sqrt{2F(s)}}ds=\infty,\end{cases} \tag{1.14}$$

where $t_0>0$ is irrelevant for our purposes. In all cases one satisfies $\varphi'(t)=-t\psi'(t)>0$. Furthermore,

$$\lim_{t\to\infty}\frac{\varphi(t)}{t}=0, \tag{1.15}$$

because it is obvious if $\displaystyle\int^\infty \frac{s}{\sqrt{2F(s)}}ds<\infty$, for which $\lim_{t\to\infty}|\varphi(t)|<\infty$, otherwise, if $\displaystyle\int^\infty \frac{s}{\sqrt{2F(s)}}ds=\infty$, by an application of the Bernoulli–L'Hôpital rule one has

$$\lim_{t\to\infty}\frac{\varphi(t)}{t}=\lim_{t\to\infty}\frac{t}{\sqrt{2F(t)}}=0$$

(see (A.1) below).

Jointly we also consider the function

$$A(t)=\frac{\sqrt{2F(t)}}{\sqrt{2F(t)}-\varphi(t)f'(t)} \quad \text{for large } t \tag{1.16}$$

and assume the following additional hypotheses on f:

(H1) there exists $\lim_{t\to\infty}A(t)\in\mathbb{R}$,

(H2) $\displaystyle\lim_{t\to\infty}\frac{\big(A(t)\varphi(t)\big)^2f''(t)}{\sqrt{F(t)}}=0$,

(H3) $\displaystyle\lim_{t\to\infty}\left(A''(t)\sqrt{2F(t)}\varphi(t)+\frac{A'(t)f(t)\varphi(t)}{\sqrt{2F(t)}}\right)=0$,

(H4) $\lim_{t\to\infty}A'(t)t=0$.

We emphasize that these hypotheses are intrinsic in the obtainment of the second terms in the explosive boundary expansion of solutions in the ordinary case, as we show in Remark 5.1.

Theorem 1.4 *(Second order terms in the ordinary case). Suppose* $\partial\Omega \in \mathcal{C}^4$ *and let* f *a function twice differentiable at infinity for which* (1.2), (1.7) *and* (H1)–(H4) *hold. If* u *is a solution of* (1.1), *then we have*

$$u(x) = \phi\big(\mathrm{d}(x)\big) + A\big(\phi\big(\mathrm{d}(x)\big)\big)\varphi\big(\phi\big(\mathrm{d}(x)\big)\big)\big(\Delta\mathrm{d}(x) + o(1)\big) \tag{1.17}$$

where $o(1) \to 0$ *as* $\mathrm{d}(x) \to 0$.

The leading term in the above expansion

$$\phi\big(\mathrm{d}(x)\big) + A\big(\phi\big(\mathrm{d}(x)\big)\big)\varphi\big(\phi\big(\mathrm{d}(x)\big)\big)\big(\Delta\mathrm{d}(x) + o(1)\big)$$

is the first summand because (H1) implies that $A(t)$ is bounded near the infinity and then

$$\lim_{\mathrm{d}(x)\to 0} \frac{A\big(\phi\big(\mathrm{d}(x)\big)\big)\varphi\big(\phi\big(\mathrm{d}(x)\big)\big)}{\phi\big(\mathrm{d}(x)\big)} = 0$$

(see (1.15)). As it will be proved below, we can use Theorem 1.4 in order to give some properties of the second term of the expansion near $\partial\Omega$. See also Corollary 2.1 where we prove that in some cases the influence of the geometry is null on the boundary $\partial\Omega$ or it appears in a nonexplosive term of the boundary expansion of the solution. Theorem 1.4 shows the eventual appearance of the mean curvature of $\partial\Omega$ (see (4.1) below) in the second order terms of the explosive boundary expansion of the solution of (1.1).

We also note that Theorem 1.4 fails in borderline cases. In this way,

What one can say about the second order terms in the borderline case?

To the best of our knowledge, a first result studying the second order terms in the explosive boundary expansion of the solutions of (1.1) under assumptions (1.2) and (1.11) is due to Anneda and Porru [6] for functions f such that for some $\alpha > 0$

$$\frac{2F(t)f'(t)}{(f(t))^2} = 1 + \big(\alpha + o(1)\big)(-\log t)^{-1}, \quad \text{where } o(1) \to 0 \text{ as } t \to \infty. \tag{1.18}$$

They proved that for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that the solution of (1.1) satisfies

$$\left|\frac{u(x)}{\phi\big(\mathrm{d}(x)\big)} - 1 - \frac{(\alpha-1)(N-1)}{2(2\alpha-1)}H(\bar{x})\mathrm{d}(x)\right| < \varepsilon\mathrm{d}(x) + C_\varepsilon\big(\mathrm{d}(x)\big)^2 \quad \text{for small } \mathrm{d}(x) \tag{1.19}$$

where $H(\bar{x})$ is the mean curvature of $\partial\Omega$ at the projection $\bar{x}$ of x at $\partial\Omega$. The condition (1.18) fails for $f(t) = t(\log t)^2(\log(\log t))^p$, if $t > 1$, $p > 2$, or $f(t) = te^{(\log t)^p}$ if $t > 1$, $0 < p < 1$, which are functions satisfying (1.2) and (1.9). The next result allows us to include the above choices in the borderline case and also gives an estimate of the second order term much more accurate than the one given in (1.19).

Theorem 1.5 *(Second order terms in the borderline case). Suppose* $\partial\Omega \in \mathcal{C}^4$ *and let* f *be a function twice differentiable at infinity satisfying* (1.2) *and* (1.9). *If* u *is a solution of* (1.1) *we have the expansion*

$$u(x) = \phi(\mathrm{d}(x))\left(1 + \frac{\sqrt{2F\big(\phi\big(\mathrm{d}(x)\big)\big)}}{f(\phi(\mathrm{d}(x)))}\big(\Delta\mathrm{d}(x) + o(1)\big)\right) \tag{1.20}$$

where $o(1) \to 0$ *as* $\mathrm{d}(x) \to 0$.

It is not difficult to check that from Theorem 1.5 the difference between a solution of (1.1) and its respective blow-up rate tends to infinity as $\mathrm{d}(x) \to 0$, whenever $\Delta\mathrm{d}(x) \neq 0$. Since (1.18) is included in assumption (1.9) (see Lemma A.3 and Remark A.2 below) the simple and sharp property (1.20) extends the results of [5,30,39]. In fact, as it is deduced from Remark 5.2, the expansion (1.20) contains the more explosive second order term where the influence of the geometry appears. We note that, as it is proved in Corollary 2.2 below, the behavior of the second term in the borderline case is always explosive.

The rest of this paper is arranged as follows. Section 2 is devoted to show examples and direct consequences where our results apply. In Section 3 we give some comments about the uniqueness and for completeness we also give a proof of Theorem 1.3 by assuming that (1.3) holds. In Section 4 we find the blow-up rate of the solutions of (1.1) which leads to validate (1.3) including borderline cases, whereas in Section 5 we prove the results associated with the second order terms in the explosive behavior expansion near the boundary of the solutions of (1.1), again including borderline cases. We end with Appendix A where we collect some technicalities to be used in previous sections.

2. Consequences, remarks and examples

The influence of the function φ, defined in (1.14), enables us to classify the difference between a solution and its respective blow-up rate. According to Theorem 1.4, it is based on

$$u(x) - \phi(\mathrm{d}(x)) = \mathcal{R}\big(\mathrm{d}(x)\big)\big(\Delta\mathrm{d}(x) + o(1)\big)$$

for $\mathcal{R}(\delta) = A\big(\phi(\delta)\big)\varphi\big(\phi(\delta)\big)$. So, from definition of the function $\varphi(t)$, one deduces the representation:

$$\mathcal{R}(\delta) = \begin{cases} -\dfrac{\displaystyle\int_{\phi(\delta)}^{\infty} \frac{s}{\sqrt{2F(s)}}ds}{1 - \dfrac{\varphi\big(\phi(\delta)\big)f'\big(\phi(\delta)\big)}{\sqrt{2F\big(\phi(\delta)\big)}}} & \text{if } \displaystyle\int^{\infty} \frac{s}{\sqrt{2F(s)}}ds < \infty \text{ and } \int_{0^+} \frac{s}{\sqrt{2F(s)}}ds = \infty, \\[4ex] \dfrac{\displaystyle\int_{0}^{\phi(\delta)} \frac{s}{\sqrt{2F(s)}}ds}{1 - \dfrac{\varphi\big(\phi(\delta)\big)f'\big(\phi(\delta)\big)}{\sqrt{2F\big(\phi(\delta)\big)}}} & \text{if } \displaystyle\int^{\infty} \frac{s}{\sqrt{2F(s)}}ds < \infty \text{ and } \int_{0^+} \frac{s}{\sqrt{2F(s)}}ds < \infty, \\[4ex] \dfrac{\displaystyle\int_{0}^{\phi(\delta)} \frac{s}{\sqrt{2F(s)}}ds}{1 - \dfrac{\varphi\big(\phi(\delta)\big)f'\big(\phi(\delta)\big)}{\sqrt{2F\big(\phi(\delta)\big)}}} & \text{if } \displaystyle\int^{\infty} \frac{s}{\sqrt{2F(s)}}ds = \infty \text{ and } \int_{0^+} \frac{s}{\sqrt{2F(s)}}ds < \infty, \\[4ex] \dfrac{\displaystyle\int_{t_0}^{\phi(\delta)} \frac{s}{\sqrt{2F(s)}}ds}{1 - \dfrac{\varphi\big(\phi(\delta)\big)f'\big(\phi(\delta)\big)}{\sqrt{2F\big(\phi(\delta)\big)}}} & \text{if } \displaystyle\int^{\infty} \frac{s}{\sqrt{2F(s)}}ds = \infty \text{ and } \int_{0^+} \frac{s}{\sqrt{2F(s)}}ds = \infty, \end{cases}$$

for small δ, where $t_0 > 0$. Then

Corollary 2.1 *(Behavior of the second term in the ordinary case). Suppose the assumptions of Theorem 1.4. Let u be a solution of (1.1).*

i) If $\int^{\infty} \frac{s}{\sqrt{2F(s)}} ds < \infty$ *and* $\int_{0^+} \frac{s}{\sqrt{2F(s)}} ds = \infty$ *one verifies*

$$u(x) - \phi\big(\mathrm{d}(x)\big) \to 0 \qquad \text{as } \mathrm{d}(x) \to 0.$$

Therefore the influence of the geometry is null on the boundary.

ii) If $\int^{\infty} \frac{s}{\sqrt{2F(s)}} ds < \infty$ *and* $\int_{0^+} \frac{s}{\sqrt{2F(s)}} ds < \infty$ *there exists a positive constant c such that*

$$|u(x) - \phi(\mathrm{d}(x))| < c \quad \text{near } \partial\Omega.$$

More precisely,

$$\frac{u(x) - \phi(\mathrm{d}(x))}{\Delta \mathrm{d}(x)} \to \left(\int_0^{\infty} \frac{s}{\sqrt{2F(s)}} ds \right) \lim_{t\to\infty} A(t) \qquad \text{as } \mathrm{d}(x) \to 0$$

if $H(\bar{x}) \neq 0$. Therefore the influence of the geometry appears in a nonexplosive term.

iii) If $\int^{\infty} \frac{s}{\sqrt{2F(s)}} ds = \infty$ *one has three cases,*

a) when $\lim_{t\to\infty} \left(A(t) \int^{t} \frac{s}{\sqrt{2F(s)}} ds \right) = 0$

$$u(x) - \phi(\mathrm{d}(x)) \to 0 \qquad \text{as } \mathrm{d}(x) \to 0.$$

Therefore the influence of the geometry is null on the boundary.

b) when $\lim_{t\to\infty} \left(A(t) \int^{t} \frac{s}{\sqrt{2F(s)}} ds \right) = L \in \mathbb{R} \setminus \{0\}$ *there exists a positive constant c such that*

$$|u(x) - \phi(\mathrm{d}(x))| < c \qquad \text{near } \partial\Omega.$$

More precisely,

$$\frac{u(x) - \phi(\mathrm{d}(x))}{\Delta \mathrm{d}(x)} \to L \qquad \text{as } \mathrm{d}(x) \to 0$$

if $H(\bar{x}) \neq 0$. Therefore the influence of the geometry appears in a nonexplosive term.

c) when $\lim_{t\to\infty} \left(A(t) \int^{t} \frac{s}{\sqrt{2F(s)}} ds \right) \in \{-\infty, +\infty\}$

$$|u(x) - \phi(\mathrm{d}(x))| \to \infty \qquad \text{as } \mathrm{d}(x) \to 0.$$

Therefore the influence of the geometry appears in an explosive term. □

Remark 2.1. Our contributions are consistent with other known studies on the difference between a solution of (1.1). Indeed, Lazer and MacKenna [29] prove $|u(x) - \phi(\mathrm{d}(x))| \to 0$ as $\mathrm{d}(x) \to 0$ if $\lim_{t\to\infty} f'(t)/\sqrt{2F(t)} = \infty$. By a different approach, the same was proved by Bandle and Marcus [11] by assuming $\limsup_{\alpha\to 1,\delta\to 0} \phi'(\alpha\delta)/\phi'(\delta) < \infty$ and $\limsup_{t\to\infty} t^4/F(t) < \infty$ in the ordinary case. On the other hand, Greco and Porru have proven that if Ω is convex, (1.2) holds, $\lim_{t\to 0} f'(t) < \infty$ and $\lim_{t\to\infty} f'(t)/\sqrt{2F(t)} = 0$, then $|u(x) - \phi(\mathrm{d}(x))| \to \infty$ as $\mathrm{d}(x) \to 0$.

The following three examples are illustrative of Theorems 1.1, 1.3 and 1.4 involving ordinary cases.

Example 2.1. The power like choice $f(t) = t^p$, $p > 1$, provides a simple presentation of our contributions. As it was also obtained in [1] or [18], the explosive boundary expansion of the unique solution u of (1.1) verifies

$$u(x) = \left(\frac{2(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}} (\mathrm{d}(x))^{-\frac{2}{p-1}} \left(1 - \frac{1}{p+3}\mathrm{d}(x)\Delta\mathrm{d}(x) + o(\mathrm{d}(x))\right), \tag{2.1}$$

hence the influence of the geometry appears in an explosive term if and only if $1 < p < 3$. More precisely, the case $1 < p < 3$ is related to the properties

$$\int^{\infty} \frac{s}{\sqrt{2F(s)}} ds = \infty \quad \text{and} \quad \int_{0^+} \frac{s}{\sqrt{2F(s)}} ds < \infty$$

while the condition $p > 3$ is related to

$$\int^{\infty} \frac{s}{\sqrt{2F(s)}} ds < \infty \quad \text{and} \quad \int_{0^+} \frac{s}{\sqrt{2F(s)}} ds = \infty.$$

In both cases

$$\varphi(t) = \frac{\sqrt{2(p+1)}}{3-p} t^{\frac{3-p}{2}} \quad \text{and} \quad A(t) \equiv \frac{p-3}{(p-1)(p+3)} \quad \text{for } t > 0,$$

lead to the explosive behavior of the first term where the influence of the geometry appears if $1 < p < 3$ while the influence of the geometry appears on a nonexplosive term of the boundary expansion vanishing on $\partial\Omega$ whenever $p > 3$. The last case $p = 3$ becomes a sharp application of our results. Indeed, it is related to

$$\int_{0^+} \frac{s}{\sqrt{2F(s)}} ds = \infty \quad \text{and} \quad \int^{\infty} \frac{s}{\sqrt{2F(s)}} ds = \infty.$$

In order to avoid the second singular integral we support the reasoning on a positive lower limit t_0 as in Corollary 2.1, for which

$$\varphi(t) = \sqrt{2}\ln\left(tt_0^{-1}\right) \quad \text{and} \quad A(t) = \frac{1}{1 - 6\ln\left(tt_0^{-1}\right)} \quad \text{for } t > t_0.$$

Here

$$A(t)\varphi(t) = \frac{\sqrt{2}\ln\left(tt_0^{-1}\right)}{1 - 6\ln\left(tt_0^{-1}\right)} = -\frac{\sqrt{2}}{6}\frac{1}{1 - \dfrac{1}{6\ln\left(tt_0^{-1}\right)}} = -\frac{\sqrt{2}}{6}\sum_{n\geq 0}\left(\frac{1}{6\ln\left(tt_0^{-1}\right)}\right)^n = -\frac{\sqrt{2}}{6}(1 + o(1))$$

where $o(1)$ goes to 0 as $t \to \infty$. Therefore, one obtains

$$u(x) = \sqrt{2}\big(\mathrm{d}(x)\big)^{-1} - \frac{\sqrt{2}}{6}\Delta \mathrm{d}(x) + o(1)$$

where $o(1)$ goes to 0 as $\mathrm{d}(x) \to 0$, according to (2.1) for $p = 3$.

Example 2.2. If $f(t) = e^t$, the explosive boundary expansion of the unique solution u of (1.1) verifies

$$u(x) = \log\left(\frac{2}{(\mathrm{d}(x))^2}\right) - \mathrm{d}(x)\Delta \mathrm{d}(x) + o(1)$$

where $o(1)$ goes to 0 as $\mathrm{d}(x) \to 0$. Indeed, this choice of $f(t)$ is related to the properties

$$\int^{\infty} \frac{s}{\sqrt{2F(s)}} ds < \infty \quad \text{and} \quad \int_{0^+} \frac{s}{\sqrt{2F(s)}} ds < \infty$$

for which

$$\varphi(t) = \sqrt{2}\left(2 - (t+2)e^{-\frac{t}{2}}\right) \quad \text{and} \quad A(t) = \frac{1}{3 + t - 2e^{\frac{t}{2}}} \quad \text{for } t > 0.$$

Therefore the influence of the geometry appears in a nonexplosive term.

Example 2.3. If $f(t) = te^{t^2}$, the explosive boundary expansion of the unique solution u of (1.1) verifies

$$u(x) = \sqrt{2}\mathrm{erfc}^{-1}\left(\frac{\mathrm{d}(x)}{\sqrt{\pi}}\right) - \frac{\sqrt{2}}{8}\frac{e^{-\left(\mathrm{erfc}^{-1}\left(\frac{\mathrm{d}(x)}{\sqrt{\pi}}\right)\right)^2}}{\left(\mathrm{erfc}^{-1}\left(\frac{\mathrm{d}(x)}{\sqrt{\pi}}\right)\right)^2}\Delta \mathrm{d}(x) + o\big(\mathrm{d}(x)\big),$$

for the complementary error function

$$\mathrm{erfc}(t) = 1 - \mathrm{erf}(t) = \frac{2}{\sqrt{\pi}}\int_t^{\infty} e^{-s^2} ds \quad \text{satisfying} \lim_{t\to\infty} \mathrm{erfc}(t) = 0,$$

as it follows from the choice of $f(t)$ and the related properties

$$\int^{\infty} \frac{s}{\sqrt{2F(s)}} ds < \infty \quad \text{and} \quad \int_{0^+} \frac{s}{\sqrt{2F(s)}} ds < \infty$$

for which

$$\varphi(t) = \sqrt{2}\left(1 - e^{-\frac{t^2}{2}}\right) \quad \text{and} \quad A(t) = \frac{1}{1 - 2(2t^2+1)\big(e^{\frac{t^2}{2}} - 1\big)} \quad \text{for } t > 0.$$

The influence of the geometry vanishes on the boundary.

Next we apply our results to the borderline case.

Corollary 2.2 *(Behavior of the second term in the borderline case). Let u be a solution of (1.1). Then, under assumptions of Theorem 1.5 one has*

$$|u(x) - \phi(\mathrm{d}(x))| = \phi(\mathrm{d}(x)) \frac{\sqrt{2F\big(\phi(\mathrm{d}(x))\big)}}{f\big(\phi(\mathrm{d}(x))\big)} \Delta \mathrm{d}(x) \to \infty \quad \textit{as} \quad \mathrm{d}(x) \to 0,$$

provided $H(\bar{x})$ does not change of sign.

Proof. We claim

$$\lim_{t\to\infty} \frac{t^2 F(t)}{(f(t))^2} = \infty. \tag{2.2}$$

Indeed, if (2.2) fails we deduce

$$\liminf_{t\to\infty} \frac{t^3}{f(t)} \text{ does not go to infinity} \tag{2.3}$$

from

$$\frac{t^2 F(t)}{(f(t))^2} = \frac{t^3}{f(t)} \frac{F(t)}{t f(t)},$$

because assumption (1.9) implies

$$\lim_{t\to\infty} \frac{2F(t)}{t f(t)} = 1$$

(see (A.4)). On the other hand, (A.4) also implies

$$\left(\frac{t^3}{f(t)}\right)' = \frac{3t^2 f(t) - t^3 f'(t)}{(f(t))^2} = \frac{t^2}{f(t)} \left(3 - \frac{t f'(t)}{f(t)}\right) > 0 \quad \text{for large } t.$$

Then (2.3) proves $\limsup\limits_{t\to\infty} \dfrac{t^3}{f(t)} = C \in \mathbb{R}$. Applying the Bernoulli–L'Hôpital rule and (A.4) we get

$$\lim_{t\to\infty} \frac{\log f(t)}{\log t} = \lim_{t\to\infty} \frac{t f'(t)}{f(t)} = 1,$$

for which

$$\log(f(t)) \leq (1+\varepsilon) \log t \quad \text{for large } t$$

for any $\varepsilon \in (0, 2)$ fixed, or equivalently

$$f(t) \leq t^{1+\varepsilon} \quad \text{for large } t.$$

Then we obtain the contradiction

$$t^3 \leq (C+\varepsilon) f(t) \leq (C+\varepsilon) t^{1+\varepsilon} \quad \text{for large } t.$$

After the proof of the claim (2.2) the result follows. $\square$

Assuming (1.2) and (A.4), the most intrinsic models, in the borderline case, follow the representation $f(t) = \dfrac{t}{\widetilde{B}(t)}$ for some positive function $\widetilde{B} \in \mathcal{C}(\mathbb{R}_+)$ verifying

$$\begin{cases} \lim\limits_{t\to\infty} \widetilde{B}(t) = \lim\limits_{t\to\infty} \dfrac{t}{f(t)} = 0 & \text{(see (A.1))}, \\ \lim\limits_{t\to\infty} \dfrac{t\widetilde{B}'(t)}{\widetilde{B}(t)} = \lim\limits_{t\to\infty} \left(1 - \dfrac{tf'(t)}{f(t)}\right) = 0 & \text{(see (A.4))}. \end{cases}$$

Moreover, $\widetilde{B}(t)$ is a decreasing function for large t under assumption (1.12) or (1.13). So that, we illustrate Theorems 1.2, 1.3 and 1.5.

Example 2.4. If $f(t) = t(\log t)^p$, $t > 1$, $p > 2$, the explosive boundary expansion of the unique solution u of (1.1) verifies

$$u(x) = e^{\left(\frac{2}{(p-2)\mathrm{d}(x)}\right)^{\frac{2}{p-2}}+\frac{1}{2}} \left(1 + \left(\frac{p-2}{2}\right)^{\frac{p}{p-2}} \big(\mathrm{d}(x)\big)^{\frac{p}{p-2}} \Delta \mathrm{d}(x) + o\left(\big(\mathrm{d}(x)\big)^{\frac{p}{p-2}}\right)\right) \tag{2.4}$$

(see Theorem 1.5). Indeed, one proves that for this choices of $f(t)$ the function

$$B(t) = \frac{\sqrt{2F(t)}}{f(t)}$$

verifies

$$\lim_{t\to\infty} B(t)\big(\log t\big)^{\frac{p}{2}} = 1$$

and

$$\lim_{\mathrm{d}(x)\to 0} B\big(\phi(\mathrm{d}(x))\big)\big(\mathrm{d}(x)\big)^{\frac{p}{2-p}} = \left(\frac{p-2}{2}\right)^{\frac{p}{p-2}}.$$

According to Corollary 2.2, the influence of the geometry appears in an explosive term. The representation (2.4) is sharper than the relative behavior of [6] and also extends some results of [14,30,39]. As it is commented in Remark 5.2, we also may obtain the property

$$u(x) = e^{\left(\frac{2}{(p-2)\mathrm{d}(x)}\right)^{\frac{2}{p-2}}+\frac{1}{2}} \left(1 + \left(\frac{p-2}{2}\right)^{\frac{2p}{p-2}} \big(\mathrm{d}(x)\big)^{\frac{2p}{p-2}} \Delta \mathrm{d}(x) + o\left(\big(\mathrm{d}(x)\big)^{\frac{2p}{p-2}}\right)\right),$$

when we replace $B(t)$ by the function

$$\widetilde{B}(t) = \frac{t}{f(t)},$$

but the behavior (2.4) is sharper because has a second term more explosive.

Remark 2.2 *(On uniqueness).* Costin and Dupaigne [16] have proven uniqueness on a ball by assuming only that f verifies (1.2). On the other hand, for domains that are not regular on $\partial\Omega$, Marcus and Véron [32,33] have proven uniqueness by assuming some conditions on f stronger than (1.12). Both mentioned works, and the related to uniqueness cited in the introduction, suggest possibly that while less regularly possesses $\partial\Omega$, more conditions should be imposed on f.

3. Uniqueness: proof of Theorem 1.3

In this section we give some details on our result of uniqueness. First we obtain the uniqueness of solutions of (1.1) in a direct way when (1.3) and

$$\frac{f(t)}{t} \text{ is increasing for } t > 0, \tag{3.1}$$

are assumed. Later we will prove that (3.1) can be replaced by the more general condition (1.12).

Proposition 3.1 *(Comparison principle). Suppose (3.1). If $u, v \in \mathcal{C}^2(\Omega)$ are two nonnegative functions verifying*

$$-\Delta u + f(u) \leq 0 \leq -\Delta v + f(v) \quad \textit{in } \Omega$$

and

$$\limsup_{x \to \partial\Omega} \frac{u(x)}{v(x)} \leq 1, \tag{3.2}$$

then $v > 0$ near $\partial\Omega$ implies

$$u \leq v \quad \textit{in } \Omega. \tag{3.3}$$

Proof. First we study the case

$$\limsup_{x \to \partial\Omega} \frac{u(x)}{v(x)} < 1$$

for which we only require

$$-\Delta u + f(u) \leq -\Delta v + f(v) \quad \text{in } \Omega.$$

If (3.3) fails the continuous function $u - v$ admits some point $x_0 \in \Omega$ such that

$$(u - v)(x_0) = \max_{\Omega} \ (u - v) > 0,$$

but it implies the contradiction

$$0 \geq \Delta(u - v)(x_0) \geq f\big(u(x_0)\big) - f\big(v(x_0)\big) > 0.$$

For the general case, (3.2) implies

$$\limsup_{x \to \partial\Omega} \frac{u(x)}{(1 + \varepsilon)v(x)} < 1,$$

for all $\varepsilon > 0$. Moreover, from (3.1) we have

$$-\Delta(1 + \varepsilon)v + f\big((1 + \varepsilon)v\big) \geq (1 + \varepsilon)\left(-\Delta v + f(v)\right) \geq 0 \quad \text{in } \Omega.$$

The above reasoning leads to

$$u(x) \le (1+\varepsilon)v(x), \quad x \in \Omega,$$

and the result follows by letting $\varepsilon \to 0$. $\square$

A simple consequence proves that condition (1.3) implies uniqueness.

Corollary 3.1. *Suppose (3.1). If $u, v \in \mathcal{C}^2(\Omega)$ are two nonnegative functions for which*

$$-\Delta u + f(u) = 0 = -\Delta v + f(v) \quad \textit{in } \Omega$$

and (1.3) hold, then $u = v$ on Ω. $\square$

Condition (3.1) follows from an assumption of convexity strict on f. Indeed, it implies

$$-f(t) = f(0^+) - f(t) > -tf'(t),$$

whence

$$\left(\frac{f(t)}{t}\right)' = \frac{tf'(t) - f(t)}{t^2} > 0 \quad \text{for all } t.$$

A sharp refinement using the Strong Maximum Principle on the Laplacian operator enables us to replace (3.1) for the more general assumption (1.12).

Theorem 3.1. *Let us assume that Ω is smooth as well as (1.2) and (1.12). If $u, v \in \mathcal{C}^2(\Omega)$ are two nonnegative large solutions of (1.1) verifying (1.3) then $u = v$ on Ω.*

Proof. First of all, as it is well known, the function $v_{\min}(x) = \lim_{n\to\infty} v_n(x)$, where $v_n \in \mathcal{C}^2(\Omega)$ solves

$$\begin{cases} -\Delta v_n + f(v_n) = 0 & \text{in } \Omega, \\ v_n = n & \text{on } \partial\Omega, \end{cases}$$

is the minimal positive large solution of (1.1). For the existence of $v_{\min}$ we require the extra condition on Ω (see for instance the reasoning of [25] or [19,27]). Therefore, $v_{\min} \le u$ in Ω for any large solution u of (1.1). We claim that in fact $v_{\min} \equiv u$ in Ω. Indeed, let us assume that there exists $x_0 \in \Omega$ such that $v_{\min}(x_0) < u(x_0)$, otherwise the result follows. Then, since Ω is bounded, the continuity of u and $v_{\min}$ imply that

$$\mathcal{O}_\varepsilon = \left\{x \in \Omega : (1+\varepsilon)v_{\min}(x) < u(x)\right\} \subset\subset \Omega,$$

is a nonempty open subset, provided $\varepsilon > 0$ small. In order to prove the claim we rewrite (1.12) as

$$\frac{f(t)}{t} \text{ is increasing for } t \text{ greater than some large } t^*,$$

and choose $\mu > 0$ so small that $v_{\min}(x) \ge t^*$ holds in the open set

$$\Omega^\mu = \{x \in \Omega : \ \mathrm{d}(x) < \mu\}.$$

Then as $\varepsilon > 0$ is small

$$\mathcal{O}_\varepsilon^\mu = \big\{x \in \Omega : \ \mathrm{d}(x) < \mu, \ (1+\varepsilon)v_{\min}(x) < u(x)\big\}$$

is a nonempty open subset where

$$-\Delta(1+\varepsilon)v_{\min}(x) + f\big((1+\varepsilon)v_{\min}(x)\big) \geq (1+\varepsilon)\left(-\Delta v_{\min}(x) + f\big(v_{\min}(x)\big)\right) \geq 0, \quad x \in \mathcal{O}_\varepsilon^\mu.$$

So, an easy adaptation of the proof of Proposition 3.1 shows

$$u(x) - (1+\varepsilon)v_{\min}(x) \leq \max_{\partial \mathcal{O}_\varepsilon^\mu} \big(u - (1+\varepsilon)v_{\min}\big), \quad x \in \mathcal{O}_\varepsilon^\mu.$$

By construction the maximum of $u - (1+\varepsilon)v_{\min}$ cannot be achieved on $\partial\mathcal{O}_\varepsilon$, it implies

$$u(x) - (1+\varepsilon)v_{\min}(x) \leq \max_{\{y \in \mathcal{O}_\varepsilon, \ \delta(y)=\mu\}} \big(u - (1+\varepsilon)v_{\min}(y)\big), \quad x \in \mathcal{O}_\varepsilon^\mu,$$

and

$$u(x) - v_{\min}(x) \leq \max_{\{y \in \Omega: \ \delta(y)=\mu\}} \big(u - v_{\min}\big)(y) \equiv C_\mu, \quad x \in \Omega^\mu,$$

when ε goes to 0. On the other hand, $f\big(v_{\min}\big) \leq f(u)$ in Ω gives

$$-\Delta\big(u - v_{\min}\big) \leq 0 \quad \text{in } \Omega$$

whence the Weak Maximum Principle implies

$$u(x) - v_{\min}(x) \leq \max_{\{y \in \Omega: \ \delta(y)=\mu\}} \big(u - v_{\min}\big)(y) = C_\mu, \quad x \in \Omega \setminus \Omega^\mu,$$

thus

$$u(x) - v_{\min}(x) \leq C_\mu, \quad x \in \Omega.$$

We have proved that the subharmonic function $u - v_{\min}$ in Ω attains the maximum at an interior point, hence by the Strong Maximum Principle (see [23])

$$u(x) - v_{\min}(x) \equiv C_\mu, \quad x \in \Omega.$$

Finally,

$$f\big(v_{\min} + C_\mu\big) = f(u) = \Delta u = \Delta v_{\min} = f\big(v_{\min}\big) \quad \text{in } \Omega$$

and the monotonicity of the function f concludes $C_\mu = 0$, thus $u \equiv v_{\min}$ in Ω. $\square$

Remark 3.1. The proof of Theorem 3.1 uses reasonings of [22] (see also [3]). Other versions of Theorem 3.1 in the framework of the variational solutions are available (see [21]).

So that, the next goal is to prove the property

$$\lim_{x \to \partial\Omega} \frac{u(x)}{v(x)} = 1$$

for any couple of large solutions of (1.1) (see (1.3)).

4. Blow-up rate: proofs of Theorems 1.1 and 1.2

In this section we propose to obtain property (1.3) under the structural assumption given by the Keller–Osserman condition (1.2). It requires a very important technicality: (1.7) for the ordinary case or (1.9) for the borderline case. Following some ideas developed in [28] (see also [19]) we will prove (1.3) in the more appropriate version

$$\lim_{\mathrm{d}(x)\to 0} \frac{u(x)}{v(x)} = 1$$

where $\mathrm{d}(x) = \mathrm{dist}(x, \partial\Omega)$. As it was proved in [23], if $\partial\Omega$ is bounded the distance to boundary is a Lispchitz function on the whole space $\mathbb{R}^N$. Whenever, $\partial\Omega \in \mathcal{C}^1$ there exists $\mu_\Omega > 0$ such that $\mathrm{d}(\cdot) \in \mathcal{C}^1(\Omega^{2\mu_\Omega})$, where $\Omega^{2\mu_\Omega} = \{x \in \Omega : 0 < \mathrm{d}(x) < 2\mu_\Omega\}$. Moreover,

$$|\nabla \mathrm{d}(x)| = 1, \quad x \in \Omega^{2\mu_\Omega},$$

and so $\partial\Omega \in \mathcal{C}^k$, $k \geq 2$, implies $\delta(\cdot) \in \mathcal{C}^k(\Omega^{2\mu_\Omega})$ and

$$\Delta \mathrm{d}(x) = -(N-1)H(\bar{x}) + o(1), \quad x \in \Omega^{2\mu_\Omega}, \tag{4.1}$$

where $H(\bar{x})$ is the mean curvature of $\partial\Omega$ at the boundary point $\bar{x} \in \partial\Omega$ such that $\mathrm{d}(x) = |x - \bar{x}|$ and where $o(1) \to 0$ as $\mathrm{d}(x) \to 0$. Since we are interested in the behavior as $\mathrm{d}(x) \to 0$, we may assume, with no loss of generality, $\mu_\Omega < 1$.

We consider Ω as in Introduction with $\partial\Omega \in \mathcal{C}^2$. Moreover we consider the functions ψ and ϕ defined by (1.4) and (1.5), respectively. So, the function

$$\Phi_0(x) = \phi(\mathrm{d}(x)), \quad x \in \Omega^{\mu_\Omega}, \tag{4.2}$$

verifies

$$-\Delta\Phi_0(x) + f(\Phi_0(x)) = -\phi''(\mathrm{d}(x)) - \phi'(\mathrm{d}(x))\Delta \mathrm{d}(x) + f(\phi(\mathrm{d}(x))), \quad x \in \Omega^{\mu_\Omega}. \tag{4.3}$$

As

$$\phi''(\mathrm{d}(x)) = f(\phi(\mathrm{d}(x))), \quad x \in \Omega^{\mu_\Omega},$$

Lemma A.1 implies

$$\lim_{\mathrm{d}(x)\to 0} \frac{\phi'(\mathrm{d}(x))}{\phi''(\mathrm{d}(x))} = -\lim_{\mathrm{d}(x)\to 0} \frac{\sqrt{F(\phi(\mathrm{d}(x)))}}{f(\phi(\mathrm{d}(x)))} = 0.$$

Hence, since $\|\Delta \mathrm{d}\|_{\mathrm{L}^\infty(\Omega^{\mu_\Omega})} < \infty$, the term

$$-\phi'(\mathrm{d}(x))\Delta \mathrm{d}(x), \quad x \in \Omega^{\mu_\Omega}$$

is negligible with respect to the other terms on the right side in (4.3), and then

$$-\Delta\Phi_0(x) + f(\Phi_0(x)) = o(\phi''(\mathrm{d}(x))), \quad x \in \Omega^{\mu_\Omega}.$$

In this way, one can expect that the asymptotic behavior near the boundary of the solutions u of (1.1) let be governed by

$$u(x) = \phi\big(\mathrm{d}(x)\big) + o\big(\phi(\mathrm{d}(x))\big), \quad x \in \Omega^{\mu_\Omega}. \tag{4.4}$$

Our approach follows some ideas of [19,1–4]. It presents some differences to other approaches known in the literature.

As it was pointed above, technicality (1.7) plays an important role in our reasoning. More precisely,

Lemma 4.1. *Suppose that (1.2) and (1.7) hold. Then for every $\eta > 1$ there exist $\varepsilon_\eta > 0$ and $\delta_\eta > 0$ such that*

$$\phi(\sqrt{1-\varepsilon}\,\delta) \le \eta\phi(\delta) \quad \textit{for } \varepsilon \in [0, \varepsilon_\eta] \textit{ and } \delta \in [0, \delta_\eta]. \tag{4.5}$$

Proof. Fix $\eta > 1$. From (1.7) there exist $\varepsilon_\eta > 0$ and t_η large enough such that

$$\frac{\psi(\eta t)}{\psi(t)} < \sqrt{1-\varepsilon_\eta} \quad \text{for } t > t_\eta.$$

Therefore

$$\frac{\psi(\eta t)}{\psi(t)} < \sqrt{1-\varepsilon} \quad \text{for } \varepsilon \in [0, \varepsilon_\eta] \text{ and } t > t_\eta.$$

Since $\phi = \psi^{-1}$ is decreasing, the change of variable $\delta = \psi(t)$ concludes (4.5). □

Proposition 4.1. *Under assumptions of Lemma 4.1 any solution $u \in \mathcal{C}^2(\Omega)$ of*

$$-\Delta u + f(u) \le 0 \quad \textit{in } \Omega$$

verifies

$$\limsup_{\mathrm{d}(x)\to 0} \frac{u(x)}{\phi\big(\mathrm{d}(x)\big)} \le 1. \tag{4.6}$$

Proof. Fix an arbitrary $\eta > 1$. Since (1.7) holds, from Lemma 4.1 there exist $\varepsilon_\eta > 0$ and $\delta_\eta > 0$ such that (4.5) holds. Now, let $\varepsilon \in (0, \min\{\mu_\Omega, \varepsilon_\eta\})$ and take $\mu^* \in (0, \varepsilon/2)$ and $C^* > 0$ both to be precised later, and let $\mu \in (0, \mu^*)$. Consider the region $\Omega_\mu^{\mu^*} = \{y \in \Omega : \mu < \mathrm{d}(y) < \mu^*\}$ and the function

$$\Phi_{0,\varepsilon}^{-\mu}(x) = \phi\left(\sqrt{1-\varepsilon}\big(\mathrm{d}(x) - \mu\big)\right) + C^*, \quad x \in \Omega_\mu^{\mu^*}.$$

Then for every $x \in \Omega_\mu^{\mu^*}$ we have

$$\begin{aligned}
&-\Delta\Phi_{0,\varepsilon}^{-\mu}(x) + f\big(\Phi_{0,\varepsilon}^{-\mu}(x)\big) \\
&\quad > -(1-\varepsilon)\,\phi''\left(\sqrt{1-\varepsilon}\,(\mathrm{d}(x)-\mu)\right)|\nabla \mathrm{d}(x)|^2 - \sqrt{1-\varepsilon}\phi'\left(\sqrt{1-\varepsilon}\,(\mathrm{d}(x)-\mu)\right)\Delta \mathrm{d}(x) \\
&\qquad + f\left(\phi\left(\sqrt{1-\varepsilon}\,(\mathrm{d}(x)-\mu)\right)\right) \\
&\quad > f\left(\phi\left(\sqrt{1-\varepsilon}\,(\mathrm{d}(x)-\mu)\right)\right)\left(\varepsilon - \frac{\phi'\left(\sqrt{1-\varepsilon}\,(\mathrm{d}(x)-\mu)\right)}{f\left(\phi\left(\sqrt{1-\varepsilon}\,(\mathrm{d}(x)-\mu)\right)\right)}\|\Delta \mathrm{d}\|_\infty\right).
\end{aligned}$$

Since $\phi'(z) = -\sqrt{2F(\phi(z))}$ and $\phi(z) \to \infty$ as $z \to 0^+$, Lemma A.1 enables us to choose $\mu^* \in \big(0, \min\{\varepsilon/2, \bar{\mu}\}\big)$ for some $\bar{\mu} > 0$ small enough in order to obtain

$$-\Delta\Phi_{0,\varepsilon}^{-\mu}(x) + f\big(\Phi_{0,\varepsilon}^{-\mu}(x)\big) > 0, \quad x \in \Omega_\mu^{\mu^*}.$$

In this way, we consider the subset $\Gamma_{\mu^*} = \{y \in \Omega :\ \mathrm{d}(y) = \mu^*\}$ and the constant $C^* = \max_{\Gamma_{\mu^*}} u$ for which

$$u(x) \le \Phi_{0,\varepsilon}^{-\mu}(x), \quad x \in \Gamma_{\mu^*}.$$

Since

$$\lim_{\mathrm{d}(x)\to\mu} \Phi_{0,\varepsilon}^{-\mu}(x) = \infty$$

holds, the Comparison Principle (see Proposition 3.1) leads to

$$u(x) \le \Phi_{0,\varepsilon}^{-\mu}(x), \quad x \in \Omega_\mu^{\mu^*}$$

and then

$$\frac{u(x)}{\phi\big(\mathrm{d}(x)-\mu\big)} \le \frac{\phi\left(\sqrt{1-\varepsilon}\big(\mathrm{d}(x)-\mu\big)\right)}{\phi\big(\mathrm{d}(x)-\mu\big)} + \frac{C^*}{\phi\big(\mathrm{d}(x)-\mu\big)}, \quad x \in \Omega_\mu^{\mu^*}.$$

In consequence, letting $\mu \to 0^+$, it follows

$$\frac{u(x)}{\phi\big(\mathrm{d}(x)\big)} \le \frac{\phi\left(\sqrt{1-\varepsilon}\mathrm{d}(x)\right)}{\phi\big(\mathrm{d}(x)\big)} + \frac{C^*}{\phi\big(\mathrm{d}(x)\big)}, \quad x \in \Omega^{\mu^*}. \tag{4.7}$$

Note that (4.5) and (4.7) imply

$$\frac{u(x)}{\phi\big(\mathrm{d}(x)\big)} \le \eta + \frac{C^*}{\phi\big(\mathrm{d}(x)\big)}, \quad x \in \Omega^{\mu^*},$$

whence

$$\limsup_{\mathrm{d}(x)\to 0} \frac{u(x)}{\phi\big(\mathrm{d}(x)\big)} \le \eta.$$

Finally, taking $\eta \searrow 1$ one concludes (4.6). □

Next, we give a lower estimate of the blow-up rate of solutions of (1.1) when (1.7) holds. To this purpose, firstly we note that condition (1.7) is equivalent to

$$\liminf_{t\to\infty} \frac{\psi(\beta t)}{\psi(t)} > 1 \quad \text{for all } \beta \in (0,1),$$

whence one obtains an analogous result to Lemma 4.1. More precisely,

Lemma 4.2. *Suppose the assumptions of Lemma 4.1. Then for every $\beta \in (0,1)$ there exist $\varepsilon_\beta > 0$ and $\delta_\beta > 0$, such that*

$$\phi(\sqrt{1+\varepsilon}\,\delta) \ge \beta\phi(\delta) \quad \textit{for } \varepsilon \in [0,\varepsilon_\beta] \textit{ and } \delta \in [0,\delta_\beta]. \qquad \square \tag{4.8}$$

Proposition 4.2. *Under assumptions of Proposition 4.1 any large solution $u \in \mathcal{C}^2(\Omega)$ of (1.1) verifies*

$$\liminf_{\mathrm{d}(x)\to 0} \frac{u(x)}{\phi\big(\mathrm{d}(x)\big)} \ge 1. \tag{4.9}$$

Proof. Fix an arbitrary $\beta \in (0,1)$. Since (1.7) holds, Lemma 4.2 implies the existence of $\varepsilon_\beta > 0$ and $\delta_\beta > 0$ such that (4.8) holds. Now, let $\varepsilon \in (0, \min\{\mu_\Omega, \varepsilon_\beta\})$ and take $\mu_* \in (0, \varepsilon/2)$ and $C_* > 0$ both to be precised later, and let $\mu \in (0, \mu_*)$. Consider the function

$$\Phi_{0,\varepsilon}^{+\mu}(x) = \phi\left(\sqrt{1+\varepsilon}\,(\mathrm{d}(x)+\mu)\right) - C_*, \quad x \in \Omega^{\mu_*}.$$

Then for every $x \in \Omega_0^{\mu_*}$ we have

$$-\Delta\Phi_{0,\varepsilon}^{+\mu}(x) + f\left(\Phi_{0,\varepsilon}^{+\mu}(x)\right) < f\left(\phi\left(\sqrt{1+\varepsilon}\,(\mathrm{d}(x)+\mu)\right)\right)\left(-\varepsilon + \frac{\phi'\left(\sqrt{1+\varepsilon}\,(\mathrm{d}(x)+\mu)\right)}{f\left(\phi\left(\sqrt{1+\varepsilon}\,(\mathrm{d}(x)+\mu)\right)\right)}\,\|\Delta \mathrm{d}\|_\infty\right).$$

Since $\phi'(z) = -\sqrt{2F(\phi(z))}$ and $\phi(z) \to \infty$ as $z \to 0^+$, Lemma A.1 allows to choose $\mu_* \in \left(0, \min\{\varepsilon/2, \underline{\mu}\}\right)$ for some $\underline{\mu} > 0$ small enough for which

$$-\Delta\Phi_{0,\varepsilon}^{+\mu}(x) + f\left(\Phi_{0,\varepsilon}^{+\mu}(x)\right) < 0, \quad x \in \Omega^{\mu_*}$$

holds. Now, the choice $C_* = \phi\left(\sqrt{1+\varepsilon}\,(\mu_* + \mu)\right)$ leads to

$$\Phi_{0,\varepsilon}^{+\mu}(x) = 0 \le u(x), \quad x \in \Gamma_{\mu_*} = \{y \in \Omega : \ \mathrm{d}(y) = \mu_*\}.$$

We have $\Phi_{0,\varepsilon}^{+\mu}(x) < \infty$ for $x \in \Omega$ and

$$\lim_{\mathrm{d}(x)\to 0} u(x) = \infty,$$

then the Comparison Principle (see again Proposition 3.1) yields

$$u(x) \ge \Phi_{0,\varepsilon}^{+\mu}(x), \quad x \in \Omega^{\mu_*}$$

that leads to

$$\frac{u(x)}{\phi\left(\mathrm{d}(x)+\mu\right)} \ge \frac{\phi\left(\sqrt{1+\varepsilon}\left(\mathrm{d}(x)+\mu\right)\right)}{\phi\left(\mathrm{d}(x)+\mu\right)} - \frac{\phi\left(\sqrt{1+\varepsilon}\,(\mu_*+\mu)\right)}{\phi\left(\mathrm{d}(x)+\mu\right)}, \quad x \in \Omega^{\mu_*}.$$

Letting $\mu \to 0^+$, inequality $\phi\left(\sqrt{1+\varepsilon}\,\mu_*\right) \le \phi(\mu_*)$ implies

$$\frac{u(x)}{\phi\left(\mathrm{d}(x)\right)} \ge \frac{\phi\left(\sqrt{1+\varepsilon}\,\mathrm{d}(x)\right)}{\phi\left(\mathrm{d}(x)\right)} - \frac{\phi(\mu_*)}{\phi\left(\mathrm{d}(x)\right)}, \quad x \in \Omega^{\mu_*}. \tag{4.10}$$

Now from (4.8) and (4.10) one obtains

$$\frac{u(x)}{\phi\left(\mathrm{d}(x)\right)} \ge \beta - \frac{\phi(\mu_*)}{\phi\left(\mathrm{d}(x)\right)}, \quad x \in \Omega^{\mu_*}.$$

Therefore,

$$\liminf_{\mathrm{d}(x)\to 0} \frac{u(x)}{\phi\left(\mathrm{d}(x)\right)} \ge \beta.$$

Finally, letting $\beta \nearrow 1$ one concludes (4.9). $\square$

Propositions 4.1 and 4.2 validate the property (1.8) and therefore Theorem 1.1 holds. Moreover, from Theorem 3.1, the proof of Theorem 1.3 relative to the ordinary case follows.

The approach for studying the borderline case

$$\limsup_{t\to\infty}\frac{\psi(\eta t)}{\psi(t)}=1 \quad \text{for some } \eta\in(1,\eta_0) \quad (\text{see } (1.9))$$

is in some sense analogous to the ordinary case before studied. However, there exists a crucial difference. Indeed, in this borderline case we only obtain the estimates

$$\frac{\phi(\sqrt{1-\varepsilon}\,\delta)}{\phi(\delta)}>\frac{1}{\eta} \quad \text{and} \quad \frac{\phi(\sqrt{1+\varepsilon}\,\delta)}{\phi(\delta)}<\eta \quad \text{for } \varepsilon\in(0,\bar{\varepsilon}_\eta) \text{ and for } \delta\in(0,\bar{\delta}_\eta)$$

for some $\bar{\varepsilon}_\eta>0$ and $\bar{\delta}_\eta>0$ small enough, and then (4.7) and (4.10) do not lead to any conclusion. Therefore, the issue here is much more delicate, and it is necessary to build sub- and super-solutions highly accurate, taking advantage as much as possible of condition (1.9).

So, related to the proof of Theorem 1.2, we are interested in proving that (4.4) holds under assumptions (1.9) and (1.11). According to previous reasoning, we need some results as in Lemmas 4.1 and 4.2 respectively. More precisely,

Lemma 4.3. *Let us suppose that (1.2) and (1.9) hold whenever f is differentiable for large t. Then for each $\ell\in(0,1)$ one verifies*

$$\lim_{\delta\to 0}\frac{\psi\left(\left(1\pm\delta^{1-\ell}\right)\phi(\delta)\right)}{\delta}=1. \tag{4.11}$$

Moreover, the function

$$\Upsilon(\delta)=\frac{\phi(\delta)}{2\delta^\ell\sqrt{2F\left(\phi(\delta)\right)}} \tag{4.12}$$

verifies

$$\lim_{\delta\to 0}\Upsilon(\delta)=0$$

and the inequalities

$$\begin{cases} 1<\dfrac{\phi\left(\zeta_+(\delta)\delta\right)}{\phi(\delta)}<1+\delta^{1-\ell} & \textit{for small } \delta>0,\\ 1-\delta^{1-\ell}<\dfrac{\phi\left(\zeta_-(\delta)\delta\right)}{\phi(\delta)}<1 & \textit{for small } \delta>0, \end{cases} \tag{4.13}$$

hold for the functions

$$\zeta_\pm(\delta)=1\mp\Upsilon(\delta).$$

We send to Appendix A for the proof of Lemma 4.3. We start finding an upper estimate of solutions of (1.1) for nonlinearities in the borderline case.

Proposition 4.3. *Suppose the assumptions of Lemma 4.3 with (1.9) whenever f is differentiable for large t. Then any solution $u \in \mathcal{C}^2(\Omega)$ of*

$$-\Delta u + f(u) \leq 0 \quad \textit{in } \Omega$$

verifies

$$\limsup_{\mathrm{d}(x)\to 0} \frac{u(x)}{\phi(\mathrm{d}(x))} \leq 1. \tag{4.14}$$

Proof. Fixed $\ell \in (0,1)$ we consider the function $\zeta_+(\delta) = 1 - \Upsilon(\delta)$, $\delta \in (0, \delta^*)$, where $\delta^* > 0$ is small enough (see Lemma 4.3 above). Next, let $\mu^* \in (0, \min\{\mu_\Omega, \delta^*\})$ and $C^* > 0$ both to be precised later. So, if $\mu \in (0, \mu^*)$ we consider the function

$$\Phi_{0,+}^{-\mu}(x) = \phi\big(\zeta_+(\mathrm{d}(x)-\mu)(\mathrm{d}(x)-\mu)\big) + C^*, \quad x \in \Omega_\mu^{\mu^*}, \tag{4.15}$$

where $\Omega_\mu^{\mu^*} = \{x \in \Omega : \mu < \mathrm{d}(x) < \mu^*\}$. Straightforward computations lead to

$$\begin{cases} \nabla\Phi_{0,+}^{-\mu}(x) = \phi'\big(\zeta_+(\mathrm{d}(x)-\mu)(\mathrm{d}(x)-\mu)\big)\big(\zeta_+'(\mathrm{d}(x)-\mu)(\mathrm{d}(x)-\mu) + \zeta_+(\mathrm{d}(x)-\mu)\big)\nabla \mathrm{d}(x), \\ \Delta\Phi_{0,+}^{-\mu}(x) = \phi''\big(\zeta_+(\mathrm{d}(x)-\mu)(\mathrm{d}(x)-\mu)\big(\zeta_+'(\mathrm{d}(x)-\mu)\big)(\mathrm{d}(x)-\mu) + \zeta_+(\mathrm{d}(x)-\mu)\big)^2 \\ \qquad + \phi'(\zeta_+(\mathrm{d}(x)-\mu)(\mathrm{d}(x)-\mu))\Big(\zeta_+''(\mathrm{d}(x)-\mu)(\mathrm{d}(x)-\mu) + 2\zeta_+'(\mathrm{d}(x)-\mu) \\ \qquad + (\zeta_+'(\mathrm{d}(x)-\mu)(\mathrm{d}(x)-\mu) + \zeta_+(\mathrm{d}(x)-\mu))\Delta \mathrm{d}(x)\Big) \end{cases}$$

for $x \in \Omega_\mu^{\mu_*}$. Since

$$\big(\zeta_+'(\mathrm{d}(x)-\mu)(\mathrm{d}(x)-\mu) + \zeta_+(\mathrm{d}(x)-\mu))\Delta \mathrm{d}(x) = o\big(\zeta_+''(\mathrm{d}(x)-\mu)(\mathrm{d}(x)-\mu) + 2\zeta_+'(\mathrm{d}(x)-\mu))\big)$$

and $\|\Delta \mathrm{d}\|_\infty < \infty$ hold, if one chooses $\mu* \in \big(0, \min\{\mu_\Omega, \bar{\mu}, \delta^*\}\big)$, for some $\bar{\mu} > 0$ validating Lemmas A.5 and A.6, we obtain

$$-\Delta\Phi_{0,+}^{-\mu} + f(\Phi_{0,+}^{-\mu}) > 0 \quad \text{in } \Omega_\mu^{\mu^*}.$$

Denoting $\Gamma_{\mu^*} = \{x \in \Omega : \mathrm{d}(x) = \mu^*\}$, the choice $C^* = \max\limits_{\Gamma_{\mu^*}} u$ implies

$$u(x) \leq \Phi_{0,+}^{-\mu}(x), \quad x \in \Gamma_{\mu^*}.$$

Since

$$\lim_{\mathrm{d}(x)\to\mu} \Phi_{0,+}^{-\mu}(x) = \infty$$

holds, the Comparison Principle (see Proposition 3.1) implies

$$u(x) \leq \Phi_{0,+}^{-\mu}(x), \quad x \in \Omega_\mu^{\mu^*},$$

that leads to

$$\frac{u(x)}{\phi\big(\mathrm{d}(x)-\mu\big)} \leq \frac{\phi\left(\zeta_+(\mathrm{d}(x)-\mu)\big(\mathrm{d}(x)-\mu\big)\right)}{\phi\big(\mathrm{d}(x)-\mu\big)} + \frac{C^*}{\phi\big(\mathrm{d}(x)-\mu\big)}, \quad x \in \Omega_\mu^{\mu^*}.$$

In consequence, letting $\mu \to 0^+$, it follows that

$$\frac{u(x)}{\phi\big(\mathrm{d}(x)\big)} \le \frac{\phi\big(\zeta_+(\mathrm{d}(x))\,\mathrm{d}(x)\big)}{\phi\big(\mathrm{d}(x)\big)} + \frac{C^*}{\phi\big(\mathrm{d}(x)\big)}, \quad x \in \Omega_0^{\mu*}.$$

Observe now that (4.13) leads to

$$\frac{u(x)}{\phi\big(\mathrm{d}(x)\big)} \le 1 + \big(\mathrm{d}(x)\big)^{1-\ell} + \frac{C^*}{\phi\big(\mathrm{d}(x)\big)}, \quad x \in \Omega_0^{\mu*},$$

for which (4.14) holds after sending $\mathrm{d}(x)$ to 0 in the previous inequality. □

Our next step is to obtain a lower estimate of the solutions of (1.1) for nonlinearities in the borderline case.

Proposition 4.4. *Under assumptions of Proposition 4.3 for any large solution $u \in \mathcal{C}^2(\Omega)$ of problem (1.1) one has*

$$\liminf_{\mathrm{d}(x)\to 0} \frac{u(x)}{\phi(\mathrm{d}(x))} \ge 1. \tag{4.16}$$

Proof. Fixed $\ell \in (0, 1)$ we consider the function $\zeta_-(\delta) = 1 + \Upsilon(\delta)$, $\delta \in (0, \delta_*)$, where $\delta_* > 0$ is small enough (see again Lemma 4.3). Next, let $\mu_* \in (0, \min\{\mu_\Omega, \delta_*\})$ and $C_* > 0$ both to be precised later. For every $\mu \in (0, \mu_*)$ we consider the function

$$\Phi_{0,-}^{+\mu}(x) = \phi\left(\zeta_-\big(\mathrm{d}(x) + \mu\big)\left(\mathrm{d}(x) + \mu\right)\right) - C_*, \quad x \in \Omega^{\mu_*}. \tag{4.17}$$

Then by means of straightforward computations one proves

$$\begin{cases} \nabla\Phi_{0,-}^{+\mu}(x) = \phi'(\zeta_-\big(\mathrm{d}(x)+\mu\big)(\mathrm{d}(x)+\mu))\big(\zeta_-'(\mathrm{d}(x)+\mu)(\mathrm{d}(x)+\mu) + \zeta_-(\mathrm{d}(x)+\mu)\big)\nabla \mathrm{d}(x), \\ \Delta\Phi_{0,-}^{+\mu}(x) = \phi''\big(\zeta_-(\mathrm{d}(x)+\mu)(\mathrm{d}(x)+\mu)\big(\zeta_-'(\mathrm{d}(x)+\mu)\big)(\mathrm{d}(x)+\mu) + \zeta_-(\mathrm{d}(x)+\mu)\big)^2 \\ \qquad + \phi'(\zeta_-(\mathrm{d}(x)+\mu)(\mathrm{d}(x)+\mu))\Big(\zeta_-''(\mathrm{d}(x)+\mu)(\mathrm{d}(x)+\mu) + 2\zeta_-'(\mathrm{d}(x)+\mu) \\ \qquad + (\zeta_-'(\mathrm{d}(x)+\mu)(\mathrm{d}(x)+\mu) + \zeta_-(\mathrm{d}(x)+\mu))\Delta \mathrm{d}(x)\Big) \end{cases}$$

for $x \in \Omega^{\mu_*}$. Since

$$(\zeta_-'(\mathrm{d}(x)+\mu)(\mathrm{d}(x)+\mu) + \zeta_-(\mathrm{d}(x)+\mu))\Delta \mathrm{d}(x) = o\big(\zeta_-''(\mathrm{d}(x)+\mu)(\mathrm{d}(x)+\mu) + 2\zeta_-'(\mathrm{d}(x)+\mu))\big)$$

and $\|\Delta \mathrm{d}\|_\infty < \infty$ hold, if one chooses $\mu_* \in \big(0, \min\{\mu_\Omega, \underline{\mu}, \delta_*\}\big)$, for some $\underline{\mu} > 0$ validating Lemmas A.5 and A.6, we obtain

$$-\Delta\Phi_{0,-}^{+\mu} + f(\Phi_{0,-}^{+\mu}) < 0 \quad \text{in } \Omega^{\mu^*}.$$

Considering $C_* = \phi\left(\zeta_-(\mu_* + \mu)\big(\mu_* + \mu\big)\right)$ one has

$$\Phi_{0,-}^{+\mu}(x) = \phi\left(\zeta_-(\mu_* + \mu)\big(\mu_* + \mu\big)\right) - C_* = 0 \le u(x) \quad \text{on } \Gamma_{\mu_*}$$

where $\Gamma_{\mu_*} = \{x \in \Omega : \mathrm{d}(x) = \mu_*\}$. Since $\Phi_{0,-}^{+\mu}(x) < \infty$ on $\partial\Omega$ and

$$\lim_{\mathrm{d}(x)\to 0} u(x) = \infty$$

hold, the Comparison Principle (see Proposition 3.1) yields

$$u(x) \geq \Phi_{0,-}^{+\mu}(x), \quad x \in \Omega^{\mu_*},$$

that leads to

$$\frac{u(x)}{\phi\left(\mathrm{d}(x)+\mu\right)} \geq \frac{\phi\left(\zeta_-(\mathrm{d}(x)+\mu)\left(\mathrm{d}(x)+\mu\right)\right)}{\phi\left(\mathrm{d}(x)+\mu\right)} - \frac{\phi\left(\zeta_-(\mu_*+\mu)(\mu_*+\mu)\right)}{\phi\left(\mathrm{d}(x)+\mu\right)}, \quad x \in \Omega^{\mu_*}.$$

Letting $\mu \to 0^+$, it follows that

$$\frac{u(x)}{\phi\left(\mathrm{d}(x)\right)} \geq \frac{\phi\left(\zeta_-(\mathrm{d}(x))\mathrm{d}(x)\right)}{\phi\left(\mathrm{d}(x)\right)} - \frac{\phi\left(\zeta_-(\mu_*)\mu_*\right)}{\phi\left(\mathrm{d}(x)\right)}, \quad x \in \Omega^{\mu_*}.$$

Note that (4.13) implies

$$\frac{u(x)}{\phi\left(\mathrm{d}(x)\right)} \geq 1 - \left(\mathrm{d}(x)\right)^{1-\ell} - \frac{\phi\left(\zeta_-(\mu_*)\,\mu_*\right)}{\phi\left(\mathrm{d}(x)\right)}, \quad x \in \Omega^{\mu_*},$$

whence (4.16) holds after passing to the limit as $\mathrm{d}(x)$ goes to 0 in the previous inequality. $\square$

Now from Propositions 4.3 and 4.4 the proof of Theorem 1.2 follows. Moreover, it concludes also the proof of Theorem 1.3 relative to the borderline case.

5. Second order estimates: proofs of Theorems 1.4 and 1.5

In order to obtain a second term in the expansion near the boundary of the solution u of (1.1) we require the additional assumption $\partial\Omega \in \mathcal{C}^4$. Sometimes, for simplicity, we will use the notation $\mathrm{d} = \mathrm{d}(x)$. Here we define

$$\Phi_1(x) = \Phi_0(x) + A\big(\Phi_0(x)\big)\varphi\big(\Phi_0(x)\big)\Delta\mathrm{d}(x), \quad x \in \Omega^{\mu_\Omega},$$

where Φ_0, A and φ are the functions given in (4.2), (1.16) and (1.14), respectively. Since $|\nabla\mathrm{d}|^2 = 1$ and $\Delta|\nabla\mathrm{d}|^2 = 0$ in Ω^{μ_Ω}, straightforward computations lead to

$$f(\Phi_1) = f\big(\phi(\mathrm{d})\big) + A\big(\phi(\mathrm{d})\big)\varphi\big(\phi(\mathrm{d})\big)\Delta\mathrm{d}f'\big(\phi(\mathrm{d})\big) + \frac{\big(A\big(\phi(\mathrm{d})\big)\varphi\big(\phi(\mathrm{d})\big)\Delta\mathrm{d}\big)^2}{2}f''(\xi)$$

for some ξ between $\phi(\mathrm{d})$ and $\phi(\mathrm{d}) + A\big(\phi(\mathrm{d})\big)\varphi\big(\phi(\mathrm{d})\big)\Delta\mathrm{d}$ and

$$\begin{aligned} -\Delta\Phi_1 + f(\Phi_1) = &-\Big(\phi'(\mathrm{d}) + A\big(\phi(\mathrm{d})\big)\Big(\phi''(\mathrm{d})\varphi'\big(\phi(\mathrm{d})\big) + \big(\phi'(\mathrm{d})\big)^2\varphi''\big(\phi(\mathrm{d})\big) - f'\big(\phi(\mathrm{d})\big)\varphi\big(\phi(\mathrm{d})\big)\Big)\Big)\,\Delta\mathrm{d} \\ &+ \mathcal{R}_1\big(\phi(\mathrm{d})\big)\Delta\mathrm{d} + \mathcal{R}_2\big(\phi(\mathrm{d})\big)\big(\Delta\mathrm{d}\big)^2 + \mathcal{R}_3\big(\phi(\mathrm{d})\big)f''(\xi)\big(\Delta\mathrm{d}\big)^2 + \mathcal{R}_4\big(\phi(\mathrm{d})\big)\Delta^2\mathrm{d} \end{aligned}$$

for

$$\begin{cases} \mathcal{R}_1(t) = -\left(A''(t)2F(t)\varphi(t) + 2A'(t)2F(t)\varphi'(t) + A'(t)f(t)\varphi(t)\right) \\ \qquad = -\left(A''(t)\sqrt{2F(t)}\varphi(t) + \dfrac{A'(t)\varphi(t)f(t)}{\sqrt{2F(t)}} + 2A'(t)t\right)\sqrt{2F(t)} \\ \mathcal{R}_2(t) = -\left(A'(t)\sqrt{2F(t)}\varphi(t) + A(t)\sqrt{2F(t)}\varphi'(t)\right) = -\left(A'(t)t\dfrac{\varphi(t)}{t} + A(t)\dfrac{t}{\sqrt{2F(t)}}\right)\sqrt{2F(t)} \\ \mathcal{R}_3(t) = \dfrac{1}{2}\big(A(t)\varphi(t)\big)^2 = \left(\dfrac{1}{2}\dfrac{\big(A(t)\varphi(t)\big)^2}{\sqrt{2F(t)}}\right)\sqrt{2F(t)} \\ \mathcal{R}_4(t) = -A(t)\varphi(t) = -\left(A(t)\dfrac{\varphi(t)}{t}\dfrac{t}{\sqrt{2F(t)}}\right)\sqrt{2F(t)} \end{cases} \tag{5.1}$$

where we have used $\phi'(\mathrm{d}) = -\sqrt{2F\big(\phi(\mathrm{d})\big)}$, $\phi''(\mathrm{d}) = f\big(\phi(\mathrm{d})\big)$ and the variable $t = \phi(\mathrm{d})$. Then, since (H1) proves that $A(t)$ is bounded at the infinity, we deduce that

$$\begin{cases} \text{(H3) and (H4) imply} & \mathcal{R}_1(t) = o\big(\sqrt{2F(t)}\big), \\ \text{(H1), (H4), Lemma A.1 and (1.15) imply} & \mathcal{R}_2(t) = o\big(\sqrt{2F(t)}\big), \\ \text{(H1), Lemma A.1 and (1.15) imply} & \mathcal{R}_4(t) = o\big(\sqrt{2F(t)}\big). \end{cases} \tag{5.2}$$

Moreover,

$$\lim_{\mathrm{d}\to 0} \frac{\mathcal{R}_3\big(\phi(\mathrm{d})\big)f''(\xi)}{\phi'(\mathrm{d})} = -\lim_{t\to\infty} \frac{\mathcal{R}_3(t)f''(t)}{\sqrt{2F(t)}} = -\lim_{t\to\infty} \frac{1}{2}\frac{\big(A(t)\varphi(t)\big)^2 f''(t)}{\sqrt{2F(t)}} = 0 \tag{5.3}$$

provided (H2). Therefore, under hypotheses (H1)–(H4) one has

$$\begin{aligned} -\Delta\Phi_1 + f(\Phi_1) = &-\Big(\phi'(\mathrm{d}) + A\big(\phi(\mathrm{d})\big)\Big(\phi''(\mathrm{d})\varphi'\big(\phi(\mathrm{d})\big) + \big(\phi'(\mathrm{d})\big)^2\varphi''\big(\phi(\mathrm{d})\big) - f'\big(\phi(\mathrm{d})\big)\varphi\big(\phi(\mathrm{d})\big)\Big)\Big)\Delta\mathrm{d} \\ &+ o\big(\phi'(\mathrm{d})\big) \end{aligned}$$

in Ω^{μ_Ω}, whence one finds

$$-\Delta\Phi_1 + f(\Phi_1) = o\big(\phi'(\mathrm{d})\big) \quad \text{in } \Omega^{\mu_\Omega}$$

provided

$$A\big(\phi(\mathrm{d})\big)\Big(\phi''(\mathrm{d})\varphi'\big(\phi(\mathrm{d})\big) + \big(\phi'(\mathrm{d})\big)^2\varphi''\big(\phi(\mathrm{d})\big) - f'\big(\phi(\mathrm{d})\big)\varphi\big(\phi(\mathrm{d})\big)\Big) = -\phi'(\mathrm{d}). \tag{5.4}$$

By using the change of variable $t = \phi(\mathrm{d})$ we deduce that

$$A(t)\left(f(t)\varphi'(t) + 2F(t)\varphi''(t) - f'(t)\varphi(t)\right) = \sqrt{2F(t)}$$

holds for the choice of function $A(t)$ given in (1.16). Indeed, since $\varphi'(t) = -t\psi'(t)$, one has

$$f(t)\varphi'(t) + 2F(t)\varphi''(t) = -f(t)t\psi'(t) - \frac{1}{\big(\psi'(t)\big)^2}\Big(\psi'(t) - f(t)t\big(\psi'(t)\big)^3\Big) = -\frac{1}{\psi'(t)} = \sqrt{2F(t)}$$

whence the choice

$$A(t)=\frac{\sqrt{2F(t)}}{\sqrt{2F(t)}-f'(t)\varphi(t)},$$

given in (1.16), satisfies (5.4). Moreover, $A\big(\phi(\mathrm{d})\big)\in\mathcal{C}^2\big(\Omega^{\mu_\Omega}\big)\cap L^\infty\big(\Omega^{\mu_\Omega}\big)$.

Remark 5.1. As it was pointed out in the Introduction, assumptions (H2)–(H4) are intrinsic and direct in the above reasonings. Since (1.15) holds by construction, we note that property (H2) follows when

$$\big(A(t)\big)^2\frac{t}{\sqrt{F(t)}}tf''(t)\quad\text{is bounded,}$$

but this last condition is more restrictive. Also we note that property (H3) follows from (H4) when

$$A''(t)\sqrt{F(t)}t\quad\text{and}\quad\frac{\varphi(t)}{t}\frac{f(t)}{\sqrt{F(t)}}\quad\text{are bounded,}$$

but again these last conditions are more restrictive.

So that, from the definition of function $\Phi_1(x)$ above, it is reasonable to expect that the asymptotic behavior of the solution of (1.1) near $\partial\Omega$ is of the form

$$u(x)=\phi\big(\mathrm{d}(x)\big)+A\big(\phi(\mathrm{d}(x))\big)\varphi\big(\phi(\mathrm{d}(x))\big)\big(\Delta\mathrm{d}(x)+o(1)\big),$$

including the influence of the mean curvature in terms of second order on the explosive expansion of the solution u of (1.1) near the boundary. Here $o(1)\to 0$ as $\mathrm{d}(x)\to 0$.

As above, we want to construct suitable sub- and super-solutions. We begin with the ordinary case.

Proposition 5.1. *Suppose $\partial\Omega\in\mathcal{C}^4$ and f is twice differentiable at infinity verifying* (H1)–(H4), (1.2) *and* (1.7). *Then, for every solution $u\in\mathcal{C}^2(\Omega)$ of*

$$-\Delta u+f(u)\le 0\quad\text{in }\Omega$$

one has

$$\limsup_{\mathrm{d}(x)\to 0}\frac{u(x)}{\phi\big(\mathrm{d}(x)\big)+A\big(\phi(\mathrm{d}(x)\big)\varphi\big(\phi(\mathrm{d}(x))\big)\Delta\mathrm{d}(x)}\le 1. \tag{5.5}$$

Proof. Fix $\eta>1$. Since (1.7) holds, one can consider $\varepsilon_\eta>0$ and $\delta_\eta>0$ given in Lemma 4.1. Now, let $\varepsilon\in(0,\min\{\mu_\Omega,\varepsilon_\eta\})$ given, and take $\mu^*\in(0,\varepsilon/2)$ and $C^*>0$ both to be precised later, and let $\mu\in(0,\mu^*)$. Consider the function

$$\begin{aligned}\Phi^{-\mu}_{1,\varepsilon}(x)&=\phi\big(\sqrt{1-\varepsilon}\big(\mathrm{d}(x)-\mu\big)\big)\\&\quad+A\Big(\phi\big(\sqrt{1-\varepsilon}(\mathrm{d}(x)-\mu)\big)\Big)\varphi\Big(\phi\big(\sqrt{1-\varepsilon}\big(\mathrm{d}(x)-\mu)\big)\Big)\Delta\mathrm{d}(x)+C^*,\quad x\in\Omega^{\mu^*}_{\mu}.\end{aligned}$$

In order to simplify we use the notation $\mathrm{d}=\mathrm{d}(x)$ and $z=\phi\big(\sqrt{1-\varepsilon}(\mathrm{d}(x)-\mu)\big)$. Then

$$\begin{cases} f\big(\Phi^{-\mu}_{1,\varepsilon}(x)\big) = f\big(\phi(z)\big) + \Big(A\big(\phi(z)\big)\varphi\big(\phi(z)\big)\Delta\mathrm{d} + C^*\Big) f'\big(\phi(z)\big) + \dfrac{\big(A\big(\phi(z)\big)\varphi\big(\phi(z)\big)\Delta\mathrm{d} + C^*\big)^2}{2} f''(\xi), \\ \nabla\Phi^{-\mu}_{1,\varepsilon}(x) = \sqrt{1-\varepsilon}\Big[1 + \Big(A'\big(\phi(z)\big)\varphi\big(\phi(z)\big) + A\big(\phi(z)\big)\varphi'\big(\phi(z)\big)\Big)\Delta\mathrm{d}\Big]\phi'(z)\nabla\mathrm{d} \\ \qquad + A\big(\phi(z)\big)\varphi\big(\phi(z)\big)\nabla\Delta\mathrm{d}, \\ \Delta\Phi^{-\mu}_{1,\varepsilon}(x) = (1-\varepsilon)\Big[1 + \Big(A'\big(\phi(z)\big)\varphi\big(\phi(z)\big) + A\big(\phi(z)\big)\varphi'\big(\phi(z)\big)\Big)\Delta\mathrm{d}\Big]\Big(\phi''(z) + \dfrac{\phi'(z)}{\sqrt{1-\varepsilon}}\Delta\mathrm{d}\Big) \\ \qquad + (1-\varepsilon)\Big(A''\big(\phi(z)\big)\varphi\big(\phi(z)\big) + 2A'\big(\phi(z)\big)\varphi'\big(\phi(z)\big) + A\big(\phi(z)\big)\varphi''\big(\phi(z)\big)\Big)\big(\phi'(z)\big)^2\Delta\mathrm{d} \\ \qquad + A\big(\phi(z)\big)\varphi\big(\phi(z)\big)\Delta^2\mathrm{d} \end{cases}$$

for some ξ between $\phi(z)$ and $\phi(z) + A\big(\phi(z)\big)\varphi(z)\Delta\mathrm{d} + C^*$, because $|\nabla\mathrm{d}|^2 = 1$ and $\Delta|\nabla\mathrm{d}|^2 = 1$. So, by using the rests introduced in (5.1), one gets

$$\begin{aligned} -\Delta\Phi^{-\mu}_{1,\varepsilon} + f\big(\Phi^{-\mu}_{1,\varepsilon}\big) \geq{}& \varepsilon f\big(\phi(z)\big) - \sqrt{1-\varepsilon}\Big[\phi'(z) + \sqrt{1-\varepsilon}A\big(\phi(z)\big)\Big(\phi''(z)\varphi'\big(\phi(z)\big) + \big(\phi'(z)\big)^2\varphi''\big(\phi(z)\big) \\ & - f'\big(\phi(z)\big)\varphi\big(\phi(z)\big)\Big]\Delta\mathrm{d} + (1-\varepsilon)\mathcal{R}_1\big(\phi(z)\big)\Delta\mathrm{d} + \sqrt{1-\varepsilon}\mathcal{R}_2\big(\phi(z)\big)\big(\Delta\mathrm{d}\big)^2 \\ & + \left(1 + \frac{C^*}{A\big(\phi(z)\big)\varphi\big(\phi(z)\big)}\right)^2 \mathcal{R}_3\big(\phi(z)\big)f''(\xi)(\Delta\mathrm{d})^2 + \mathcal{R}_4\big(\phi(z)\big)\Delta^2\mathrm{d} \end{aligned}$$

in $\Omega^{\mu*}_{\mu}$. In this point, the definition of the function $A(t)$ (see (5.4)) leads to

$$\begin{aligned} -\Delta\Phi^{-\mu}_{1,\varepsilon} + f\big(\Phi^{-\mu}_{1,\varepsilon}\big) \geq{}& f\big(\phi(z)\big)\Bigg\{\varepsilon - \Bigg[\left(1 - \sqrt{1-\varepsilon} + (1-\varepsilon)\frac{\mathcal{R}_1\big(\phi(z)\big)}{\phi'(z)}\right)\Delta\mathrm{d} + \sqrt{1-\varepsilon}\frac{\mathcal{R}_2\big(\phi(z)\big)}{\phi'(z)}(\Delta\mathrm{d})^2 \\ & + \left(1 + \frac{C^*}{A\big(\phi(z)\big)\varphi\big(\phi(z)\big)}\right)^2 \frac{\mathcal{R}_3\big(\phi(z)\big)}{\phi'(z)}f''(\xi)(\Delta\mathrm{d})^2 + \frac{\mathcal{R}_4\big(\phi(z)\big)}{\phi'(z)}\Delta^2\mathrm{d}\Bigg]\frac{\phi'(z)}{\phi''(z)}\Bigg\} \\ ={}& f\big(\phi(z)\big)\big(\varepsilon + o(1)\big) \end{aligned}$$

where $o(1) \to 0$ as $\mathrm{d}(x) - \mu \to 0$, due to the properties (5.2) and (5.3) and

$$\frac{\phi'(z)}{\phi''(z)} = -\frac{\sqrt{2F\big(\phi(z)\big)}}{f\big(\phi(z)\big)} \to 0 \quad \text{as } z \to 0$$

(see (A.1)). Therefore, one can choose $\mu^* \in \big(0, \min\{\varepsilon/2, \bar{\mu}\}\big)$, with $\bar{\mu} > 0$ small enough such that

$$-\Delta\Phi^{-\mu}_{1,\varepsilon} + f(\Phi^{-\mu}_{1,\varepsilon}) \geq 0 \quad \text{in } \Omega^{\mu*}_{\mu}.$$

Considering the set $\Gamma_{\mu^*} = \{x \in \Omega : \mathrm{d}(x) = \mu^*\}$ and $C^* = \max\limits_{\Gamma_{\mu^*}} u(y)$, one has

$$\Phi^{-\mu}_{1,\varepsilon}(x) = \phi\left(\sqrt{1-\varepsilon}(\mu^* - \mu)\right) + A\Big(\phi\big(\sqrt{1-\varepsilon}(\mu^* - \mu)\big)\varphi\Big(\phi\big(\sqrt{1-\varepsilon}(\mu^* - \mu)\big)\Big) + C^* \geq u(x), \quad x \in \Gamma_{\mu^*}.$$

Then

$$\lim_{\mathrm{d}(x)\to\mu} \Phi^{-\mu}_{1,\varepsilon}(x) = \infty$$

implies, by the Comparison Principle (see Proposition 3.1),

$$u(x) \le \Phi_{1,\varepsilon}^{-\mu}(x), \quad x \in \Omega_{\mu}^{\mu^*}.$$

Next, we divide each side on the previous inequality by

$$\phi\big(\mathrm{d}(x)-\mu\big) + A\Big(\phi\big(\mathrm{d}(x)-\mu\big)\Big)\varphi\big(\phi\big(\mathrm{d}(x)-\mu\big)\big)\Delta \mathrm{d}(x)$$

in order to use reasonings as in the end of the proof of Proposition 4.1 for which letting $\mu \searrow 0$ and then $\eta \searrow 1$ one concludes (5.5). □

Analogously, we obtain

Proposition 5.2. *Suppose $\partial\Omega \in \mathcal{C}^4$ and let f be twice differentiable at infinity satisfying* (H1)–(H4)*,* (1.2) *and* (1.7)*. Then, for every solution u of the problem* (1.1) *one has*

$$\liminf_{\mathrm{d}(x)\to 0} \frac{u(x)}{\phi\big(\mathrm{d}(x) + A\big(\phi\big(\mathrm{d}(x)\big)\big)\,\varphi\big(\phi\big(\mathrm{d}(x)\big)\big)\,\Delta \mathrm{d}(x)} \ge 1. \tag{5.6}$$

Proof. Let $\beta < 1$ given. Since (1.7) holds, one can consider numbers $\varepsilon_\beta > 0$ and $\delta_\beta > 0$ given in Lemma 4.2. Now, let $\varepsilon \in (0, \min\{\mu_\Omega, \varepsilon_\beta\})$ given, and take $\mu_* \in (0, \varepsilon/2)$ and $C_* > 0$ both to be precised later, and let $\mu \in (0, \mu_*)$. Consider the function

$$\begin{aligned}\Phi_{1,\varepsilon}^{+\mu}(x) &= \phi\big(\sqrt{1+\varepsilon}(\mathrm{d}(x)+\mu)\big)\\ &\quad + A\Big(\phi\big(\sqrt{1+\varepsilon}(\mathrm{d}(x)+\mu)\big)\Big)\varphi\Big(\phi\big(\sqrt{1+\varepsilon}(\mathrm{d}(x)+\mu)\big)\Big)\Delta \mathrm{d}(x) - C_*, \quad x \in \Omega^{\mu_*}.\end{aligned}$$

Then one can choose $\mu_* < \min\{\varepsilon/2, \underline{\mu}\}$ with $\underline{\mu} > 0$ so small that a reasoning as in the above proof leads to

$$-\Delta\Phi_{1,\varepsilon}^{+\mu}(x) + f(\Phi_{1,\varepsilon}^{+\mu})(x) \le f\left(\phi\left(\sqrt{1+\varepsilon}(\mathrm{d}(x)+\mu)\right)\right)\left(-\varepsilon + o\left(1\right)\right) \le 0, \quad x \in \Omega^{\mu_*},$$

where $o(1) \to 0$ as $\mathrm{d}(x) + \mu \to 0$. Choosing

$$C_* = \phi\big(\sqrt{1+\varepsilon}(\mu_*+\mu)\big) + A\Big(\phi\big(\sqrt{1+\varepsilon}(\mu_*+\mu)\big)\Big)\varphi\Big(\phi\big(\sqrt{1+\varepsilon}(\mu_*+\mu)\big)\Big),$$

one has

$$\Phi_{1,\varepsilon}^{+\mu}(x) = 0 \le u(x), \quad x \in \Gamma_{\mu_*} = \{y \in \Omega : \mathrm{d}(y) = \mu_*\}.$$

Since $\Phi_{1,\varepsilon}^{+\mu} < \infty$ on $\partial\Omega$ and

$$\lim_{\mathrm{d}(x)\to 0} u(x) = \infty,$$

the Comparison Principle (see Proposition 3.1) leads to

$$\Phi_{1,\varepsilon}^{+\mu}(x) \le u(x), \quad x \in \Omega^{\mu_*}.$$

Therefore, dividing each side of the previous inequality by

$$\phi\left(\mathrm{d}(x)+\mu\right) + A\Big(\phi\big(\mathrm{d}(x)+\mu\big)\Big)\varphi\Big(\phi\big(\mathrm{d}(x)+\mu\big)\Big)\Delta \mathrm{d}(x)$$

and then taking $\mu \searrow 0$ and later $\varepsilon \to 0$, one obtains (5.6) after applying Lemma 4.2 and by taking $\beta \nearrow 1$. □

Now from Propositions 5.1 and 5.2, the proof of Theorem 1.4 follows validating (1.17) in the ordinary case.

Finally, we deal with borderline cases. In order to prove Theorem 1.5 we need the following result

Lemma 5.1. *Let f be a function twice differentiable at infinity satisfying (1.2) and (1.9). Then the function*

$$B(t)=\frac{\sqrt{2F(t)}}{f(t)},\quad t>0,$$

verifies

$$\Delta\Big(B(\phi(\mathrm{d}(x)))\phi(\mathrm{d}(x))\Delta\mathrm{d}(x)\Big)=B\Big(\phi(\mathrm{d}(x))\Big)\Delta\mathrm{d}(x)f(\phi(\mathrm{d}(x)))+o\big(\phi'(\mathrm{d}(x))\big). \tag{5.7}$$

Proof. Since

$$\begin{cases} B'(t)=\dfrac{1}{\sqrt{2F(t)}}\left(1-\dfrac{2f'(t)F(t)}{\big(f(t)\big)^2}\right),\\[2ex] B''(t)=-\dfrac{f(t)}{\big(2F(t)\big)^{\frac{3}{2}}}\left(1-\dfrac{2f'(t)F(t)}{\big(f(t)\big)^2}\right)-\dfrac{1}{\sqrt{2F(t)}}\left[\dfrac{f''(t)}{f'(t)}\dfrac{2f'(t)F(t)}{\big(f(t)\big)^2}+\dfrac{2f'(t)}{f(t)}\left(1-\dfrac{2f'(t)F(t)}{\big(f(t)\big)^2}\right)\right]\end{cases}$$

and $|\nabla \mathrm{d}|^2=1$ and $\Delta|\nabla\mathrm{d}|^2=0$ in Ω^{μ_Ω}, straightforward computations lead to

$$\begin{cases}\nabla\big(B(\phi(\mathrm{d}))\phi(\mathrm{d})\Delta\mathrm{d}\big)=\left(B'(\phi(\mathrm{d}))\phi'(\mathrm{d})\phi(\mathrm{d})+B(\phi(\mathrm{d}))\phi'(\mathrm{d})\right)\Delta\mathrm{d}\nabla\mathrm{d}+B(\phi(\mathrm{d}))\phi(\mathrm{d})\nabla\Delta\mathrm{d},\\ \Delta\big(B(\phi(\mathrm{d}))\phi(\mathrm{d})\Delta\mathrm{d}\big)=B(\phi(\mathrm{d}))\phi''(\mathrm{d})\Delta\mathrm{d}+\mathcal{R}_1\big(\phi(\mathrm{d})\big)\Delta\mathrm{d}+\mathcal{R}_2\big(\phi(\mathrm{d})\big)\big(\Delta\mathrm{d}\big)^2+\mathcal{R}_3\big(\phi(\mathrm{d})\big)\Delta^2\mathrm{d},\end{cases}$$

in Ω^{μ_Ω}, for

$$\begin{cases}\mathcal{R}_1(t)=B'(t)f(t)t+2B'(t)\big(2F(t)\big)+B''(t)\big(2F(t)\big)t\\ \qquad =2\sqrt{2F(t)}\left(1-\dfrac{2f'(t)F(t)}{\big(f(t)\big)^2}\right)-\sqrt{2F(t)}\left(\dfrac{tf''(t)}{f'(t)}\dfrac{2f'(t)F(t)}{\big(f(t)\big)^2}+\dfrac{2tf'(t)}{f(t)}\left(1-\dfrac{2f'(t)F(t)}{\big(f(t)\big)^2}\right)\right)\\ \qquad =\left[2\left(1-\dfrac{2f'(t)F(t)}{\big(f(t)\big)^2}\right)-\dfrac{tf''(t)}{f'(t)}\dfrac{2f'(t)F(t)}{\big(f(t)\big)^2}-\dfrac{2tf'(t)}{f(t)}\left(1-\dfrac{2f'(t)F(t)}{\big(f(t)\big)^2}\right)\right]\sqrt{2F(t)},\\ \mathcal{R}_2(t)=-\big(B(t)+B'(t)t\big)\sqrt{2F(t)}=-\left[\dfrac{\sqrt{2F(t)}}{f(t)}+\dfrac{t}{\sqrt{2F(t)}}\left(1-\dfrac{2f'(t)F(t)}{\big(f(t)\big)^2}\right)\right]\sqrt{2F(t)}\\ \mathcal{R}_3(t)=B(t)t=\left[\dfrac{B(t)t}{\sqrt{2F(t)}}\right]\sqrt{2F(t)}=\left[\dfrac{t}{f(t)}\right]\sqrt{2F(t)}\end{cases}$$

where we have used $\phi'(\mathrm{d})=-\sqrt{2F\big(\phi(\mathrm{d})\big)}$, $\phi''(\mathrm{d})=f\big(\phi(\mathrm{d})\big)$ and the variable $t=\phi(\mathrm{d})$. Due to (1.2) and (1.9) imply (A.1), (A.4) and (A.5), we deduce

$$\Delta\big(B(\phi(\mathrm{d}))\phi(\mathrm{d})\Delta\mathrm{d}\big)=B(\phi(\mathrm{d}))f\big(\phi(\mathrm{d})\big)\Delta\mathrm{d}+o\big(\phi'(\mathrm{d})\big).\qquad\square$$

Remark 5.2. Straightforward computations enable us to prove that the functions

$$\widehat{B}(t)=\frac{t}{\sqrt{2F(t)}}\quad\text{and}\quad\widetilde{B}(t)=\frac{t}{f(t)}$$

also satisfy (5.7). Moreover $B(t)$ and $\widehat{B}(t)$ have the same behavior for large t because

$$\frac{\widehat{B}(t)}{B(t)} = \frac{tf(t)}{2F(t)} \to 1 \quad \text{as } t \to \infty$$

(see (A.4)). On the other hand, (A.2) says that $\widetilde{B}(t)$ tends to zero faster than $B(t)$. Consequently

$$B\big(\phi\big(\mathrm{d}(x)\big)\big)\phi\big(\mathrm{d}(x)\big)\Delta \mathrm{d}(x)$$

is more explosive near $\partial\Omega$ than

$$\widetilde{B}\big(\phi\big(\mathrm{d}(x)\big)\big)\phi\big(\mathrm{d}(x)\big)\Delta \mathrm{d}(x).$$

Proposition 5.3. *Suppose $\partial\Omega \in \mathcal{C}^4$ and let f be a function twice differentiable at infinity satisfying (1.2) and (1.9). Then, for every solution u of*

$$-\Delta u + f(u) \le 0 \quad \text{in } \Omega$$

one has

$$\limsup_{\mathrm{d}(x)\to 0} \frac{u(x)}{\phi(\mathrm{d}(x))\Big(1 + B\big(\phi\big(\mathrm{d}(x)\big)\big)\Delta \mathrm{d}(x)\Big)} \le 1.$$

Proof. Fixed $\ell \in (0, 1)$ we consider the function $\zeta_+(\delta) = 1 - \Upsilon(\delta)$, $\delta \in (0, \delta^*)$, where $\delta^* > 0$ is small enough (see Lemma 4.3). Also one considers $\mu \in (0, \mu^*)$, with $\mu^* \in (0, \min\{\mu_\Omega, \tilde{\mu}_0, \delta^*\})$ to be precised as well as the constant $C^* > 0$. To conclude the proof it suffices to apply Lemma 5.1 and to argue as in the proof of Proposition 4.3 on the perturbed function

$$\widehat{\Phi}_{1,+}^{-\mu}(x) = \Phi_{0,+}^{-\mu}(x) + \phi\big(\zeta_+(\mathrm{d}(x) - \mu)(\mathrm{d}(x) - \mu)\big)B\big(\phi\big(\zeta_+(\mathrm{d}(x) - \mu)(\mathrm{d}(x) - \mu)\big)\big)\Delta \mathrm{d}(x), \quad x \in \Omega_\mu^{\mu^*}$$

(see (4.15)). $\square$

Proposition 5.4. *Suppose $\partial\Omega \in \mathcal{C}^4$ and let f be a function twice differentiable at infinity satisfying (1.2) and (1.9). Then, for every solution u of the problem (1.1) one has*

$$\liminf_{\mathrm{d}(x)\to 0} \frac{u(x)}{\phi(\mathrm{d}(x))\Big(1 + B(\phi(\mathrm{d}(x)))\Delta \mathrm{d}(x)\Big)} \ge 1.$$

Proof. Fixed $\ell \in (0, 1)$ we consider the function $\zeta_-(\delta) = 1 + \Upsilon(\delta)$, $\delta \in (0, \delta_*)$, where $\delta_* > 0$ is small enough (see Lemma 4.3). Here we also consider $\mu \in (0, \mu_*)$, with $\mu_* \in (0, \min\{\mu_\Omega, \underline{\mu}_0, \delta_*\})$ suitably chosen as well as a constant $C_* > 0$. Then one argues with the function

$$\widehat{\Phi}_{1,-}^{+\mu}(x) = \Phi_{0,-}^{+\mu}(x) + \phi\big(\zeta_-(\mathrm{d}(x) + \mu)(\mathrm{d}(x) + \mu)\big)B\big(\phi\big(\zeta_-(\mathrm{d}(x) + \mu)(\mathrm{d}(x) + \mu)\big)\big)\Delta \mathrm{d}(x), \quad x \in \Omega^{\mu_*}$$

(see (4.17)). So that, as in the proof of Proposition 5.2, one concludes the result by using Lemma 5.1 and reasoning as in the proof of Proposition 4.4. $\square$

Clearly, from Propositions 5.3 and 5.4 the estimate (1.20) is validated.

Acknowledgments

S.A. was partially supported by FONDECYT Grant No. 11110482, Millennium Nucleus Center for Analysis of PDE NC130017 and Programa Basal, CMM, U. de Chile. G.D. was supported by the Research Group MOMAT (Ref. 910480) supported by Banco Santander and UCM and by the projects MTM2011-26119 (DGISPI, Spain) and MTM2014-57113 (DGISPI, Spain). The work of J.M.R. has been done in the framework of project MTM2011-22658 of the Spanish Ministry of Science and Innovation and supported by the Research Group MOMAT (Ref. 910480) from Banco Santander and UCM.

Appendix A. Some basic estimates and technical results

A first basic result, that we have used in proving all our results, is the following

Lemma A.1. *Let us assume (1.2). Then*

$$\lim_{t\to\infty}\frac{t}{\sqrt{F(t)}}=0,\quad \lim_{t\to\infty}\frac{t}{f(t)}=0\quad \textit{and}\quad \lim_{t\to\infty}\frac{\sqrt{F(t)}}{f(t)}=0. \tag{A.1}$$

Hence

$$\frac{t}{f(t)}=o\left(\frac{t}{\sqrt{F(t)}}\right)\quad \textit{and}\quad \frac{t}{f(t)}=o\left(\frac{\sqrt{F(t)}}{f(t)}\right) \tag{A.2}$$

hold. Moreover,

$$\liminf_{t\to\infty}\frac{tf(t)}{2F(t)}\geq 1,\quad \liminf_{t\to\infty}\frac{tf'(t)}{f(t)}\geq 1\quad \textit{and}\quad \liminf_{t\to\infty}\frac{2f'(t)F(t)}{\big(f(t)\big)^2}\geq 1. \tag{A.3}$$

Proof. The Mean Value Integral Theorem provides the inequality

$$0\leq \frac{1}{2}\frac{t}{\sqrt{F(t)}}\leq \int_{\frac{t}{2}}^{t}\frac{ds}{\sqrt{F(s)}}\quad \text{for large } t.$$

Since (1.2) implies

$$\lim_{t\to\infty}\int_{\frac{t}{2}}^{t}\frac{ds}{\sqrt{F(s)}}=0,$$

we obtain

$$\lim_{t\to\infty}\frac{t}{\sqrt{F(t)}}=0$$

after letting $t\to\infty$. On the other hand, it is clear that by monotonicity and the Fundamental Calculus Theorem one has $F(t)\leq tf(t)$ for all $t>0$, which implies

$$0\leq\frac{t}{f(t)}\leq\frac{t^2}{F(t)}.$$

Then, letting $t \to \infty$, one gets

$$\lim_{t\to\infty} \frac{t}{f(t)} = 0$$

by the above contribution. This property implies

$$0 \leq \frac{F(t)}{(f(t))^2} \leq \frac{t}{f(t)}$$

and (A.1) concludes. Then the behavior (A.2) follows from the equality

$$\frac{t}{f(t)} = \frac{t}{\sqrt{F(t)}} \frac{\sqrt{F(t)}}{f(t)}.$$

On the other hand, from $\lim_{t\to\infty} \dfrac{t}{f(t)} = 0$ we deduce $t \leq f(t)$ for large t. Taking logarithm one obtains

$$1 \leq \liminf_{t\to\infty} \frac{\log f(t)}{\log t}.$$

Therefore we use the Bernoulli–L'Hôpital rule for obtaining

$$1 \leq \liminf_{t\to\infty} \frac{\log f(t)}{\log t} = \liminf_{t\to\infty} \frac{tf'(t)}{f(t)}.$$

Analogous reasoning imply

$$1 \leq \liminf_{t\to\infty} \frac{tf(t)}{2F(t)} \quad \text{and} \quad 1 \leq \liminf_{t\to\infty} \frac{2f'(t)F(t)}{\big(f(t)\big)^2}. \qquad \square$$

The next technicalities claim the obtainment of a simple way to describe the borderline case assumption. In order to do it, we are interested in to study the behavior of the quotients

$$\frac{\psi(\eta t)}{\psi(t)}, \quad \frac{\eta f(t)}{f(\eta t)} \quad \text{and} \quad \frac{tf'(t)}{f(t)} \quad \text{with } \eta > 1$$

for large t, where ψ is the function defined by (1.4). These quotients involve regularly varying properties.

Lemma A.2. *Suppose that f verifies (1.2). Then for $\eta > 1$ one verifies*

$$\limsup_{t\to\infty} \frac{\eta f(t)}{f(\eta t)} = \left(\limsup_{t\to\infty} \frac{\psi(\eta t)}{\psi(t)}\right)^2 \leq 1.$$

In particular, (1.9) is equivalent to (1.10)

$$\limsup_{t\to\infty} \frac{\eta f(t)}{f(\eta t)} = 1 \quad \textit{for } \eta \in [1, \eta_0].$$

Proof. It is clear that any continuous and increasing function g on $\mathbb{R}_+$, with $g(0) = 0$, satisfies

$$0 \leq \frac{\eta g(t)}{g(\eta t)} \leq \eta \quad \text{for all } t \text{ sufficiently large,}$$

provided $\eta > 1$, whence the properties

$$0 \leq \liminf_{t\to\infty} \frac{\eta g(t)}{g(\eta t)} \leq \limsup_{t\to\infty} \frac{\eta g(t)}{g(\eta t)} \leq \eta$$

hold. In particular, the choice $g(t) = \sqrt{F(t)}$ shows

$$0 \leq \liminf_{t\to\infty} \frac{\eta\sqrt{F(t)}}{\sqrt{F(\eta t)}} \leq \limsup_{t\to\infty} \frac{\eta\sqrt{F(t)}}{\sqrt{F(\eta t)}} \leq \eta$$

again for $\eta > 1$. In fact, by the Bernoulli–L'Hôpital rule we have

$$\limsup_{t\to\infty} \frac{\eta\sqrt{F(t)}}{\sqrt{F(\eta t)}} = \limsup_{t\to\infty} \frac{\psi(\eta t)}{\psi(t)} \leq 1 \quad \text{with } \eta > 1.$$

Analogously, the above monotonicity reasoning implies

$$0 \leq \liminf_{t\to\infty} \frac{\eta f(t)}{f(\eta t)} \leq \limsup_{t\to\infty} \frac{\eta f(t)}{f(\eta t)} \leq \eta,$$

whence now the Bernoulli–L'Hôpital rule leads to

$$1 \geq \left(\limsup_{t\to\infty} \frac{\psi(\eta t)}{\psi(t)}\right)^2 = \limsup_{t\to\infty} \frac{\eta^2 F(t)}{F(\eta t)} = \limsup_{t\to\infty} \frac{\eta f(t)}{f(\eta t)} \quad \text{with } \eta > 1. \qquad \square$$

Whenever f is differentiable for large t we obtain another characterization more useful to the reasoning of the paper.

Lemma A.3. *Suppose that f is differentiable for large t and verifies (1.2) and (1.9). Then*

$$\lim_{t\to\infty} \frac{tf(t)}{2F(t)} = \lim_{t\to\infty} \frac{tf'(t)}{f(t)} = \lim_{t\to\infty} \frac{2f'(t)F(t)}{\big(f(t)\big)^2} = 1. \tag{A.4}$$

Furthermore, if f is twice differentiable at infinity one has

$$\lim_{t\to\infty} \frac{tf''(t)}{f'(t)} = 0. \tag{A.5}$$

Proof. From (A.3) we have

$$1 \leq \liminf_{t\to\infty} \frac{tf(t)}{2F(t)}.$$

On the other hand, since f is increasing and positive, for $\eta > 1$ we may construct the inequalities

$$(\eta - 1)tf(t) \leq \int_t^{\eta t} f(s)ds = F(\eta t) - F(t)$$

and

$$\frac{\eta - 1}{\eta^2} + \frac{F(t)}{\eta^2 tf(t)} \leq \frac{F(\eta t)}{\eta t f(\eta t)} \frac{f(\eta t)}{\eta f(t)},$$

for which

$$\frac{1}{\eta+1}=\frac{\eta-1}{\eta^2-1}\leq \liminf_{t\to\infty}\frac{F(t)}{tf(t)}$$

holds, provided $\eta\in(1,\eta_0]$ (see (1.10)). Letting $\eta\searrow 1$ one has

$$\frac{1}{2}\leq \liminf_{t\to\infty}\frac{F(t)}{tf(t)},$$

thus

$$2\leq \liminf_{t\to\infty}\frac{tf(t)}{F(t)}\leq \limsup_{t\to\infty}\frac{tf(t)}{F(t)}\leq 2.$$

By applying the Bernoulli–L'Hôpital rule we deduce

$$\lim_{t\to\infty}\frac{tf(t)}{F(t)}=\lim_{t\to\infty}\left(1+\frac{tf'(t)}{f(t)}\right),$$

and so by

$$\frac{2f'(t)F(t)}{\left(f(t)\right)^2}=\frac{tf'(t)}{f(t)}\frac{2F(t)}{tf(t)}$$

we conclude (A.4). Finally, applying again the Bernoulli–L'Hôpital rule we deduce

$$1=\lim_{t\to\infty}\frac{tf'(t)}{f(t)}=\lim_{t\to\infty}\left(1+\frac{tf''(t)}{f'(t)}\right),$$

whence (A.5) holds. $\square$

Remark A.1. We also note that from (1.2) and (1.9) we deduce $\lim_{t\to\infty}f'(t)=\infty$. In fact, if f is a function of class $\mathcal{C}^k$ at infinity, one proves

$$\lim_{t\to\infty}\frac{tf^{j)}(t)}{f^{j-1)}(t)}=2-j\quad\text{for } 1\leq j\leq k,$$

whence $\lim_{t\to\infty}f^{j)}(t)=\infty$, $1\leq j\leq k$.

Lemma A.4. *The condition*

$$\liminf_{t\to\infty}\frac{tf'(t)}{f(t)}=\alpha>1 \tag{A.6}$$

implies

$$\frac{f(t)}{t^\beta}\ \textit{is increasing for large } t, \tag{A.7}$$

provided $\beta<\alpha$. Moreover, (1.2) *and*

$$\limsup_{t\to\infty}\frac{\psi(\eta t)}{\psi(t)}<1$$

also follow.

Proof. Straightforward computations lead to

$$\left(\frac{f(t)}{t^\beta}\right)' = t^{-(\beta+1)} f(t) \left(\frac{tf'(t)}{f(t)} - \beta\right)$$

and (A.7) follows. On the other hand, given $\varepsilon \in (0, \alpha - 1)$ the assumption (A.6) implies

$$\frac{\alpha - \varepsilon}{t} \leq \frac{f'(t)}{f(t)} \quad \text{for large } t$$

whence an integration gives

$$(\alpha - \varepsilon) \ln \frac{t}{t_1} \leq \ln \frac{f(t)}{f(t_1)}, \quad t > t_1 \quad \text{for large } t_1,$$

thus

$$Ct^{\alpha-\varepsilon} \leq f(t), \quad t > t_1 \quad \text{for large } t \tag{A.8}$$

and

$$\sqrt{\frac{C}{1 + \alpha - \varepsilon}} t^{\frac{1+\alpha-\varepsilon}{2}} \leq \sqrt{F(t)} \quad \text{for large } t$$

hold for $C = \frac{f(t_1)}{t_1^{\alpha-\varepsilon}} > 0$. So that, we have

$$\int_{t_2}^{t} \frac{ds}{\sqrt{F(s)}} \leq \frac{2}{\alpha - 1 - \varepsilon} \sqrt{\frac{1 + \alpha - \varepsilon}{C}} \left(t_2^{\frac{1-\alpha+\varepsilon}{2}} - t^{\frac{1-\alpha+\varepsilon}{2}}\right), \quad t > t_2 > t_1 \quad \text{for large } t.$$

Since $\varepsilon \in (0, \alpha - 1)$ the Keller–Osserman condition (1.2) is concluded by letting $t \to \infty$. Finally, it follows

$$\limsup_{t\to\infty} \frac{\psi(\eta t)}{\psi(t)} < 1$$

otherwise Lemma A.3 implies

$$\liminf_{t\to\infty} \frac{tf'(t)}{f(t)} = 1$$

contrary to (A.6). □

Remark A.2. Under condition (1.2), Lemmas A.2 and A.3 prove that (1.9) implies (1.11). On the other hand, Lemma A.4 proves that (1.11) implies (1.9), provided (1.2), because whenever (1.9) fails the same happens with (1.11). So that, in the class of functions satisfying the Keller–Osserman condition, the properties (1.9) and (1.11) are equivalent. Therefore, under condition (1.2) the couple (A.6)–(1.11) is an equivalent alternative to the couple (1.7)–(1.9). From Lemmas A.1, A.2 and A.3 analogous alternative can be constructed. So, the ordinary case is governed by the properties

$$\liminf_{t\to\infty} \frac{tf(t)}{2F(t)} > 1 \quad \text{or} \quad \liminf_{t\to\infty} \frac{2f'(t)F(t)}{\big(f(t)\big)^2} > 1$$

and the borderline case by

$$\lim_{t\to\infty}\frac{tf(t)}{2F(t)}=1 \quad \text{or} \quad \lim_{t\to\infty}\frac{2f'(t)F(t)}{\big(f(t)\big)^2}=1.$$

We note that, in general,

$$\lim_{t\to\infty}\frac{tf'(t)}{f(t)}=1$$

does not imply the Keller–Osserman condition (1.2), as it follows from the choice $f(t)=t\log t$, $t>1$, but it is an irrelevant question in this paper because, as it was proved by Keller [25] and Osserman [38], the property (1.2) is a necessary condition for the existence of large solutions.

In the rest of Appendix A we denote by ϕ to the function defined by (1.5).

Proof of Lemma 4.3. Let $\eta_0>1$. It is clear that

$$\begin{cases} \eta_0>1+\delta^{1-\ell} & \text{for small } \delta>0,\\ \eta_0^{-1}<1-\delta^{1-\ell} & \text{for small } \delta>0. \end{cases}$$

Since ψ is decreasing at infinity we obtain

$$\begin{cases} \dfrac{\psi\big(\eta_0\phi(\delta)\big)}{\delta}<\dfrac{\psi\big((1+\delta^{1-\ell})\phi(\delta)\big)}{\delta}<\dfrac{\psi(\phi(\delta))}{\delta}=1 & \text{for small } \delta>0,\\ 1=\dfrac{\psi(\phi(\delta))}{\delta}<\dfrac{\psi\big((1-\delta^{1-\ell})\phi(\delta)\big)}{\delta}<\dfrac{\psi\big(\eta_0^{-1}\phi(\delta)\big)}{\delta} & \text{for small } \delta>0, \end{cases}$$

respectively, whence (1.9) implies (4.11). Next, we rewrite property (A.1) as

$$\lim_{t\to\infty}t\psi'(t)=-\lim_{t\to\infty}\frac{t}{\sqrt{2F(t)}}=0$$

(see Lemma A.1). Since (A.4) implies that given $k>1$ the bound

$$1-\frac{k}{2}\le 1-\frac{tf(t)}{2F(t)}\le\frac{1}{2}\quad\text{for large } t$$

(see Lemma A.3) the Bernoulli–L'Hôpital rule enables us to obtain

$$\lim_{t\to\infty}\frac{t\psi'(t)}{(\psi(t))^{\ell}}=\lim_{t\to\infty}\frac{(\psi(t))^{1-\ell}}{\ell}\left(1-\frac{tf(t)}{2F(t)}\right)=0. \tag{A.9}$$

Then, there exist $t_\pm$ for which we deduce the expansion

$$\psi\big((1\pm(\psi(t))^{1-\ell})t\big)=\psi(t)\pm t\big(\psi(t)\big)^{1-\ell}\psi'(t)+\frac{\big(t\big(\psi(t)\big)^{1-\ell}\big)^2}{2}\psi''(t_\pm)$$

or equivalently

$$\frac{\psi\big((1\pm(\psi(t))^{1-\ell})t\big)}{\psi(t)}=1\pm\frac{t\psi'(t)}{\big(\psi(t)\big)^{\ell}}+\frac{t^2(\psi(t))^{1-2\ell}}{2}\psi''(t_\pm)$$

where $t < t_+ < (1 + (\psi(t))^{1-\ell})t$ and $(1 - (\psi(t))^{1-\ell})t < t_- < t$. The monotonicity of the function f gives

$$\begin{cases} F(t) = F(t_+) + (t - t_+)f(\widetilde{t}_+) \geq F(t_+) + (t - t_+)f(t_+) \\ \qquad \geq F(t_+) - (\psi(t))^{1-\ell} t f(t_+) \geq F(t_+) - (\psi(t))^{1-\ell} t_+ f(t_+) \\ F(t) = F(t_-) + (t - t_-)f(\widetilde{t}_-) \leq F(t_-) + (t - t_-)f(t) \\ \qquad \leq F(t_-) + (\psi(t))^{1-\ell} t f(t) \end{cases}$$

for some $\widetilde{t}_+ \in (t, t_+)$ and $\widetilde{t}_- \in (t_-, t)$, whence

$$\begin{cases} 1 \geq \dfrac{F(t)}{F(t_+)} \geq 1 - (\psi(t))^{1-\ell} \dfrac{t_+ f(t_+)}{F(t_+)} \\ 1 \leq \dfrac{F(t)}{F(t_-)} \leq \dfrac{1}{1 - (\psi(t))^{1-\ell} \dfrac{t f(t)}{F(t)}} \end{cases}$$

hold, then the property (A.4) (see again Lemma A.3) implies

$$\lim_{t\to\infty} \frac{F(t)}{F(t_\pm)} = 1.$$

So that, the bounds

$$\begin{cases} 0 \leq -t^2 \big(\psi(t)\big)^{1-2\ell} \psi''(t_+) \dfrac{\big(\psi(t)\big)^{\ell}}{t\psi'(t)} = t(\psi(t))^{1-\ell} \dfrac{f(t_+)}{2F(t_+)} \sqrt{\dfrac{F(t)}{F(t_+)}} \leq (\psi(t))^{1-\ell} \dfrac{t_+ f(t_+)}{2F(t_+)} \\ 0 \geq t^2 \big(\psi(t)\big)^{1-2\ell} \psi''(t_-) \dfrac{\big(\psi(t)\big)^{\ell}}{t\psi'(t)} = -t(\psi(t))^{1-\ell} \dfrac{f(t_-)}{2F(t_-)} \sqrt{\dfrac{F(t)}{F(t_-)}} \geq -\dfrac{(\psi(t))^{1-\ell}}{1 - (\psi(t))^{1-\ell}} \dfrac{t_- f(t_-)}{2F(t_-)} \sqrt{\dfrac{F(t)}{F(t_-)}} \end{cases}$$

imply

$$\lim_{t\to\infty} t^2 \big(\psi(t)\big)^{1-2\ell} \psi''(t_\pm) \frac{\big(\psi(t)\big)^{\ell}}{t\psi'(t)} = 0.$$

Therefore one deduces that the functions

$$\zeta_\pm\big(\psi(t)\big) = 1 \pm \frac{t\psi'(t)}{2\big(\psi(t)\big)^{\ell}} \quad \text{for large } t$$

verify

$$\begin{cases} \dfrac{\psi\big((1 + (\psi(t))^{1-\ell})t\big)}{\psi(t)} < \zeta_+\big(\psi(t)\big) < 1 \quad \text{for large } t, \\ 1 < \zeta_-\big(\psi(t)\big) < \dfrac{\psi\big((1 - (\psi(t))^{1-\ell})t\big)}{\psi(t)} \quad \text{for large } t. \end{cases}$$

The change of variable $t = \phi(\delta)$ and the monotonicity of ψ conclude

$$\begin{cases} 1 < \dfrac{\phi\big(\zeta_+(\delta)\delta\big)}{\phi(\delta)} < 1 + \delta^{1-\ell} \quad \text{for small } \delta > 0, \\ 1 - \delta^{1-\ell} < \dfrac{\phi\big(\zeta_-(\delta)\delta\big)}{\phi(\delta)} < 1 \quad \text{for small } \delta > 0. \end{cases} \qquad \square$$

Remark A.3. Lemmas A.4 and 4.3 enable us refine the properties given in (A.1). For instance, for the ordinary case (A.8) leads to

$$\frac{t}{f(t)} = o\big(t^{-\beta}\big)$$

for all $\beta \in (0, \alpha - 1)$, and for the borderline case (A.9) implies

$$\frac{t}{f(t)} = o\Big(\big(\psi(t)\big)^{2\ell}\Big)$$

for all $\ell \in (0, 1)$.

Lemma A.5. *Suppose the assumptions of Lemma 4.3. Then the following relation holds*

$$\frac{\sqrt{F(\phi(\zeta_\pm(\delta)\delta))}}{f(\phi(\zeta_\pm(\delta)\delta))} = \frac{\sqrt{F(\phi(\delta))}}{f(\phi(\delta))} + o(1) \tag{A.10}$$

where $o(1)$ goes to 0 as $\delta \to 0$.

Proof. Straightforward computation leads to

$$\frac{\sqrt{2F(\phi(\zeta_\pm(\delta)\delta))}}{f(\phi(\zeta_\pm(\delta)\delta))} = \frac{\sqrt{2F(\phi(\delta))}}{f(\phi(\delta))} \pm \delta\varsigma(\phi(\delta))\frac{1}{\phi(\bar\delta_\pm)f(\phi(\bar\delta_\pm))}\left(\frac{\phi(\bar\delta_\pm)f(\phi(\bar\delta_\pm))}{2F(\phi(\bar\delta_\pm))} - \frac{\phi(\bar\delta_\pm)f'(\phi(\bar\delta_\pm))}{f(\phi(\bar\delta_\pm))}\right)$$

for small $\delta > 0$, $\bar\delta_+ \in (\zeta_+(\delta)\delta, \delta)$ and $\bar\delta_- \in (\delta, \zeta_-(\delta)\delta)$, where the function

$$\varsigma(t) = \Upsilon\big(\psi(t)\big) = \frac{t}{2(\psi(t))^\ell\sqrt{2F(t)}} \quad \text{for large } t \tag{A.11}$$

(see (4.12)) verifies

$$\lim_{t\to\infty} \varsigma(t) = 0 \tag{A.12}$$

(see (A.9) and Lemma A.3). Hence, assumption (1.9) concludes (A.10). □

Lemma A.6. *Suppose the assumptions and notations of Lemma 4.3. Then the following relations hold*

$$\begin{cases} 1 - \big(\zeta_+'(\delta)\delta + \zeta_+(\delta)\big)^2 - \dfrac{\phi'(\zeta_+(\delta)\delta)}{\phi''(\zeta_+(\delta)\delta)}\big(\zeta_+''(\delta)\delta + 2\zeta_+'(\delta)\big) > 0 & \textit{for small } \delta > 0, \\ 1 - \big(\zeta_-'(\delta)\delta + \zeta_-(\delta)\big)^2 - \dfrac{\phi'(\zeta_-(\delta)\delta)}{\phi''(\zeta_-(\delta)\delta)}\big(\zeta_-''(\delta)\delta + 2\zeta_-'(\delta)\big) < 0 & \textit{for small } \delta > 0. \end{cases}$$

Proof. By simplicity, we focus only on $\zeta_+(\delta)$ (the reasoning for $\zeta_-(\delta)$ is analogous). We sketch some straightforward calculations. We begin with

$$\begin{cases} \zeta_+(\delta) = 1 + \dfrac{\phi(\delta)}{2\delta^\ell\phi'(\delta)}, \quad \ell \in (0,1), \\ \zeta_+'(\delta) = \dfrac{1}{2\delta^{\ell+1}}\left[\delta - \left(\ell + \dfrac{\delta\phi''(\delta)}{\phi'(\delta)}\right)\dfrac{\phi(\delta)}{\phi'(\delta)}\right], \\ \zeta_+''(\delta) = -\dfrac{1}{2}\left(\dfrac{2\ell}{\delta^{\ell+1}} + \dfrac{1}{\delta^\ell}\dfrac{\phi''(\delta)}{\phi'(\delta)} - \dfrac{\ell(\ell+1)}{\delta^{\ell+2}}\dfrac{\phi(\delta)}{\phi'(\delta)} - \dfrac{2\ell}{\delta^{\ell+1}}\dfrac{\phi(\delta)\phi''(\delta)}{\big(\phi'(\delta)\big)^2} + \dfrac{1}{\delta^\ell}\dfrac{\phi(\delta)\phi'''(\delta)}{\big(\phi'(\delta)\big)^2} - \dfrac{2}{\delta^\ell}\dfrac{\phi(\delta)\big(\phi''(\delta)\big)^2}{\big(\phi'(\delta)\big)^3}\right) \end{cases}$$

for which equalities

$$1-\big(\zeta_+'(\delta)\delta+\zeta_+(\delta)\big)^2=-\frac{1}{4\delta^{2(\ell-1)}}-\frac{1}{4\delta^{2\ell}}\left(\frac{\phi(\delta)}{\phi'(\delta)}\right)^2\left[(1-\ell)^2+\left(\delta\frac{\phi''(\delta)}{\phi'(\delta)}\right)^2-2(1-\ell)\delta\frac{\phi''(\delta)}{\phi'(\delta)}\right]$$
$$-\frac{1}{\delta^{\ell-1}}-\frac{1}{\delta^{\ell}}\frac{\phi(\delta)}{\phi'(\delta)}\left((1-\ell)-\delta\frac{\phi''(\delta)}{\phi'(\delta)}\right)\left(1-\frac{1}{2\delta^{\ell-1}}\right)$$

and

$$\zeta_+''(\delta)\delta+2\zeta_+'(\delta)=\frac{1-\ell}{\delta^{\ell}}+\frac{(\ell-1)\ell}{2}\frac{1}{\delta^{\ell+1}}\frac{\phi(\delta)}{\phi'(\delta)}+\frac{\ell-1}{\delta^{\ell}}\frac{\phi(\delta)\phi''(\delta)}{\big(\phi'(\delta)\big)^2}$$
$$-\frac{1}{2\delta^{\ell-1}}\left(\frac{\phi''(\delta)}{\phi'(\delta)}+\frac{\phi(\delta)\phi'''(\delta)}{\big(\phi'(\delta)\big)^2}-2\frac{\phi(\delta)\big(\phi''(\delta)\big)^2}{\big(\phi'(\delta)\big)^3}\right)$$

hold. So that, by using

$$\phi'(\delta)=-\sqrt{2F\big(\phi(\delta)\big)},\quad \phi''(\delta)=f\big(\phi(\delta)\big)\quad\text{and}\quad \phi'''(\delta)=-f'\big(\phi(\delta)\big)\sqrt{2F\big(\phi(\delta)\big)}$$

the function

$$\mathcal{F}(\delta)=1-(\zeta_+'(\delta)\delta+\zeta_+(\delta))^2+\frac{\sqrt{2F(\phi(\delta))}}{f(\phi(\delta))}\big(\zeta_+''(\delta)\delta+2\zeta_+'(\delta)\big)$$

verifies

$$\mathcal{F}(\delta)=-\frac{\delta^{1-\ell}}{2}\left(1+\frac{\delta^{1-\ell}}{2}\right)+\frac{\delta^{2(1-\ell)}}{2}\frac{\phi(\delta)f\big(\phi(\delta)\big)}{2F\big(\phi(\delta)\big)}\left(1-(1-\ell)\delta\frac{\phi(\delta)}{\sqrt{2F\big(\phi(\delta)\big)}}\right)$$
$$-\big((1-\ell)\varsigma\big(\phi(\delta)\big)\big)^2-\frac{\delta^{2(1-\ell)}}{4}\left(\frac{\phi(\delta)f\big(\phi(\delta)\big)}{2F\big(\phi(\delta)\big)}\right)^2+\frac{\delta^{1-\ell}}{2}\frac{\phi(\delta)f'\big(\phi(\delta)\big)}{f\big(\phi(\delta)\big)}$$
$$+\frac{1-\ell}{\delta^{\ell}}\frac{\sqrt{2F\big(\phi(\delta)\big)}}{\phi(\delta)}+\frac{(1-\ell)\ell}{2\delta^{\ell+1}}\frac{\phi(\delta)}{f\big(\phi(\delta)\big)}+\frac{(1-\ell)\delta^{1-2\ell}}{2}\frac{\phi(\delta)}{\sqrt{2F\big(\phi(\delta)\big)}}$$

(see (A.11)). Relative to the right hand side of the last equality we note:

$$\begin{cases}\text{the first summand goes to 0 as } \delta\to 0,\\ \text{from (A.1) and (A.4) the second summand goes to 0 as } \delta\to 0,\\ \text{from (A.12) the third summand goes to 0 as } \delta\to 0,\\ \text{from (A.4) the fourth and fifth summands go to 0 as } \delta\to 0,\\ \text{from (A.1) the sixth summand goes to } \infty \text{ as } \delta\to 0,\\ \text{the seventh and eighth summands are positive.}\end{cases}$$

Therefore, from Lemma A.5 we conclude

$$1-\big(\zeta_+'(\delta)\delta+\zeta_+(\delta)\big)^2-\frac{\phi'(\zeta_+(\delta)\delta)}{\phi''(\zeta_+(\delta)\delta)}\big(\zeta_+''(\delta)\delta+2\zeta_+'(\delta)\big)>0$$

for small $\delta>0$. □

References

[1] S. Alarcón, G. Díaz, R. Letelier, J.M. Rey, Expanding the asymptotic explosive boundary behavior of large solutions to a semilinear elliptic equation, Nonlinear Anal. 72 (2010) 2426–2443.
[2] S. Alarcón, G. Díaz, J.M. Rey, The influence of sources terms on the boundary behavior of the large solutions of quasilinear elliptic equations. The power like case, Z. Angew. Math. Phys. 64 (2013) 659–677.
[3] S. Alarcón, G. Díaz, J.M. Rey, Borderline case in the study of the large solutions of quasilinear elliptic equations, in elaboration.
[4] S. Alarcón, A. Quaas, Large viscosity solutions for some fully nonlinear equations, NoDEA Nonlinear Differential Equations Appl. 20 (2013) 1453–1472.
[5] C. Anedda, G. Porru, Second order estimates for boundary blow-up solutions of elliptic equations, Discrete Contin. Dyn. Syst. (Suppl.) (2007) 54–63.
[6] C. Anedda, G. Porru, Boundary behaviour for solutions of boundary blow-up problems in a borderline case, J. Math. Anal. Appl. 352 (2009) 35–47.
[7] C. Bandle, Asymptotic behavior of large solutions of elliptic equations, An. Univ. Craiova, Math. Comp. Sci. Ser. 32 (2005) 1–8.
[8] C. Bandle, M. Essén, Sur les solutions maximales de problèmes elliptiques nonlinéaires: bornes isopérimetriques et comportamient asymptotique, C. R. Acad. Sci. Paris Sér. I Math. 311 (1990) 91–93.
[9] C. Bandle, M. Essén, On the solutions of quasilinear elliptic problems with boundary blow-up, Symp. Math. 35 (1990) 93–111.
[10] C. Bandle, M. Marcus, Asymptotic behaviour of solutions and their derivatives, for semilinear elliptic problems with blowup on the boundary, Ann. Inst. H. Poincaré Anal. Non Linéaire 12 (1995) 155–171.
[11] C. Bandle, M. Marcus, Dependence of blowup rate of large solutions of semilinear elliptic equations on the curvature of the boundary, Complex Var. Theory Appl. 49 (2004) 555–570.
[12] L. Bieberbach, $\Delta u = e^u$ und die automorphen funktionen, Math. Ann. 77 (1916) 173–212.
[13] N.H. Bingham, C.M. Goldie, J.L. Teugels, Regular Variation, Cambridge University Press, 1987.
[14] F.C. Cîrstea, Y. Du, Large solutions of elliptic equations with a weakly superlinear nonlinearity, J. Anal. Math. 103 (2007) 261–277.
[15] F.C. Cîrstea, V. Radulescu, Nonlinear problems with boundary blow-up: a Karamata regular variation approach, Asymptot. Anal. 45 (2006) 275–298.
[16] O. Costin, L. Dupaigne, Boundary blow-up solutions in the unit ball: asymptotics, uniqueness and symmetry, J. Differential Equations 249 (2010) 931–964.
[17] O. Costin, L. Dupaigne, O. Goubet, Uniqueness of large solutions, J. Math. Anal. Appl. 395 (2012) 806–812.
[18] M. Del Pino, R. Letelier, The influence of domain geometry in boundary blow-up elliptic problems, Nonlinear Anal. 48 (2002) 897–904.
[19] G. Díaz, R. Letelier, Explosive solutions of quasilinear elliptic equations: existence and uniqueness, Nonlinear Anal. 20 (1993) 97–125.
[20] H. Dong, S. Kim, M. Safonov, On uniqueness of boundary blow-up solutions of a class of nonlinear elliptic equations, Comm. Partial Differential Equations 33 (2008) 177–188.
[21] S. Dumont, L. Dupaigne, O. Goubet, V. Rădulescu, Back to the Keller–Osserman condition for boundary blow-up solutions, Adv. Nonlinear Stud. 7 (2007) 271–298.
[22] J. García-Melián, Uniqueness of positive solutions for a boundary blow-up problem, J. Math. Anal. Appl. 360 (2009) 530–536.
[23] D. Gilbarg, N.S. Trudinguer, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, 1998.
[24] S. Huang, Q. Tian, Second order estimates for large solutions of elliptic equations, Nonlinear Anal. 74 (2011) 2031–2044.
[25] J.B. Keller, On solutions of $\Delta u = f(u)$, Comm. Pure Appl. Math. 10 (1957) 503–510.
[26] V. Kondrat'ev, V. Nikishkin, Asymptotics near the boundary of a solution of a singular boundary value problems for semilinear elliptic equations, Differ. Equ. 26 (1990) 345–348.
[27] A.V. Lair, A necessary and sufficient condition for existence of large solutions to semilinear elliptic equations, J. Math. Anal. Appl. 240 (1999) 205–218.
[28] J.M. Lasry, P.L. Lions, Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints, Math. Ann. 283 (1989) 583–630.
[29] A.C. Lazer, P.J. McKenna, Asymptotic behaviour of solutions of boundary blow-up problems, Differential Integral Equations 7 (1994) 1001–1019.
[30] G.M. Lieberman, Asymptotic behavior and uniqueness of blow-up solution of quasilinear elliptic equation, J. Anal. Math. 115 (1) (2011) 213–249.
[31] C. Loewner, L. Nirenberg, Partial differential equations invariant under conformal or projective transformations, in: L. Ahlfors (Ed.), Contributions to Analysis, Academic Press, 1974, pp. 245–272.
[32] M. Marcus, L. Véron, Uniqueness and asymptotic behavior of solutions with boundary blow-up for a class of nonlinear elliptic equations, Ann. Inst. H. Poincaré Anal. Non Linéaire 14 (1997) 231–274.
[33] M. Marcus, L. Véron, Existence and uniqueness results for large solutions of general nonlinear elliptic equations, J. Evol. Equ. 3 (2004) 637–652.
[34] V. Maric, Regular Variation and Differential Equations, Lecture Notes in Math., vol. 1726, 2000.
[35] M. Marras, G. Porru, Estimates and uniqueness for boundary blow-up solutions of p-Laplacian equations, Electron. J. Differential Equations 2011 (119) (2011) 1–10.
[36] J. Matero, Quasilinear elliptic equations with boundary blow-up, J. Anal. Math. 96 (1996) 229–247.

[37] L. Mi, B. Liu, Second order expansion for blowup solutions of semilinear elliptic problems, Nonlinear Anal. 75 (2012) 2591–2613.
[38] R. Osserman, On the inequality $\Delta u \geq f(u)$, Pacific J. Math. 7 (1957) 1641–1647.
[39] Z. Zhang, Boundary behavior of large solutions for semilinear elliptic equations in borderline cases, Electron. J. Differential Equations 2012 (136) (2012) 1–11.