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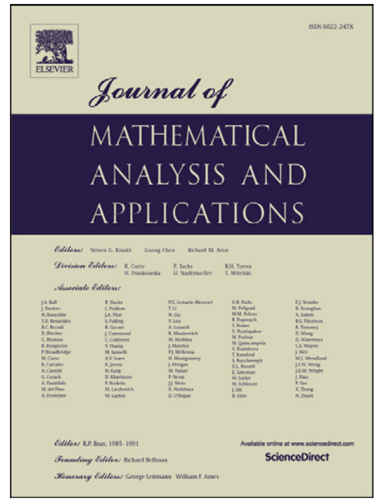

A unified approach for the Hankel determinants of classical combinatorial numbers

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A UNIFIED APPROACH FOR THE HANKEL DETERMINANTS OF CLASSICAL COMBINATORIAL NUMBERS

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Abstract. We give a general formula for the determinants of a class of Hankel matrices which arise in combinatorics theory. We revisit and extend existant results on Hankel determinants involving the sum of consecutive Catalan, Motzkin and Schroder numbers and we prove a conjecture in [20] about the recurrence relations satisfied by the Hankel transform of linear combinations of Catalans numbers.

1. Introduction

Let $a = \{a_n\}_{n\in\mathbb{N}}$ denote a sequence of numbers. The $n \times n$ matrix

$$\mathcal{H}_n(a) = (a_{i+j})_{0\le i,j\le n-1},$$

is called Hankel matrix. If $h_n = \det(\mathcal{H}_n(a))$, then the sequence $\{h_n\}_{n\ge 0}$ is referenced as the Hankel transform of the sequence a and was widely investigated in numerous papers. Hankel determinants are particularly interesting when applied to classical combinatorial sequences arising from the lattice path enumerations and has attracted an increasing amount of attention recently [2, 3, 5, 7, 15, 18]. One of the most popular themes in this context is to consider the determinant of the Hankel matrix generated by the sequence that are linear combinations of the sequences $\{a_n\}$ where $a_n = C_n$, M_n and R_n are Catalan, Motzkin or Schroder numbers respectively. For instance, Hankel determinant evaluations such as

$$\begin{aligned}
\det\left((C_{i+j})_{0\le i,j\le n-1}\right) &= 1,\\
\det\left((M_{i+j})_{0\le i,j\le n-1}\right) &= 1,\\
\det\left((R_{i+j})_{0\le i,j\le n-1}\right) &= 2^{\binom{n}{2}},
\end{aligned}$$

or these involving consecutive terms have been addressed numerous times in the literature. Among the method employed to prove such formulae we cite the combinatorial methods based on the Lindström–Gessel–Viennot lemma on non-intersecting lattice paths and orthogonal polynomials. The reader is referred to Krattenthaler papers [12, 13].

In this paper, our main focus is an overall generalization of these results. We evaluate $\det(\mathcal{H}_n(b))$ for $b = \{b_n\}$ of the form

$$b_n = \sum_{k=0}^{r} \lambda_k a_{n+k},$$

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where $\lambda_0, \lambda_1, \ldots, \lambda_{r-1}, \lambda_r$, $r \geq 1$, are complex numbers such that $\lambda_r = 1$. We shall assume that $\det\left((a_{i+j})_{0\leq i,j\leq n-1}\right) \neq 0$ for all $n \geq 0$ and we denote by $\mathcal{L}$ the linear functional on the vector space of all polynomials defined by

$$\mathcal{L}\left(x^n\right) = a_n \quad \text{for } n = 0, 1, \ldots$$

To $\{a_n\}$ we associate the monic orthogonal polynomial sequence $\{p_n(x)\}_{n\in\mathbb{N}}$ [30] such that p_n is monic of degree n and

$$\mathcal{L}\left(p_n p_m\right) = 0 \quad \text{for } n \neq m.$$

We remark that $b_n = \sum\limits_{k=0}^{r} \lambda_k a_{n+k} = \mathcal{L}\left(x^n q\right)$, where

$$q(x) = x^r + \lambda_{r-1}x^{r-1} + \ldots + \lambda_0.$$

The $r-$kernel $\mathcal{K}_{n,P}^{(r)}$ of $P = \{p_n\}_{n\in\mathbb{N}}$ is defined by

$$\mathcal{K}_{n,P}^{(r)}\left(x_1, x_2, \ldots, x_r\right) = \frac{\det\left(\left(p_{n+i-1}\left(x_j\right)\right)_{1\leq i,j\leq r}\right)}{\prod\limits_{1\leq i<j\leq r}\left(x_j - x_i\right)}$$

for $r \geq 2$ and $\mathcal{K}_{n,P}^{(1)}(x) = p_n(x)$. As it will be shown latter, $\mathcal{K}_{n,P}^{(r)}\left(x_1, x_2, \ldots, x_r\right)$ is a polynomial of the variables $x_1, x_2, \ldots$ and x_r.

The following Theorem constitutes our main result:

Theorem 1. *We have*

$$\det\left(\mathcal{H}_n(b)\right) = (-1)^{nr}\det\left(\mathcal{H}_n(a)\right)\mathcal{K}_{n,P}^{(r)}\left(\alpha_1, \alpha_2, \ldots, \alpha_r\right). \tag{1.1}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_r$ are the zeros of q.

In most examples considered in the existing literature, b_n has a specific pattern. Namely

$$b_n = a_{n+r} - c a_{n+r-1}, \text{ with } c \in \mathbb{C}.$$

Theorem 2. *We have for $c \neq 0$:*

$$\det\left(\left(a_{i+j+r} - c a_{i+j+r-1}\right)_{0\leq i,j\leq n-1}\right) = (-1)^{nr}\det\left(\mathcal{H}_n(a)\right) \tag{1.2}$$
$$\times \det\begin{pmatrix} \frac{p_n(0)}{0!} & \frac{p_n'(0)}{1!} & \cdots & \frac{p_n^{(r-2)}(0)}{(r-2)!} & \frac{p_n(c)}{c^{r-1}} \\ \frac{p_{n+1}(0)}{0!} & \frac{p_{n+1}'(0)}{1!} & \cdots & \frac{p_{n+1}^{(r-2)}(0)}{(r-2)!} & \frac{p_{n+1}(c)}{c^{r-1}} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{p_{n+r-1}(0)}{0!} & \frac{p_{n+r-1}'(0)}{1!} & \cdots & \frac{p_{n+r-1}^{(r-2)}(0)}{(r-2)!} & \frac{p_{n+r-1}(c)}{c^{r-1}} \end{pmatrix}$$

and

$$\det\left(\left(a_{i+j+r}\right)_{0\le i,j\le n-1}\right) = (-1)^{nr}\det\left(\mathcal{H}_n(a)\right) \times \det\begin{pmatrix} \frac{p_n(0)}{0!} & \frac{p_n'(0)}{1!} & \cdots & \frac{p_n^{(r-1)}(0)}{(r-1)!} \\ \frac{p_{n+1}(0)}{0!} & \frac{p_{n+1}'(0)}{1!} & \cdots & \frac{p_{n+1}^{(r-1)}(0)}{(r-1)!} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{p_{n+r-1}(0)}{0!} & \frac{p_{n+r-1}'(0)}{1!} & \cdots & \frac{p_{n+r-1}^{(r-1)}(0)}{(r-1)!} \end{pmatrix} \tag{1.3}$$

The proof of these theorems will be the object of the next section. In the second section we give various of its applications on Catalan, Motzkin and Schroder sequences. This is done by identifying the corresponding orthogonal polynomials through their generating functions. In the last section, we will prove a conjecture in [20].

2. Hankel determinant as a generalized kernel

2.1. **Proof of the Theorem 1.** The proof of the Theorem 1. follows the method in [28]. We will divide it into many lemmas but first, we will assume that $\alpha_1,\alpha_2,\ldots,\alpha_r$ are pairwise distinct.

Lemma 1. $\mathcal{K}_{n,P}^{(r)}(\alpha_1,\alpha_2,\ldots,\alpha_r)$ *is polynomial of degree* n *of the variable* $\lambda_0 = q(0)$ *with leading coefficient* $(-1)^{nr}$.

Proof. Let the alternant determinant

$$h(x_1,x_2,\ldots,x_r) = \begin{vmatrix} p_n(x_1) & p_n(x_2) & \cdots & p_n(x_r) \\ p_{n+1}(x_1) & & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ p_{n+r-1}(x_1) & p_{n+r-1}(x_2) & \cdots & p_{n+r-1}(x_r) \end{vmatrix}.$$

We have by the Leibniz formula

$$h(x_1,x_2,\ldots,x_r) = \sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\prod_{k=1}^{r} p_{n+k-1}(x_{\sigma(k)}),$$

which show that $h(x_1,x_2,\ldots,x_r)$ is a multivariate polynomial of degree at most $\sum_{k=1}^{r} n+k-1 = nr + \frac{r(r-1)}{2}$. For any $i<j$, $x_j - x_i$ divide h and hence $\mathcal{K}_{n,P}^{(r)}(x_1,x_2,\ldots,x_r)$ is a symmetric polynomial in variables $x_1,x_2,\ldots,x_r$, of degree at most nr, since the polynomial $\prod_{1\le i<j\le r}(x_j-x_i)$ is of degree $\frac{r(r-1)}{2}$. By consequent, $\mathcal{K}_{n,P}^{(r)}(x_1,x_2,\ldots,x_r)$ can be expressed in terms of elementary symmetric polynomials

$$\sigma_1 = \sum_{i=1}^{r} x_i,\ \ldots,\ \sigma_r = \prod_{i=1}^{r} x_i.$$

Let us write

$$\mathcal{K}_{n,P}^{(r)}\left(x_1, x_2, \ldots, x_r\right) = A\sigma_r^n + F,$$

where F is of degree $< n$ of σ_r and $A \in \mathbb{C}$. The coefficient of the term $x_1^n x_2^{n+1} \ldots x_r^{n+r-1} = \sigma_r^n x_2^1 \ldots x_r^{r-1}$ in $h\left(x_1, x_2, \ldots, x_r\right)$ is 1, while its coefficient in

$$\prod_{1\leq i<j\leq r}\left(x_j - x_i\right) \times \mathcal{K}_{n,P}^{(r)}\left(x_1, x_2, \ldots, x_r\right),$$

is A, this because one of the terms in polynomial $\prod_{1\leq i<j\leq r}\left(x_j - x_i\right)$ is $x_2^1 \ldots x_r^{r-1}$. This give $A = 1$. On the other hand, we have by Vieta's formulas:

$$\sigma_r\left(\alpha_1, \alpha_2, \ldots, \alpha_r\right) = (-1)^r \lambda_0.$$

The proof is completed. □

Lemma 2. $\det\left(\left(b_{i+j}\right)_{0\leq i,j\leq n-1}\right)$ *is polynomial of degree* n *of the variable* $\lambda_0 = q(0)$ *with leading coefficient* $\det\left(\mathcal{H}_n(a)\right)$.

Proof. We can write

$$\left(\left(b_{i+j}\right)_{0\leq i,j\leq n-1}\right)_{0\leq i,j\leq n-1} = \lambda_0\left(a_{i+j}\right)_{0\leq i,j\leq n-1} + B,$$

where the matrix B is independant of λ_0. The result follows. □

Lemma 3. *We have*

$$\det\left(\left(b_{i+j}\right)_{0\leq i,j\leq n-1}\right) = \det\left(\left(\mathcal{L}\left(p_i p_j q\right)\right)_{0\leq i,j\leq n-1}\right).$$

Proof. Let us write

$$p_i(x) = \sum_{k=0}^{i} c_{i,k} x^k,$$

with $c_{i,i} = 1$. The j, k entry of the matrix $\left(\mathcal{L}\left(p_i p_j q\right)\right)_{0\leq i,j\leq n-1}$ is

$$\sum_{m\leq j}\sum_{l\leq k} c_{j,m} c_{k,l} \mathcal{L}\left(x^{m+l} q\right).$$

Form this it follows that $\left(\mathcal{L}\left(p_i p_j q\right)\right)_{0\leq i,j\leq n-1}$ is a product of three matrices $T\mathcal{H}_n(b)\,T^t$ where the matrix T is lower triangular with entries $c_{j,m}, m \leq j$. It is easy to see from this that the lemma follows. □

Lemma 4. *If* $\mathcal{K}_{n,P}^{(r)}\left(\alpha_1, \alpha_2, \ldots, \alpha_r\right) = 0$ *then* $\det\left(\left(b_{i+j}\right)_{0\leq i,j\leq n-1}\right) = 0$. *The converse is true.*

Proof. Assume that $\mathcal{K}_{n,P}^{(r)}\left(\alpha_1, \alpha_2, \ldots, \alpha_r\right) = 0$. Then, the row vectors of the matrix

$$\begin{pmatrix} p_n(\alpha_1) & p_n(\alpha_2) & \cdots & p_n(\alpha_r) \\ p_{n+1}(\alpha_1) & p_{n+1}(\alpha_2) & \ddots & p_{n+1}(\alpha_r) \\ \vdots & & \ddots & \vdots \\ p_{n+r-1}(\alpha_1) & p_{n+r-1}(\alpha_2) & \cdots & p_{n+r-1}(\alpha_r) \end{pmatrix},$$

are linearly dependants, i.e, there exist scalar $c_1, c_2, \ldots, c_r$, not all zero such that

$$\sum_{i=0}^{r-1} c_i p_{n+i}(\alpha_k) = 0, \quad \text{for } k = 1, 2, \ldots, r.$$

It follows that $\alpha_1, \alpha_2, \ldots, \alpha_r$ are zeros of the polynomial $g(x) = \sum_{i=0}^{r-1} c_i p_{n+i}(x)$ and consequently q divide g :

$$g = qh.$$

Obviously h is of degree at most $n-1$, we can write

$$h = \sum_{j=0}^{n-1} \mu_j p_j, \text{ with } (\mu_1, \ldots, \mu_n) \in \mathbb{C}^n - \{0\}.$$

and then $g = \sum_{j=0}^{n-1} \mu_j q p_j$.We have for $k = 0, 1, 2, \ldots, n-1$:

$$\mathcal{L}(gp_k) = \sum_{i=0}^{r-1} c_i \mathcal{L}(p_k p_{n+i}) = 0,$$

so we obtain

$$\sum_{j=0}^{n-1} \mu_j \mathcal{L}(p_k p_j q) = 0 \text{ for } k = 0, 1, 2, \ldots, n-1,$$

which implies that the matrix $(\mathcal{L}(p_k p_j q))_{j,k=0}^{n-1}$ is singular and so is for $(b_{i+j})_{0 \le i,j \le n-1}$. $\square$

It follows from the Lemmas that $(-1)^{nr} \mathcal{K}_{n,P}^{(r)}(\alpha_1, \alpha_2, \ldots, \alpha_r)$ and $\det(\mathcal{H}_n(a))^{-1} \times \det\left((b_{i+j})_{0 \le i,j \le n-1}\right)$ are monic polynomials of the variable λ_0 with degree n, with the same distinct zeros. Consequently, if their zeros are all simple then

$$\det(\mathcal{H}_n(a))^{-1} \times \det\left((b_{i+j})_{0 \le i,j \le n-1}\right) = (-1)^{nr} \mathcal{K}_{n,P}^{(r)}(\alpha_1, \alpha_2, \ldots, \alpha_r).$$

Since this relation is an equality between multivariate polynomials then it still valid for the general case. This completes the proof of the theorem.

Example 1. *Assume that $q(x) = x - c$. Then*

$$\det\left((a_{i+j+1} - c a_{i+j})_{0 \le i,j \le n-1}\right) = (-1)^n \det(\mathcal{H}_n(a)) \times p_n(c).$$

This formula can derived by rows operations from the formula:

$$p_n(c) = \frac{1}{\det(\mathcal{H}_n(a))} \times \begin{vmatrix} a_0 & a_1 & \ldots & a_{n-1} & 1 \\ a_1 & a_2 & \ldots & a_n & c \\ a_2 & a_3 & \ldots & a_{n+1} & c^2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_n & a_{n+1} & \ldots & a_{2n-1} & c^n \end{vmatrix}$$

Example 2. *Assume that* $q(x) = (x-c)(x-d)$. *Then*

$$\det\left(\left(a_{i+j+2} - (c+d)\,a_{i+j+1} + cda_{i+j}\right)_{0\le i,j\le n-1}\right) = \det\left(\mathcal{H}_n(a)\right) \times \mathcal{K}^{(2)}_{n,P}(c,d)$$

where $\mathcal{K}^{(2)}_{n,P}(c,d)$ *is the classical kernel of the orthogonal polynomials* P *defined by*:

$$\mathcal{K}^{(2)}_{n,P}(c,d) = \frac{1}{d-c}\begin{vmatrix} p_n(c) & p_n(d) \\ p_{n+1}(c) & p_{n+1}(d) \end{vmatrix}$$

if $c \ne d$ *and* $\mathcal{K}^{(2)}_{n,P}(c,c) = p'_{n+1}(c)\,p_n(c) - p_{n+1}(c)\,p'_n(c)$.

2.2. **Proof of the Theorem 2.** By the proof of the Theorem 1.we have seen that

$\mathcal{K}^{(r)}_{n,P}(\alpha_1,\alpha_2,\ldots,\alpha_r)$ is well defined even if $\alpha_1,\alpha_2,\ldots,\alpha_r$ are not distinct. We will explicit $\mathcal{K}^{(r)}_{n,P}(\alpha_1,\alpha_2,\ldots,\alpha_r)$ in this case.

Let assume first that $\alpha_1,\alpha_2,\ldots,\alpha_r$ are pairwise distinct and write

$$p_{n+i}(x) = d_i(x)\,q(x) + r_i(x)$$

where $d_i(x), r_i(x)$ are polynomials with $r_i(x) = \sum_{j=0}^{r-1} \beta_{i,j} x^j$ of degree $< r$. Then

$$p_{n+i}(\alpha_{k+1}) = \sum_{j=0}^{r-1} \beta_{i,j}\alpha_{k+1}^j,$$

and hence

$$\left(p_{n+i}(\alpha_{k+1})\right)_{0\le i,k\le r-1} = \left(\beta_{i,j}\right)_{0\le i,j\le r-1} \times \left(\alpha_{k+1}^j\right)_{0\le j,k\le r-1}.$$

Using the formula for the Vandermonde determinant, one can find that

$$\mathcal{K}^{(r)}_{n,P}(\alpha_1,\alpha_2,\ldots,\alpha_r) = \det\left(\left(\beta_{i,j}\right)_{0\le i,j\le r-1}\right). \tag{2.1}$$

Since $\beta_{i,j}$ are polynomial functions of the variables $\lambda_0,\lambda_1,\ldots,\lambda_{r-1}$, then the formula (2.1) is valid in the general case. We assume more generally that

$$q(x) = \prod_{k=1}^{s}(x-\alpha_k)^{m_k},$$

where $m_1,m_2,\ldots,m_s$ are non null integers with $\sum_{k=1}^{s} m_k = r$ and $\alpha_1,\alpha_2,\ldots,\alpha_s$ are pairwise distinct. For $k=1,\ldots,s$, $0\le i\le r-1$ and $0\le l\le m_k-1$, we have

$$p^{(l)}_{n+i}(\alpha_k) = \sum_{j=l}^{r-1} \beta_{i,j}\frac{j!}{(j-l)!}\alpha_k^{j-l}.$$

It follows that if we denote by A the $r \times r$ matrix

$$\left(\begin{array}{cccc|cccc|cc}
p_n(\alpha_1) & p_n'(\alpha_1) & \ldots & p_n^{(m_1-1)}(\alpha_1) & p_n(\alpha_2) & p_n'(\alpha_2) & \ldots & p_n^{(m_2-1)}(\alpha_2) & p_n(\alpha_3) & \cdots \\
p_{n+1}(\alpha_1) & p_{n+1}'(\alpha_1) & \ldots & p_{n+1}^{(m_1-1)}(\alpha_1) & p_{n+1}(\alpha_2) & p_{n+1}'(\alpha_2) & \ldots & p_{n+1}^{(m_2-1)}(\alpha_2) & p_{n+1}(\alpha_3) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
p_{n+r-1}(\alpha_1) & p_{n+r-1}'(\alpha_1) & \ldots & p_{n+r-1}^{(m_1-1)}(\alpha_1) & p_{n+r-1}(\alpha_2) & p_{n+r-1}'(\alpha_2) & \ldots & p_{n+r-1}^{(m_2-1)}(\alpha_2) & p_{n+r-1}(\alpha_3) & \cdots
\end{array}\right)$$

and by V the confluent Vandermonde matrix

$$\left(\begin{array}{ccccc|ccccc|cc}
1 & 0 & \cdots & & 0 & 1 & 0 & \cdots & & 0 & 1 & \cdots \\
\alpha_1 & 1 & \ddots & & \vdots & \alpha_2 & 1 & \ddots & & \vdots & \alpha_3 & \cdots \\
\alpha_1^2 & 2\alpha_1 & 2 & & & \alpha_2^2 & 2\alpha_2 & 2 & & 0 & \alpha_3^2 & \cdots \\
\vdots & & \ddots & \ddots & 0 & \vdots & & \ddots & \ddots & 0 & \vdots & \\
\alpha_1^{m_1-1} & \cdots & & & \frac{(m_1-1)!}{1!} & \alpha_2^{m_2-1} & \cdots & & & \frac{(m_2-1)!}{1!} & & \cdots \\
 & & & & \frac{m_1!}{2!}\alpha_1 & \vdots & & & & \vdots & \alpha_3^{m_3-1} & \cdots \\
\vdots & & & & \vdots & & & & & & \vdots & \\
\alpha_1^{r-1} & (r-1)\alpha_1^{r-2} & \cdots & \cdots & \frac{(r-1)!}{(r-m_1)!}\alpha_1^{r-m_1} & \alpha_2^{r-1} & (r-1)\alpha_2^{r-2} & \cdots & \cdots & \frac{(r-1)!}{(r-m_2)!}\alpha_2^{r-m_2} & \alpha_3^{r-1} & \cdots
\end{array}\right),$$

then $A = \left(\beta_{i,j}\right)_{0 \le i,j \le r-1} \times V$ and consequently

$$\mathcal{K}_{n,P}^{(r)}(\alpha_1, \alpha_2, \ldots, \alpha_r) = \frac{\det(A)}{\det(V)}.$$

If $q(x) = x^{r-1}(x-c)$, $c \neq 0$, then

$$A = \begin{pmatrix}
p_n(0) & p_n'(0) & \ldots & p_n^{(r-2)}(0) & p_n(c) \\
p_{n+1}(0) & p_{n+1}'(0) & \ldots & p_{n+1}^{(r-2)}(0) & p_{n+1}(c) \\
\vdots & \vdots & \ldots & \vdots & \vdots \\
p_{n+r-1}(0) & p_{n+r-1}'(0) & \ldots & p_{n+r-1}^{(r-2)}(0) & p_{n+r-1}(c)
\end{pmatrix},$$

and

$$V=\begin{pmatrix} 1 & 0 & \cdots & & 1 \\ 0 & 1! & & & c \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & (r-2)! & c^{r-2} \\ 0 & \cdots & \cdots & 0 & c^{r-1} \end{pmatrix}.$$

The formula (1.2) follows immediately. Similarly, if $q(x)=x^r$ then

$$A=\begin{pmatrix} p_n(0) & p_n'(0) & \dots & p_n^{(r-2)}(0) & p_n^{(r-1)}(0) \\ p_{n+1}(0) & p_{n+1}'(0) & \dots & p_{n+1}^{(r-2)}(0) & p_{n+1}^{(r-1)}(0) \\ \vdots & \vdots & \dots & \vdots & \vdots \\ p_{n+r-1}(0) & p_{n+r-1}'(0) & \dots & p_{n+r-1}^{(r-2)}(0) & p_{n+r-1}^{(r-1)}(0) \end{pmatrix}.$$

and

$$V=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1! & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & (r-1)! \end{pmatrix},$$

and so we obtain the formula (1.3).

Remark 1. *Let for $u\in\mathbb{C}$, $\mathcal{L}_u$ be the linear functional on the vector space of all polynomials defined by*

$$\mathcal{L}_u(x^n)=\mathcal{L}\left((x-u)^n\right) \quad \text{for } n=0,1,\dots$$

Hence, the moments sequence of $\mathcal{L}_u$ is $q_u=\{q_n(u)\}_n$ such that

$$q_n(u)=\sum_{k=0}^{n}a_k\binom{n}{k}(-u)^{n-k}.$$

One can see that the monic orthogonal polynomials $\{p_{n,u}(x)\}$ associated with the moments sequence $\{q_n(u)\}$ are given by

$$p_{n,u}(x)=p_n(x+u),$$

this because

$$\mathcal{L}_u\left(p_n(x+u)\,p_m(x+u)\right)=\mathcal{L}\left(p_n(x)\,p_m(x)\right)=0 \quad \text{for } n\neq m.$$

Then it follows form (1.3):

$$\det\left(\left(q_{r+i+j}(u)\right)_{0\le i,j\le n-1}\right)=(-1)^{nr}\det\left(\mathcal{H}_n(q_u)\right)\times\det\left(\left(\frac{p_{n+j-1}^{(i-1)}(u)}{(i-1)!}\right)_{1\le i,j\le n}\right).$$

On the other hand, we have for $u\neq 0$,

$$q_n(u)=u^n a_n^*(u),$$

where $a_u^*=\{a_n^*(u)\}_n$ is the inverse binomial transform of the sequence $a_u=$

$\{ a_n u^{-n}\}_n$ [26, 27]. *It is known* [26, 27] *that*

$$\det\left(\mathcal{H}_n\left(a_u^*\right)\right) = \det\left(\mathcal{H}_n\left(a_u\right)\right),$$

and hence

$$\begin{aligned} \det\left(\mathcal{H}_n\left(q_u\right)\right) &= \det\left(\left(u^{i+j} a_{i+j}^*\left(u\right)\right)_{0\le i,j\le n-1}\right) \\ &= u^{n(n-1)} \det\left(\mathcal{H}_n\left(a_u^*\right)\right) \\ &= \det\left(\mathcal{H}_n\left(a\right)\right). \end{aligned}$$

Since we have

$$p_n\left(x\right) = \frac{1}{\det\left(\mathcal{H}_n\left(a\right)\right)} \times \begin{vmatrix} a_0 & a_1 & \dots & a_{n-1} & 1 \\ a_1 & a_2 & \dots & a_n & x \\ a_2 & a_3 & \dots & a_{n+1} & x^2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_n & a_{n+1} & \dots & a_{2n-1} & x^n \end{vmatrix},$$

then we obtain the following formula due to Bernard Leclerc [23]*:*

$$\det\left(\left(q_{r+i+j}\left(u\right)\right)_{0\le i,j\le n-1}\right) = (-1)^{nr} \det\left(\mathcal{H}_n\left(a\right)\right) \times \det\left(\left(\frac{p_{n+j-1}^{(i-1)}\left(u\right)}{(i-1)!}\right)_{1\le i,j\le n}\right).$$

A generalization of this formula was given by Antonio Duran [25]*. Reciprocally, the formula (1.3) follows from the above by choosing* $u = 0$*.*

3. Hankel determinants with classical sequences

It is a common knowledge that the monic orthogonal polynomials associated with the moments sequence $\{a_n\}$ can be obtained in the following way:

Let

$$f(x) = \sum_{n=0}^{+\infty} a_n x^n = \cfrac{a_0}{1-\alpha_0 x - \cfrac{\beta_1 x^2}{1-\alpha_1 x - \cfrac{\beta_2 x^2}{1-\alpha_2 x - \cfrac{\beta_3 x^2}{1-\alpha_3 x - \cdots}}}},$$

be the generating function of the sequence $\{a_n\}$. Then $\{p_n\left(x\right)\}$ satisfies the three-term recurrence:

$$p_{n+1}\left(x\right) = (x-\alpha_n)p_n(x) - \beta_n p_{n-1}(x), \text{ for } n \ge 1,$$

with $p_0 = 1$ and $p_1\left(x\right) = x - \alpha_0$ [21]. All sequences considered here have generating functions satisfying a certain type of quadratic equations of the form

$$f\left(x\right) = a + bxf\left(x\right) + cx^2 f\left(x\right)^2,$$

or

$$f\left(x\right) = a + bxf\left(x\right) + cxf\left(x\right)^2,$$

for some constants $a, b, c,$ $a \neq 0$.

Lemma 5. (1) *Assume that* $f(x) = a + bxf(x) + cx^2 f(x)^2$, *for some constants* $a, b, c,\ a \neq 0$. *Then*

$$f(x) = \cfrac{a}{1 - bx - \cfrac{acx^2}{1 - bx - \cfrac{acx^2}{1 - bx - \cdots}}} \tag{3.1}$$

(2) *Assume that* $f(x) = a + bxf(x) + cxf(x)^2$, *for some constants* $a, b, c,\ a \neq 0$. *Then*

$$f(x) = \cfrac{a}{1 - (b + ac)x - \cfrac{a'cx^2}{1 - b'x - \cfrac{a'cx^2}{1 - b'x - \cfrac{a'cx^2}{1 - b'x - \cdots}}}} \tag{3.2}$$

where $a' = a(ac + b)$ *and* $b' = 2ac + b$.

Proof. 1. We have

$$\begin{aligned} f(x) &= \frac{a}{1 - bx - cx^2 f(x)} \\ &= \cfrac{a}{1 - bx - \cfrac{acx^2}{1 - bx - cx^2 f(x)}} \\ &= \cfrac{a}{1 - bx - \cfrac{acx^2}{1 - bx - \cfrac{acx^2}{1 - bx - \cdots}}}. \end{aligned}$$

2. We put $g(x) = \dfrac{f(x) - a}{x}$, thus g is the generating function of the sequence $\{a_{n+1}\}$. We have

$$\begin{aligned} g(x) &= bf(x) + cf(x)^2 \\ &= b(xg(x) + a) + c(xg(x) + a)^2 \\ &= a' + b'xg(x) + cx^2 g(x)^2, \end{aligned}$$

where $a' = a(ac + b)$ and $b' = 2ac + b$. According to (3.1)we get

$$g(x) = \cfrac{a'}{1 - b'x - \cfrac{a'cx^2}{1 - b'x - \cfrac{a'cx^2}{1 - b'x - \cdots}}}$$

and then

$$\begin{aligned} f(x) &= \frac{a}{1-bx-cxf(x)} \\ &= \frac{a}{1-(b+ac)x-cx^2g(x)} \\ &= \cfrac{a}{1-(b+ac)x-\cfrac{a'cx^2}{1-b'x-\cfrac{a'cx^2}{1-b'x-\cfrac{a'cx^2}{1-b'x-\cdots}}}} \end{aligned}$$

□

3.1. **Hankel determinants of Catalan numbers.** The Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ counts the number of Dyck paths of length n, which are the lattice paths in the plane $\mathbb{Z}\times\mathbb{Z}$ from $(0,0)$ to $(2n,0)$ using steps $U=(1,1)$, $D=(1,-1)$ that never pass below the x-axis. It is a folklore that $\det\left((C_{i+j})_{0\le i,j\le n-1}\right)=1$ [16] , on the other hand , $f(x)=\sum_{n=0}^{+\infty} C_n x^n$ satisfies

$$f(x) = 1 + xf(x)^2.$$

Hence the monic orthogonal polynomials $\{p_n(x)\}$ satisfies the three-term recurrence:

$$p_{n+1}(x) = (x-2)p_n(x) - p_{n-1}(x), \text{ for } n \ge 1,$$

with $p_0 = 1$ and $p_1(x) = x-1$. We see that $p_n(x) = W_n\left(\frac{x}{2}-1\right)$, where W_n is the Chebychev polynomials of the fourth kinds [29].

It follows immediately from the Theorem 2. that

$$\det\left((C_{i+j+1}+C_{i+j})_{0\le i,j\le n-1}\right) = (-1)^n p_n(-1) = (-1)^n W_n\left(-\frac{3}{2}\right).$$

and

$$\begin{aligned} \det\left((C_{i+j+2}+C_{i+j+1})_{0\le i,j\le n-1}\right) &= \begin{vmatrix} p_n(-1) & p_n(0) \\ p_{n+1}(-1) & p_{n+1}(0) \end{vmatrix} \\ &= \begin{vmatrix} W_n\left(-\frac{3}{2}\right) & (-1)^n \\ W_{n+1}\left(-\frac{3}{2}\right) & (-1)^{n+1} \end{vmatrix}. \end{aligned}$$

One can check by recurrence that $(-1)^n W_n\left(-\frac{3}{2}\right) = F_{2n+1}$, where $\{F_n\}$ are the Fibonnacci numbers. It follows that $\det\left((C_{i+j+1}+C_{i+j})_{0\le i,j\le n-1}\right) = F_{2n+1}$ and

$$\det\left((C_{i+j+2}+C_{i+j+1})_{0\le i,j\le n-1}\right) = F_{2n+3} - F_{2n+1} = F_{2n+2}.$$

This results is due to Cvetkovic, Rajkovic and Ivkovic [8]. On the other hand, it can be shown [22] that

$$p_n(x) = \sum_{k=0}^{n} (-1)^{n+k}\binom{n+k}{n-k}x^k$$

and then we obtain form the formula (1.3)

$$\det\left((C_{i+j+r})_{0\le i,j\le n-1}\right) = (-1)^{nr}\det\left((-1)^{n+i+j}\binom{n+i+j}{n+i-j}\right)_{0\le i,j\le r-1} = \det\left(\binom{n+i+j}{n+i-j}\right)_{0\le i,j\le r-1}. \tag{3.3}$$

Note that De Sainte-Catherine and Viennot [6] proved that

$$\det\left((C_{i+j+r})_{0\le i,j\le n-1}\right) = \prod_{1\le i\le j\le r-1}\frac{i+j+2n}{i+j}$$

so we obtain the following formula:

$$\det\left(\binom{n+i+j}{n+i-j}\right)_{0\le i,j\le r-1} = \prod_{1\le i\le j\le r-1}\frac{i+j+2n}{i+j}.$$

By applying the Theorem 2 using the value of $p_n(1) = W_n\left(-\frac{1}{2}\right)$ where

$$\begin{aligned} W_n\left(-\frac{1}{2}\right) &= W_n\left(\cos\left(\frac{2\pi}{3}\right)\right) \\ &= \frac{\sin(n+1/2)\frac{2\pi}{3}}{\sin\left(\frac{2\pi}{3}\right)} \\ &= \frac{2}{\sqrt{3}}\sin\frac{(2n+1)\pi}{3}, \end{aligned}$$

we obtain the following theorem:

Theorem 3. *We have.*

(1)

$$\det\left((C_{i+j+r} - C_{i+j+r-1})_{0\le i,j\le n-1}\right) = \frac{(-1)^{n+r-1}2}{\sqrt{3}} \tag{3.4}$$
$$\times\det\begin{pmatrix} \binom{n}{n} & \binom{n+1}{n-1} & \cdots & \binom{n+r-2}{n-r+2} & \sin\frac{(2n+1)\pi}{3} \\ \binom{n+1}{n+1} & \binom{n+2}{n} & \cdots & \binom{n+r-1}{n-r+3} & -\sin\frac{(2n+3)\pi}{3} \\ \binom{n+2}{n+2} & \binom{n+3}{n+1} & \cdots & \binom{n+r}{n-r+4} & \sin\frac{(2n+1)\pi}{3} \\ \vdots & \vdots & & \vdots & \vdots \\ \binom{n+r-1}{n+r-1} & \binom{n+r}{n+r-2} & \cdots & \binom{n+2r-3}{n+3} & (-1)^{r-1}\sin\frac{(2n+2r-1)\pi}{3} \end{pmatrix}.$$

(2)

$$\det\left((C_{i+j+r} + C_{i+j+r-1})_{0\le i,j\le n-1}\right) = \tag{3.5}$$
$$\det\begin{pmatrix} \binom{n}{n} & \binom{n+1}{n-1} & \cdots & \binom{n+r-2}{n-r+2} & F_{2n+1} \\ \binom{n+1}{n+1} & \binom{n+2}{n} & \cdots & \binom{n+r-1}{n-r+3} & F_{2n+3} \\ \binom{n+2}{n+2} & \binom{n+3}{n+1} & \cdots & \binom{n+r}{n-r+4} & F_{2n+5} \\ \vdots & \vdots & & \vdots & \vdots \\ \binom{n+r-1}{n+r-1} & \binom{n+r}{n+r-2} & \cdots & \binom{n+2r-3}{n+3} & F_{2n+2r-1} \end{pmatrix}$$

where F_n are the Fibonnacci numbers.

Remark 2. *Krattenthaler provided a formula for* $\det\left(\left(C_{\beta_i+j}+C_{\beta_{i+1}+j}\right)_{0\le i,j\le n-1}\right)$ *, where* $\beta_0,\beta_1,\dots,\beta_n$ *are non-negative integers* [13].

Example 3. *We have the following formulae:*

$$\det\left(\left(C_{i+j+3}-C_{i+j+2}\right)_{0\le i,j\le n-1}\right)=\begin{cases}(-1)^k & \text{if} \quad n=3k\\ (-1)^k\,3\,(k+1) & \text{if} \quad n=3k+1 \text{ or } n=3k+2\end{cases}$$

and

$$\det\left(\left(C_{i+j+4}-C_{i+j+3}\right)_{0\le i,j\le n-1}\right)=\begin{cases}(-1)^{k+1}\,\dfrac{(k+1)(3k+2)(6k-1)}{2} & \text{if} \quad n=3k\\ (-1)^k\,9\,(k+1)^2 & \text{if} \quad n=3k+1\\ (-1)^k\,\dfrac{(k+1)(3k+4)(6k+13)}{2} & \text{if} \quad n=3k+2\end{cases}$$

3.2. **Hankel determinants of Motzkin numbers.** The Motzkin numbers M_n count the number of Motzkin paths of length n, which are the lattice paths in the plane $\mathbb{Z}\times\mathbb{Z}$ from $(0,0)$ to $(n,0)$ using steps $U=(1,1), D=(1,-1), L=(1,0)$ that never pass below the x-axis. The t-Motzkin numbers $\{M_n^t\}$ is a generalization of these numbers, thus $M_n=M_n^1$ [5]. The evaluation of Hankel determinants with Motzkin numbers are performed in [5], for instance we have $\det\left(\left(M_{i+j}\right)_{0\le i,j\le n-1}\right)=1$ and Cameron and Yip [5] had evaluated by combinatorial methods $\det\left(\left(M_{i+j+r}^t+M_{i+j+r-1}^t\right)_{0\le i,j\le n-1}\right)$ and $\det\left(\left(M_{i+j+r}^t\right)_{0\le i,j\le n-1}\right)$ for $r=1,2$.If $f(x)=\sum_{n=0}^{+\infty}M_n^t x^n$ then

$$f(x)=1+txf(x)+x^2f(x)^2,$$

and by consequent the corresponding monic orthogonals polynomials satisfy the three-term recurrence:

$$p_{n+1}(x)=(x-t)p_n(x)-p_{n-1}(x),\ \text{for } n\ge 1,$$

with $p_{n0}(x)=1$ and $p_1(x)=x-t$. We obtain

$$p_n(x)=U_n\left(\frac{x-t}{2}\right)$$

where U_n is the Chebyshev polynomial of the second kind [29]. After this we get by the Theorem 2:

$$\det\left(\left(M_{i+j+1}^t\right)_{0\le i,j\le n-1}\right)=(-1)^n\,U_n\left(\frac{-t}{2}\right)=U_n\left(\frac{t}{2}\right)$$

$$\begin{aligned}\det\left(\left(M_{i+j+2}^t\right)_{0\le i,j\le n-1}\right) &= \begin{vmatrix} p_n(0) & p_n'(0)\\ p_{n+1}(0) & p_{n+1}'(0)\end{vmatrix}\\ &= \frac{1}{2}\begin{vmatrix} U_n\left(\frac{-t}{2}\right) & U_n'\left(\frac{-t}{2}\right)\\ U_{n+1}\left(\frac{-t}{2}\right) & U_{n+1}'\left(\frac{-t}{2}\right)\end{vmatrix}\end{aligned}$$

$$\det\left(\left(M_{i+j+1}^t+M_{i+j}^t\right)_{0\le i,j\le n-1}\right)=(-1)^n\,U_n\left(\frac{-1-t}{2}\right)=U_n\left(\frac{1+t}{2}\right)$$

and

$$\begin{aligned}\det\left(\left(M^t_{i+j+2}+M^t_{i+j+1}\right)_{0\le i,j\le n-1}\right) &= \mathcal{K}^{(2)}_{n,P}(-1,0)\\ &= \begin{vmatrix} U_n\left(\frac{-1-t}{2}\right) & U_n\left(\frac{-t}{2}\right)\\ U_{n+1}\left(\frac{-1-t}{2}\right) & U_{n+1}\left(\frac{-t}{2}\right)\end{vmatrix}.\end{aligned}$$

This is in agreement with [5]. Furthermore:

$$\begin{aligned}\det\left(\left(M^t_{i+j+3}+M^t_{i+j+2}\right)_{0\le i,j\le n-1}\right) &= \begin{vmatrix} p_n(0) & p'_n(0) & p_n(-1)\\ p_{n+1}(0) & p'_{n+1}(0) & p_n(-1)\\ p_{n+2}(0) & p'_{n+2}(0) & p_n(-1)\end{vmatrix}\\ &= \frac{1}{2}\begin{vmatrix} U_n\left(\frac{-t}{2}\right) & U'_n\left(\frac{-t}{2}\right) & U_n\left(\frac{-1-t}{2}\right)\\ U_{n+1}\left(\frac{-t}{2}\right) & U'_{n+1}\left(\frac{-t}{2}\right) & U_{n+1}\left(\frac{-1-t}{2}\right)\\ U_{n+2}\left(\frac{-t}{2}\right) & U'_{n+2}\left(\frac{-t}{2}\right) & U_{n+2}\left(\frac{-1-t}{2}\right)\end{vmatrix}.\end{aligned}$$

Since we have

$$U'_k(x)=\frac{xU_k(x)-(k+1)\,T_{k+1}(x)}{1-x^2}, \tag{3.6}$$

where T_{k+1} is the Chebyshev polynomial of the first kind [29], then we obtain finally

$$\det\left(\left(M^t_{i+j+3}+M^t_{i+j+2}\right)_{0\le i,j\le n-1}\right)=\frac{-2}{4-t^2}\begin{vmatrix} U_n\left(\frac{-t}{2}\right) & (n+1)\,T_{n+1}\left(\frac{-t}{2}\right) & U_n\left(\frac{-1-t}{2}\right)\\ U_{n+1}\left(\frac{-t}{2}\right) & (n+2)\,T_{n+2}\left(\frac{-t}{2}\right) & U_{n+1}\left(\frac{-1-t}{2}\right)\\ U_{n+2}\left(\frac{-t}{2}\right) & (n+3)\,T_{n+3}\left(\frac{-t}{2}\right) & U_{n+2}\left(\frac{-1-t}{2}\right)\end{vmatrix}.$$

Let us put $t=1$. We will give a method to compute explicitly the first coefficients of the polynomial p_n. It is well known that the polynomial U_n satisfies the second order linear differential equation:

$$\left(1-x^2\right)U''_n(x)-3xU'_n(x)+n(n+2)\,U_n(x)=0. \tag{3.7}$$

If we note for $k\in\mathbb{N}$, $P_{n,k}(x)=U_n^{(k)}(x)$ then differentiating (3.7) up to order $k-1$, $k\ge 1$, we obtain

$$\begin{aligned}&\left(1-x^2\right)P_{n,k+1}(x)-2(k-1)\,xP_{n,k}(x)+2\binom{k-1}{2}P_{n,k-1}(x)\\ &-3xP_{n,k}(x)-3(k-1)\,P_{n,k-1}(x)+n(n+2)\,P_{n,k-1}(x)=0,\end{aligned}$$

which can be transformed into the three term recurrence

$$\left(1-x^2\right)P_{n,k+1}(x)-(2k+1)\,xP_{n,k}(x)+(n+k+1)(n-k+1)\,P_{n,k-1}(x)=0. \tag{3.8}$$

We denote by $a_{n,k}=\dfrac{p_n^{(k)}(0)}{k!}=\dfrac{1}{2^k\times k!}U_n^{(k)}\left(\frac{-1}{2}\right)$. Thus $\{a_{n,k}\}_k$ verifies the three term recurrence

$$3a_{n,k+1}+\frac{2k+1}{k+1}a_{n,k}+\frac{(n+k+1)(n-k+1)}{k(k+1)}a_{n,k-1}=0.$$

The initial values can be computed as follows:

$$a_{n,0}=U_n\left(-\frac{1}{2}\right)=U_n\left(\cos\left(\frac{2\pi}{3}\right)\right)=\frac{2}{\sqrt{3}}\sin\left(\frac{2(n+1)\pi}{3}\right)$$

where we have used the relation

$$\sin(\theta)\,U_n(\cos(\theta))=\sin((n+1)\,\theta) \tag{3.9}$$

By using (3.6) we get

$$a_{n,1} = \frac{1}{2} U_n' \left(-\frac{1}{2} \right) = -\frac{1}{3} a_{n,0} - \frac{2(n+1)}{3} \cos \left(\frac{2(n+1)\pi}{3} \right).$$

Using the expressions for $p_n(1)$ and $p_n(-1)$:

$$\begin{aligned} p_n(1) &= U_n(0) = \sin \left(\frac{(n+1)\pi}{2} \right) \\ p_n(-1) &= U_n(-1) = (-1)^n (n+1), \end{aligned}$$

we obtain the following theorem:

Theorem 4. *Let for $n \in \mathbb{N}$ the sequence $\{a_{n,k}\}_k$ that verifies the three term recurrence*

$$3a_{n,k+1} + \frac{2k+1}{k+1} a_{n,k} + \frac{(n+k+1)(n-k+1)}{k(k+1)} a_{n,k-1} = 0.$$

with the initial conditions $a_{n,0} = \frac{2}{\sqrt{3}} \sin \left(\frac{2(n+1)\pi}{3} \right)$ and

$$a_{n,1} = -\frac{1}{3} a_{n,0} - \frac{2(n+1)}{3} \cos \left(\frac{2(n+1)\pi}{3} \right).$$

Then we have

(1)

$$\begin{aligned} &\det \left((M_{i+j+r} + M_{i+j+r-1})_{0 \le i,j \le n-1} \right) \\ = \ & (-1)^{(n+1)(r+1)} \det \begin{pmatrix} a_{n,0} & a_{n,1} & \cdots & a_{n,r-2} & (n+1) \\ a_{n+1,0} & a_{n+1,1} & \cdots & a_{n+1,r-2} & -(n+2) \\ a_{n+2,0} & a_{n+2,1} & \cdots & a_{n+2,r-2} & (n+3) \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n+r-1,0} & a_{n+r-1,1} & \cdots & a_{n+r-1,r-2} & (-1)^{r-1}(n+r) \end{pmatrix} \end{aligned}$$

(2)

$$\begin{aligned} &\det \left((M_{i+j+r} - M_{i+j+r-1})_{0 \le i,j \le n-1} \right) \\ = \ & (-1)^{nr} \det \begin{pmatrix} a_{n,0} & a_{n,1} & \cdots & a_{n,r-2} & \sin\left(\frac{(n+1)\pi}{2}\right) \\ a_{n+1,0} & a_{n+1,1} & \cdots & a_{n+1,r-2} & \sin\left(\frac{(n+2)\pi}{2}\right) \\ a_{n+2,0} & a_{n+2,1} & \cdots & a_{n+2,r-2} & \sin\left(\frac{(n+3)\pi}{2}\right) \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n+r-1,0} & a_{n+r-1,1} & \cdots & a_{n+r-1,r-2} & \sin\left(\frac{(n+r)\pi}{2}\right) \end{pmatrix} \end{aligned}$$

(3)

$$\det \left((M_{i+j+r})_{0 \le i,j \le n-1} \right) = (-1)^{nr} \det \begin{pmatrix} a_{n,0} & a_{n,1} & \cdots & a_{n,r-1} \\ a_{n+1,0} & a_{n+1,1} & \cdots & a_{n+1,r-1} \\ a_{n+2,0} & a_{n+2,1} & \cdots & a_{n+2,r-1} \\ \vdots & \vdots & & \vdots \\ a_{n+r-1,0} & a_{n+r-1,1} & \cdots & a_{n+r-1,r-1} \end{pmatrix}.$$

Example 4. *By using the Maple software, we obtained the following formulae which some are new:*

$$\det\left((M_{i+j+2})_{0\le i,j\le n-1}\right) = \begin{cases} 2k+1 & \text{if} \quad n=3k \\ 2k+2 & \text{if} \quad n=3k+1 \quad \text{or } n=3k+2 \end{cases}.$$

$$\det\left((M_{i+j+3})_{0\le i,j\le n-1}\right) = \begin{cases} (-1)^k(k+1)(2k+1) & \text{if} \quad n=3k \\ (-)^k 4(k+1)^2 & \text{if} \quad n=3k+1 \\ (-1)^k(k+1)(2k+3) & \text{if} \quad n=3k+2 \end{cases}$$

$$\det\left((M_{i+j+4})_{0\le i,j\le n-1}\right) = \begin{cases} (k+1)^2(2k+1)^2 & \text{if} \quad n=3k \\ (2k+3)^2(k+1)^2 & \text{if} \quad n=3k+1 \\ (k+2)(k+1)(2k+3)^2 & \text{if} \quad n=3k+2 \end{cases}$$

$$\det\left((M_{i+j+2}+M_{i+j+1})_{0\le i,j\le n-1}\right) = \begin{cases} (-1)^k & \text{if} \quad n=3k \\ (-1)^k 3(k+1) & \text{if} \quad n=3k+1 \quad \text{or } n=3k+2 \end{cases}$$

$$\det\left((M_{i+j+3}+M_{i+j+2})_{0\le i,j\le n-1}\right) = \begin{cases} (k+1)(6k+1) & \text{if} \quad n=3k \\ 6(k+1)^2 & \text{if} \quad n=3k+1 \\ (6k+11)(k+1) & \text{if} \quad n=3k+2 \end{cases}$$

$$\det\left((M_{i+j+4}+M_{i+j+3})_{0\le i,j\le n-1}\right) = \begin{cases} (-1)^{k+1}(6k-1)(k+1)^2 & \text{if} \quad n=3k \\ (-1)^k(6k+13)(k+1)^2 & \text{if} \quad n=3k+1 \\ (-1)^k(2k+3)(k+2)(k+1) & \text{if} \quad n=3k+2 \end{cases}$$

3.3. **Hankel determinants of Schroder numbers.** The (large) Schroder numbers R_n count the number of large Schroder paths of length n, which are the paths in the plane $\mathbb{Z}\times\mathbb{Z}$ from $(0,0)$ to $(2n,0)$ using $U=(1,1), D=(1,-1), L=(2,0)$ that never pass below the x-axis. Many determinants evaluation with Schroder numbers are known [4, 9, 24], for example we have $\det\left((R_{i+j})_{0\le i,j\le n-1}\right)=2^{\binom{n}{2}}$.

Let $f(x)=\sum_{n=0}^{+\infty} R_n x^n$ then

$$f(x)=1+xf(x)+xf(x)^2.$$

The monic orthogonal polynomials associated to the sequences $\{R_n\}$ satisfy satisfies the three-term recurrence:

$$p_n(x)=(x-3)\,p_{n-1}(x)-2p_{n-2}(x) \quad \text{for } n\ge 2, \tag{3.10}$$

with $p_0(x)=1$ and $p_1(x)=x-2$. We get easily the following values:

$$\begin{aligned} p_n(0) &= (-2)^n \\ p_n(1) &= (-1)^n 2^{\frac{n}{2}}\cos\left(\frac{n\pi}{4}\right) \\ p_n(-1) &= \frac{(-1)^n}{4}\left(\left(2+\sqrt{2}\right)^{n+1}+\left(2-\sqrt{2}\right)^{n+1}\right), \end{aligned}$$

from which follows immediately the formulas:

$$\begin{aligned}
\det\left((R_{i+j+1})_{0\le i,j\le n-1}\right) &= (-1)^n\, 2^{\binom{n}{2}} p_n(0) = 2^{\binom{n+1}{2}},\\
\det\left((R_{i+j+1}-R_{i+j})_{0\le i,j\le n-1}\right) &= (-1)^n\, 2^{\binom{n}{2}} p_n(1)\\
&= 2^{\binom{n}{2}+\frac{n}{2}} \cos\left(\frac{n\pi}{4}\right)\\
\det\left((R_{i+j+1}+R_{i+j})_{0\le i,j\le n-1}\right) &= (-1)^n\, 2^{\binom{n}{2}} p_n(-1)\\
&= 2^{\binom{n}{2}-2}\left(\left(2+\sqrt{2}\right)^{n+1}+\left(2-\sqrt{2}\right)^{n+1}\right).
\end{aligned}$$

The polynomial p_n seems to have a complex form to be of any use. Fortunately, this is not the case of the shifted sequence $\{R_{n+1}\}_n$. Indeed, if g is the generating function of the sequence $\{R_{n+1}\}_n$, then $g(x) = \dfrac{f(x)-1}{x}$ and verifies

$$g(x) = 2 + 3xg(x) + x^2 g(x)^2.$$

Therefore, the monic orthogonal polynomials $\{q_n\}$ associated with $\{R_{n+1}\}$ verify

$$q_n(x) = (x-3)\, q_{n-1}(x) - 2q_{n-2}(x) \quad \text{for } n \ge 2,$$

with $q_0(x) = 1$ and $q_1(x) = x-3$. This give

$$q_n(x) = \sqrt{2}^n U_n\left(\frac{x-3}{2\sqrt{2}}\right).$$

The values $q_n^{(k)}(0)$ can been computed as for the Motzkin case. Let us denotes by

$$a_{n,k} = \frac{q_n^{(k)}(0)}{k!} = \frac{\sqrt{2}^{n-k}}{2^k \times k!} U_n^{(k)}\left(\frac{-3}{2\sqrt{2}}\right).$$

Through the relation (3.8), $U_n^{(k)}\left(\frac{-3}{2\sqrt{2}}\right)$ verifies the three term recurrence

$$\frac{-1}{8} U_n^{(k+1)}\left(\frac{-3}{2\sqrt{2}}\right) + \frac{3(2k+1)}{2\sqrt{2}} U_n^{(k)}\left(\frac{-3}{2\sqrt{2}}\right) + (n+k+1)(n-k+1)\, U_n^{(k-1)}\left(\frac{-3}{2\sqrt{2}}\right) = 0\,,$$

for $k \ge 1$.Hence for $k \ge 1$:

$$-a_{n,k+1} + \frac{3(2k+1)}{k+1} a_{n,k} + \frac{(n+k+1)(n-k+1)}{k(k+1)} a_{n,k-1} = 0.$$

Using the fact that for $|x| > 1$:

$$U_n(x) = \frac{\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}}{2\sqrt{x^2-1}}$$

and

$$T_n(x) = \frac{\left(x+\sqrt{x^2-1}\right)^{n} + \left(x-\sqrt{x^2-1}\right)^{n}}{2},$$

one can obtain the initial values $a_{n,0}$ and $a_{n,1}$ as follows:

$$\begin{aligned} a_{n,0} &= \sqrt{2}^n U_n\left(\frac{-3}{2\sqrt{2}}\right) \\ &= \sqrt{2}^n \frac{\left(\frac{-3}{2\sqrt{2}}+\sqrt{\left(\frac{-3}{2\sqrt{2}}\right)^2-1}\right)^{n+1}-\left(\frac{-3}{2\sqrt{2}}-\sqrt{\left(\frac{-3}{2\sqrt{2}}\right)^2-1}\right)^{n+1}}{2\sqrt{\left(\frac{-3}{2\sqrt{2}}\right)^2-1}} \\ &= (-1)^n\left(2^{n+1}-1\right), \end{aligned}$$

and

$$\begin{aligned} a_{n,1} &= \frac{\sqrt{2}^{n-1}}{2}U_n'\left(\frac{-3}{2\sqrt{2}}\right) \\ &= \frac{\sqrt{2}^{n-1}}{2}\frac{\frac{-3}{2\sqrt{2}}U_n\left(\frac{-3}{2\sqrt{2}}\right)-(n+1)\,T_{n+1}\left(\frac{-3}{2\sqrt{2}}\right)}{1-\frac{9}{8}} \\ &= -4\sqrt{2}^{n-1}\left(\frac{-3}{2\sqrt{2}}U_n\left(\frac{-3}{2\sqrt{2}}\right)-(n+1)\,T_{n+1}\left(\frac{-3}{2\sqrt{2}}\right)\right) \end{aligned}$$

with

$$\begin{aligned} T_{n+1}\left(\frac{-3}{2\sqrt{2}}\right) &= \frac{\left(\frac{-3}{2\sqrt{2}}+\sqrt{\left(\frac{-3}{2\sqrt{2}}\right)^2-1}\right)^{n+1}+\left(\frac{-3}{2\sqrt{2}}-\sqrt{\left(\frac{-3}{2\sqrt{2}}\right)^2-1}\right)^{n+1}}{2} \\ &= \frac{(-1)^{n+1}}{2\sqrt{2}^{n+1}}\left(2^{n+1}+1\right). \end{aligned}$$

Finally we get

$$\begin{aligned} a_{n,1} &= -4\sqrt{2}^{n-1}\frac{-3}{2\sqrt{2}}U_n\left(\frac{-3}{2\sqrt{2}}\right)+(n+1)\,4\sqrt{2}^{n-1}\frac{(-1)^{n+1}}{2\sqrt{2}^{n+1}}\left(1+2^{n+1}\right) \\ &= 3a_{n,0}+(-1)^{n+1}(n+1)\left(1+2^{n+1}\right). \end{aligned}$$

Since we have

$$\begin{aligned} q_n(1) &= \sqrt{2}^n U_n\left(\frac{-1}{\sqrt{2}}\right) \\ &= (-1)^n\sqrt{2}^n U_n\left(\cos\frac{\pi}{4}\right) \\ &= (-1)^n\sqrt{2}^{n+1}\sin\left(\frac{(n+1)\,\pi}{4}\right), \end{aligned}$$

then we obtain:

Theorem 5. *Let for $n\in\mathbb{N}$ the sequence $\{a_{n,k}\}_k$ that verifies the three term recurrence*

$$-a_{n,k+1}+\frac{3\,(2k+1)}{k+1}a_{n,k}+\frac{(n+k+1)\,(n-k+1)}{k\,(k+1)}a_{n,k-1}=0 \text{ for } k\geq 1,$$

with the initial conditions $a_{n,0}=(-1)^n\left(2^{n+1}-1\right)$ and

$$a_{n,1}=3a_{n,0}+(-1)^{n+1}(n+1)\left(2^{n+1}+1\right).$$

Then we have

(1)

$$\begin{aligned}
&\det\left((R_{i+j+r+1})_{0\le i,j\le n-1}\right)\\
=\;& (-1)^{nr}2^{\binom{n+1}{2}}\det\begin{pmatrix} a_{n,0} & a_{n,1} & \cdots & a_{n,r-1}\\ a_{n+1,0} & a_{n+1,1} & \cdots & a_{n+1,r-1}\\ a_{n+2,0} & a_{n+2,1} & \cdots & a_{n+2,r-1}\\ \vdots & \vdots & & \vdots\\ a_{n+r-1,0} & a_{n+r-1,1} & \cdots & a_{n+r-1,r-1}\end{pmatrix}.
\end{aligned}$$

(2)

$$\det\left((R_{i+j+r+2}-R_{i+j+r+1})_{0\le i,j\le n-1}\right) = (-1)^{nr+n}2^{\binom{n+1}{2}}\times$$

$$\det\begin{pmatrix} a_{n,0} & a_{n,1} & \cdots & a_{n,r-2} & \sqrt{2}^{n+1}\sin\left(\frac{(n+1)\pi}{4}\right)\\ a_{n+1,0} & a_{n+1,1} & \cdots & a_{n+1,r-2} & -\sqrt{2}^{n+2}\sin\left(\frac{(n+2)\pi}{4}\right)\\ a_{n+2,0} & a_{n+2,1} & \cdots & a_{n+2,r-2} & \sqrt{2}^{n+3}\sin\left(\frac{(n+2)\pi}{4}\right)\\ \vdots & \vdots & & \vdots & \\ a_{n+r-1,0} & a_{n+r-1,1} & \cdots & a_{n+r-1,r-2} & (-1)^{r-1}\sqrt{2}^{n+r}\sin\left(\frac{(n+r)\pi}{4}\right)\end{pmatrix}.$$

Example 5. *Here some examples computed with the Maple software:*

$$\begin{aligned}
\det\left((R_{i+j+2})_{0\le i,j\le n-1}\right) &= 2^{\binom{n+1}{2}}\left(2^{n+1}-1\right)\\
\det\left((R_{i+j+3})_{0\le i,j\le n-1}\right) &= 2^{\binom{n+1}{2}}\left(2^{2n+3}-(4n+6)2^{n}-1\right)
\end{aligned}$$

$$\det\left((R_{i+j+3}-R_{i+j+2})_{0\le i,j\le n-1}\right) = \begin{cases} (-1)^k\, 2^{8k^2+4k} & \textit{if} \quad n=4k\\ (-)^k\, 2^{8k^2+8k+2} & \textit{if} \quad n=4k+1\\ (-)^k\, 2^{8k^2+12k+4} & \textit{if} \quad n=4k+2\\ 0 & \textit{if} \quad n=4k+3\end{cases}$$

$$\begin{aligned}
&\det\left((R_{i+j+4}-R_{i+j+3})_{0\le i,j\le n-1}\right)\\
=&\begin{cases} (-1)^k\, 2^{8k^2+4k} & \textit{if} \quad n=4k\\ (-)^k\, 2^{8k^2+12k+4} & \textit{if} \quad n=4k+1\\ (-)^k\, 2^{8k^2+12k+4}\left(2^{k+1}-1\right)\left(2^{k+1}+1\right)\left(2^{2k+2}+1\right) & \textit{if} \quad n=4k+2\\ (-)^k\, 2^{8k^2+12k+8}\left(2^{k+1}-1\right)\left(2^{k+1}+1\right)\left(2^{2k+2}+1\right) & \textit{if} \quad n=4k+3\end{cases}
\end{aligned}$$

4. Hankel Transforms of Linear Combinations of Catalan and Motzkin Numbers

Many authors has carried out the computation of Hankel determinant of more than tow consecutives Catalan or Motzkin numbers [17, 20]. It is conjectured [20] that for Catalan numbers, the Hankel transform $\{h_n\}_n$ satisfies a homogeneous linear recurrence relation of order 2^r. We shall give a positive answer to this conjecture and we give explicitly the coefficients of such relation.

Theorem 6. *Let for* $n \in \mathbb{N}$

$$b_n = \sum_{k=0}^{r} \lambda_k C_{n+k}$$

where C_m *are Catalan numbers. Let us write for* $j = 1, \ldots, r$:

$$\frac{\alpha_j}{2} - 1 = \frac{1}{2}\left(\zeta_j + \frac{1}{\zeta_j}\right)$$

where $\zeta_j, \dfrac{1}{\zeta_j}$ *are assumed pairwise distinct. If*

$$B(z) = \prod_{J \subset \{1,\ldots,r\}} \left(1 - (-1)^r \omega_J z\right) = \sum_{k=0}^{2^r} \gamma_k z^k$$

where $\omega_J = \prod_{k=1}^{r} \zeta_k^{I_J(k)}$ *with* $I_J(k) = 1$ *if* $k \in J$, -1 *otherwise, then* $\{h_n\}_n$, $h_n = \det(\mathcal{H}_n(b))$, *satisfies the following homogeneous linear recurrence relation*

$$\sum_{k=0}^{2^r} \gamma_k h_{n+2^r-k} = 0$$

Proof. For $1 \leq i \leq r$ we have

$$\begin{aligned} p_{n+i-1}(\alpha_j) &= W_{n+i-1}\left(\frac{\alpha_j}{2} - 1\right) \\ &= W_{n+i-1}\left(\frac{1}{2}\left(\zeta_j + \frac{1}{\zeta_j}\right)\right) \\ &= \frac{\zeta_j^{(n+i-1)+1/2} - \zeta_j^{-(n+i-1)-1/2}}{\zeta_j^{1/2} - \zeta_j^{-1/2}}. \end{aligned}$$

Using the formula (1.1) and following the method in [28] we obtain

$$h_n = (-1)^{nr} K \sum_{J \subset \{1,2,\ldots,r\}} (-1)^{|J|} \gamma_J \omega_J^{n+1/2}$$

where

$$K = \prod_{1 \leq i < j \leq r} (\alpha_j - \alpha_i)^{-1} \prod_{k=1}^{r} \left(\zeta_k^{1/2} - \zeta_k^{-1/2}\right)^{-1},$$

and for $J \subset \{1, 2, \ldots, r\}$:

$$\omega_J = \prod_{k=1}^{r} \zeta_k^{I_J(k)} \quad \text{with } I_J(k) = \begin{cases} 1 & \text{if} \quad k \in J \\ -1 & \text{if} \quad k \notin J \end{cases},$$

and $\gamma_J = \prod_{1\leq j<k\leq r} \left(\zeta_k^{I_J(j)} - \zeta_j^{I_J(k)}\right)$.By consequent we obtain :

$$\begin{aligned} f(z) &= \sum_{n=0}^{+\infty} h_n z^n = K \sum_J (-1)^{|J|} \gamma_J \omega_J^{1/2} \left(\sum_{n=0}^{+\infty} ((-1)^r \omega_J z)^n\right) \\ &= K \sum_J \frac{(-1)^{|J|} \gamma_J \omega_J^{1/2}}{1-(-1)^r \omega_J z} \\ &= \frac{A(z)}{B(z)}, \end{aligned}$$

which show that $f(z)$ is a rational function. □

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