

On conjugacies between asymmetric Bernoulli shifts*

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Abstract

This paper investigates conjugacies between asymmetric Bernoulli shifts. It is shown that there exists a unique increasing conjugacy and a unique decreasing conjugacy. We respectively construct a sequence of functions to approximate these two conjugacies, and give an estimation for the error of the approximation. We also present explicit formulae of these two conjugacies. It is shown that these two conjugacies are singular, Hölder continuous and not differentiable.

Keywords: asymmetric Bernoulli shift; topological conjugacy; B_a -expansion; singular function; Hölder continuity.

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1 Introduction

One of the central questions in iteration theory or dynamical systems is to decide whether two self-maps $f : I \rightarrow I$ and $g : J \rightarrow J$ are topologically conjugate, i.e., whether there exists a homeomorphism $\varphi : I \rightarrow J$ such that $\varphi \circ f = g \circ \varphi$. Such a homeomorphism φ is called a topological conjugacy or *conjugacy* from f to g . In

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dynamical systems, topological conjugation defines an equivalence relation, which is useful in topological classification of systems.

The conjugacy problem first arose from linearization problem. There are many classic results, for instance, Denjoy's theory [7, 8, 16], Herman's theory [14, 19] for circle diffeomorphisms conjugate to a rigid rotation; Grobman-Hartman theorem [27] for topological linearization of local dynamical systems; Parry's theory [32, 25, 26] and Milnor-Thurston's theory [22] respectively for continuous piecewise monotone interval maps conjugate or semi-conjugate to a piecewise linear map. There are few works [1, 4, 5] on the conjugacy problem for piecewise continuous piecewise monotone interval maps. A Lorenz map, considered by Glendinning [13], is a piecewise continuous piecewise monotone interval map. He gave necessary and sufficient conditions for a Lorenz map to be conjugate to a linear mod one transformation, and the regularity of some conjugacies is recently investigated in [6].

Consider the asymmetric Bernoulli shift $B_a : [0, 1] \rightarrow [0, 1]$ with a parameter $0 < a < 1$, defined by

$$B_a = \begin{cases} \frac{x}{a}, & 0 \leq x \leq a, \\ \frac{x-a}{1-a}, & a < x \leq 1, \end{cases}$$

which is a simple piecewise continuous piecewise monotone interval map, but different from the class of Lorenz maps in [6]. Specially, if $a = 1/2$, then $B_{1/2}$ is the Bernoulli Shift, also known as *doubling map*. Palmore [23] proved that B_a is chaotic for $0 < a < 1$.

In this paper, we study these conjugacies from B_{a_1} to B_{a_2} where $a_1, a_2 \in (0, 1)$ and $a_1 \neq a_2$. The existence and uniqueness are proved for increasing and decreasing conjugacies in the next section. In Section 3, we respectively construct a sequence of functions to approximate the increasing and decreasing conjugacies, and give an estimation for the error of the approximation. Section 4 presents explicit formulae of these two conjugacies. In Section 5, we show that these conjugacies are singular, Hölder continuous, but not differentiable. In addition, we calculate the arc-length of a conjugacy curve and the area under a conjugacy curve.

2 Existence of conjugacies

First, we will give two necessary conditions of topological conjugation between B_{a_1} and B_{a_2} .

Lemma 2.1 *If there exists an increasing conjugacy φ from B_{a_1} to B_{a_2} , then:*

(i) $\varphi(0) = 0$, $\varphi(a_1) = a_2$, and $\varphi(1) = 1$;

(ii) φ is a solution of a system of functional equations

$$\varphi(x) = \begin{cases} a_2\varphi\left(\frac{x}{a_1}\right), & 0 \leq x \leq a_1, \\ (1 - a_2)\varphi\left(\frac{x-a_1}{1-a_1}\right) + a_2, & a_1 < x \leq 1. \end{cases} \quad (2.1)$$

Proof. (i) Substituting $x = 0$ into the equation $\varphi \circ f(x) = g \circ \varphi(x)$, we have $\varphi(0) = \varphi \circ f(0) = g \circ \varphi(0)$. Then $\varphi(0)$ is a fixed point of g . So $\varphi(0) = 0$ or 1 . Since φ is strictly increasing, we have $\varphi(0) = 0$. Similarly, one has the fact $\varphi(1) = 1$.

Now we prove that $\varphi(a_1) = a_2$ by contradiction. Assume that there exists a point $x_0 \neq a_1$ such that $\varphi(x_0) = a_2$. Then there is a small enough neighborhood of x_0 , denoted by $O(x_0, \delta)$, such that $a_1 \notin O(x_0, \delta)$. On the one hand, it is clear that $\varphi \circ f(x)$ is continuous on $O(x_0, \delta)$. On the other hand, one can see that $g \circ \varphi(x)$ is not continuous on $O(x_0, \delta)$ since $a_2 \in \varphi(O(x_0, \delta))$. This contradicts the equation $\varphi \circ f(x) = g \circ \varphi(x)$. Therefore $\varphi(a_1) = a_2$.

(ii) If φ is an increasing conjugacy from B_{a_1} to B_{a_2} , then $\varphi([0, a_1]) = [0, a_2]$ and $\varphi([a_1, 1]) = [a_2, 1]$. Thus

$$\begin{aligned} \varphi\left(\frac{x}{a_1}\right) &= \frac{\varphi(x)}{a_2}, & \text{if } 0 \leq x \leq a_1, \\ \varphi\left(\frac{x-a_1}{1-a_1}\right) &= \frac{\varphi(x) - a_2}{1-a_2}, & \text{if } a_1 < x \leq 1. \end{aligned}$$

Consequently, φ is a solution of a system (2.1). \square

Remark that when $a_1 = 1/2$ and $a_2 = a$, the solution of the system (2.1) is de Rham's function, which is extensively investigated by many authors, i.e., [2, 3, 9, 20].

With the similar argument to Lemma 2.1, we have a similar result for the decreasing conjugacy.

Lemma 2.2 *If there exists a decreasing conjugacy φ from B_{a_1} to B_{a_2} , then:*

(i) $\varphi(0) = 1$, $\varphi(a_1) = a_2$, and $\varphi(1) = 0$;

(ii) φ is a solution of a system of functional equations

$$\varphi(x) = \begin{cases} (1 - a_2)\varphi\left(\frac{x}{a_1}\right) + a_2, & 0 \leq x \leq a_1, \\ a_2\varphi\left(\frac{x-a_1}{1-a_1}\right), & a_1 < x \leq 1. \end{cases} \quad (2.2)$$

Theorem 2.1 *For any pair of $a_1 \neq a_2$ in $(0, 1)$, there exists a unique increasing (decreasing, respectively), continuous and surjective map $\varphi : [0, 1] \rightarrow [0, 1]$ such that $\varphi \circ B_{a_1} = B_{a_2} \circ \varphi$.*

Proof. Now we only consider the decreasing case. The proof for the increasing case follows from [5, Theorem 1].

Denote by $C_{bd}^0([0, 1])$ the Banach space of bounded, decreasing and continuous maps $\varphi : [0, 1] \rightarrow [0, 1]$, $\varphi(1) = 0$, and $\varphi(0) = 1$ endowed with the norm

$$\|\varphi\| = \sup_{x \in [0, 1]} |\varphi(x)|.$$

Denote by \mathcal{T} the map from the right of (2.2). It acts in the space $C_{bd}^0([0, 1])$. Moreover,

$$\|\mathcal{T}\varphi - \mathcal{T}\psi\| \leq \max\{1 - a_2, a_2\} \|\varphi - \psi\|.$$

Since $\max\{1 - a_2, a_2\} < 1$, the map \mathcal{T} is a contraction in $C_{bd}^0([0, 1])$. Therefore, (2.2) has a unique solution $\varphi \in C_{bd}^0([0, 1])$. \square

In Section 4, one can see that this map φ is a homomorphism, thus φ is the unique decreasing conjugacy from B_{a_1} to B_{a_2} .

3 Approximation of conjugacies

In order to find the increasing and decreasing conjugacies, we construct two iterated function systems for $n = 0, 1, \dots$

$$\varphi_{n+1}(x) = \begin{cases} a_2 \varphi_n\left(\frac{x}{a_1}\right), & 0 \leq x \leq a_1, \\ (1 - a_2) \varphi_n\left(\frac{x - a_1}{1 - a_1}\right) + a_2, & a_1 < x \leq 1 \end{cases} \quad (3.3)$$

and

$$\varphi_{n+1}(x) = \begin{cases} (1 - a_2) \varphi_n\left(\frac{x}{a_1}\right) + a_2, & 0 \leq x \leq a_1, \\ a_2 \varphi_n\left(\frac{x - a_1}{1 - a_1}\right), & a_1 < x \leq 1. \end{cases} \quad (3.4)$$

for all $x \in [0, 1]$.

For example, given $a_1 = \frac{7}{16}$, $a_2 = \frac{7}{8}$, φ_n 's are calculated recursively by (3.3) and (3.4) with an initial function $\varphi_0(x) = x$ and $\varphi_0(x) = 1 - x$, respectively. See Fig.1 and Fig.2 .

In what follows, we only consider the system (3.4). The other system (3.3) is similar.

Put $\lambda := \max\{a_2, 1 - a_2\}$. It is obvious that $0 < \lambda < 1$.

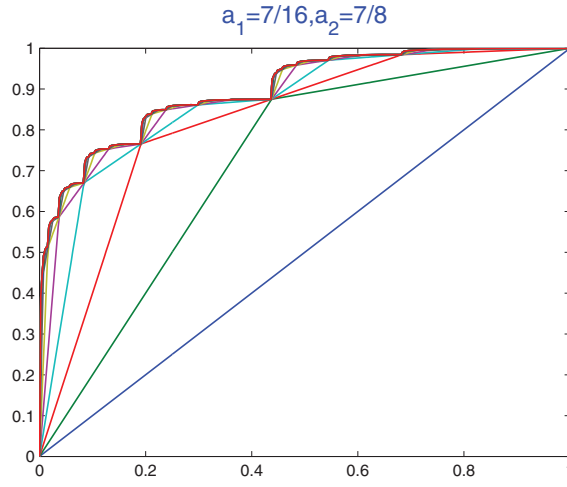


Fig. 1: These increasing conjugacies from $B_{7/16}$ to $B_{7/8}$ are calculated by (3.3) with the initial function $\varphi_0(x) = x$ for all $x \in [0, 1]$.

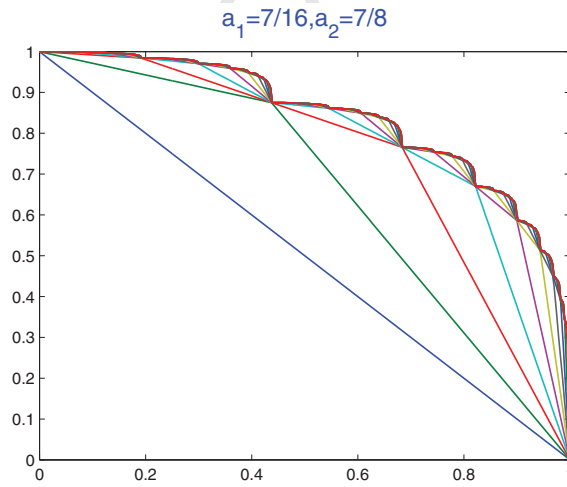


Fig. 2: These decreasing approximate conjugacies from $B_{7/16}$ to $B_{7/8}$ are calculated by (3.4) with the initial function $\varphi_0(x) = 1 - x$ for all $x \in [0, 1]$.

Lemma 3.1 Suppose φ_n is defined by (3.4) with an initial function $\varphi_0(x) = 1 - x$. Then

(i) φ_n is a continuous and strictly decreasing map from $[0, 1]$ onto $[0, 1]$ with

$$\varphi_n(1) = 0, \quad \varphi_n(0) = 1, \quad \forall n \in \mathbb{N}$$

(ii) $|\varphi_m(x) - \varphi_n(x)| \leq \frac{\lambda^n - \lambda^m}{1 - \lambda} |a_1 - (1 - a_2)|, \quad \forall x \in [0, 1], m > n \geq 0;$

Proof. (i) Since $\varphi_0(0) = 1$ and $\varphi_0(1) = 0$, we have by induction $\varphi_n(0) = (1 - a_2)\varphi_{n-1}(0) + a_2 = 1$, and $\varphi_n(1) = a_2\varphi_{n-1}(1) = 0$ for each $n \in \mathbb{N}$.

We shall prove by induction that φ_n is continuous and strictly decreasing. Assume that φ_k is a continuous and strictly decreasing map from $[0, 1]$ onto itself. By (3.4), φ_{k+1} is strictly decreasing on both the interval $[0, a_1]$ and $[a_1, 1]$. Thus φ_{k+1} is strictly decreasing on $[0, 1]$. Now it suffices to show that φ_{k+1} is continuous at the point $x = a_1$. By (3.4), we have

$$\begin{aligned} \varphi_{k+1}(a_1 - 0) &= (1 - a_2)\varphi_k(1) + a_2 = a_2, \\ \varphi_{k+1}(a_1 + 0) &= a_2\varphi_k(0) = a_2. \end{aligned}$$

Thus φ_{k+1} is continuous on the domain $[0, 1]$.

(ii) By induction, we have

$$|\varphi_{n+1}(x) - \varphi_n(x)| \leq \lambda^n |a_1 - (1 - a_2)|, \quad \forall x \in [0, 1].$$

Thus

$$\begin{aligned} |\varphi_m(x) - \varphi_n(x)| &= |(\varphi_m(x) - \varphi_{m-1}(x)) + \cdots + (\varphi_{n+1}(x) - \varphi_n(x))| \\ &\leq (\lambda^{m-1} + \cdots + \lambda^n) |a_1 - (1 - a_2)| \\ &= \frac{\lambda^n - \lambda^m}{1 - \lambda} |a_1 - (1 - a_2)|. \end{aligned} \tag{3.5}$$

□

Theorem 3.1 Suppose φ_n is defined by (3.4) and φ is the unique decreasing and continuous solution $\varphi : [0, 1] \rightarrow [0, 1]$ of the functional equation $\varphi \circ B_{a_1} = B_{a_2} \circ \varphi$. Then

(I) $\{\varphi_n\}_{n=0}^{\infty}$ is uniformly convergent to φ ;

(II) $|\varphi(x) - \varphi_n(x)| \leq \frac{\lambda^n}{1 - \lambda} |a_1 - (1 - a_2)|, \quad \forall x \in [0, 1], \forall n \in \mathbb{N}.$

Proof. (I) For every $\varepsilon > 0$ and $\varepsilon < 1$, choose $N = \lceil \frac{\ln \varepsilon}{\ln \lambda} \rceil + 1$, $\forall m, n > N$, $\forall x \in [0, 1]$, we have

$$\begin{aligned} |\varphi_n(x) - \varphi_m(x)| &\leq \max\{a_2, 1 - a_2\} |\varphi_{n-1} \circ B_{a_1}(x) - \varphi_{m-1} \circ B_{a_1}(x)| \\ &= \lambda |\varphi_{n-1}(y_1) - \varphi_{m-1}(y_1)| \quad \text{let } y_k := B_{a_1}^k(x) \\ &\leq \lambda^N |\varphi_{n-N}(y_N) - \varphi_{m-N}(y_N)| \\ &\leq \lambda^N < \varepsilon. \end{aligned}$$

Thus $\{\varphi_n\}$ converges uniformly to φ by Theorem 2.1.

(II) The result is straight forward. Just let m tend to ∞ in (3.5). \square

4 Expression of conjugacies

To present explicit formulae for these two conjugacies, we need B_a -expansions of a real number in $[0, 1]$, also named $(\tau, \tau - 1)$ -expansions in [24]. This expansion is a particular case of an f -expansion as defined by Rényi [28]. Now we introduce the B_a -expansions.

Given $a \in (0, 1)$, consider the unit interval $[0, 1]$ and its partition into two parts by a :

$$[0, 1] = [0, a] \cup (a, 1].$$

Any real number $x \in [0, 1]$ has the following expression:

$$x = \begin{cases} ax_1, & x \leq a, \\ a + (1 - a)x_1, & x > a, \end{cases}$$

where, in both cases, $x_1 \in [0, 1]$.

Iteration of the previous process, leads to a sequence $\{x_n\}$ where $x_1 = x$ and for $n \geq 1$

$$x_{n+1} = B_a(x_n) = \begin{cases} \frac{x_n}{a}, & x_n \leq a, \\ \frac{x_n - a}{1 - a}, & x_n > a. \end{cases}$$

And one has an expansion for x in powers of the numbers a and $1 - a$:

$$x = \sum_{n=1}^{\infty} \varepsilon_n a^{n-s_{n-1}} (1-a)^{s_{n-1}} = \sum_{n=1}^{\infty} \varepsilon_n a^n \left(\frac{1-a}{a} \right)^{s_{n-1}},$$

where $s_0 = 0$, $s_n := \sum_{k=1}^n \varepsilon_k$ for $n \geq 1$ and

$$\varepsilon_n = \begin{cases} 0, & x_n \leq a, \\ 1, & x_n > a. \end{cases}$$

Now, every $x \in [0, 1]$ can be represented through its digit sequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$, denoted by $x = [\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \dots]_a$ for short. It is readily seen that $x = 0$ has a B_a -expansion with $\varepsilon_n = 0$ for all $n \in \mathbb{N}$, and $x = 1$ has a B_a -expansion with $\varepsilon_n = 1$ all $n \in \mathbb{N}$.

It is easy to see that, in case $x \in [0, 1)$ has a finite B_a -expansion $[\varepsilon_1, \dots, \varepsilon_k]_a$ where $\varepsilon_k = 1$, then x has also an infinite equivalent expansion simply by changing the digit ε_k to 0 and making $\varepsilon_{k+j} = 1$, $j = 1, 2, \dots$.

Note that if $x = [\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots, \varepsilon_n, \dots]_a \in [0, 1]$, then $B_a(x) = [\varepsilon_2, \varepsilon_3, \dots, \varepsilon_n, \dots]_a$. This shows that, topologically, B_a corresponds to the shift map on the space $\{0, 1\}^{\mathbb{N}}$, at least for those points with an infinite B_a -expansion.

The map $B_a(x_n) = x_{n+1}$ is ergodic (cf. [28]). As a result, the orbit $\{T^n(x)\}$ of almost all $x \in [0, 1]$ is uniformly distributed. Consequently, we have,

$$\lim_{n \rightarrow \infty} \frac{s_n}{n} = 1 - a. \quad (4.6)$$

Now we present explicit expressions of these two conjugacies.

Theorem 4.1 *For each $x = [\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \dots]_{a_1} \in [0, 1]$, the increasing conjugacy and decreasing conjugacy from B_{a_1} to B_{a_2} are respectively given by*

$$\varphi(x) = [\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \dots]_{a_2} \quad (4.7)$$

and

$$\varphi(x) = [\hat{\varepsilon}_1, \hat{\varepsilon}_2, \dots, \hat{\varepsilon}_n, \dots]_{a_2} \quad (4.8)$$

where $\hat{\varepsilon}_k := 1 - \varepsilon_k$.

Proof. Now we only consider the decreasing conjugacy. The other case is similar.

It is clear that φ defined by (4.8) is a strictly decreasing and continuous function with $\varphi(0) = 1$ and $\varphi(1) = 0$. According to the existence and uniqueness of the conjugacy in Theorem 2.1, it suffices to check φ given by (4.8) is a solution of conjugacy equation $\varphi \circ B_{a_1} = B_{a_2} \circ \varphi$.

If $x \in [0, a_1]$, then $x \in [0, 1]$. Let $x/a_1 = [\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \dots]_{a_1}$. Then $x = [0, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \dots]_{a_1}$. Thus

$$\varphi(x) = [1, \hat{\varepsilon}_1, \hat{\varepsilon}_2, \dots, \hat{\varepsilon}_n, \dots]_{a_2}.$$

It follows from Lemma 2.2 that $\varphi(x) \in [a_2, 1]$. Thus

$$B_{a_2}(\varphi(x)) = \frac{\varphi(x) - a_2}{1 - a_2} = \sum_{n=1}^{\infty} (1 - \varepsilon_n) a_2^n \left(\frac{1 - a_2}{a_2} \right)^{\sum_{j=1}^{n-1} (1 - \varepsilon_j)} = [\hat{\varepsilon}_1, \hat{\varepsilon}_2, \dots, \hat{\varepsilon}_n, \dots]_{a_2}.$$

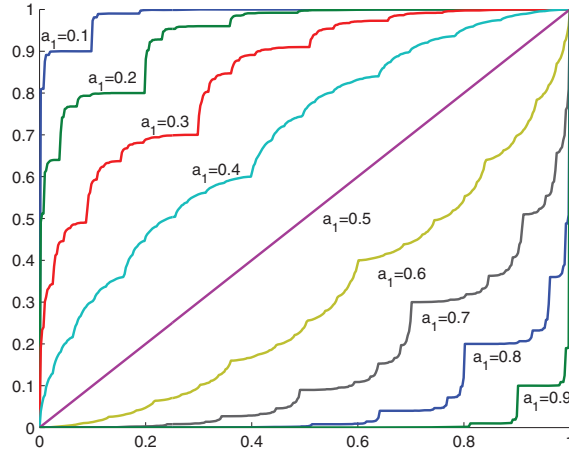


Fig. 3: The curves of increasing conjugacies from B_{a_1} to B_{1-a_1} with $a_1 = 0.1, 0.2, \dots, 0.9$.

On the other hand, we have

$$\varphi(B_{a_1}(x)) = \varphi\left(\frac{x}{a_1}\right) = [\hat{\varepsilon}_1, \hat{\varepsilon}_2, \dots, \hat{\varepsilon}_n, \dots]_{a_2}.$$

Therefore $\varphi(B_{a_1}(x)) = B_{a_2}(\varphi(x))$ for all $x \in [0, a_1]$.

If $x \in (a_1, 1]$, then $\frac{x-a_1}{1-a_1} \in (0, 1]$. Let

$$\frac{x-a_1}{1-a_1} = [\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \dots]_{a_1}.$$

Then $x = [1, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \dots]_{a_1}$. Consequently, $\varphi(x) = [0, \hat{\varepsilon}_1, \hat{\varepsilon}_2, \dots, \hat{\varepsilon}_n, \dots]_{a_2}$. It follows from Lemma 2.2 that $\varphi(x) \in [0, a_1]$. Thus

$$B_{a_2}(\varphi(x)) = \frac{\varphi(x)}{a_2} = [\hat{\varepsilon}_1, \hat{\varepsilon}_2, \dots, \hat{\varepsilon}_n, \dots]_{a_2}.$$

On the other hand, we have

$$\varphi(B_{a_1}(x)) = \varphi\left(\frac{x-a_1}{1-a_1}\right) = [\hat{\varepsilon}_1, \hat{\varepsilon}_2, \dots, \hat{\varepsilon}_n, \dots]_{a_2}.$$

Therefore $\varphi(B_{a_1}(x)) = B_{a_2}(\varphi(x))$ for all $x \in (a_1, 1]$. \square

For example, we present these increasing and decreasing conjugacies from B_{a_1} to B_{1-a_1} with $a_1 = 0.1, 0.2, \dots, 0.9$, see Figs. 3, 4.

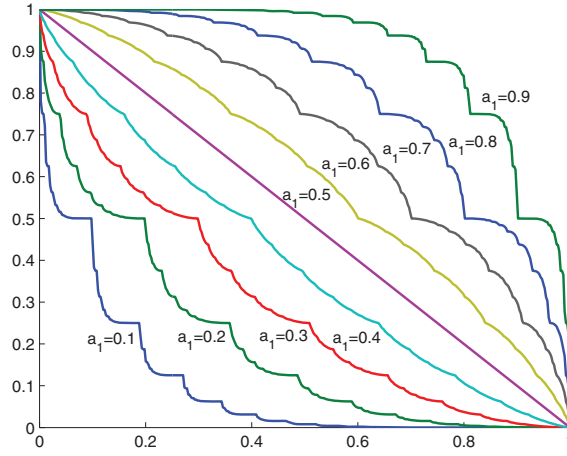


Fig. 4: The curves of decreasing conjugacies from B_{a_1} to B_{1-a_1} with $a_1 = 0.1, 0.2, \dots, 0.9$.

5 Regularity of conjugacies

In this section, we shall discuss the singularity, Hölder continuity and differentiability of these two conjugacies.

We will now define the cylinder sets associated with the map B_a . For each k -tuple $(\varepsilon_1, \dots, \varepsilon_k)$ of 0 and 1, define the B_a cylinder set $C_a(\varepsilon_1, \dots, \varepsilon_k)$ associated with the B_a -expansion to be

$$C_a(\varepsilon_1, \dots, \varepsilon_k) := \{[y_1, y_2, \dots]_a : y_i = \varepsilon_i \text{ for } 1 \leq i \leq k\}.$$

Observe that these sets are closed intervals with the left endpoint $[\varepsilon_1, \dots, \varepsilon_k]_a$ and the right endpoint $[\varepsilon_1, \dots, \varepsilon_k, 1, 1, \dots]_a$. For the length of these closed intervals we have that

$$|C_a(\varepsilon_1, \dots, \varepsilon_k)| = a^k \left(\frac{1-a}{a} \right)^{s_k}. \quad (5.9)$$

Define a -sum-level sets for each $n \in \mathbb{N} \cup \{0\}$ by

$$\mathcal{L}_n^a := \{x \in C_{a_1}(\varepsilon_1, \dots, \varepsilon_k) : s_k = n, \text{ for some } n \in \mathbb{N}\}.$$

Remark that every B_a cylinder set $C_a(\varepsilon_1, \dots, \varepsilon_k)$ is an a -sum set of level n where $n = \sum_{j=1}^k \varepsilon_j$.

Theorem 5.1 *Both the increasing and decreasing conjugacies from T_{c_1} to T_{c_2} are singular.*

Proof. Assume that φ is the increasing conjugacy from B_{a_1} to B_{a_2} . If $x = [\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \dots]_{a_1}$, then x must belong to each B_{a_1} cylinder set $C_{a_1}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)$ for $k = 1, 2, \dots$, namely,

$$[\varepsilon_1, \dots, \varepsilon_k]_{a_1} \leq x \leq [\varepsilon_1, \dots, \varepsilon_k, 1, 1, \dots]_{a_1}.$$

Since φ is strictly increasing, the images corresponding to the endpoints will satisfy

$$[\varepsilon_1, \dots, \varepsilon_k]_{a_2} \leq \varphi(x) \leq [\varepsilon_1, \dots, \varepsilon_k, 1, 1, \dots]_{a_2}.$$

If $\varphi'(x)$ exists, it will have to verify

$$\begin{aligned} \varphi'(x) &= \lim_{k \rightarrow \infty} \frac{|C_{a_2}(\varepsilon_1, \dots, \varepsilon_k)|}{|C_{a_1}(\varepsilon_1, \dots, \varepsilon_k)|} \\ &= \lim_{k \rightarrow \infty} \frac{a_2^k \left(\frac{1-a_2}{a_2}\right)^{s_k}}{a_1^k \left(\frac{1-a_1}{a_1}\right)^{s_k}} \\ &= \lim_{k \rightarrow \infty} \left(\left(\frac{a_2}{a_1}\right)^{1-\frac{s_k}{k}} \cdot \left(\frac{1-a_2}{1-a_1}\right)^{\frac{s_k}{k}} \right)^k. \end{aligned}$$

For almost all $x \in [0, 1]$, it follows from (4.6) that

$$\lim_{k \rightarrow \infty} \frac{s_k}{k} = 1 - a_1.$$

And for all $a_1, a_2 \in (0, 1)$, $a_1 \neq a_2$, we have the following inequality:

$$\left(\frac{a_2}{a_1}\right)^{a_1} \cdot \left(\frac{1-a_2}{1-a_1}\right)^{1-a_1} < 1.$$

So the derivative of φ is zero almost everywhere, by a theorem of H. Lebesgue [21, p. 128] that every monotonic function possesses a finite derivative at every point with the possible exception of the points of a set of measure zero.

Assume that φ is the decreasing conjugacy from B_{a_1} to B_{a_2} . If $x = [\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \dots]_{a_1}$, then x must belong to each B_{a_1} cylinder set $C_{a_1}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)$ for $k = 1, 2, \dots$, namely,

$$[\varepsilon_1, \dots, \varepsilon_k]_{a_1} \leq x \leq [\varepsilon_1, \dots, \varepsilon_k, 1, 1, \dots]_{a_1}.$$

Since φ is strictly decreasing, the images corresponding to the endpoints will satisfy

$$[\hat{\varepsilon}_1, \dots, \hat{\varepsilon}_k]_{a_2} \leq \varphi(x) \leq [\hat{\varepsilon}_1, \dots, \hat{\varepsilon}_k, 1, 1, \dots]_{a_2}.$$

If $\varphi'(x)$ exists, it will have to verify

$$\begin{aligned}
 \varphi'(x) &= - \lim_{k \rightarrow \infty} \frac{|C_{a_2}(\hat{\varepsilon}_1, \dots, \hat{\varepsilon}_k)|}{|C_{a_1}(\varepsilon_1, \dots, \varepsilon_k)|} \\
 &= - \lim_{k \rightarrow \infty} \frac{a_2^k \left(\frac{1-a_2}{a_2}\right)^{k-s_k}}{a_1^k \left(\frac{1-a_1}{a_1}\right)^{s_k}} \\
 &= - \lim_{k \rightarrow \infty} \left(\left(\frac{1-a_2}{a_1}\right)^{1-\frac{s_k}{k}} \cdot \left(\frac{a_2}{1-a_1}\right)^{\frac{s_k}{k}} \right)^k. \quad (5.10)
 \end{aligned}$$

For almost all $x \in [0, 1]$, $\lim_{k \rightarrow \infty} \frac{s_k}{k} = 1 - a_1$. Thus for all $a_1, a_2 \in (0, 1)$, $a_1 \neq a_2$, we have the following inequality:

$$\left(\frac{1-a_2}{a_1}\right)^{a_1} \cdot \left(\frac{a_2}{1-a_1}\right)^{1-a_1} < 1.$$

Therefore the derivative of φ is zero almost everywhere. \square

Our next aim is to determine the Hölder exponent and sub-Hölder exponent of these two conjugacies.

For $\alpha \in (0, \infty)$, a map $f : I \rightarrow I$ is called α -sub-Hölder continuous if there exists a constant c such that $|f(x) - f(y)| \geq c|x - y|^\alpha$ for all $x, y \in I$.

Theorem 5.2 *The increasing conjugacy from T_{c_1} to T_{c_2} is α_+ -Hölder continuous and α_- -sub-Hölder continuous where*

$$\alpha_+ = \min \left\{ \frac{\ln a_2}{\ln a_1}, \frac{\ln(1-a_2)}{\ln(1-a_1)} \right\} \quad \text{and} \quad \alpha_- = \max \left\{ \frac{\ln a_2}{\ln a_1}, \frac{\ln(1-a_2)}{\ln(1-a_1)} \right\}.$$

Proof. It follows from (5.9) that

$$|C_{a_1}(\varepsilon_1, \dots, \varepsilon_k)| = a_1^k \left(\frac{1-a_1}{a_1}\right)^{s_k} = a_1^{k-s_k} (1-a_1)^{s_k}.$$

In order to calculate the Hölder exponent of the conjugacy φ , first note that

$$|\varphi(C_{a_1}(\varepsilon_1, \dots, \varepsilon_k))| = a_2^k \left(\frac{1-a_2}{a_2}\right)^{s_k} = a_2^{k-s_k} (1-a_2)^{s_k}.$$

This can be seen by simply calculating the image of the endpoints of this cylinder. Set $\alpha_1 := \frac{\ln a_2}{\ln a_1}$ and $\alpha_2 := \frac{\ln(1-a_2)}{\ln(1-a_1)}$. Thus we have

$$|C_{a_1}(\varepsilon_1, \dots, \varepsilon_k)| = a_1^{k-s_k} (1-a_1)^{s_k} = a_2^{(k-s_k)/\alpha_1} (1-a_2)^{(s_k)/\alpha_2}.$$

$$\begin{aligned}
 |C_{a_1}(\varepsilon_1, \dots, \varepsilon_k)| &= a_1^{k-s_k}(1-a_1)^{s_k} \\
 &= a_2^{(k-s_k)/\alpha_1}(1-a_2)^{s_k/\alpha_2} \\
 &\geq \left(a_2^{k-s_k}(1-a_2)^{s_k}\right)^{1/\alpha_+} \\
 &= |\varphi(C_{a_1}(\varepsilon_1, \dots, \varepsilon_k))|^{1/\alpha_+}.
 \end{aligned}$$

Let $x, y \in [0, 1]$ be two arbitrary numbers. There must be the biggest B_a cylinder set contained in the interval $[x, y]$. Let this B_a cylinder set be denoted by $C_{a_1}(\varepsilon_1, \dots, \varepsilon_k)$ where $\sum_{j=1}^k \varepsilon_j = p+1$. This leads to the observation that, since the interval $C_{a_1}(\varepsilon_1, \dots, \varepsilon_k)$ is contained in $[x, y]$,

$$|x - y|^{\alpha_+} \geq |C_{a_1}(\varepsilon_1, \dots, \varepsilon_k)|^{\alpha_+} \geq |\varphi(C_{a_1}(\varepsilon_1, \dots, \varepsilon_k))| = a_2^k \left(\frac{1-a_2}{a_2}\right)^{p+1}.$$

Consider the interval $[x, y]$ again. It is contained inside two neighbouring $(p-1)$ -th level B_a cylinder sets, and so

$$|\varphi(x) - \varphi(y)| \leq 2a_2^k \left(\frac{1-a_2}{a_2}\right)^{p-1} = \frac{2a_2^2}{(1-a_2)^2} |x - y|^{\alpha_+}.$$

The proof of the α_- -sub-Hölder continuity of the conjugacy follows by similar means. \square

With the similar argument to the above, we have a similar result for the decreasing conjugacy.

Theorem 5.3 *The decreasing conjugacy from T_{c_1} to T_{c_2} is α_+ -Hölder continuous and α_- -sub-Hölder continuous where*

$$\alpha_+ = \min \left\{ \frac{\ln(1-a_2)}{\ln a_1}, \frac{\ln a_2}{\ln(1-a_1)} \right\} \quad \text{and} \quad \alpha_- = \max \left\{ \frac{\ln(1-a_2)}{\ln a_1}, \frac{\ln a_2}{\ln(1-a_1)} \right\}.$$

Theorem 5.4 *Let φ be the decreasing conjugacy from B_{a_1} to B_{a_2} ,*

$$R = R(a_1, a_2) = \frac{\ln\left(\frac{a_1}{1-a_2}\right)}{\ln\left(\frac{a_1 a_2}{(1-a_1)(1-a_2)}\right)},$$

and $x = [\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \dots]_{a_1} \in [0, 1]$. If $\frac{a_2}{1-a_2} > \frac{1-a_1}{a_1}$ (resp. $\frac{a_2}{1-a_2} < \frac{1-a_1}{a_1}$), and there exists a value r such that

$$\liminf_{k \rightarrow \infty} \frac{s_k}{k} \geq r > R \quad (\text{resp.} \quad \limsup_{k \rightarrow \infty} \frac{s_k}{k} \leq r < R),$$

then, if $\varphi'(x)$ exists in a wide sense, it has to be infinite.

Proof. Case $\frac{a_2}{1-a_2} > \frac{1-a_1}{a_1}$. If $\liminf_{k \rightarrow \infty} \frac{s_k}{k} \geq r > R$, there exists a k_0 such that for $k > k_0$ we have $\frac{s_k}{k} \geq r > R$. For such k ,

$$\begin{aligned} \frac{1-a_2}{a_1} \cdot \left(\frac{a_1 a_2}{(1-a_1)(1-a_2)} \right)^{\frac{s_k}{k}} &\geq \frac{1-a_2}{a_1} \cdot \left(\frac{a_1 a_2}{(1-a_1)(1-a_2)} \right)^r \\ &> \frac{1-a_2}{a_1} \cdot \left(\frac{a_1 a_2}{(1-a_1)(1-a_2)} \right)^R = 1 \end{aligned}$$

Thus the limit in (5.10) is ∞ .

Case $\frac{a_2}{1-a_2} < \frac{1-a_1}{a_1}$. If $\limsup_{k \rightarrow \infty} \frac{s_k}{k} \geq r > R$, there exists a k_0 such that for $k > k_0$ we have $\frac{s_k}{k} \geq r > R$. For such k ,

$$\begin{aligned} \frac{1-a_2}{a_1} \cdot \left(\frac{a_1 a_2}{(1-a_1)(1-a_2)} \right)^{\frac{s_k}{k}} &\geq \frac{1-a_2}{a_1} \cdot \left(\frac{a_1 a_2}{(1-a_1)(1-a_2)} \right)^r \\ &> \frac{1-a_2}{a_1} \cdot \left(\frac{a_1 a_2}{(1-a_1)(1-a_2)} \right)^R = 1 \end{aligned}$$

Thus the limit in (5.10) is ∞ . \square

With the similar argument to the above, we have a similar result for the decreasing conjugacy.

Theorem 5.5 *Let φ be the increasing conjugacy from B_{a_1} to B_{a_2} ,*

$$R = R(a_1, a_2) = \frac{\ln \left(\frac{1-a_1}{1-a_2} \right)}{\ln \left(\frac{a_2(1-a_1)}{a_1(1-a_2)} \right)},$$

and $x = [\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \dots]_{a_1} \in [0, 1]$. If $\frac{a_2}{1-a_2} > \frac{a_1}{1-a_1}$ (resp. $\frac{a_2}{1-a_2} < \frac{a_1}{1-a_1}$), and there exists a value r such that

$$\liminf_{k \rightarrow \infty} \frac{s_k}{k} \geq r > R \quad (\text{resp.} \quad \limsup_{k \rightarrow \infty} \frac{s_k}{k} \leq r < R),$$

then, if $\varphi'(x)$ exists in a wide sense, it has to be infinite.

Proposition 5.1 *Let φ be the increasing (decreasing) conjugacy from B_{a_1} to B_{a_2} . Then*

$$\int_0^1 \varphi(x) dx = \frac{a_2(1-a_1)}{a_1 + a_2 - 2a_1 a_2} \quad \left(\frac{a_1 a_2}{1 + 2a_1 a_2 - a_1 - a_2}, \quad \text{resp.} \right)$$

and the arc-length of the curve $y = \varphi(x)$ between the points $(0, 0)$ and $(1, 1)$ is 2.

Proof. Assume that φ is the increasing conjugacy. By (2.1), we have

$$\begin{aligned}
 \int_0^1 \varphi(x) dx &= \int_0^{a_1} a_2 \varphi\left(\frac{x}{a_1}\right) dx + \int_{a_1}^1 \left((1-a_2) \varphi\left(\frac{x-a_1}{1-a_1}\right) + a_2 \right) dx \\
 &= \int_0^1 a_1 a_2 \varphi(t) dt + \int_{a_1}^1 (1-a_2) \varphi\left(\frac{x-a_1}{1-a_1}\right) dx + \int_{a_1}^1 a_2 dx \\
 &= a_1 a_2 \int_0^1 \varphi(t) dt + \int_1^0 (1-a_2)(1-a_1) \varphi(u) du + a_2(1-a_1) \\
 &= (a_1 a_2 + (1-a_2)(1-a_1)) \int_0^1 \varphi(t) dt + a_2(1-a_1).
 \end{aligned}$$

Consequently, $\int_0^1 \varphi(x) dx = \frac{a_2(1-a_1)}{a_1+a_2-2a_1a_2}$.

Assume that φ is the decreasing conjugacy. By (2.2), we have

$$\begin{aligned}
 \int_0^1 \varphi(x) dx &= \int_0^{a_1} \left((1-a_2) \varphi\left(\frac{x}{a_1}\right) + a_2 \right) dx + \int_{a_1}^1 a_2 \varphi\left(\frac{x-a_1}{1-a_1}\right) dx \\
 &= \int_0^{a_1} (1-a_2) \varphi\left(\frac{x}{a_1}\right) dx + \int_0^{a_1} a_2 dx + \int_0^1 a_2(1-a_1) \varphi(u) du \\
 &= a_1(1-a_2) \int_0^1 \varphi(t) dt + a_1 a_2 + a_2(1-a_1) \int_0^1 \varphi(u) du.
 \end{aligned}$$

Consequently, $\int_0^1 \varphi(x) dx = \frac{a_1 a_2}{1+2a_1 a_2 - a_1 - a_2}$.

Since φ is a singular function, by [10, Corollary 2.10], the arc-length of the curve $y = \varphi(x)$ is 2. \square

6 Concluding Remarks

One can see that the increasing conjugacy from B_{a_1} to B_{a_2} is a new generalization of de Rham's function, although there are some kinds of generalizations, e.g. [11, 12, 15, 17, 18, 33]. The other new discovery is the fact that de Rham's function is coincident with Salem's function [29].

It is natural for expressions of conjugacies to be closely related with a system for real number representation, which is called B_a -expansions. These real number expansions contribute significantly to studying the properties of conjugacies, including explicit expressions, singularity and Hölder continuity of conjugacies

In all the cases, the metrical properties of the real number expansions make the study of conjugacies much easier, and allow us to obtain further results on the properties of conjugacies, cf. [31, 30].

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