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On the drawdowns and drawups in diffusion-type models with running maxima and minima

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ABSTRACT

We obtain closed-form expressions for the values of joint Laplace transforms of the running maximum and minimum of a diffusion-type process stopped at the first time at which the associated drawdown and drawup processes hit constant levels. It is assumed that the coefficients of the diffusion-type process are regular functions of the running values of the process itself, its maximum and minimum, as well as its maximum drawdown and maximum drawup processes. The proof is based on the solution to the equivalent boundary-value problems and application of the normal-reflection conditions for the value functions at the edges of the state space of the resulting five-dimensional Markov process. We show that the joint Laplace transforms represent linear combinations of solutions to the systems of first-order partial differential equations arising from the application of the normal-reflection conditions.

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1. Introduction

The main aim of this paper is to derive closed-form expressions for the Laplace transform (2.5) of the first time to a given drawdown occurring before a fixed drawup of the diffusion-type process X and its running maximum and minimum S and Q, defined in (2.1)–(2.3), stopped at that time. The running *maximum drawdown* process Y is defined as the maximum of the difference between the running maximum and the current value of the initial process (this difference is sometimes called *reflected* process), while the running *maximum drawup* process Z is defined as the maximum of the difference between the current value and the running minimum of the process (this difference is sometimes called *rally* process). Such extremum processes have been intensively studied in the recent literature and found subsequent applications in queueing theory,

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risk and option pricing theory, change-point detection and many others (see, e.g. Asmussen [1], Peskir and Shiryaev [36], and Poor and Hadjiliadis [37] for extensive overviews and further references).

The Laplace transform of the first time to a given drawdown of a Brownian motion with linear drift and the running maximum stopped at that time was computed by Taylor [42], and the joint law of those variables was obtained by Lehoczky [27]. Some explicit expressions for other related characteristics such as the expectation and the density of the maximum drawdown of the Brownian motion with linear drift were derived by Douady, Shiryaev, and Yor [8] and Magdon-Ismail et al. [29], respectively. More recently, Pospisil, Vecer, and Hadjiliadis [38] computed the probability of a drawdown of a given size occurring before a drawup of a fixed size in several one-dimensional diffusion models. Mijatović and Pistorius [30] obtained the laws of the first-passage times of spectrally positive and negative Lévy processes over constant levels as well as analytically explicit identities for a number of characteristics of drawdowns and drawups in those models.

In the present paper, closed-form expressions are derived for the joint Laplace transforms of the first time to a given drawdown or drawup and the maximum and minimum values at that time of a diffusion-type process with coefficients depending on the running values of the process itself, its maximum and minimum, as well as its maximum drawdown and maximum drawup. Such diffusion-type processes can be considered as immediate generalisations of diffusion processes particularly arising in the so-called local volatility models introduced by Dupire [10], where the local drift and diffusion coefficients depend only on the running value of the initial process. Other extensions with diffusion coefficients depending on the running values of the initial processes and their running minima were constructed by Forde [12] for given joint laws of the terminal level and supremum at an independent exponential time (see also Forde, Pogudin, and Zhang [14] and Zhang [43] for other important probability characteristics of processes of such type). Cont and Fournié [6] and Fournié [15] obtained the valuation functional equations for general functional path-dependent volatility models and considered the sensitivity analysis of path-dependent financial derivative securities.

The dependence of the local drift and diffusion coefficients on the past dynamics of the observable process through certain sufficient statistics is often used in financial practice as well as well studied in the related literature. For instance, an increase of the maximum drawdown or drawup of a risky asset price normally causes a structural change in the local drift representing its expected return and dividend policy. It also triggers changes in the diffusion coefficient representing the volatility rate of an asset price with a higher impact under a maximum drawdown increase rather than a maximum drawup increase. Such sufficient statistics transparently exhibit the risk levels of the assets and therefore usually influence the decisions taken by the market participants. The demand for option pricing in models with stochastic interest rates and volatility initiated the development and subsequent calibration of these models, based on diffusion-type processes with tractable path-dependent coefficients, which were realised by Henry-Labordère [23] and Ren, Madan, and Qian [39] among others.

The problem of finding the Markovian projections of continuous semimartingales in order to mimic their marginal distributions was studied by Gyöngy [21] and then generalised by Bentata and Cont [2] for the discontinuous case. The resulting Markov processes were given as weak solutions of stochastic differential equations with coefficients depending on the running value of the initial process. These arguments were further developed by Brunick and Shreve [5] and Forde [13] by extending the projections conditioned on path-dependent functionals of the initial process along with its running value. The purpose of the resulting *mimicking processes* was to give the opportunity to apply Markovian techniques and tackle both analytical and computational aspects of the initial processes with path-dependent distributional characteristics. Such processes were efficiently used by Dupire [10] and Klebaner [26] for solving option pricing problems in models in which the dynamics of the risky assets are described by general continuous semimartingales and by Bentata and Cont [3] for the discontinuous case. Guyon [20] showed that diffusion-type processes with path-dependent coefficients are conveniently helpful in order to replicate the spot volatility dynamics of the financial market, particularly through the extremum processes such as the running maximum. In this respect, the coefficients of the diffusion-type model considered in this paper can be interpreted as the Markovian projection of a continuous semimartingale conditioned on the current state of the associated five-dimensional Markov process with four path-dependent components. Other similar Markovian projections were studied by Bremaud [4] for queues and by Cont and Minca [7] for marked point processes with path-dependent intensities. Calibrational aspects of such models with path-dependent distributional characteristics were recently studied by Hambly, Mariapragassam, and Reisinger [22] among others. We provide closed-form solutions to the boundary-value problems associated with the values of joint Laplace transforms as stopping problems for the five-dimensional continuous Markov process.

Optimal stopping problems for running maxima of some diffusion processes were studied by Jacka [24], Dubins, Shepp, and Shiryaev [9], and Peskir [32] among others. Discounted optimal stopping problems for certain payoff functions depending on the running maxima of geometric Brownian motions were initiated by Shepp and Shiryaev [41] and then taken further by Pedersen [31], Guo and Shepp [18], Guo and Zervos [19], Glover, Hulley, and Peskir [17], and [16] among others. The main feature of the resulting optimal stopping problems and their equivalent free-boundary problems was the application of the normal-reflection condition for the value functions at the diagonal of the two-dimensional state space to derive first-order ordinary differential equations for the optimal stopping boundaries depending on the current value of the running maximum process. These properties follow directly from the definition of the infinitesimal operator of the two-dimensional continuous Markov process having the initial process and the running maximum as its state space components. More recently, Peskir [34,35] studied optimal stopping problems for three-dimensional Markov processes having the initial diffusion process as well as its maximum and minimum as state space components.

The paper is organised as follows. In Section 2, we first introduce the setting and notation of the model with a five-dimensional continuous Markov process, whose state space components are the initial process, the running values of its maximum, minimum, maximum drawdown and maximum drawup. We define the value function of the joint Laplace transform of the first time to a given drawdown occurring before the first time of a fixed drawup together with the running maximum and minimum stopped at that time. In Section 3, we obtain a closed-form solution to the associated boundary-value problem and show that the value function represents a linear combination of the solutions to the systems of first-order partial differential equations which arise from the application of the normal-reflection conditions for this function at the edges of the five-dimensional state space. In Section 4, we verify that the resulting solution to the boundary-value problem provides the joint Laplace transform. The main result of the paper is stated in Theorem 4.1.

2. Preliminaries

In this section, we give a precise probabilistic formulation of the model and the five-dimensional stopping problem as well as its equivalent boundary-value problem.

2.1. Formulation of the problem

Let us consider a probability space $(\Omega, \mathcal{F}, P)$ with a standard Brownian motion $B = (B_t)_{t \geq 0}$. Assume that there exists a diffusion-type process $X = (X_t)_{t \geq 0}$ solving the stochastic differential equation

$$dX_t = \mu(X_t, S_t, Q_t, Y_t, Z_t)\, dt + \sigma(X_t, S_t, Q_t, Y_t, Z_t)\, dB_t \quad (X_0 = x) \tag{2.1}$$

where $x \in \mathbb{R}$ is fixed, and $\mu(x, s, q, y, z)$ and $\sigma(x, s, q, y, z) > 0$ are continuously differentiable functions on $[-\infty, \infty]^5$ which are of at most linear growth in x and uniformly bounded in all other variables. Here, the associated with X *running maximum* process $S = (S_t)_{t \geq 0}$ and the *running minimum* process $Q = (Q_t)_{t \geq 0}$ are defined by

$$S_t = s \vee \max_{0 \le u \le t} X_u \quad \text{and} \quad Q_t = q \wedge \min_{0 \le u \le t} X_u \tag{2.2}$$

as well as the *running maximum drawdown* process $Y = (Y_t)_{t \ge 0}$ and the *running maximum drawup* process $Z = (Z_t)_{t \ge 0}$ are given by

$$Y_t = y \vee \max_{0 \le u \le t} (S_u - X_u) \quad \text{and} \quad Z_t = z \vee \max_{0 \le u \le t} (X_u - Q_u) \tag{2.3}$$

for arbitrary $(s - y) \vee q \le x \le s \wedge (q + z)$. It follows from the result of [28, Chapter IV, Theorem 4.8] that the equation in (2.1) admits a pathwise unique (strong) solution. We also define the associated first hitting (stopping) times

$$\tau_a = \inf\{t \ge 0 \,|\, y \vee (S_t - X_t) \ge a\} \quad \text{and} \quad \zeta_b = \inf\{t \ge 0 \,|\, z \vee (X_t - Q_t) \ge b\} \tag{2.4}$$

for some $a, b > 0$ fixed.

The main purpose of the present paper is to derive closed-form expressions for the joint Laplace transform of the random variables $\tau_a \wedge \zeta_b$, $S_{\tau_a \wedge \zeta_b}$, and $Q_{\tau_a \wedge \zeta_b}$. We therefore need to compute the value function of the following stopping problem for the time-homogeneous (strong) Markov process $(X, S, Q, Y, Z) = (X_t, S_t, Q_t, Y_t, Z_t)_{t \ge 0}$ given by

$$V^*(x, s, q, y, z) = E_{x,s,q,y,z}\big[e^{-\lambda(\tau_a \wedge \zeta_b) - \theta S_{\tau_a \wedge \zeta_b} - \varkappa Q_{\tau_a \wedge \zeta_b}}\, I(\tau_a < \zeta_b)\big] \tag{2.5}$$

for any $(x, s, q, y, z) \in E^5$ and some $\lambda, \theta, \varkappa > 0$ fixed, where $I(\cdot)$ denotes the indicator function. Here, $E_{x,s,q,y,z}$ denotes the expectation under the assumption that the five-dimensional time-homogeneous (strong) Markov process (X, S, Q, Y, Z) defined in (2.1)–(2.3) starts at $(x, s, q, y, z) \in E^5$, where we assume that the state space of (X, S, Q, Y, Z) is essentially $E^5 = \{(x, s, q, y, z) \in \mathbb{R}^5 \,|\, (s - y) \vee q \le x \le s \wedge (q + z)\}$.

2.2. The boundary-value problem

By means of standard arguments based on the application of Itô's formula, it is shown that the infinitesimal operator $\mathbb{L}$ of the process (X, S, Q, Y, Z) acts on a function $F(x, s, q, y, z)$ from the class $C^{2,1,1,1,1}$ on the interior of E^5 according to the rule

$$(\mathbb{L}F)(x, s, q, y, z) = \mu(x, s, q, y, z)\, \partial_x F(x, s, q, y, z) + \sigma^2(x, s, q, y, z)\, \partial^2_{xx} F(x, s, q, y, z)/2 \tag{2.6}$$

for all $(s - y) \vee q < x < s \wedge (q + z)$. In order to find analytic expressions for the unknown value function $V^*(x, s, q, y, z)$ in (2.5), let us build on the results of general theory of Markov processes (see, e.g. [11, Chapter V]). The value $V^*(x, s, q, y, z)$ from (2.5) solves the equivalent boundary-value problem

$$(\mathbb{L}V)(x, s, q, y, z) = \lambda V(x, s, q, y, z) \quad \text{for} \quad s - a < (s - y) \vee q < x < s \wedge (q + z) < q + b \tag{2.7}$$

$$V(x, s, q, y, z)\big|_{x=(s-a)+,\, y=a-} = e^{-\theta s - \varkappa q} \quad \text{for} \quad s - q \ge a \quad \text{and} \quad 0 < z < b \tag{2.8}$$

$$V(x, s, q, y, z)\big|_{x=(q+b)-,\, z=b-} = 0 \quad \text{for} \quad s - q \ge b \quad \text{and} \quad 0 < y < a \tag{2.9}$$

$$\partial_q V(x, s, q, y, z)\big|_{x=q+} = 0 \quad \text{for} \quad 0 < s - q < y < a \tag{2.10}$$

$$\partial_s V(x, s, q, y, z)\big|_{x=s-} = 0 \quad \text{for} \quad 0 < s - q < z < b \tag{2.11}$$

$$\partial_y V(x, s, q, y, z)\big|_{x=(s-y)+} = 0 \quad \text{for} \quad 0 < y < (s - q) \wedge a \tag{2.12}$$

$$\partial_z V(x, s, q, y, z)\big|_{x=(q+z)-} = 0 \quad \text{for} \quad 0 < z < (s - q) \wedge b \tag{2.13}$$

for $a, b > 0$ fixed.

3. Solutions to the boundary-value problem

In this section, we obtain closed-form solutions to the boundary-value problem in (2.7)–(2.13) under various relations of the parameters of the model.

3.1. The general solution of the ordinary differential equation

We first observe that the general solution of the equation in (2.7) has the form

$$V(x,s,q,y,z) = C_1(s,q,y,z)\,\Psi_1(x,s,q,y,z) + C_2(s,q,y,z)\,\Psi_2(x,s,q,y,z) \tag{3.1}$$

where $C_i(s,q,y,z)$, $i=1,2$, are some arbitrary continuously differentiable functions and $\Psi_i(x,s,q,y,z)$, $i=1,2$, are the two fundamental positive solutions (i.e. nontrivial linearly independent particular solutions) of the second-order ordinary differential equation in (2.7). Without loss of generality, we may assume that $\Psi_1(x,s,q,y,z)$ and $\Psi_2(x,s,q,y,z)$ are the (strictly) increasing and decreasing (convex) functions, respectively. Note that these solutions should satisfy the properties $\Psi_1(r,r,r,\varepsilon,\varepsilon)\uparrow\infty$ and $\Psi_2(r,r,r,\varepsilon,\varepsilon)\downarrow 0$ as $r\uparrow\infty$ and $\Psi_1(r,r,r,\varepsilon,\varepsilon)\downarrow 0$ and $\Psi_2(r,r,r,\varepsilon,\varepsilon)\uparrow\infty$ as $r\downarrow -\infty$, for any sufficiently small $\varepsilon>0$, on the state space E^5 of the process (X,S,Q,Y,Z). These functions can be represented as the functionals

$$\Psi_1(x,s,q,y,z) = \begin{cases} E_{x,s,q,y,z}[e^{-\lambda\xi'}I(\xi'<\infty)], & \text{if} \quad x\le x' \\ 1/E_{x',s,q,y,z}[e^{-\lambda\xi}I(\xi<\infty)], & \text{if} \quad x\ge x' \end{cases} \tag{3.2}$$

and

$$\Psi_2(x,s,q,y,z) = \begin{cases} 1/E_{x',s,q,y,z}[e^{-\lambda\xi}I(\xi<\infty)], & \text{if} \quad x\le x' \\ E_{x,s,q,y,z}[e^{-\lambda\xi'}I(\xi'<\infty)], & \text{if} \quad x\ge x' \end{cases} \tag{3.3}$$

of the first hitting times $\xi=\inf\{t\ge 0\,|\,X_t=x\}$ and $\xi'=\inf\{t\ge 0\,|\,X_t=x'\}$ of the process X solving the stochastic differential equation in (2.1) and started at x and x' such that $(x,s,q,y,z),(x',s,q,y,z)\in E^5$, respectively (see, e.g. [40, Chapter V, Section 50] for further details).

Then, by applying the conditions of (2.8)–(2.13) to the function in (3.1), we obtain the equalities

$$C_1(s,q,a,z)\,\Psi_1(s-a,s,q,a,z) + C_2(s,q,a,z)\,\Psi_2(s-a,s,q,a,z) = e^{-\theta s-\varkappa q} \tag{3.4}$$

for $s-q\ge a$ and $0<z<b$,

$$C_1(s,q,y,b)\,\Psi_1(q+b,s,q,a,z) + C_2(s,q,y,b)\,\Psi_2(q+b,s,q,a,z) = 0 \tag{3.5}$$

for $s-q\ge b$ and $0<y<a$,

$$\sum_{i=1}^{2}\Big(\partial_q C_i(s,q,y,z)\,\Psi_i(q,s,q,y,z) + C_i(s,q,y,z)\,\partial_q\Psi_i(x,s,q,y,z)\big|_{x=q}\Big) = 0 \tag{3.6}$$

for $0<s-q<y<a$,

$$\sum_{i=1}^{2}\Big(\partial_s C_i(s,q,y,z)\,\Psi_i(s,s,q,y,z) + C_i(s,q,y,z)\,\partial_s\Psi_i(x,s,q,y,z)\big|_{x=s}\Big) = 0 \tag{3.7}$$

for $0<s-q<z<b$,

$$\sum_{i=1}^{2} \Big(\partial_y C_i(s,q,y,z)\, \Psi_i(s-y,s,q,y,z) + C_i(s,q,y,z)\, \partial_y \Psi_i(x,s,q,y,z)\big|_{x=s-y} \Big) = 0 \tag{3.8}$$

for $0 < y < (s-q) \wedge a$, and

$$\sum_{i=1}^{2} \Big(\partial_z C_i(s,q,y,z)\, \Psi_i(q+z,s,q,y,z) + C_i(s,q,y,z)\, \partial_z \Psi_i(x,s,q,y,z)\big|_{x=q+z} \Big) = 0 \tag{3.9}$$

for $0 < z < (s-q) \wedge b$.

3.2. The solution to the boundary-value problem in the (X, S, Q)-setting

We begin with the case in which $\mu(x,s,q,y,z) = \mu(x,s,q)$ and $\sigma(x,s,q,y,z) = \sigma(x,s,q)$ in (2.1) and put $y = s - x$ and $z = x - q$ into (2.4). Then, the general solution $V(x,s,q,y,z) = U(x,s,q)$ of the equation in (2.7) has the form of (3.1) with $C_i(s,q,y,z) = D_i(s,q)$ and $\Psi_i(x,s,q,y,z) = \Phi_i(x,s,q)$, $i = 1,2$, in (3.1). We further denote the state space of the three-dimensional (strong) Markov process (X,S,Q) by $E^3 = \{(x,s,q) \in \mathbb{R}^3 \,|\, q \le x \le s\}$ and its border planes by $d_1^3 = \{(x,s,q) \in \mathbb{R}^3 \,|\, x = s\}$ and $d_2^3 = \{(x,s,q) \in \mathbb{R}^3 \,|\, x = q\}$. We also recall that the second and third components of the process (X,S,Q) can increase and decrease only at the planes d_1^3 and d_2^3, that is, when $X_t = S_t$ and $X_t = Q_t$ for $t \ge 0$, respectively.

(i) Let us first consider the domain $a \vee b \le s - q \le a + b$. In this case, solving the system of equations in (3.4) and (3.5), we conclude that the candidate value function admits the representation

$$U(x,s,q;\infty) = D_1(s,q;\infty)\, \Phi_1(x,s,q) + D_2(s,q;\infty)\, \Phi_2(x,s,q) \tag{3.10}$$

in the region $R^3(\infty) = \{(x,s,q) \in E^3 \,|\, q \le s - a \le x \le q + b \le s\}$, with

$$D_1(s,q;\infty) = \frac{e^{-\theta s - \varkappa q}\Phi_2(q+b,s,q)}{\Phi_1(s-a,s,q)\Phi_2(q+b,s,q) - \Phi_1(q+b,s,q)\Phi_2(s-a,s,q)} \tag{3.11}$$

and

$$D_2(s,q;\infty) = \frac{e^{-\theta s - \varkappa q}\Phi_1(q+b,s,q)}{\Phi_1(s-a,s,q)\Phi_2(q+b,s,q) - \Phi_1(q+b,s,q)\Phi_2(s-a,s,q)} \tag{3.12}$$

for all $q + a \vee b \le s \le q + a + b$ (see Figs. 1 and 2).

(ii) Let us now consider the domain $a \le s - q < b$. In this case, it follows from the equations in (3.4) and (3.7) that the candidate value function admits the representation

$$U(x,s,q;a) = D_1(s,q;a)\, \Phi_1(x,s,q) + D_2(s,q;a)\, \Phi_2(x,s,q) \tag{3.13}$$

in the region $R^3(a) = \{(x,s,q) \in E^3 \,|\, q \le s - a \le x \le s < q + b\}$, with

$$D_2(s,q;a) = \big(e^{-\theta s - \varkappa q} - D_1(s,q;a)\, \Phi_1(s-a,s,q)\big)/\Phi_2(s-a,s,q) \tag{3.14}$$

for $q + a \le s < q + b$, where $D_1(s,q;a)$ solves the first-order linear ordinary differential equation

$$\partial_s D_1(s,q;a)\, H_{1,2}(s,q;a) + D_1(s,q;a)\, H_{1,1}(s,q;a) = H_{1,0}(s,q;a) \tag{3.15}$$

with

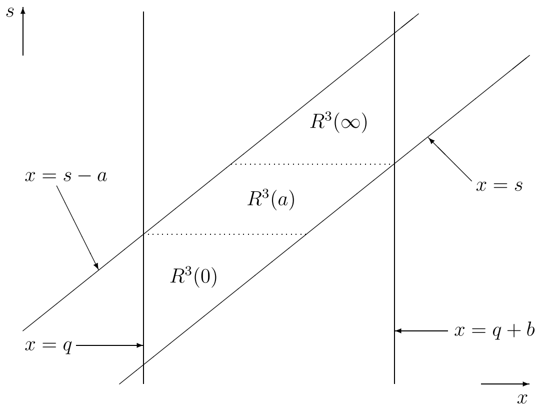

Fig. 1. A computer drawing of the state space of the process (X, S, Q), for some $q \in \mathbb{R}$ fixed and $a < b$.

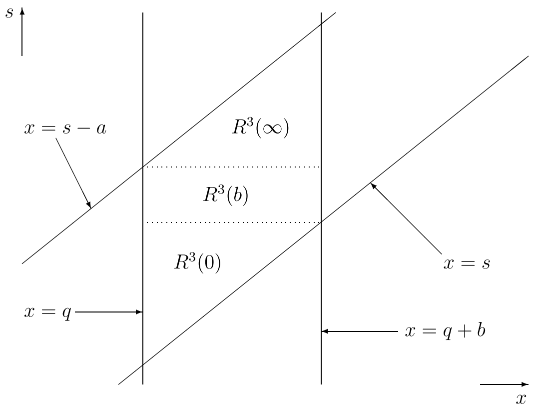

Fig. 2. A computer drawing of the state space of the process (X, S, Q), for some $q \in \mathbb{R}$ fixed and $b \le a$.

$$H_{1,2}(s, q; a) = \Phi_1(s, s, q) - \Phi_1(s - a, s, q)\, \Phi_2(s, s, q)/\Phi_2(s - a, s, q) \tag{3.16}$$

$$H_{1,1}(s, q; a) = \partial_s\big(\Phi_1(x, s, q) - \Phi_1(s - a, s, q)\, \Phi_2(x, s, q)/\Phi_2(s - a, s, q)\big)\Big|_{x=s} \tag{3.17}$$

$$H_{1,0}(s, q; a) = e^{-\theta s - \varkappa q} \left(\theta\, \frac{\Phi_2(s, s, q)}{\Phi_2(s - a, s, q)} - \partial_s \left(\frac{\Phi_2(x, s, q)}{\Phi_2(s - a, s, q)} \right) \bigg|_{x=s} \right) \tag{3.18}$$

for all $q + a \le s < q + b$. Observe that the process (X, S, Q) can exit the region $R^3(a)$ by passing to the region $R^3(\infty)$ in part (i) of this subsection only through the point $x = s = q + b$, by hitting the plane d_1^3 so that increasing its second component S. Thus, the candidate function $U(x, s, q)$ should be continuous at the point $(q + b, q + b, q)$, that is expressed by the equality

$$D_1(q + b, q; a)\, \Phi_1(q + b, q + b, q) + D_2(q + b, q; a)\, \Phi_2(q + b, q + b, q) = 0 \tag{3.19}$$

for all $q \in \mathbb{R}$ (see Fig. 1). Hence, solving the differential equation in (3.15) together with the system of equations in (3.14) with $s = q + b$ and (3.19), we obtain

$$D_1(s, q; a) = D_1(q + b, q; a) \exp\left(\int_s^{q+b} \frac{H_{1,1}(u, q; a)}{H_{1,2}(u, q; a)}\, du \right) - \int_s^{q+b} \frac{H_{1,0}(u, q; a)}{H_{1,2}(u, q; a)} \exp\left(\int_s^u \frac{H_{1,1}(v, q; a)}{H_{1,2}(v, q; a)}\, dv \right) du \tag{3.20}$$

for all $q + a \le s < q + b$, where $D_1(q + b, q; a)$ is given by

$$D_1(q+b,q;a) = \frac{e^{-\theta(q+b)-\varkappa q}\Phi_2(q+b,q+b,q)}{\Phi_1(q+b-a,q+b,q)\Phi_2(q+b,q+b,q) - \Phi_1(q+b,q+b,q)\Phi_2(q+b-a,q+b,q)} \tag{3.21}$$

for all $q \in \mathbb{R}$.

(iii) Let us now consider the domain $b \le s - q < a$. In this case, it follows from the equations in (3.5) and (3.6) that the candidate value function admits the representation

$$U(x,s,q;b) = D_1(s,q;b)\,\Phi_1(x,s,q) + D_2(s,q;b)\,\Phi_2(x,s,q) \tag{3.22}$$

in the region $R^3(b) = \{(x,s,q) \in E^3 \,|\, s-a < q \le x \le q+b \le s\}$, with

$$D_2(s,q;b) = -D_1(s,q;b)\,\Phi_1(q+b,s,q)/\Phi_2(q+b,s,q) \tag{3.23}$$

for $q + b \le s < q + a$, where $D_1(s, q; b)$ solves the first-order linear ordinary differential equation

$$\partial_q D_1(s,q;b)\,H_{2,2}(s,q;b) + D_1(s,q;b)\,H_{2,1}(s,q;b) = 0 \tag{3.24}$$

with

$$H_{2,2}(s,q;b) = \Phi_1(q,s,q) - \Phi_1(q+b,s,q)\,\Phi_2(q,s,q)/\Phi_2(q+b,s,q) \tag{3.25}$$

$$H_{2,1}(s,q;b) = \partial_q\big(\Phi_1(x,s,q) - \Phi_1(q+b,s,q)\,\Phi_2(x,s,q)/\Phi_2(q+b,s,q)\big)\Big|_{x=q} \tag{3.26}$$

for all $q + b \le s < q + a$. Observe that the process (X, S, Q) can exit $R^3(b)$ by passing to the region $R^3(\infty)$ in part (i) of this subsection only through the point $x = q = s - a$, by hitting the plane d_2^3 so that decreasing its third component Q. Then, the candidate value function should be continuous at the point $(s-a, s, s-a)$, that is expressed by the equality

$$D_1(s,s-a;b)\,\Phi_1(s-a,s,s-a) + D_2(s,s-a;b)\,\Phi_2(s-a,s,s-a) = e^{-\theta s-\varkappa(s-a)} \tag{3.27}$$

for all $s \in \mathbb{R}$ (see Fig. 2). Hence, solving the differential equation in (3.24) together with the system of equations in (3.23) with $q = s - a$ and (3.27), we obtain

$$D_1(s,q;b) = D_1(s,s-a;b)\,\exp\left(-\int_{s-a}^{q}\frac{H_{2,1}(s,u;b)}{H_{2,2}(s,u;b)}\,du\right) \tag{3.28}$$

for all $q + b \le s < q + a$, where $D_1(s, s - a; b)$ is given by

$$D_1(s,s-a;b) = \frac{e^{-\theta s-\varkappa(s-a)}\Phi_2(s-a+b,s,s-a)}{\Phi_1(s-a,s,s-a)\Phi_2(s-a+b,s,s-a) - \Phi_1(s-a+b,s,s-a)\Phi_2(s-a,s,s-a)} \tag{3.29}$$

for $s \in \mathbb{R}$.

(iv) Let us now consider the domain $0 \le s - q < a \wedge b$. In this case, it follows that the candidate value function admits the representation

$$U(x,s,q;0) = D_1(s,q;0)\,\Phi_1(x,s,q) + D_2(s,q;0)\,\Phi_2(x,s,q) \tag{3.30}$$

in the region $R^3(0) = \{(x,s,q) \in E^3 \,|\, s-a < q \le x \le s < q+b\}$, where $D_i(s,q;0)$, $i=1,2$, solve the first-order linear partial differential equations in (3.6) and (3.7), for all $0 < s-q < a \wedge b$. Observe that, the process (X,S,Q) can exit $R^3(0)$ by passing to the region $R^3(a \wedge b)$ in part (ii) or (iii) of this subsection only through the points $x = s = q + a \wedge b$ and $x = q = s - a \wedge b$, by hitting the plane d_1^3 or d_2^3, so that increasing its second or third components, S or Q, respectively. Then, the candidate value function should be continuous at the points $(q + a \wedge b, q + a \wedge b, q)$ and $(s - a \wedge b, s, s - a \wedge b)$, that is expressed by the equalities

$$\begin{aligned} &D_1(q+a\wedge b, q; 0)\,\Phi_1(q+a\wedge b, q+a\wedge b, q) \\ &\quad + D_2(q+a\wedge b, q; 0)\,\Phi_2(q+a\wedge b, q+a\wedge b, q) \\ &= D_1(q+a\wedge b, q; a\wedge b)\,\Phi_1(q+a\wedge b, q+a\wedge b, q) \\ &\quad + D_2(q+a\wedge b, q; a\wedge b)\,\Phi_2(q+a\wedge b, q+a\wedge b, q) \end{aligned} \tag{3.31}$$

for all $q \in \mathbb{R}$ and

$$\begin{aligned} &D_1(s, s-a\wedge b; 0)\,\Phi_1(s-a\wedge b, s, s-a\wedge b) \\ &\quad + D_2(s, s-a\wedge b; 0)\,\Phi_2(s-a\wedge b, s, s-a\wedge b) \\ &= D_1(s, s-a\wedge b; a\wedge b)\,\Phi_1(s-a\wedge b, s, s-a\wedge b) \\ &\quad + D_2(s, s-a\wedge b; a\wedge b)\,\Phi_2(s-a\wedge b, s, s-a\wedge b) \end{aligned} \tag{3.32}$$

for all $s \in \mathbb{R}$, where $D_i(q + a \wedge b, q; a \wedge b)$ and $D_i(s, s - a \wedge b; a \wedge b)$, $i = 1,2$, are found in (3.14)+(3.20) or (3.23)+(3.28). Moreover, we have the property $D_2(r,r;0) \to 0$ as $r \downarrow -\infty$, since otherwise $U(r,r,r;0) \to \pm\infty$, that must be excluded by virtue of the obvious fact that the value function in (2.5) is bounded (see Figs. 1 and 2). We may therefore conclude that the candidate value function admits the representation of (3.30) in the region $R^3(0)$ above, where $D_i(s,q;0)$, $i=1,2$, provide a unique solution of the two-dimensional system of first-order linear partial differential equations in (2.10) and (2.11) with the boundary conditions of (3.31)–(3.32) and $D_2(r,r;0) \to 0$ as $r \downarrow -\infty$. Hereafter, the existence and uniqueness of solutions to such special kinds of systems of equations follow from the classical existence and uniqueness results of solutions to appropriate boundary-value problems for first-order linear partial differential equations.

3.3. The solution to the boundary-value problem in the (X,S,Q,Y)-setting

We now continue with the case in which $\mu(x,s,q,y,z) = \mu(x,s,q,y)$ and $\sigma(x,s,q,y,z) = \sigma(x,s,q,y)$ in (2.1) and put $z = x - q$ into (2.4). Then, the general solution $V(x,s,q,y,z) = W_1(x,s,q,y)$ of the equation in (2.7) has the form of (3.1) with $C_i(s,q,y,z) = A_{1,i}(s,q,y)$ and $\Psi_i(x,s,q,y,z) = \Upsilon_{1,i}(x,s,q,y)$, $i=1,2$, in (3.1). We further denote the state space of the four-dimensional (strong) Markov process (X,S,Q,Y) by $E_1^4 = \{(x,s,q,y) \in \mathbb{R}^4 \,|\, (s-y) \vee q \le x \le s\}$ and its border hyperplanes by $d_{1,1}^4 = \{(x,s,q,y) \in \mathbb{R}^4 \,|\, x = s\}$, $d_{1,2}^4 = \{(x,s,q,y) \in \mathbb{R}^4 \,|\, x = q\}$, and $d_{1,3}^4 = \{(x,s,q,y) \in \mathbb{R}^4 \,|\, x = s - y\}$. We also recall that the second, third, and fourth components of the process (X,S,Q,Y) can increase or decrease only at the planes $d_{1,1}^4$, $d_{1,2}^4$, and $d_{1,3}^4$, that is, when $X_t = S_t$, $X_t = Q_t$, and $X_t = S_t - Y_t$ for $t \ge 0$, respectively. Finally, we introduce the stopping time $\nu_a = \inf\{t \ge 0 \,|\, S_t - Y_t = Q_t\}$ and observe that $Y_t = S_t - Q_t$ holds for all $t \ge \nu_a$.

(i) Let us first consider the domain $b \vee y \le s - q \le y + b$. In this case, it follows from the equations in (3.5) and (3.8) that the candidate value function admits the representation

$$W_1(x,s,q,y;\infty) = A_{1,1}(s,q,y;\infty)\,\Upsilon_{1,1}(x,s,q,y) + A_{1,2}(s,q,y;\infty)\,\Upsilon_{1,2}(x,s,q,y) \tag{3.33}$$

in the region $R_1^4(\infty) = \{(x,s,q,y) \in E_1^4 \,|\, (s-a) \vee q < s-y \le x \le q+b \le s\}$, with

$$A_{1,2}(s,q,y;\infty) = -A_{1,1}(s,q,y;\infty)\, \Upsilon_{1,1}(q+b,s,q,y)/\Upsilon_{1,2}(q+b,s,q,y) \tag{3.34}$$

for $q+b \le s \le q+y+b < q+a+b$, where $A_{1,1}(s,q,y;\infty)$ solves the first-order linear ordinary differential equation

$$\partial_y A_{1,1}(s,q,y;\infty)\, G_{1,2}(s,q,y;\infty) + A_{1,1}(s,q,y;\infty)\, G_{1,1}(s,q,y;\infty) = 0 \tag{3.35}$$

with

$$G_{1,2}(s,q,y;\infty) = \Upsilon_{1,1}(s-y,s,q,y) - \frac{\Upsilon_{1,1}(q+b,s,q,y)}{\Upsilon_{1,2}(q+b,s,q,y)}\, \Upsilon_{1,2}(s-y,s,q,y) \tag{3.36}$$

$$G_{1,1}(s,q,y;\infty) = \partial_y \left(\Upsilon_{1,1}(x,s,q,y) - \frac{\Upsilon_{1,1}(q+b,s,q,y)}{\Upsilon_{1,2}(q+b,s,q,y)}\, \Upsilon_{1,2}(x,s,q,y) \right) \bigg|_{x=s-y} \tag{3.37}$$

for all $q+b \le s \le q+y+b < q+a+b$. Observe that the process (X,S,Q,Y) can reach the edge of the region $R_1^4(\infty)$ only through the point $x = s-y = (s-a) \vee q$, by hitting the hyperplane $d_{1,3}^4$, so that increasing its fourth component Y. Then, the component Y becomes either equal to the value a or is set to $S-Q$ and the region $R_1^4(\infty)$ is identified with $R^3(b)$ in part (iii) of Subsection 3.2. Thus, the candidate value function should be continuous at the point $((s-a)\vee q, s, q, (s-q)\wedge a)$, that is expressed by the equality

$$\begin{aligned} &A_{1,1}(s,q,(s-q)\wedge a;\infty)\, \Upsilon_{1,1}((s-a)\vee q, s,q,(s-q)\wedge a) \\ &\quad + A_{1,2}(s,q,(s-q)\wedge a;\infty)\, \Upsilon_{1,2}((s-a)\vee q,s,q,(s-q)\wedge a) \\ &= U(q,s,q;b)\, I(q > s-a) + e^{-\theta s - \varkappa q}\, I(q \le s-a) \end{aligned} \tag{3.38}$$

for all $q+b \le s \le q+y+b < q+a+b$, where $U(q,s,q;b)$ is determined in part (iii) of Subsection 3.2 (see Figs. 3 and 4). Hence, solving the differential equation in (3.35) together with the system of equations in (3.34) and (3.38), we obtain

$$A_{1,1}(s,q,y;\infty) = A_{1,1}(s,q,(s-q)\wedge a;\infty)\, \exp\left(\int_y^{(s-q)\wedge a} \frac{G_{1,1}(s,q,u;\infty)}{G_{1,2}(s,q,u;\infty)}\, du \right) \tag{3.39}$$

for all $q+b \le s \le q+y+b < q+a+b$, where $A_{1,1}(s,q,(s-q)\wedge a;\infty)$ is given by

$$\begin{aligned} &A_{1,1}(s,q,(s-q)\wedge a;\infty) \\ &= \frac{(U(q,s,q;b) I(q > s-a) + e^{-\theta s - \varkappa q} I(q \le s-a))/\Upsilon_{1,2}((s-a)\vee q,s,q,(s-q)\wedge a)}{(\Upsilon_{1,1}/\Upsilon_{1,2})((s-a)\vee q,s,q,(s-q)\wedge a) - (\Upsilon_{1,1}/\Upsilon_{1,2})(q+b,s,q,(s-q)\wedge a)} \end{aligned} \tag{3.40}$$

for $q \le s$.

(ii) Let us now consider the domain $0 < y < s-q < b$. In this case, it follows that the candidate value function admits the representation

$$W_1(x,s,q,y;a) = A_{1,1}(s,q,y;a)\, \Upsilon_{1,1}(x,s,q,y) + A_{1,2}(s,q,y;a)\, \Upsilon_{1,2}(x,s,q,y) \tag{3.41}$$

in the region $R_1^4(a) = \{(x,s,q,y) \in E_1^4 \,|\, (s-a)\vee q < s-y \le x \le s < q+b\}$, where $A_{1,i}(s,q,y;a)$, $i = 1,2$, solve the system of first-order linear ordinary differential equations (3.7) and (3.8), for all $q+y < s < q+b$. Observe that on one hand, the process (X,S,Q,Y) can exit the region $R_1^4(a)$ by passing to the region $R_1^4(\infty)$

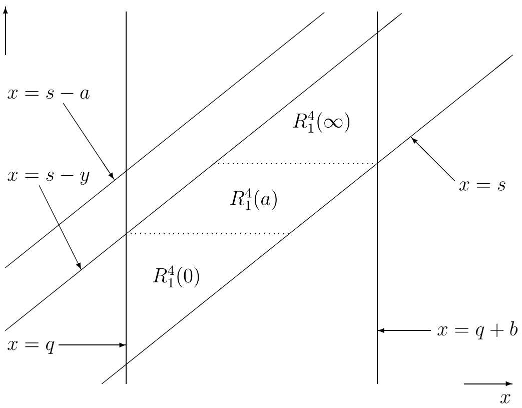

Fig. 3. A computer drawing of the state space of the process (X, S, Q, Y), for some $q, y \in \mathbb{R}$ fixed and $y < b$.

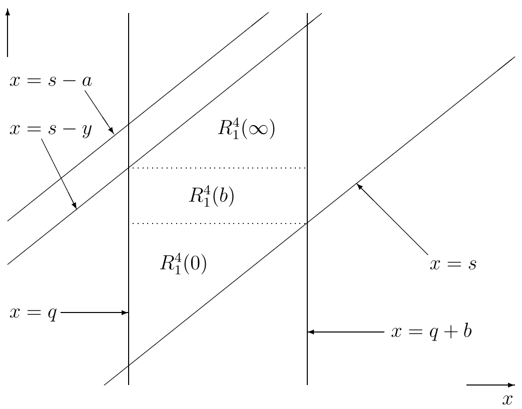

Fig. 4. A computer drawing of the state space of the process (X, S, Q, Y), for some $q, y \in \mathbb{R}$ fixed and $b \le y$.

in part (i) of this subsection, only through the point $x = s = q + b$, by hitting the hyperplane $d_{1,1}^4$ so that increasing its second component S. On the other hand, the process (X, S, Q, Y) can reach the edge of the region $R_1^4(a)$ through the point $x = s - y = (s - a) \vee q$, by hitting the hyperplane $d_{1,3}^4$ so that increasing its fourth component Y. Then, the component Y becomes either equal to the value a or is set to $S - Q$ and the region $R_1^4(a)$ is identified with $R^3(0)$ in part (iv) of Subsection 3.2. Thus, the candidate value function should be continuous at the points $(q + b, q + b, q, y)$ and $((s - a) \vee q, s, q, (s - q) \wedge a)$, that is expressed by the equalities

$$A_{1,1}(q + b, q, y; a)\, \Upsilon_{1,1}(q + b, q + b, q, y) + A_{1,2}(q + b, q, y; a)\, \Upsilon_{1,2}(q + b, q + b, q, y) = 0 \tag{3.42}$$

for all $q \in \mathbb{R}$ and $0 < y < b$, and

$$\begin{aligned} &A_{1,1}(s, q, (s - q) \wedge a; a)\, \Upsilon_{1,1}((s - a) \vee q, s, q, (s - q) \wedge a) \\ &\quad + A_{1,2}(s, q, (s - q) \wedge a; a)\, \Upsilon_{1,2}((s - a) \vee q, s, q, (s - q) \wedge a) \\ &= U(q, s, q; 0)\, I(q > s - a) + e^{-\theta s - \varkappa q}\, I(q \le s - a) \end{aligned} \tag{3.43}$$

for all $q < s < q+b$, where $U(q, s, q; 0)$ is found in part (iv) of Subsection 3.2. Moreover, we have the property $A_{1,2}(r, r, \varepsilon; a) \to 0$ as $r \downarrow -\infty$, since otherwise $W_1(r, r, r, \varepsilon; a) \to \pm\infty$, for any sufficiently small $\varepsilon > 0$, that must be excluded by virtue of the obvious fact that the value function in (2.5) is bounded (see Fig. 3). We may therefore conclude that the candidate value function admits the representation of (3.41) in the region

$R_1^4(a)$, where $A_{1,i}(s,q,y;a)$, $i=1,2$, provide a unique solution of the two-dimensional system of first-order linear partial differential equations in (3.7) and (3.8) with the boundary conditions of (3.42)–(3.43) and $A_{1,2}(r,r,\varepsilon;a) \to 0$ as $r \downarrow -\infty$, for any sufficiently small $\varepsilon > 0$.

(iii) Let us finally consider the domain $0 \le s-q \le y$. Observe that since the fourth component Y is set to $S-Q$ after the process (X,S,Q,Y) hits both hyperplanes $d_{1,2}^4$ and $d_{1,3}^4$, we may conclude that the candidate value function takes the form

$$W_1(x,s,q,y;b) = W_1(x,s,q,s-q;b) = U(x,s,q;b) \tag{3.44}$$

in the region $R_1^4(b) = \{(x,s,q,y) \in E_1^4 \,|\, s-a < s-y \le q \le x \le q+b \le s\}$ (see Fig. 4) and

$$W_1(x,s,q,y;0) = W_1(x,s,q,s-q;0) = U(x,s,q;0) \tag{3.45}$$

in the region $R_1^4(0) = \{(x,s,q,y) \in E_1^4 \,|\, s-a < s-y \le q \le x \le s < q+b\}$ (see Figs. 3 and 4), where the functions $U(x,s,q;b)$ and $U(x,s,q;0)$ are determined in parts (iii) and (iv) of Subsection 3.2, respectively.

3.4. The solution to the boundary-value problem in the (X,S,Q,Z)-setting

We now continue with the case in which $\mu(x,s,q,y,z) = \mu(x,s,q,z)$ and $\sigma(x,s,q,y,z) = \sigma(x,s,q,z)$ in (2.1) and put $y = s-x$ into (2.4). Then, the general solution $V(x,s,q,y,z) = W_2(x,s,q,z)$ of the equation in (2.7) has the form of (3.1) with $C_i(s,q,y,z) = A_{2,i}(s,q,z)$ and $\Psi_i(x,s,q,y,z) = \Upsilon_{2,i}(x,s,q,z)$, $i=1,2$, in (3.1). We further denote the state space of the four-dimensional (strong) Markov process (X,S,Q,Z) by $E_2^4 = \{(x,s,q,z) \in \mathbb{R}^4 \,|\, q \le x \le s \wedge (q+z)\}$ and its border hyperplanes by $d_{2,1}^4 = \{(x,s,q,z) \in \mathbb{R}^4 \,|\, x = s\}$, $d_{2,2}^4 = \{(x,s,q,z) \in \mathbb{R}^4 \,|\, x = q\}$, and $d_{2,3}^4 = \{(x,s,q,z) \in \mathbb{R}^4 \,|\, x = q+z\}$. We also recall that the second, third, and fourth components of the process (X,S,Q,Z) can increase or decrease only at the hyperplanes $d_{2,1}^4$, $d_{2,2}^4$, and $d_{2,3}^4$, that is, when $X_t = S_t$, $X_t = Q_t$, and $X_t = Q_t + Z_t$ for $t \ge 0$, respectively. Finally, we introduce the stopping time $\eta_b = \inf\{t \ge 0 \,|\, Q_t + Z_t = S_t\}$ and observe that $Z_t = S_t - Q_t$ holds for all $t \ge \eta_b$.

(i) Let us first consider the domain $a \vee z \le s-q \le a+z$. In this case, it follows from the equations in (3.4) and (3.9) that the candidate value function admits the representation

$$W_2(x,s,q,z;\infty) = A_{2,1}(s,q,z;\infty)\,\Upsilon_{2,1}(x,s,q,z) + A_{2,2}(s,q,z;\infty)\,\Upsilon_{2,2}(x,s,q,z) \tag{3.46}$$

in the region $R_2^4(\infty) = \{(x,s,q,z) \in E_2^4 \,|\, q \le s-a \le x \le q+z < s \wedge (q+b)\}$, with

$$A_{2,2}(s,q,z;\infty) = \big(e^{-\theta s - \varkappa q} - A_{2,1}(s,q,z;\infty)\,\Upsilon_{2,1}(s-a,s,q,z)\big)/\Upsilon_{2,2}(s-a,s,q,z) \tag{3.47}$$

for $q+a \le s \le q+a+z < q+a+b$, where $A_{2,1}(s,q,z;\infty)$ solves the first-order linear ordinary differential equation

$$\partial_z A_{2,1}(s,q,z;\infty)\, G_{2,2}(s,q,z;\infty) + A_{2,1}(s,q,z;\infty)\, G_{2,1}(s,q,z;\infty) = G_{2,0}(s,q,z;\infty) \tag{3.48}$$

with

$$G_{2,2}(s,q,z;\infty) = \Upsilon_{2,1}(q+z,s,q,z) - \frac{\Upsilon_{2,1}(s-a,s,q,z)}{\Upsilon_{2,2}(s-a,s,q,z)}\,\Upsilon_{2,2}(q+z,s,q,z) \tag{3.49}$$

$$G_{2,1}(s,q,z;\infty) = \partial_z\left(\Upsilon_{2,1}(x,s,q,z) - \frac{\Upsilon_{2,1}(s-a,s,q,z)}{\Upsilon_{2,2}(s-a,s,q,z)}\,\Upsilon_{2,2}(x,s,q,z)\right)\Bigg|_{x=q+z} \tag{3.50}$$

$$G_{2,0}(s,q,z;\infty) = e^{-\theta s - \varkappa q}\left(\theta\,\frac{\Upsilon_{2,2}(q+z,s,q,z)}{\Upsilon_{2,2}(s-a,s,q,z)} - \partial_z\left(\frac{\Upsilon_{2,2}(x,s,q,z)}{\Upsilon_{2,2}(s-a,s,q,z)}\right)\Bigg|_{x=q+z}\right) \tag{3.51}$$

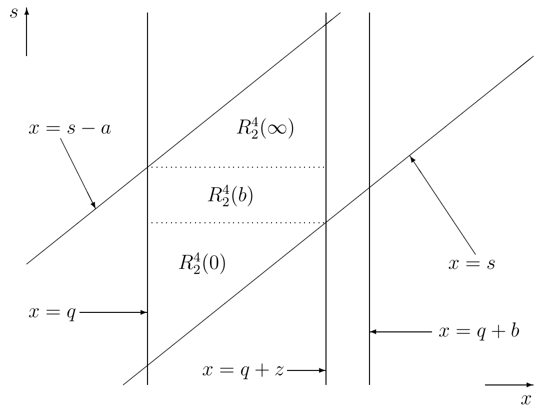

Fig. 5. A computer drawing of the state space of the process (X, S, Q, Z), for some $q, z \in \mathbb{R}$ fixed and $z \leq a$.

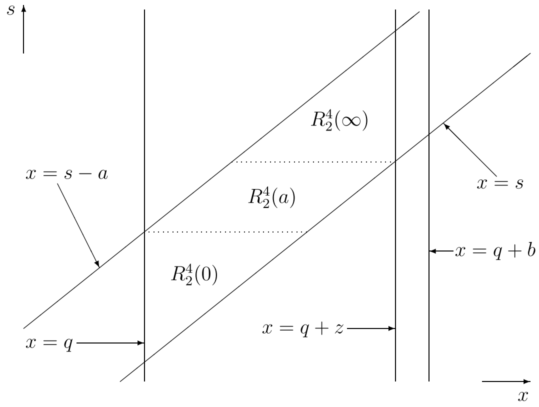

Fig. 6. A computer drawing of the state space of the process (X, S, Q, Z), for some $q, z \in \mathbb{R}$ fixed and $a < z$.

for $q + a \leq s \leq q + a + z < q + a + b$. Observe that the process (X, S, Q, Z) can reach the edge of the region $R_2^4(\infty)$, only through the point $x = q + z = (q + b) \wedge s$, by hitting the hyperplane $d_{2,3}^4$, so that increasing its fourth component Z. Then, the component Z becomes either equal to the value b or is set to $S - Q$ and the region $R_2^4(\infty)$ is identified with $R^3(a)$ in part (ii) of Subsection 3.2. Thus, the candidate value function should be continuous at the point $(s \wedge (q + b), s, q, (s - q) \wedge b)$, that is expressed by the equality

$$A_{2,1}(s, q, (s-q) \wedge b; \infty)\, \Upsilon_{2,1}(s \wedge (q+b), s, q, (s-q) \wedge b) \\ + A_{2,2}(s, q, (s-q) \wedge b; \infty)\, \Upsilon_{2,2}(s \wedge (q+b), s, q, (s-q) \wedge b) = U(s, s, q; a)\, I(s < q+b) \tag{3.52}$$

for all $q + a \leq s \leq q + a + z < q + a + b$, where $U(s, s, q; a)$ is determined in part (ii) of Subsection 3.2 (see Figs. 5 and 6). Hence, solving the differential equation in (3.48) together with the system of equations in (3.47) with $z = (s - q) \wedge b$ and (3.52), we obtain

$$A_{2,1}(s, q, z; \infty) = A_{2,1}(s, q, (s-q) \wedge b; \infty) \exp\left(\int_z^{(s-q)\wedge b} \frac{G_{2,1}(s, q, u; \infty)}{G_{2,2}(s, q, u; \infty)}\, du \right) \\ - \int_z^{(s-q)\wedge b} \frac{G_{2,0}(s, q, u; \infty)}{G_{2,2}(s, q, u; \infty)} \exp\left(\int_z^{u} \frac{G_{2,1}(s, q, v; \infty)}{G_{2,2}(s, q, v; \infty)}\, dv \right) du \tag{3.53}$$

for all $q + a \leq s \leq q + a + z < q + a + b$, where $A_{2,1}(s, q, (s - q) \wedge b; \infty)$ is given by

$$A_{2,1}(s,q,(s-q)\wedge b;\infty)$$
$$= \frac{e^{-\theta s-\varkappa q}/\Upsilon_{2,2}(s-a,s,q,(s-q)\wedge b) - U(s,s,q;a)I(s<q+b)/\Upsilon_{2,2}(s\wedge(q+b),s,q,(s-q)\wedge b)}{(\Upsilon_{2,1}/\Upsilon_{2,2})(s-a,s,q,(s-q)\wedge b) - (\Upsilon_{2,1}/\Upsilon_{2,2})(s\wedge(q+b),s,q,(s-q)\wedge b)} \tag{3.54}$$

for all $q \le s$.

(ii) Let us now consider the domain $0 < z < s-q < a$. In this case, it follows that the candidate value function admits the representation

$$W_2(x,s,q,z;b) = A_{2,1}(s,q,z;b)\,\Upsilon_{2,1}(x,s,q,z) + A_{2,2}(s,q,z;b)\,\Upsilon_{2,2}(x,s,q,z) \tag{3.55}$$

in the region $R_2^4(b) = \{(x,s,q,z) \in E_2^4 \,|\, s-a < q \le x \le q+z < s\wedge(q+b)\}$, where $A_{2,i}(s,q,z;b)$, $i=1,2$, solve the system of first-order linear partial differential equations in (3.6) and (3.9), for all $q+z<s<q+a$. Observe that on one hand, the process (X,S,Q,Z) can exit the region $R_2^4(b)$ by passing to the region $R_2^4(\infty)$ in part (i) of this subsection, only through the point $x=q=s-a$, by hitting the hyperplane $d_{2,2}^4$ so that decreasing its third component Q. On the other hand, the process (X,S,Q,Z) can reach the edge of the region $R_2^4(b)$ through the point $x=q+z=(q+b)\wedge s$, by hitting the hyperplane $d_{2,3}^4$, so that increasing its fourth component Z. Then, the component Z becomes either equal to the value b or is set to $S-Q$ and the region $R_2^4(b)$ is identified with $R^3(0)$ in part (iv) of Subsection 3.2. Thus, the candidate value function should be continuous at the points $(s-a,s,s-a,z)$ and $(s\wedge(q+b),s,q,(s-q)\wedge b)$, that is expressed by the equalities

$$\begin{aligned}&A_{2,1}(s,s-a,z;b)\,\Upsilon_{2,1}(s-a,s,s-a,z)\\&\quad + A_{2,2}(s,s-a,z;b)\,\Upsilon_{2,2}(s-a,s,s-a,z) = e^{-\theta s-\varkappa(s-a)}\end{aligned} \tag{3.56}$$

for all $s \in \mathbb{R}$ and $0<z<a$, and

$$\begin{aligned}&A_{2,1}(s,q,(s-q)\wedge b;b)\,\Upsilon_{2,1}(s\wedge(q+b),s,q,(s-q)\wedge b)\\&\quad + A_{2,2}(s,q,(s-q)\wedge b;b)\,\Upsilon_{2,2}(s\wedge(q+b),s,q,(s-q)\wedge b) = U(s,s,q;0)\,I(s<q+b)\end{aligned} \tag{3.57}$$

for all $q<s<q+a$, where $U(s,s,q;0)$ is determined in part (iv) of Subsection 3.2. Moreover, we have the property $A_{2,2}(r,r,\varepsilon;b) \to 0$ as $r \downarrow -\infty$, since otherwise $W_2(r,r,r,\varepsilon;b) \to \pm\infty$, for any sufficiently small $\varepsilon>0$, which must be excluded by virtue of the obvious fact that the value function in (2.5) is bounded (see Fig. 5). We may therefore conclude that the candidate value function admits the representation of (3.41) in the region $R_2^4(b)$, where $A_{2,i}(s,q,z;b)$, $i=1,2$, provide a unique solution of the two-dimensional system of first-order linear partial differential equations in (3.6) and (3.9) with the boundary conditions of (3.56)–(3.57) and $A_{2,2}(r,r,\varepsilon;b) \to 0$ as $r\downarrow -\infty$, for any sufficiently small $\varepsilon>0$.

(iii) Let us finally consider the domain $0 \le s-q \le z$. Observe that since the fourth component Z is set to $S-Q$ after the process (X,S,Q,Z) hits both hyperplanes $d_{2,1}^4$ and $d_{2,3}^4$, we may conclude that the candidate value function has the form

$$W_2(x,s,q,z;a) = W_2(x,s,q,s-q;a) = U(x,s,q;a) \tag{3.58}$$

in the region $R_2^4(a) = \{(x,s,q,z) \in E_2^4 \,|\, q \le s-a \le x \le s \le q+z < q+b\}$ (see Fig. 6) and

$$W_2(x,s,q,z;0) = W_2(x,s,q,s-q;0) = U(x,s,q;0) \tag{3.59}$$

in the region $R_2^4(0) = \{(x,s,q,z) \in E_2^4 \,|\, s-a < q \le x \le s \le q+z < q+b\}$ (see Figs. 5 and 6), where the functions $U(x,s,q;a)$ and $U(x,s,q;0)$ are determined in parts (ii) and (iv) of Subsection 3.2, respectively.

3.5. The solution to the boundary-value problem in the (X,S,Q,Y,Z)-setting

We finally consider the general form of the coefficients $\mu(x,s,q,y,z)$ and $\sigma(x,s,q,y,z)$ in (2.1), and thus, of the functions $\Psi_i(x,s,q,y,z)$, $i=1,2$, in (3.1). We denote the border hyperplanes of the state space E^5 by $d_1^5 = \{(x,s,q,y,z) \in \mathbb{R}^5 \,|\, x=s\}$, $d_2^5 = \{(x,s,q,y,z) \in \mathbb{R}^5 \,|\, x=q\}$, $d_3^5 = \{(x,s,q,y,z) \in \mathbb{R}^5 \,|\, x=s-y\}$, and $d_4^5 = \{(x,s,q,y,z) \in \mathbb{R}^5 \,|\, x=q+z\}$. We also recall that the second, third, fourth, and fifth components of the process (X,S,Q,Y,Z) can increase or decrease only at the hyperplanes d_1^5, d_2^5, d_3^5, and d_4^5, that is, when $X_t = S_t$, $X_t = Q_t$, $X_t = S_t - Y_t$, and $X_t = Q_t + Z_t$ for $t \ge 0$, respectively.

(i) Let is now consider the domain $0 < y \vee z < s-q \le y+z$. In this case, it follows that the candidate value function admits the representation

$$V(x,s,q,y,z;\infty) = C_1(s,q,y,z;\infty)\,\Psi_1(x,s,q,y,z) + C_2(s,q,y,z;\infty)\,\Psi_2(x,s,q,y,z) \tag{3.60}$$

in the region $R^5(\infty) = \{(x,s,q,y,z) \in E^5 \,|\, (s-a) < q \le s-y \le x \le q+z < s \wedge (q+b)\}$, where $C_i(s,q,y,z;\infty)$, $i=1,2$, solve the first-order linear ordinary differential equations (3.8) and (3.9) for all $q < q+y \vee z < s < q+y+z$. Observe that, on one hand, the process (X,S,Q,Y,Z) can reach the edge of the region $R^5(\infty)$ through the point $x = q+z = (q+b)\wedge s$, by hitting the hyperplane d_4^5 so that increasing its fifth component Z. Then, the component Z becomes either equal to the value b or is set to $S-Q$ and the region $R^5(\infty)$ is identified with $R_1^4(a)$ in part (ii) of Subsection 3.3. On the other hand, the process (X,S,Q,Y,Z) can reach the edge of the region $R^5(\infty)$ through the point $x = s-y = (s-a)\vee q$, by hitting the hyperplane d_3^5 so that increasing its fourth component Y. Then, the component Y becomes either equal to the value a or is set to $S-Q$ and the region $R^5(\infty)$ is identified with $R_2^4(b)$ in part (ii) of Subsection 3.4. Thus, the candidate value function should be continuous at the points $(s\wedge(q+b), s, q, y, (s-q)\wedge b)$ and $((s-a)\vee q, s, q, (s-q)\wedge a, z)$, that is expressed by the equalities

$$\begin{aligned} &C_1(s,q,y,(s-q)\wedge b;\infty)\,\Psi_1(s\wedge(q+b),s,q,y,(s-q)\wedge b) \\ &\qquad + C_2(s,q,y,(s-q)\wedge b;\infty)\,\Psi_2(s\wedge(q+b),s,q,y,(s-q)\wedge b) \\ &\quad = W_1(s,s,q,y;a)\,I(s<q+b) \end{aligned} \tag{3.61}$$

for all $q \le s$ and $0 < y < a$, and

$$\begin{aligned} &C_1(s,q,(s-q)\wedge a,z;\infty)\,\Psi_1((s-a)\vee q,s,q,(s-q)\wedge a,z) \\ &\qquad + C_2(s,q,(s-q)\wedge a,z;\infty)\,\Psi_2((s-a)\vee q,s,q,(s-q)\wedge a,z) \\ &\quad = W_2(q,s,q,z;b)\,I(q>s-a) + e^{-\theta s - \varkappa q}\,I(q \le s-a) \end{aligned} \tag{3.62}$$

for all $q \le s$ and $0 < z < b$, where $W_1(s,s,q,y;a)$ and $W_2(q,s,q,z;b)$ are determined in parts (ii) of Subsections 3.3 and 3.4. Moreover, we have the property $C_2(r,r,\varepsilon,\varepsilon;\infty) \to 0$ as $r \downarrow -\infty$, since otherwise $V(r,r,r,\varepsilon,\varepsilon;\infty) \to \pm\infty$, for any sufficiently small $\varepsilon > 0$, that must be excluded by virtue of the obvious fact that the value function in (2.5) is bounded (see Figs. 7 and 8). We may therefore conclude that the candidate value function admits the representation of (3.60) in the region $R^5(\infty)$, where $C_i(s,q,y,z;\infty)$, $i=1,2$, provide a unique solution of the two-dimensional system of first-order linear partial differential equations in (3.8) and (3.9) with the boundary conditions of (3.61)–(3.62) and $C_2(r,r,\varepsilon,\varepsilon;\infty) \to 0$ as $r \downarrow -\infty$, for any sufficiently small $\varepsilon > 0$.

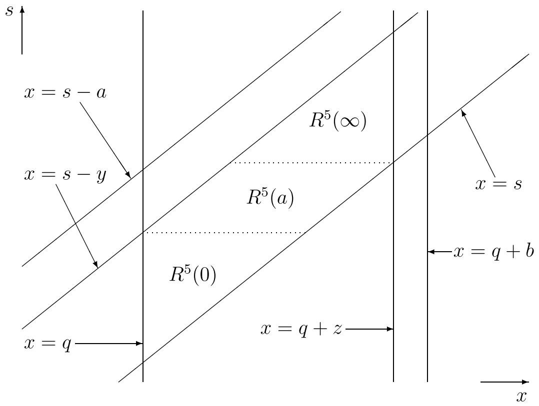

Fig. 7. A computer drawing of the state space of the process (X, S, Q, Y, Z), for some $q, y, z \in \mathbb{R}$ fixed and $y < z$.

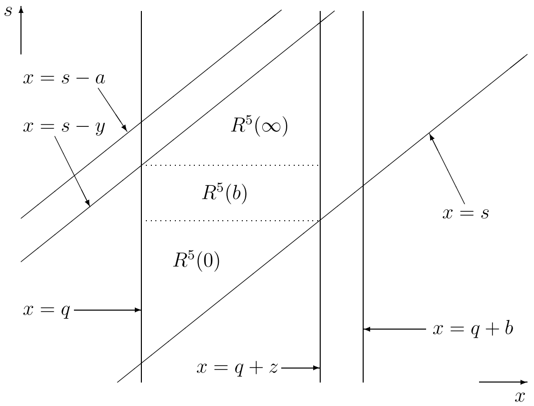

Fig. 8. A computer drawing of the state space of the process (X, S, Q, Y, Z), for some $q, y, z \in \mathbb{R}$ fixed and $z \leq y$.

(ii) Let us finally consider the domain $0 \leq s - q \leq y \vee z$. Observe that since the fourth or fifth component, Y or Z, is set to $S - Q$ after the process (X, S, Q, Y, Z) hits both hyperplanes d_2^5 and d_3^5, or d_1^5 and d_4^5, respectively, we may conclude that the candidate value function takes the form

$$V(x, s, q, y, z; a) = V(x, s, q, y, s - q; a) = W_1(x, s, q, y; a) \tag{3.63}$$

in the region $R^5(a) = \{(x, s, q, y, z) \in E^5 \,|\, (s - a) \vee q < s - y \leq x \leq s \leq q + z < q + b\}$ (see Fig. 7),

$$V(x, s, q, y, z; b) = V(x, s, q, s - q, z; b) = W_2(x, s, q, z; b) \tag{3.64}$$

in the region $R^5(b) = \{(x, s, q, y, z) \in E^5 \,|\, s - a < s - y \leq q \leq x \leq q + z < s \wedge (q + b)\}$ (see Fig. 8), and

$$V(x, s, q, y, z; 0) = V(x, s, q, s - q, s - q; 0) = U(x, s, q; 0) \tag{3.65}$$

in the region $R^5(0) = \{(x, s, q, y, z) \in E^5 \,|\, s - a < s - y \leq q \leq x \leq s \leq q + z < q + b\}$ (see Figs. 7 and 8), where the functions $W_1(x, s, q, y; a)$ and $W_2(x, s, q, z; b)$ are determined in parts (ii) of Subsections 3.3 and 3.4, and the function $U(x, s, q; 0)$ is determined in part (iv) of Subsection 3.2.

4. The main result and proof

In this section, taking into account the facts proved above, we formulate and prove the main result of the paper.

Theorem 4.1. *Suppose that the coefficients $\mu(x,s,q,y,z)$ and $\sigma(x,s,q,y,z)$ of the diffusion-type process X given by (2.1)–(2.3) are of their general form. Then the joint Laplace transform $V^*(x,s,q,y,z)$ from (2.5) of the associated with X random variables τ_a, S_{τ_a}, and Q_{τ_a} such that $\tau_a < \zeta_b$ from (2.4), admits the representation*

$$V^*(x,s,q,y,z) = \begin{cases} V(x,s,q,y,z;\infty), & \text{if } (s-a)\vee q < s-y \le x \le q+z < s \wedge (q+b) \\ V(x,s,q,y,z;a), & \text{if } s-a < s-y \le q \le x \le q+z < s \wedge (q+b) \\ V(x,s,q,y,z;b), & \text{if } (s-a)\vee q < s-y \le x \le s \le q+z < q+b \\ V(x,s,q,y,z;0), & \text{if } s-a < s-y \le q \le x \le s \le q+z < q+b \end{cases} \tag{4.1}$$

for any $a,b>0$ fixed. Here, the function $V(x,s,q,y,z;\infty)$ takes the form of (3.60) with $C_i(s,q,y,z;\infty)$, $i=1,2$, being a unique solution of the two-dimensional system of first-order partial differential equations in (3.8)+(3.9) and satisfying the conditions of (3.61)–(3.62) together with the property $C_2(r,r,\varepsilon,\varepsilon;\infty)\to 0$ as $r\downarrow -\infty$, for any sufficiently small $\varepsilon>0$, and the functions $V(x,s,q,y,z;a)$, $V(x,s,q,y,z;b)$, and $V(x,s,q,y,z;0)$ are given by (3.63), (3.64), and (3.65), respectively.

Proof. In order to verify the assertion stated above, it remains to show that the function defined in (4.1) coincides with the value function in (2.5). For this, let us denote by $V(x,s,q,y,z)$ the right-hand side of the expression in (4.1). Then, taking into account the fact that the function $V(x,s,q,y,z)$ is $C^{2,1,1,1,1}$ on E^5, by applying the change-of-variable formula from [33, Theorem 3.1] to $e^{-\lambda t}V(X_t,S_t,Q_t,Y_t,Z_t)$, we obtain that the expression

$$\begin{aligned} & e^{-\lambda(\tau_a\wedge\zeta_b\wedge t)}\, V(X_{\tau_a\wedge\zeta_b\wedge t}, S_{\tau_a\wedge\zeta_b\wedge t}, Q_{\tau_a\wedge\zeta_b\wedge t}, Y_{\tau_a\wedge\zeta_b\wedge t}, Z_{\tau_a\wedge\zeta_b\wedge t}) \\ &\quad = V(x,s,q,y,z) + M_{\tau_a\wedge\zeta_b\wedge t} \\ &\qquad + \int_0^{\tau_a\wedge\zeta_b\wedge t} e^{-\lambda u}\,(\mathbb{L}V-\lambda V)(X_u,S_u,Q_u,Y_u,Z_u) \\ &\qquad\qquad \times I(X_u\ne S_u, X_u\ne Q_u, X_u \ne S_u - Y_u, X_u\ne Q_u+Z_u)\, du \\ &\qquad + \int_0^{\tau_a\wedge\zeta_b\wedge t} e^{-\lambda u}\,\partial_s V(X_u,S_u,Q_u,Y_u,Z_u)\, I(X_u=S_u)\, dS_u \\ &\qquad + \int_0^{\tau_a\wedge\zeta_b\wedge t} e^{-\lambda u}\,\partial_q V(X_u,S_u,Q_u,Y_u,Z_u)\, I(X_u=Q_u)\, dQ_u \\ &\qquad + \int_0^{\tau_a\wedge\zeta_b\wedge t} e^{-\lambda u}\,\partial_y V(X_u,S_u,Q_u,Y_u,Z_u)\, I(X_u=S_u-Y_u)\, dY_u \\ &\qquad + \int_0^{\tau_a\wedge\zeta_b\wedge t} e^{-\lambda u}\,\partial_z V(X_u,S_u,Q_u,Y_u,Z_u)\, I(X_u=Q_u+Z_u)\, dZ_u \end{aligned} \tag{4.2}$$

holds for all $t \geq 0$ and the stopping times τ_a and ζ_b given by (2.4). Here, the process $M = (M_t)_{t \geq 0}$ defined by

$$M_t = \int_0^t e^{-\lambda u}\, \partial_x V(X_u, S_u, Q_u, Y_u, Z_u) \times I(X_u \neq S_u, X_u \neq Q_u, X_u \neq S_u - Y_u, X_u \neq Q_u + Z_u)\, \sigma(X_u, S_u, Q_u, Y_u, Z_u)\, dB_u \tag{4.3}$$

is a continuous local martingale under $P_{x,s,q,y,z}$. Note that, since the time spent by the process X at the hyperplanes d^5_k, $k = 1, 2, 3, 4$, is of Lebesgue measure zero, the indicators in the fourth line of formula (4.2) and in formula (4.3) can be ignored. Moreover, since the processes S, Q, Y, and Z change their values only on the hyperplanes d^5_1, d^5_2, d^5_3, and d^5_4, respectively, the indicators appearing in the fifth to eighth lines of (4.2) can be set equal to one.

By virtue of straightforward calculations and the arguments of the previous section, it is verified that the function $V(x, s, q, y, z)$ solves the ordinary differential equation in (2.7) and satisfies the normal-reflection conditions in (2.10)–(2.13). Observe that the process $(M_{\tau_a \wedge \zeta_b \wedge t})_{t \geq 0}$ is a uniformly integrable martingale, since the derivative and the coefficient in (4.3) are bounded functions on the compact set $\{(x, s, q, y, z) \in \mathbb{R}^5 \mid a \vee (s - y) \vee q \leq x \leq s \wedge (q + z) \wedge b\}$. Then, using the properties of the indicators mentioned above and taking the expectation with respect to $P_{x,s,q,y,z}$ in (4.2), by means of the optional sampling theorem (see, e.g. [28, Chapter III, Theorem 3.6] or [25, Chapter I, Theorem 3.22]), we get

$$\begin{aligned} &E_{x,s,q,y,z}\big[e^{-\lambda(\tau_a \wedge \zeta_b \wedge t)}\, V(X_{\tau_a \wedge \zeta_b \wedge t}, S_{\tau_a \wedge \zeta_b \wedge t}, Q_{\tau_a \wedge \zeta_b \wedge t}, Y_{\tau_a \wedge \zeta_b \wedge t}, Z_{\tau_a \wedge \zeta_b \wedge t})\big] \\ &\quad = V(x, s, q, y, z) + E_{x,s,q,y,z}\big[M_{\tau_a \wedge \zeta_b \wedge t}\big] = V(x, s, q, y, z) \end{aligned} \tag{4.4}$$

for all $(x, s, q, y, z) \in E^5$. Therefore, letting t go to infinity and using the instantaneous-stopping conditions in (2.8)–(2.9) as well as the fact that $e^{-\lambda(\tau_a \wedge \zeta_b)}\, V(X_{\tau_a \wedge \zeta_b}, S_{\tau_a \wedge \zeta_b}, Q_{\tau_a \wedge \zeta_b}, Y_{\tau_a \wedge \zeta_b}, Z_{\tau_a \wedge \zeta_b}) = 0$ on $\{\tau_a \wedge \zeta_b = \infty\}$ ($P_{x,s,q,y,z}$-a.s.), we can apply the Lebesgue dominated convergence theorem for (4.4) to obtain the equalities

$$\begin{aligned} &E_{x,s,q,y,z}\big[e^{-\lambda(\tau_a \wedge \zeta_b) - \theta S_{\tau_a \wedge \zeta_b} - \varkappa Q_{\tau_a \wedge \zeta_b}}\, I(\tau_a < \zeta_b)\big] \\ &\quad = E_{x,s,q,y,z}\big[e^{-\lambda(\tau_a \wedge \zeta_b)}\, V(X_{\tau_a \wedge \zeta_b}, S_{\tau_a \wedge \zeta_b}, Q_{\tau_a \wedge \zeta_b}, Y_{\tau_a \wedge \zeta_b}, Z_{\tau_a \wedge \zeta_b})\big] = V(x, s, q, y, z) \end{aligned} \tag{4.5}$$

for all $(x, s, q, y, z) \in E^5$, which directly implies the desired assertion. □

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