



Global existence and exponential growth of solution for the logarithmic Boussinesq-type equation



Qingying Hu, Hongwei Zhang*, Gongwei Liu

Department of Mathematics, Henan University of Technology, Zhengzhou 450001, PR China

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ABSTRACT

In this paper we consider the Boussinesq-type equation associated with logarithmic nonlinear term and initial boundary value conditions. By using potential well method combined with the logarithmic Sobolev inequality, we obtain the existence of global solution. At last we show that the L^2 -norm of the solution will grow up as an exponential function as time goes to infinity under some suitable conditions for initial data. The result shows that the polynomial nonlinearity is a critical condition of blow-up in finite time for the solutions of Boussinesq equations.

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1. Introduction

In this paper, we study the following initial boundary value problem:

$$v_{tt} - v_{xx} + v_{xxx} + (v_x \log |v_x|^k)_x = 0, \quad 0 < x < l, \quad t > 0, \quad (1.1)$$

$$v(0, t) = v(l, t) = 0, \quad v_x(0, t) = v_x(l, t) = 0, \quad t \geq 0, \quad (1.2)$$

$$v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), \quad x \in \Omega, \quad (1.3)$$

where $k \geq 1$, $\Omega = (0, l)$, $v(x, t)$ denotes the unknown function, the subscripts x and t denote partial derivatives with respect to x and t , $v_0(x)$ and $v_1(x)$ are given initial value functions and satisfy the boundary condition (1.2).

Equation (1.1) arises from a recent paper by Wazwaz [25], where the author firstly introduced the logarithmic Boussinesq equation (log-BE) in the form

$$u_{tt} + u_{xx} + u_{xxx} + (u \log |u|^k)_{xx} = 0 \quad (1.4)$$

* Corresponding author.

E-mail addresses: slxhqy@163.com (Q. Hu), whz661@163.com (H. Zhang), gongweiliu@126.com (G. Liu).

for analysis of the Gaussian solitary wave solutions based on the so-called logarithmic-KdV (log-KdV) equation defined by

$$v_t + v_{xxx} + (v \log|v|)_{xx} = 0.$$

As indicated in [29,7,18], let $u = v_x$ in (1.4), then (1.4) can be transformed into equation (1.1). It is well known that Boussinesq equation

$$u_{tt} - u_{xx} + bu_{xxxx} = a(u^2)_{xx} \quad (1.5)$$

and improved Boussinesq equation

$$u_{tt} - u_{xx} - bu_{xxtt} = a(u^2)_{xx} \quad (1.6)$$

are very important and famous high order strongly nonlinear mathematical physics model equations, where a, b are constants and $b > 0$. The Boussinesq equations (1.5), (1.6) appeared not only in the study of the dynamics of thin inviscid layers with free surface but also in the study of the nonlinear string, the shape-memory alloys, the propagation of waves in elastic rods and in the continuum limit of lattice dynamics or coupled electrical circuits. These equations also arise in other physical applications such as nonlinear lattice waves, ion sound waves in a plasma, and in vibrations in a nonlinear string. Moreover, it was applied to problems in the percolation of water in porous subsurface strata (see [25] and references therein). The study of Cauchy problem for Boussinesq-type equation has attracted considerable attention from many mathematicians and physicists over the last couple of decades. There are many Boussinesq-type equations, such as the generalized Boussinesq equation (or Bq equation)

$$u_{tt} - u_{xx} + bu_{xxxx} = f(u)_{xx}, \quad (1.7)$$

the improved Boussinesq equation (or IMBq equation)

$$u_{tt} - u_{xx} - u_{xxtt} = g(u)_{xx}, \quad (1.8)$$

its higher order generalizations (or generalized double dispersion equations)

$$u_{tt} - u_{xx} + bu_{xxxx} - u_{xxtt} = f(u)_{xx}, \quad (1.9)$$

and the multi-dimensional version of Boussinesq equation

$$u_{tt} - \Delta u - a\Delta u_{tt} + b\Delta^2 u = \Delta f(u), \quad a \geq 0. \quad (1.10)$$

A great deal of efforts has been made to establish the sufficient condition for the existence or nonexistence of global solution of Cauchy problem for Boussinesq type equations with various nonlinear terms, such as power-type nonlinearities $f(s) = \pm a|s|^p$, $f(s) = a|s|^p s$, $p > 1$, or more general nonlinearities satisfying either sign properties or some additional structure growth conditions such as $f'(s) \geq C$ (bounded below) or $f(s) = \sum_{k=1}^m a_k |s|^p s$ or $f(s) = e^{ks^2}$. As the literature on the Boussinesq-type equation is too extensive to mention all of them, we only partially refer the reader to [31,24,11,21,16,27,19,1,15,33] and references therein.

The practical, quantitative use of the Boussinesq-type equation and its relatives does not always involve the pure initial-value problem. Instead, IBVP on a finite domain or on the half line often come to the fore. To the authors' best knowledge, however, it is seldom discussed on global solutions of the initial boundary value problem for Boussinesq-type equation in the literature. Bona and Sachs [2] studied the

local well-posedness of the initial boundary value problem for Boussinesq equation (1.5) with small initial data. In [29,6], the authors proved that the initial boundary value problem for (1.8) has a unique global generalized or classical solution and gave the sufficient conditions of the nonexistence of global solution. Then the results were extended to a system of generalized IMBq equation [8] and the generalized cubic double dispersion equation [7]. In [32,22,23,12,28], the authors studied exponential stabilization and global solution of an initial boundary value problem for the damped Boussinesq equation. In [18], the initial boundary value problem of the generalized double dispersion equations (1.9) was studied by potential well method. The authors in [13,10,30] established the existence of the global attractor and an exponential attractor for the related initial boundary value problem of Boussinesq equation. However, although all of the above-mentioned paper considered initial boundary value problem of Boussinesq equation, the obtained results are only for power-type nonlinearities or some additional structure growth conditions and the results of [2,29,6,8,7,32,22,23,12,28,18,13,10,30] are not applicable for the logarithmic Boussinesq equation (1.1).

We also mention some related mathematical work involving power-type nonlinear term $f(v_x)_x$ in (1.1) in the literature (see, for example, [3,26] and references therein).

In this paper we consider the existence and blow-up of the solution for the initial boundary value problem (1.1)–(1.3). As far as we are concerned, this is the first work in the literature that takes into account existence and blow-up of global solution for the logarithmic Boussinesq equation. The motivation of studying problem (1.1)–(1.3) comes from the papers [4,5] which studied the following initial boundary value problem of a semilinear heat equation with logarithmic nonlinearity

$$u_t - \Delta u = u \log|u|, \quad x \in \Omega, \quad t > 0, \quad (1.11)$$

$$u(x, t) = 0, \quad x \in \partial\Omega, \quad t > 0, \quad (1.12)$$

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad (1.13)$$

where Ω is a bounded domain of R^N with smooth boundary. The ingredients used in their proof [4] are essentially the logarithmic Sobolev inequality (see [14,9]) and the potential well theory [20,17]. The potential well approach to stability of hyperbolic equations was originally developed by Payne and Sattinger [20]. Since then the technique has been employed by many authors to analyze hyperbolic or parabolic equation (e.g. [17]). We can see that the potential well method has been widely used and now it is a very popular and important way to prove the global nonexistence of solution for nonlinear hyperbolic or parabolic equations which has positive definite energy. And this potential well method also plays a very important role in deriving the threshold between the global existence and nonexistence of solutions. The main difficulty in the case of logarithmic nonlinearity term is that the normal potential well method in [20,17] will be not suitable for the problem (1.11)–(1.13), since the nonlinear function $f(u) = u \log|u|$ doesn't satisfy the conditions of $f(u)$ in [20,17]. To handle the logarithmic nonlinear term $u \log|u|$, the author in [4] introduced the logarithmic Sobolev inequality.

The main purpose of this article is to establish the sufficient conditions for existence and blow-up of global solution to the logarithmic Boussinesq equation (1.1)–(1.3). By using the logarithmic Sobolev inequality and potential well method, we obtain the existence of global solution for the problem (1.1)–(1.3). The ingredients used in our proof are essentially the logarithmic Sobolev inequality and Galerkin methods combined with potential well method. We find that the normal concavity method (e.g. [18]) no longer applies in this particular situation when the logarithmic nonlinear term appears, for blow-up in finite time of the solutions. So, we adapt the normal concavity method to get the blow-up property in infinity time, i.e. exponential growth. That is to say, we show that the L^2 -norm of the solution will grow up as an exponential function as time goes to infinity under some suitable conditions for initial data. The result shows that the polynomial nonlinearity (see [29,7,18]) is a critical condition of blow-up in finite time for the solutions of Boussinesq equations.

This article is organized as follows. Section 2 is concerned with some notation and some lemmas. In Section 3, we prove the existence of global solutions to problem (1.1)–(1.3) by using a Galerkin approximation scheme combined with the potential well theory. In Section 4, we prove the blow-up at $+\infty$ of the solution for the problem (1.1)–(1.3).

2. Preliminaries and main results

In this paper, we denote $\Omega = (0, l)$, $L^p(\Omega)$ by L^p , $H^s(\Omega)$ by H^s , $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$, $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$ and $(u, v) = \int_{\Omega} u(x)v(x)dx$.

We call $v(x, t)$ a weak solution of problem (1.1)–(1.3) on $\Omega \times [0, T)$, if $v \in L^\infty(0, T; H_0^2)$, $v_t \in L^\infty(0, T; L^2)$ satisfy equation (1.1) in the distribution sense, i.e.

$$(v_t, w) + \int_0^t [(v_x, w_x) + (v_{xx}, w_{xx})] ds - \int_0^t (v_x \log |v_x|^k, w_x) ds = (v_1, w), \quad \forall w \in H_0^2, \quad \forall t \in [0, T), \quad (2.1)$$

$$v(x, 0) = v_0(x) \text{ in } H_0^2, \quad v_t(x, 0) = v_0(x) \text{ in } L^2 \quad (2.2)$$

and the conservation law of the full energy of the weak solutions v holds, i.e.

$$E(t) \leq E(0), \quad \text{for all } t \in [0, T), \quad (2.3)$$

where

$$E(t) = E(v) = \frac{1}{2}(\|v_t\|^2 + \|v_x\|^2 + \|v_{xx}\|^2 - \int_0^l v_x^2 \log |v_x|^k dx) + \frac{k}{4}\|v_x\|^2,$$

$$E(0) = \frac{1}{2}(\|v_1\|^2 + \|v_{0x}\|^2 + \|v_{0xx}\|^2 - \int_0^l v_{0x}^2 \log |v_{0x}|^k dx) + \frac{k}{4}\|v_{0x}\|^2$$

Throughout this paper we will make repeated use of the following logarithmic Sobolev inequality, which can be founded in [4,14,9].

Lemma 2.1 (Logarithmic Sobolev inequality). (See [4,14,9].) Let u be any function in $H_0^1(\Omega)$ and $a > 0$ be any number. Then

$$2 \int_{\Omega} |u|^2 \log \frac{|u|}{\|u\|} dx + (1 + \log a) \|u\|^2 \leq \frac{a^2}{\pi} \int_{\Omega} |u_x|^2 dx. \quad (2.4)$$

Now, we introduce two functionals $J(v)$ and $I(v)$:

$$J(v) = J(v(t)) = J(t) = \frac{1}{2}(\|v_x\|^2 + \|v_{xx}\|^2 - \int_0^l v_x^2 \log |v_x|^k dx) + \frac{k}{4}\|v_x\|^2, \quad (2.5)$$

$$I(v) = I(v(t)) = I(t) = \|v_x\|^2 + \|v_{xx}\|^2 - \int_0^l v_x^2 \log |v_x|^k dx. \quad (2.6)$$

Then, it is obvious that

$$J(v) = \frac{1}{2}I(v) + \frac{k}{4}\|v_x\|^2, \quad (2.7)$$

$$E(t) = \frac{1}{2}\|v_t\|^2 + J(v). \quad (2.8)$$

According to the logarithmic Sobolev inequality (Lemma 2.1), $J(v)$ and $I(v)$ are well defined on H_0^2 . Furthermore, $E(v) = E(t)$ is well defined on H_0^2 .

Lemma 2.2. For any $v \in H_0^2(\Omega) \setminus \{0\}$, then:

(i) $I(\lambda v) = \lambda \frac{d}{d\lambda} J(\lambda v)$ and $\lim_{\lambda \rightarrow 0} J(\lambda v) = 0$, $\lim_{\lambda \rightarrow +\infty} J(\lambda v) = -\infty$.

(ii) There exists a unique $\lambda^* = \lambda^*(v)$ such that $\frac{d}{d\lambda} J(\lambda v)|_{\lambda=\lambda^*} = 0$, and $J(\lambda v)$ is increasing on $0 < \lambda \leq \lambda^*$, decreasing on $\lambda^* \leq \lambda < \infty$ and takes the maximum at $\lambda = \lambda^*$. In other words, that is there exists a unique $\lambda^* \in (0, +\infty)$ such that

$$I(\lambda v) = \lambda \frac{d}{d\lambda} J(\lambda v) > 0, \text{ for } 0 < \lambda < \lambda^*; \quad I(\lambda v) = 0, \text{ for } \lambda = \lambda^*; \quad I(\lambda v) < 0, \text{ for } \lambda > \lambda^*, \quad (2.9)$$

where $\lambda^* = \exp\left(\frac{\|v_x\|^2 + \|v_{xx}\|^2 - \int_0^l v_x^2 \log|v_x|^k dx}{k\|v_x\|^2}\right)$.

Proof. It is obvious that

$$\begin{aligned} J(\lambda v) &= \frac{1}{2}\lambda^2(\|v_x\|^2 + \|v_{xx}\|^2) + \frac{k}{4}\lambda^2\|v_x\|^2 - \frac{1}{2}\int_0^l \lambda^2 v_x^2 \log|\lambda v_x|^k dx \\ &= \frac{1}{2}\lambda^2(\|v_x\|^2 + \|v_{xx}\|^2) + \frac{k}{4}\lambda^2\|v_x\|^2 - \frac{k}{2}\lambda^2 \int_0^l v_x^2 \log|v_x| dx - \frac{k}{2}\lambda^2(\log|\lambda|)\|v_x\|^2 \end{aligned}$$

and

$$\begin{aligned} \frac{d}{d\lambda} J(\lambda v) &= \lambda(\|v_x\|^2 + \|v_{xx}\|^2) - \lambda \int_0^l v_x^2 \log|v_x|^k dx - \lambda(\log|\lambda|^k)\|v_x\|^2 \\ &= \lambda(\|v_x\|^2 + \|v_{xx}\|^2) - \lambda \int_0^l v_x^2 \log|\lambda v_x|^k dx. \end{aligned}$$

Then we have $I(\lambda v) = \lambda \frac{d}{d\lambda} J(\lambda v)$ and $\lim_{\lambda \rightarrow 0} J(\lambda v) = 0$, $\lim_{\lambda \rightarrow +\infty} J(\lambda v) = -\infty$ since $\|v_x\| \neq 0$. Furthermore, we have

$$I(\lambda v) > 0, \quad 0 < \lambda < \lambda^*; \quad I(\lambda v) = 0, \quad \lambda = \lambda^*; \quad I(\lambda v) < 0, \quad \lambda > \lambda^*,$$

where $\lambda^* = \exp\left(\frac{\|v_x\|^2 + \|v_{xx}\|^2 - \int_0^l v_x^2 \log|v_x|^k dx}{k\|v_x\|^2}\right)$, and $J(\lambda v)$ is increasing on $0 < \lambda < \lambda^*$, decreasing on $\lambda > \lambda^*$ and takes the maximum at $\lambda = \lambda^*$. $\square$

We introduce the well-known Nehari manifold and well depth d

$$N = \{v | v \in H_0^2 \setminus \{0\}, I(v) = 0\}, \quad (2.10)$$

$$d = \inf_{\lambda \geq 0} \{\sup J(\lambda v) | v \in H_0^2 \setminus \{0\}\}. \quad (2.11)$$

Then, [Lemma 2.2](#) implies that $0 < d = \inf_{v \in N} J(v)$. Now we define two subsets of H_0^2 related to problem (1.1)–(1.3). Set

$$W_1 = \{v \in H_0^2 \mid J(v) < d, I(v) > 0\} \cup \{0\}, \quad (2.12)$$

$$W_2 = \{v \in H_0^2 \mid J(v) < d, I(v) < 0\}. \quad (2.13)$$

Lemma 2.3. Let $v \in H_0^2$ and $r_0 = (\frac{2\pi}{k})^{\frac{1}{4}} e^{\frac{1}{2}}$,

(i) if $0 < \|v_x\| < r_0$, then $I(v) > 0$;

(ii) if $I(v) < 0$, then $\|v_x\| > r_0$;

(iii) if $I(v) = 0$ and $\|v_x\| \neq 0$, i.e. $v \in N$, then $\|v_x\| > r_0$.

Proof. By the logarithmic Sobolev inequality, for any $a > 0$, we have

$$\begin{aligned} I(v) &= \|v_x\|^2 + \|v_{xx}\|^2 - k \int_0^l v_x^2 (\log \frac{|v_x|}{\|v_x\|} + \log \|v_x\|) dx \\ &\geq (1 - \frac{ka^2}{2\pi})(\|v_x\|^2 + \|v_{xx}\|^2) + \frac{k(1 + \log a)}{2} \|v_x\|^2 - k \|v_x\|^2 \log \|v_x\|. \end{aligned} \quad (2.14)$$

Taking $a = \sqrt{\frac{2\pi}{k}}$ in (2.14), we gain

$$I(v) \geq k(\frac{2 + \log \frac{2\pi}{k}}{4} - \log \|v_x\|) \|v_x\|^2. \quad (2.15)$$

If $0 < \|v_x\| < r_0$, then $I(v) > 0$ from (2.15). If $I(v) < 0$, then by (2.15), we have

$$\frac{2 + \log \frac{2\pi}{k}}{4} - \log \|v_x\| < 0,$$

this means that $\|v_x\| > (\frac{2\pi}{k})^{\frac{1}{4}} e^{\frac{1}{2}} = r_0$. If $I(v) = 0$ and $\|v_x\| \neq 0$, then by (2.15), we have

$$\log \|v_x\| \geq \frac{2 + \log(\frac{2\pi}{k})}{4}, \quad (2.16)$$

that is $\|v_x\| > r_0$. $\square$

Lemma 2.4. (i) $d \geq \frac{k}{4} (\frac{2\pi}{k})^{\frac{1}{2}} e = \frac{k}{4} r_0^2$;

(ii) If $v \in H_0^2$ and $I(v) < 0$, then

$$I(v) < 2(J(v) - d). \quad (2.17)$$

Proof. (i) If $I(v) = 0$ and $\|v_x\| \neq 0$, then by [Lemma 2.3](#) we have $\|v_x\| \geq r_0$. Together with (2.7), we get

$$J(v) = \frac{1}{2} I(v) + \frac{k}{4} \|v_x\|^2 \geq \frac{k}{4} (\frac{2\pi}{k})^{\frac{1}{2}} e,$$

then, we have $d \geq \frac{k}{4} (\frac{2\pi}{k})^{\frac{1}{2}} e$.

(ii) If $v \in H_0^2$ and $I(v) < 0$, then from [Lemma 2.2](#) it follows that there exists a λ^* such that $0 < \lambda^* < 1$ and $I(\lambda^* v) = 0$. From the definition of d in (2.11), we have

$$\begin{aligned} d &\leq J(\lambda^*v) = \frac{1}{2}I(\lambda^*v) + \frac{k}{4}\|\lambda^*v_x\|^2 \\ &= \frac{k}{4}\|\lambda^*v_x\|^2 < \frac{k}{4}\|v_x\|^2, \end{aligned}$$

from this and the definition of $J(v)$ and $I(v)$, we get

$$d < \frac{k}{4}\|v_x\|^2 = J(v) - \frac{1}{2}I(v),$$

that is to say

$$I(v) < 2(J(v) - d). \quad \square$$

Lemma 2.5. *If $v_0 \in H_0^2, v_1 \in L^2$ and $0 < E(0) < d$, v is a weak solution of problem (1.1)–(1.3), then:*

- (i) $v \in W_1$ if $I(v_0) > 0$;
- (ii) $v \in W_2$ if $I(v_0) < 0$.

Proof. Let T be the maximal existence time of weak solution of v .

(i) From the definition of weak solution and (2.3), we have

$$\frac{1}{2}\|v_t\|^2 + J(v) \leq \frac{1}{2}\|v_1\|^2 + J(v_0) < d, \quad \text{for any } t \in [0, T]. \quad (2.18)$$

Then we claim that $v(t) \in W_1$ for all $t \in [0, T]$. If it is false, then there is a $t_0 \in (0, T)$ such that $v(t_0) \in \partial W_1$, so we have

(a) $I(v(t_0)) = 0$ and $\|v_x(t_0)\| \neq 0$, or (b) $J(v(t_0)) = d$.

By (2.18), (b) is impossible, thus we have $I(v(t_0)) = 0$ and $\|v_x(t_0)\| \neq 0$. However, from the definition of d , one has $J(v(t_0)) \geq d$, which is contractive with (2.18). Then we have $v(t) \in W_1$ for all $t \in [0, T]$.

(ii) If there is a $t_0 \in (0, T)$ such that $v(x, t) \in W_2$ for $0 \leq t < t_0$, and $v(x, t_0) \in \partial W_2$, thus

(c) $I(v(t_0)) = 0$, or (d) $J(v(t_0)) = d$.

From (2.18), (d) is not true. If $I(v(t_0)) = 0$ and $I(v(t)) < 0$ for $0 < t < t_0$, then $\|v_x\| > r_0$ for $0 < t \leq t_0$ by Lemma 2.3(ii). Hence, from the definition of d , we get $J(v(t_0)) \geq d$, which is contractive with (2.18). This implies that $v(t) \in W_2$ for all $t \in [0, T]$. $\square$

3. Existence of global solution

In this section we study the existence of global solution for problem (1.1)–(1.3).

Theorem 3.1. *Let $v_0 \in H_0^2, v_1 \in L^2$. Assume that $E(0) < d$, and $I(v_0) > 0$ or $\|v_0\| = 0$, then the problem (1.1)–(1.3) admits a global weak solution $v(t) \in L^\infty(0, \infty; H_0^2)$ with $v_t(t) \in L^\infty(0, \infty; L^2)$ and $v(\cdot) \in W_1$ for $0 \leq t < \infty$.*

Proof. We prove the existence of global solution by using the Fadeko–Galerkin approximations. Let $\{w_j\}_{j=1}^\infty$ be a basis function system in H_0^2 . Construct the approximate solutions of problem (1.1)–(1.3)

$$v_m(x, t) = \sum_{j=1}^m g_{jm}(t)w_j(x), \quad m = 1, 2, \dots, \quad (3.1)$$

satisfying

$$(v_{mtt}, w_s) + (v_{mx}, w_{sx}) + (v_{mxx}, w_{sxx}) - (v_{mx} \log |v_{mx}|^k, w_{sx}) = 0, \quad s = 1, 2, \dots, m, \quad (3.2)$$

$$v_m(x, 0) = \sum_{j=1}^m a_{jm}(t) w_j(x) \rightarrow v_0(x) \quad \text{in } H_0^2, \quad (3.3)$$

$$v_{mt}(x, 0) = \sum_{j=1}^m b_{jm}(t) w_j(x) \rightarrow v_1(x) \quad \text{in } L^2. \quad (3.4)$$

Multiplying (3.2) by $g'_{sm}(t)$ and summing for s we get

$$\frac{d}{dt} E_m(t) = 0 \quad (3.5)$$

and

$$E_m(t) = E_m(0), \quad (3.6)$$

where

$$E_m(t) = E(v_m) = \frac{1}{2} (\|v_{mt}\|^2 + \|v_{mx}\|^2 + \|v_{mxx}\|^2 - \int_0^l v_{mx}^2 \log |v_{mx}|^k dx) + \frac{k}{4} \|v_{mx}\|^2.$$

It is obvious that it follows from (3.3) and dominated convergence theorem that

$$\int_0^l v_{0mx}^2 \log |v_{0mx}|^k dx \rightarrow \int_0^l v_{0x}^2 \log |v_{0x}|^k dx$$

as $m \rightarrow \infty$ and then

$$E_m(0) \rightarrow E(0) \quad \text{as } m \rightarrow \infty. \quad (3.7)$$

Next we claim that for the approximate solutions v_m defined by (3.1) there holds $v_m(t) \in W_1$ for all $t \in [0, T)$ and sufficiently large m . Arguing by contradiction, we assume that there exists $\bar{t} \in (0, T)$ such that $v_m(\bar{t}) \notin W_1$. By $E_m(0) \rightarrow E(0)$ as $m \rightarrow \infty$, and $E(0) < d$, we have $J(v_m(t)) \leq E_m(0) < d$ for all $t \in [0, T)$. According to the continuity of $I_m(t) = I(v_m(t))$ with respect to t , it follows that there is a $0 < t_0 < \bar{t}$ such that $v_m(t_0) \in \partial W_1$. Since $J(v_m(t_0)) = d$ is impossible, hence $I(v_m(t_0)) = 0$ and $\|v_{mx}(t_0)\| = 0$ for some sufficiently large m . From the definition of d we get $J(v_m(t_0)) \geq d$, which contradicts (by (3.7))

$$E_m(t) = \frac{1}{2} \|v_{mt}\|^2 + J(v_m) = E_m(0) < d, \quad 0 \leq t < T \quad (3.8)$$

for sufficiently large m . Thus we conclude that $v_m(t) \in W_1$ for all $t \in [0, T)$ and sufficiently large m .

From (3.8), using (2.7) and $I(v_m) > 0$, we get that for sufficiently large m there holds

$$\begin{aligned} d &> E_m(0) = \frac{1}{2} \|v_{mt}\|^2 + J(v_m) \\ &\geq J(v_m) = I(v_m) + \frac{k}{4} \|v_{mx}\|^2 \end{aligned} \quad (3.9)$$

then we have

$$\|v_{mx}\|^2 \leq 4d, \quad \|v_{mt}\|^2 \leq 2d. \quad (3.10)$$

Furthermore, taking $a = \sqrt{\frac{k}{\pi}}$ and $v = v_m$ in (2.14), using (2.7) and (3.10) and noting $J(v_m) < d$, we have

$$\begin{aligned} \|v_{mx}\|^2 + \|v_{mxx}\|^2 &\leq 2I(v_m) - k(1 + \log\sqrt{\frac{k}{\pi}})\|v_{mx}\|^2 + k\|v_{mx}\|^2 \log\|v_{mx}\| \\ &\leq 4J(v_m) - k(1 + \log\sqrt{\frac{k}{\pi}})\|v_{mx}\|^2 + k\|v_{mx}\|^2 \log\|v_{mx}\| < C. \end{aligned} \quad (3.11)$$

By a direct calculation and the fact that $(\log|z|)^2 \leq z^2$ for $|z| > 1$, we have

$$\begin{aligned} \int_0^l |v_{mx} \log|v_{mx}||^k dx &= \int_{\{0 \leq x \leq l: |v_{mx}| \leq 1\}} k^2 |v_{mx} \log|v_{mx}||^2 dx + \int_{\{0 \leq x \leq l: |v_{mx}| > 1\}} k^2 |v_{mx} \log|v_{mx}||^2 dx \\ &\leq le^{-2} + lk^2 \|v_{mx}\|_4^4 \leq le^{-2} + lk^2 C_* \|v_{mxx}\|^4 < C, \end{aligned} \quad (3.12)$$

where we have used (3.11) and C_* is the constant of Sobolev embedding H_0^2 into L^4 .

From (3.10), (3.11) and (3.12), it follows that there exists a subsequence of $\{v_m\}$, denote by $\{v_m\}$, such that

$$\begin{aligned} v_m &\rightarrow v \text{ in } L^\infty(0, T; H_0^2) \text{ weakly star;} \\ v_{mt} &\rightarrow v_t \text{ in } L^\infty(0, T; L^2) \text{ weakly star;} \\ v_{mx} \log|v_{mx}|^k &\rightarrow v_x \log|v_x|^k \text{ in } L^\infty(0, T; L^2) \text{ weakly star.} \end{aligned}$$

Integrating (3.2) with respect to t from 0 to t , and letting $m \rightarrow \infty$ for the fixed s , we get

$$(v_t, w_s) + \int_0^t ((v_x, w_{sx}) + (v_{xx}, w_{sxx})) ds - \int_0^t (v_x \log|v_x|^k, w_{sx}) ds = (v_1, w_s)$$

and (2.1) in definition of the solution. On the other hand, (3.3) and (3.4) give $v(x, 0) = v_0(x)$, $v_t(x, 0) = v_1(x)$, i.e. (2.2) holds. From (3.6) and (3.7), we know that the above v satisfies (2.3). Therefore above $v(x, t)$ is a global weak solution of problem (1.1)–(1.3). Finally from Lemma 2.5 we get $v(t) \in W_1$ for $0 \leq t < T$. $\square$

4. Exponential growth of the solution

In this section, we shall prove blowup property in infinity time of the solution for problem (1.1)–(1.3). In fact, we show that the L^2 -norm of the solution will grow up as an exponential function as time goes to infinity under some suitable conditions for initial data.

Theorem 4.1. *Let $v_0 \in H_0^2$, $v_1 \in L^2$. Assume further that $v_0 \in W_2$, $0 < E(0) < d$, then the L^2 -norm of the solution of problem (1.1)–(1.3) will grow up as an exponential function as time goes to infinity.*

Proof. Let $v(t, x)$ is a weak solution of problem (1.1)–(1.3) with $J(v_0) < E(0) < d$, $I(v_0) < 0$. Then we consider the function $\phi(t) : [0, +\infty) \rightarrow R^+$ defined by

$$\phi(t) = \|v\|^2. \quad (4.1)$$

Direct computations show that

$$\phi'(t) = 2(v, v_t). \quad (4.2)$$

Therefore, due to the equation (1.1), and the definition of $I(v)$, we have

$$\begin{aligned}\phi''(t) &= 2\|v_t\|^2 + 2(v, v_{tt}) \\ &= 2\|v_t\|^2 - 2(\|v_x\|^2 + \|v_{xx}\|^2 - \int_0^l v_x^2 \log|v_x|^k dx) \\ &= 2\|v_t\|^2 - 2I(v).\end{aligned}\tag{4.3}$$

Using the Cauchy–Schwarz inequality, it follows from (4.2) that for $t \in [0, \infty)$

$$|\phi'(t)|^2 \leq 4\phi(t)\|v_t\|^2.$$

Then we have for each $t \in [0, \infty)$ that, by (2.8) and (2.3)

$$\begin{aligned}\phi''(t)\phi(t) - [\phi'(t)]^2 &\geq 2\phi(t)[\|v_t\|^2 - I(v(t))] - 4\phi(t)\|v_t\|^2 \\ &= -2\phi(t)[\|v_t\|^2 + I(v(t))] \\ &\geq -2\phi(t)[E(0) - J(v(t)) + I(v(t))].\end{aligned}\tag{4.4}$$

Since $v_0 \in W_2$, $E(0) < d$, it follows from Lemma 2.5 that $I(v(t)) < 0$. Thus, using Lemma 2.4(2), we can obtain that

$$E(0) - J(v(t)) + I(v(t)) < d - J(v(t)) + 2(J(v(t)) - d) = J(v(t)) - d < 0$$

and

$$\phi''(t)\phi(t) - [\phi'(t)]^2 > 0.\tag{4.5}$$

On the other hand, by direct calculation, we can see that

$$(\log|\phi(t)|)' = \frac{\phi'(t)}{\phi(t)},\tag{4.6}$$

$$(\log|\phi(t)|)'' = \left(\frac{\phi'(t)}{\phi(t)}\right)' = \frac{\phi''(t)\phi(t) - [\phi'(t)]^2}{\phi^2(t)} > 0.\tag{4.7}$$

By (4.7), we know that $(\log|\phi(t)|)' = \frac{\phi'(t)}{\phi(t)}$ is increasing on t , using this fact, integrating (4.6) from t_0 to t , we have

$$\log|\phi(t)| - \log|\phi(t_0)| = \int_{t_0}^t (\log|\phi(\tau)|)' d\tau = \int_{t_0}^t \frac{\phi'(\tau)}{\phi(\tau)} d\tau \geq \frac{\phi'(t_0)}{\phi(t_0)}(t - t_0),$$

where $0 \leq t_0 < t$. Then

$$\phi(t) \geq \phi(t_0) \exp\left(\frac{\phi'(t_0)}{\phi(t_0)}(t - t_0)\right).\tag{4.8}$$

If we can take t_0 sufficiently small such that $\phi'(t_0) > 0$, $\phi(t_0) > 0$, then, from (4.8), we have $\lim_{t \rightarrow +\infty} \phi(t) = \infty$, that is to say that the L^2 -norm of the solutions of problem (1.1)–(1.3) grows up as an exponential function as time goes to infinity.

Indeed, from (4.3), Lemma 2.4(ii), and the conservation law of the energy (2.3), it follows that $\phi''(t) \geq 2(d - E(0)) > 0$. After twice integrations of this inequality one get $\phi(t_0) > 0, \phi'(t_0) > 0$ for sufficiently large t_0 . Then we complete the proof. $\square$

Remark. The result of Theorem 4.1 tells us the blowup property at infinity time of the solution, but we don't know whether the solution blows up in finite time. We find that the normal concavity method (e.g. [18]) no longer applies in this particular situation when the logarithmic nonlinear term appears, for blow-up in finite time of the solutions. In (1.1), if the nonlinear term is power-type, such as $u|u|^p$, under the similar conditions with that in Theorem 4.1, the results about blowup of weak solution in finite time have been proved by many authors (see [29,7,18]). However, if the nonlinear term is the logarithmic nonlinear term $u \log|u|$, then Theorem 4.1 tells us that there is quite a difference in cases of the equations with polynomial nonlinear term or logarithmic nonlinear term respectively and that the polynomial nonlinearity is a critical condition of blow-up in finite time for the solutions of Boussinesq equations.

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