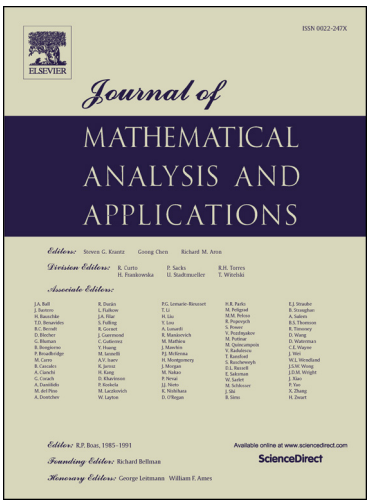
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On Maximal Regularity for a Class of Evolutionary Equations.

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The issue of so-called maximal regularity is discussed within a Hilbert space framework for a class of evolutionary equations. Viewing evolutionary equations as a sum of two unbounded operators, showing maximal regularity amounts to establishing that the operator sum considered with its natural domain is already closed. For this we use structural constraints of the coefficients rather than semi-group strategies or sesqui-linear form methods, which would be difficult to come by for our general problem class. Our approach, although limited to the Hilbert space case, complements known strategies for approaching maximal regularity and extends them in a different direction. The abstract findings are illustrated by re-considering some known maximal regularity results within the framework presented.

Keywords and phrases: maximal regularity, evolutionary equations, material laws, coupled systems

Mathematics subject classification 2010: 35B65, 35D35, 58D25

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0 Introduction

The issue of maximal regularity has received much attention as an important property of certain partial differential equations and more abstractly as a feature of a class of evolution equations. In a Hilbert space setting, the typical situation thus refers to an abstract operator equation in $L^{2,\mathrm{loc}}([0,\infty[,H)$ of the form

$$u' + \mathcal{A}u = f, \tag{1}$$

for some given $f \in L^{2,\mathrm{loc}}([0,\infty[,H)$, H a Hilbert space. Moreover, $u\colon]0,\infty[\to H$ is a measurable function with u' being its weak derivative and $\mathcal{A}$ is the (abstract) linear operator on $L^{2,\mathrm{loc}}([0,\infty[,H)$ induced by an operator A, assumed to be the infinitesimal generator of a one-parameter C_0-semi-group, i.e. $(\mathcal{A}u)(t) \coloneqq A(u(t))$. If we solve equation (1) for u subject to homogeneous initial conditions, we can expect u to be at best only continuous. Thus, u is a so-called *mild solution of* (1), that is, u solves the equation in question in an integrated form. To obtain better regularity behaviour one is interested in the case where for any given f, the corresponding solution u is such that u' and $\mathcal{A}u$ both belong to $L^{2,\mathrm{loc}}([0,\infty[,H)$ and, hence, u *literally* solves (1) in $L^{2,\mathrm{loc}}([0,\infty[,H)$. This property is commonly attributed to the semi-group generator A and one says in this case that A *admits maximal L^2-regularity*. A standard situation is that $\mathcal{A}$ is a non-negative selfadjoint operator. In this case, $\mathcal{A} = C^*C$ for some closed and densely defined linear operator C. Then the corresponding evolution equation admits maximal regularity as can be easily seen with the help of the spectral theorem for A. We refer to the seminal paper [7] as a standard reference for maximal regularity. We also refer the reader to [5, 6, 3] for the L^p-maximal regularity of second-order Cauchy problems, to [1, 2] for maximal regularity for non-autonomous problems, to [14, 17] for integro-differential equations and to [4, 13] for fractional differential equations.

In this article, we revisit the standard Hilbert space case $\mathcal{A} = C^*C$ under a system perspective: By setting $v \coloneqq -Cu$ we deduce from (1), writing ∂_0 for the time derivative, that

$$\begin{pmatrix} \partial_0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

Now, we ask for the maximal regularity when the coefficients $\begin{pmatrix} \partial_0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are replaced by more general operators $\begin{pmatrix} \partial_0\mathcal{M} & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathcal{N}$ acting in space time. Under suitable

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conditions on $\mathcal{M}$ and $\mathcal{N}$ we will show, in our main Theorem 2.4, that for a given L^2-right-hand side f, the solution (u, v) has the following properties. We have that u is weakly L^2-differentiable with respect to time and that $(u, v) \in D\left(\begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}\right)$. Moreover, the equation

$$\begin{pmatrix} \partial_0 \mathcal{M} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \mathcal{N} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

is satisfied *literally*. This remains true if we alter the right-hand side $\begin{pmatrix} f \\ 0 \end{pmatrix}$ to $\begin{pmatrix} f \\ g \end{pmatrix}$ for any weakly differentiable g. Our first order approach complements known results on maximal regularity by allowing for quite general coefficients $\mathcal{M}, \mathcal{N}$. With this generalization, we enter the realm of so-called evolutionary equations, which we briefly introduce in the next section. This class comprises of the standard initial boundary value problems of mathematical physics in a unified setting, we refer to [12] for a survey. After having introduced the mathematical framework, we will provide our main result in Section 2. We conclude this article with several illustrative examples in the last section. The more involved examples are (abstract) second order problems (in both time and space) (adapted from [3, 14]) as well as problems with a fractional time derivative, which is an adaptation from [13].

1 A brief description of the framework of evolutionary equations

We recall the notion of evolutionary equations, as introduced in [8, Solution Theory], a term we use in distinction to classical evolution equations, which are a special case. For this, let throughout ϱ be a positive, real parameter and H a Hilbert space. Define

$$L^2_\varrho(\mathbb{R}, H) := \{f \in L^2_{\mathrm{loc}}(\mathbb{R}, H) | (t \mapsto e^{-\varrho t} f(t)) \in L^2(\mathbb{R}, H)\},$$

which endowed with the natural scalar product

$$\langle f, g \rangle := \int_{\mathbb{R}} \langle f(t), g(t) \rangle_H e^{-2\varrho t} \, \mathrm{d}t \quad (f, g \in L^2_\varrho(\mathbb{R}, H))$$

is a Hilbert space. The operator

$$\partial_0 \colon D(\partial_0) \subseteq L^2_\varrho(\mathbb{R}, H) \to L^2_\varrho(\mathbb{R}, H), f \mapsto f',$$

with f' being the distributional derivative and $D(\partial_0) = \{f \in L^2_\varrho(\mathbb{R}, H) | f' \in L^2_\varrho(\mathbb{R}, H)\}$, defines a normal operator with $\mathfrak{Re}\, \partial_0 = \varrho$ (see e.g [12, Section 2.2]). Indeed, ∂_0 is unitarily equivalent to the operator $\mathrm{i}m + \varrho$, of multiplication by the function $\xi \mapsto \mathrm{i}\xi + \varrho$, considered as an operator in $L^2(\mathbb{R}, H)$. This spectral representation result is realized by the so-called

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Fourier–Laplace transformation $\mathcal{L}_\varrho\colon L^2_\varrho(\mathbb{R},H)\to L^2(\mathbb{R},H)$, that is, the unitary extension of the integral operator given by

$$\mathcal{L}_\varrho\phi(\xi):=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\mathrm{i}t\xi-\varrho t}\phi(t)\,\mathrm{d}t\quad(\xi\in\mathbb{R})$$

for bounded, measurable and compactly supported functions $\phi\colon\mathbb{R}\to H$. In particular, since $\varrho>0$, we read off that ∂_0 is boundedly invertible on $L^2_\varrho(\mathbb{R},H)$ with $\|\partial_0^{-1}\|\leq\frac{1}{\varrho}$. It is clear that the spectrum of $\mathrm{i}m+\varrho$ is given by the set $\mathrm{i}\,[\mathbb{R}]+\varrho$. Hence, $\sigma(\partial_0^{-1})=\sigma((\mathrm{i}m+\varrho)^{-1})=\partial B_{\mathbb{C}}(r,r)$ with $r=1/(2\varrho)$. Thus, the said spectral representation gives rise to a functional calculus for ∂_0^{-1}: Let $r'>r$. For an analytic bounded function $M\colon B_{\mathbb{C}}(r',r')\to L(H)$ we define

$$M(\partial_0^{-1}):=\mathcal{L}_\varrho^* M\left((\mathrm{i}m+\varrho)^{-1}\right)\mathcal{L}_\varrho,$$

where $\left(M\left((\mathrm{i}m+\varrho)^{-1}\right)\phi\right)(t):=M\left((\mathrm{i}t+\varrho)^{-1}\right)\phi(t)$ for all $t\in\mathbb{R}$ and $\phi\in L^2(\mathbb{R},H)$. Again, we refer to [12] for several examples of analytic operator-valued functions of ∂_0^{-1} and their occurrence in the context of partial differential equations.

The solution theory, that is, the unique existence of solutions and continuous dependence on the data, for many linear equations of mathematical physics is covered by the following theorem. For this, note that we do not distinguish between operators defined on H and their respective lifts to the space $L^2_\varrho(\mathbb{R},H)$. Also the explicit dependence on ϱ is frequently suppressed.

Theorem 1.1 ([8, Solution Theory],[9, Theorem 6.2.5])**.** *Let $A\colon D(A)\subseteq H\to H$ be skew-selfadjoint and M as above. Assume that there is $c>0$ such that*

$$\mathfrak{Re}\langle z^{-1}M(z)\phi,\phi\rangle\geq c\langle\phi,\phi\rangle\quad(\phi\in H, z\in B_{\mathbb{C}}(r',r')).$$

Then, the operator $B:=\partial_0 M(\partial_0^{-1})+A$ defined on its natural domain is closable and the closure is continuously invertible, that is, $S:=\overline{B}^{-1}\in L(L^2_\varrho(\mathbb{R},H))$. Moreover, S commutes with ∂_0^{-1} and for all $u\in D(\overline{B})$, and $\varepsilon>0$, we have $(1+\varepsilon\partial_0)^{-1}u\in D(B)=D(\partial_0 M(\partial_0^{-1}))\cap D(A)$.

For the last statement of the theorem one may also consult [16, Lemma 5.2]. We have purposely left out here the reference to causality, which also holds and is essential for well-posedness of evolutionary equations in general, but plays a lesser role in this paper. We note the following corollary to Theorem 1.1.

Corollary 1.2. *With the assumptions and notation in the last theorem, the following is true. Let $u\in D(\overline{B})$. If $u\in D(\partial_0 M(\partial_0^{-1}))$, then $u\in D(A)$ and $\overline{B}u=Bu=\partial_0 M(\partial_0^{-1})u+Au$.*

Proof. Let $\varepsilon>0$ and define $u_\varepsilon:=(1+\varepsilon\partial_0)^{-1}u$. By Theorem 1.1, we get $u_\varepsilon\in D(B)$ and, since S commutes with ∂_0^{-1}, $(1+\varepsilon\partial_0)^{-1}\overline{B}u=Bu_\varepsilon$. Thus, since $(1+\varepsilon\partial_0)^{-1}\to 1$ as $\varepsilon\to 0$

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in the strong operator topology, we infer $u_\varepsilon \to u$ and $Bu_\varepsilon \to \overline{B}u$ in $L^2_\varrho(\mathbb{R}, H)$ as $\varepsilon \to 0$. Furthermore, from

$$Bu_\varepsilon = \left(\partial_0 M(\partial_0^{-1}) + A\right) u_\varepsilon = \partial_0 M(\partial_0^{-1}) u_\varepsilon + A u_\varepsilon = (1+\varepsilon\partial_0)^{-1}\partial_0 M(\partial_0^{-1}) u + A u_\varepsilon,$$

and by the closedness of A, we infer that $u \in D(A)$ and $\overline{B}u = \partial_0 M(\partial_0^{-1})u + Au$. □

2 The main result

In this section, we show a maximal regularity result for a prototype equation (see also Corollary 2.5 below). Let throughout this section $C\colon D(C) \subseteq H_0 \to H_1$ be a densely defined, closed linear operator between Hilbert spaces H_0 and H_1, $r > 0$. Moreover, let $M\colon B_{\mathbb{C}}(r,r) \to L(H_0)$, $N_{ij}\colon B_{\mathbb{C}}(r,r) \to L(H_j, H_i)$ be analytic and bounded, $i,j \in \{0,1\}$. The prototype operator to study in the following is

$$B \coloneqq \left(\partial_0 \begin{pmatrix} M(\partial_0^{-1}) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} N_{00}(\partial_0^{-1}) & N_{01}(\partial_0^{-1}) \\ N_{10}(\partial_0^{-1}) & N_{11}(\partial_0^{-1}) \end{pmatrix}\right) + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix} \tag{2}$$

with domain $D(\partial_0 M(\partial_0^{-1})) \cap D\left(\begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}\right)$ in the space $L^2_\varrho(\mathbb{R}, H_0 \oplus H_1)$, where $\varrho > 1/2r$.

We will use the following assumptions:

1. There is $c_0 > 0$ such that for all $z \in B_{\mathbb{C}}(r,r)$ and $(\phi,\psi) \in H_0 \oplus H_1$ the estimate

$$\mathfrak{Re}\left\langle \left(z^{-1}\begin{pmatrix} M(z) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} N_{00}(z) & N_{01}(z) \\ N_{10}(z) & N_{11}(z) \end{pmatrix}\right)\begin{pmatrix} \phi \\ \psi \end{pmatrix}, \begin{pmatrix} \phi \\ \psi \end{pmatrix}\right\rangle \geq c_0 \left\langle \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \begin{pmatrix} \phi \\ \psi \end{pmatrix}\right\rangle$$

   is satisfied.

2. For some $\beta \in]0,1]$ we have

   a) There is $c_1 > 0$ such that for all $z \in B_{\mathbb{C}}(r,r)$ and $\phi \in H_0$ the estimate

$$\mathfrak{Re}\langle z^{\beta-1} M(z)\phi, \phi\rangle \geq c_1 \langle \phi, \phi\rangle$$

   is satisfied and the mapping $B_{\mathbb{C}}(r,r) \ni z \mapsto z^{\beta-1}M(z)$ is bounded.

   b) If for all $z \in B_{\mathbb{C}}(r,r)$, we have $(N_{11}(z))^{-1} \in L(H_1)$ then there is $c_2 \in \mathbb{R}$ such that

$$\mathfrak{Re}\left\langle \left((z^*)^\beta N_{11}(z)\right)^{-1}\psi, \psi\right\rangle \geq c_2 \langle \psi, \psi\rangle$$

   for all $\psi \in H_1$.

Some consequences of the latter assumptions are in order.

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Lemma 2.1. *Assume that condition* (2a) *holds. Then* $D(\partial_0 M(\partial_0^{-1})) = H^\beta_\varrho(\mathbb{R}, H_0) \coloneqq D(\partial_0^\beta)$.

Proof. We first show $\partial_0 M(\partial_0^{-1}) = \overline{\partial_0^{1-\beta} M(\partial_0^{-1})\partial_0^\beta}$. Since $\partial_0^{1-\beta} M(\partial_0^{-1})\partial_0^\beta \subseteq \partial_0 M(\partial_0^{-1})$ and $\partial_0 M(\partial_0^{-1})$ is closed, as it is a product of a bounded and a closed operator, we get

$$\overline{\partial_0^{1-\beta} M(\partial_0^{-1})\partial_0^\beta} \subseteq \partial_0 M(\partial_0^{-1}).$$

Now, let $u \in D(\partial_0 M(\partial_0^{-1}))$ and set $u_\varepsilon \coloneqq (1+\varepsilon\partial_0)^{-1}u \in D(\partial_0) \subseteq D(\partial_0^\beta)$ for $\varepsilon > 0$. Then $u_\varepsilon \to u$ in $L^2_\varrho(\mathbb{R}, H_0)$ as $\varepsilon \to 0$ and

$$\begin{aligned}\partial_0^{1-\beta} M(\partial_0^{-1})\partial_0^\beta u_\varepsilon &= \partial_0 M(\partial_0^{-1}) u_\varepsilon \\ &= (1+\varepsilon\partial_0)^{-1}\partial_0 M(\partial_0^{-1})u \to \partial_0 M(\partial_0^{-1})u \quad (\varepsilon \to 0).\end{aligned}$$

Thus, $u \in D\left(\overline{\partial_0^{1-\beta} M(\partial_0^{-1})\partial_0^\beta}\right)$ with $\overline{\partial_0^{1-\beta} M(\partial_0^{-1})\partial_0^\beta} u = \partial_0 M(\partial_0^{-1})u$, which proves the asserted equality. Now, by condition (2a), the operator $\partial_0^{1-\beta} M(\partial_0^{-1})$ is boundedly invertible on $L^2_\varrho(\mathbb{R}, H)$ and hence, the operator $\partial_0^{1-\beta} M(\partial_0^{-1})\partial_0^\beta$ is closed. The latter yields $\partial_0^{1-\beta} M(\partial_0^{-1})\partial_0^\beta = \partial_0 M(\partial_0^{-1})$. Next $\partial_0^{1-\beta} M(\partial_0^{-1})$ is a bounded operator, since $z \mapsto z^{\beta-1}M(z)$ is bounded by condition (2a). Hence, $D\left(\partial_0^{1-\beta} M(\partial_0^{-1})\partial_0^\beta\right) = H^\beta_\varrho(\mathbb{R}, H_0)$ and the assertion follows. □

Lemma 2.2. *Assume condition* (1). *Then for all* $z \in B_{\mathbb{C}}(r, r)$, *the operator* $N_{11}(z)$ *is continuously invertible.*

Proof. The claim is immediate by putting $(\phi, \psi) = (0, \psi)$ in the positivity estimate in condition (1). □

Lemma 2.3. *Let* $u \in L^2_\varrho(\mathbb{R}, H_0)$. *Then* $u \in D(\partial_0^\beta)$ *if and only if* $\sup_{\varepsilon>0} \|\partial_0^\beta (1+\varepsilon\partial_0)^{-1}u\| < \infty$.

Proof. From $\|(1+\varepsilon\partial_0)^{-1}\| \leq 1$ for all $\varepsilon > 0$, it follows that the assertion $u \in D(\partial_0^\beta)$ is a sufficient condition for the supremum to be finite. On the other hand, assume that the supremum is finite. Then there is a sequence $(\varepsilon_n)_n$ in $(0, \infty)$ such that $\varepsilon_n \to 0$ as $n \to \infty$ and $v \coloneqq \lim_{n\to\infty} \partial_0^\beta (1+\varepsilon_n\partial_0)^{-1}u$ exists in the weak topology of $L^2_\varrho(\mathbb{R}, H_0)$. By the (weak) closedness of ∂_0^β and the fact that $(1+\varepsilon_n\partial_0)^{-1}u \to u$ as $n \to \infty$, we infer $u \in D(\partial_0^\beta)$. □

Theorem 2.4. *Assume conditions* (1), (2a), *and* (2b). *Then, for each* $\varrho > \frac{1}{2r}$, B *given in* (2) *is continuously invertible on* $L^2_\varrho(\mathbb{R}, H_0 \oplus H_1)$ *and for* $(f, g) \in L^2_\varrho(\mathbb{R}, H_0) \oplus H^\beta_\varrho(\mathbb{R}, H_1)$, *we have* $\overline{B}^{-1}(f, g) \in \left(H^\beta_\varrho(\mathbb{R}, H_0) \oplus L^2_\varrho(\mathbb{R}, H_1)\right) \cap \left(D\left(\begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}\right)\right)$.

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Proof. We want to apply Corollary 1.2. For this, we have to show that

$$(u,v) \coloneqq \overline{B}^{-1}(f,g) \in D\left(\begin{pmatrix} \partial_0 M(\partial_0^{-1}) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} N_{00}(\partial_0^{-1}) & N_{01}(\partial_0^{-1}) \\ N_{10}(\partial_0^{-1}) & N_{11}(\partial_0^{-1}) \end{pmatrix}\right).$$

By the boundedness of $\begin{pmatrix} N_{00}(\partial_0^{-1}) & N_{01}(\partial_0^{-1}) \\ N_{10}(\partial_0^{-1}) & N_{11}(\partial_0^{-1}) \end{pmatrix}$, we are left with showing that $u \in D(\partial_0 M(\partial_0^{-1}))$. By Lemma 2.1, we need to show that $u \in D(\partial_0^{\beta})$. Invoking Lemma 2.3, it suffices to show that

$$\sup_{\varepsilon>0} \|\partial_0^{\beta}(1+\varepsilon\partial_0)^{-1}u\| < \infty.$$

So, let $\varepsilon > 0$ and define $u_\varepsilon \coloneqq (1+\varepsilon\partial_0)^{-1}u$. We further set $v_\varepsilon \coloneqq (1+\varepsilon\partial_0)^{-1}v$. By Theorem 1.1 (note that $\begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}$ is skew-selfadjoint; and that the needed inequality for the application of Theorem 1.1 is warranted by (1)), we have that

$$(u_\varepsilon, v_\varepsilon) \in D(B) = D\left(\begin{pmatrix} \partial_0 M(\partial_0^{-1}) & 0 \\ 0 & 0 \end{pmatrix}\right) \cap D\left(\begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}\right).$$

Thus, we read off $v_\varepsilon \in D(C^*)$ as well as $u_\varepsilon \in D(C) \cap D(\partial_0 M(\partial_0^{-1}))$. Moreover, we have the equalities

$$\begin{aligned} \partial_0 M(\partial_0^{-1})u_\varepsilon + N_{00}(\partial_0^{-1})u_\varepsilon + N_{01}(\partial_0^{-1})v_\varepsilon - C^* v_\varepsilon &= f_\varepsilon, \\ N_{11}(\partial_0^{-1})v_\varepsilon + N_{10}(\partial_0^{-1})u_\varepsilon + Cu_\varepsilon &= g_\varepsilon, \end{aligned}$$

where $f_\varepsilon \coloneqq (1+\varepsilon\partial_0)^{-1}f$ and $g_\varepsilon \coloneqq (1+\varepsilon\partial_0)^{-1}g$. Next, letting $\varepsilon \to 0$ in the second equality, we infer by the closedness of C that $u \in D(C)$ and

$$N_{11}(\partial_0^{-1})v + N_{10}(\partial_0^{-1})u + Cu = g.$$

Furthermore, we get

$$\begin{aligned} \|Cu\| &\leq \|g\| + \|N_{11}(\partial_0^{-1})\|\|v\| + \|N_{10}(\partial_0^{-1})\|\|u\| \\ &\leq \left(1 + \frac{1}{c}\left(\|N_{11}(\partial_0^{-1})\| + \|N_{10}(\partial_0^{-1})\|\right)\right)\left(\|g\| + \|f\|\right), \quad (3) \end{aligned}$$

where we have used condition (1). By Lemma 2.2, we also get

$$v_\varepsilon = \left(N_{11}(\partial_0^{-1})\right)^{-1}\left(-N_{10}(\partial_0^{-1})u_\varepsilon - Cu_\varepsilon + g_\varepsilon\right).$$

Substituting the latter equation into the first one, we arrive at

$$\begin{aligned} \partial_0 M(\partial_0^{-1})u_\varepsilon + N_{00}(\partial_0^{-1})u_\varepsilon + N_{01}(\partial_0^{-1})\left(N_{11}(\partial_0^{-1})\right)^{-1}\left(-N_{10}(\partial_0^{-1})u_\varepsilon - Cu_\varepsilon + g_\varepsilon\right) \\ - C^*\left(N_{11}(\partial_0^{-1})\right)^{-1}\left(-N_{10}(\partial_0^{-1})u_\varepsilon - Cu_\varepsilon + g_\varepsilon\right) = f_\varepsilon. \end{aligned}$$

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Hence,

$$\begin{aligned}\partial_0 M(\partial_0^{-1})u_\varepsilon = &-N_{00}(\partial_0^{-1})u_\varepsilon + N_{01}(\partial_0^{-1})\left(N_{11}(\partial_0^{-1})\right)^{-1}N_{10}(\partial_0^{-1})u_\varepsilon\\ &+ N_{01}(\partial_0^{-1})\left(N_{11}(\partial_0^{-1})\right)^{-1}Cu_\varepsilon - N_{01}(\partial_0^{-1})\left(N_{11}(\partial_0^{-1})\right)^{-1}g_\varepsilon\\ &- C^*\left(N_{11}(\partial_0^{-1})\right)^{-1}\left(N_{10}(\partial_0^{-1})u_\varepsilon + Cu_\varepsilon - g_\varepsilon\right) + f_\varepsilon.\end{aligned}$$

We apply $\langle\cdot,\partial_0^\beta u_\varepsilon\rangle_{L^2_\varrho}$ to the latter equation, take real parts and use condition (2a) to get

$$\begin{aligned}c_1\,\mathfrak{Re}\langle\partial_0^\beta u_\varepsilon,\partial_0^\beta u_\varepsilon\rangle &\le \mathfrak{Re}\langle\partial_0^{1-\beta}M(\partial_0^{-1})\partial_0^\beta u_\varepsilon,\partial_0^\beta u_\varepsilon\rangle\\ &= \mathfrak{Re}\langle\partial_0 M(\partial_0^{-1})u_\varepsilon,\partial_0^\beta u_\varepsilon\rangle\\ &= \mathfrak{Re}\left\langle -N_{00}(\partial_0^{-1})u_\varepsilon + N_{01}(\partial_0^{-1})\left(N_{11}(\partial_0^{-1})\right)^{-1}N_{10}(\partial_0^{-1})u_\varepsilon,\partial_0^\beta u_\varepsilon\right\rangle\\ &\quad+\mathfrak{Re}\left\langle N_{01}(\partial_0^{-1})\left(N_{11}(\partial_0^{-1})\right)^{-1}Cu_\varepsilon - N_{01}(\partial_0^{-1})\left(N_{11}(\partial_0^{-1})\right)^{-1}g_\varepsilon,\partial_0^\beta u_\varepsilon\right\rangle\\ &\quad+\mathfrak{Re}\left\langle -C^*\left(N_{11}(\partial_0^{-1})\right)^{-1}\left(N_{10}(\partial_0^{-1})u_\varepsilon + Cu_\varepsilon - g_\varepsilon\right),\partial_0^\beta u_\varepsilon\right\rangle\\ &\quad+\mathfrak{Re}\left\langle f_\varepsilon,\partial_0^\beta u_\varepsilon\right\rangle.\end{aligned}$$

We recall that $u\in D(C)$ and, hence, $u_\varepsilon\in D(C)$ as well as $\partial_0^\beta u_\varepsilon\in D(C)$. Thus, we have

$$\begin{aligned}c_1\,\mathfrak{Re}\langle\partial_0^\beta u_\varepsilon,\partial_0^\beta u_\varepsilon\rangle &\le \mathfrak{Re}\left\langle -N_{00}(\partial_0^{-1})u_\varepsilon + N_{01}(\partial_0^{-1})\left(N_{11}(\partial_0^{-1})\right)^{-1}N_{10}(\partial_0^{-1})u_\varepsilon,\partial_0^\beta u_\varepsilon\right\rangle\\ &\quad+\mathfrak{Re}\left\langle N_{01}(\partial_0^{-1})\left(N_{11}(\partial_0^{-1})\right)^{-1}Cu_\varepsilon - N_{01}(\partial_0^{-1})\left(N_{11}(\partial_0^{-1})\right)^{-1}g_\varepsilon,\partial_0^\beta u_\varepsilon\right\rangle\\ &\quad-\mathfrak{Re}\left\langle\left(\partial_0^\beta\right)^*\left(N_{11}(\partial_0^{-1})\right)^{-1}\left(N_{10}(\partial_0^{-1})u_\varepsilon + Cu_\varepsilon - g_\varepsilon\right),Cu_\varepsilon\right\rangle\\ &\quad+\mathfrak{Re}\left\langle f_\varepsilon,\partial_0^\beta u_\varepsilon\right\rangle.\end{aligned}$$

We note that apart from the term $\mathfrak{Re}\left\langle\left(\partial_0^\beta\right)^*\left(N_{11}(\partial_0^{-1})\right)^{-1}\left(N_{10}(\partial_0^{-1})u_\varepsilon + Cu_\varepsilon - g_\varepsilon\right),Cu_\varepsilon\right\rangle$ the remaining terms of the right-hand side can by estimated by

$$K_1\|\partial_0^\beta u_\varepsilon\|$$

for some constant $K_1\ge 0$, where we also used (3) as well as $\|(1+\varepsilon\partial_0)^{-1}\|\le 1$. For the treatise of $\mathfrak{Re}\left\langle\left(\partial_0^\beta\right)^*\left(N_{11}(\partial_0^{-1})\right)^{-1}\left(N_{10}(\partial_0^{-1})u_\varepsilon + Cu_\varepsilon - g_\varepsilon\right),Cu_\varepsilon\right\rangle$ we estimate with the

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help of condition (2b) (note that the implication is not void by Lemma 2.2)

$$
\begin{aligned}
&-\mathfrak{Re}\left\langle \left(\partial_0^\beta\right)^* \left(N_{11}(\partial_0^{-1})\right)^{-1} \left(N_{10}(\partial_0^{-1})u_\varepsilon + Cu_\varepsilon - g_\varepsilon\right), Cu_\varepsilon \right\rangle \\
&= -\mathfrak{Re}\left\langle \left(\partial_0^\beta\right)^* \left(N_{11}(\partial_0^{-1})\right)^{-1} N_{10}(\partial_0^{-1})u_\varepsilon, Cu_\varepsilon \right\rangle \\
&\quad -\mathfrak{Re}\left\langle \left(\partial_0^\beta\right)^* \left(N_{11}(\partial_0^{-1})\right)^{-1} Cu_\varepsilon - \left(\partial_0^\beta\right)^* \left(N_{11}(\partial_0^{-1})\right)^{-1} g_\varepsilon, Cu_\varepsilon \right\rangle \\
&= -\mathfrak{Re}\left\langle \left(N_{11}(\partial_0^{-1})\right)^{-1} N_{10}(\partial_0^{-1}) \left(\partial_0^\beta\right)^* u_\varepsilon, Cu_\varepsilon \right\rangle \\
&\quad -\mathfrak{Re}\left\langle \left(\partial_0^\beta\right)^* \left(N_{11}(\partial_0^{-1})\right)^{-1} Cu_\varepsilon, Cu_\varepsilon \right\rangle + \mathfrak{Re}\left\langle \left(N_{11}(\partial_0^{-1})\right)^{-1} \left(\partial_0^\beta\right)^* g_\varepsilon, Cu_\varepsilon \right\rangle \\
&\le \left\| \left(N_{11}(\partial_0^{-1})\right)^{-1} N_{10}(\partial_0^{-1}) \right\| \left\| \left(\partial_0^\beta\right)^* u_\varepsilon \right\| \|Cu\| + |c_1| \|Cu\|^2 \\
&\quad + \left\| \left(N_{11}(\partial_0^{-1})\right)^{-1} \right\| \left\| \left(\partial_0^\beta\right)^* g_\varepsilon \right\| \|Cu\| \\
&= \left\| \left(N_{11}(\partial_0^{-1})\right)^{-1} N_{10}(\partial_0^{-1}) \right\| \left\| \partial_0^\beta u_\varepsilon \right\| \|Cu\| + |c_1| \|Cu\|^2 \\
&\quad + \left\| \left(N_{11}(\partial_0^{-1})\right)^{-1} \right\| \left\| \partial_0^\beta g \right\| \|Cu\| \\
&\le K_2 \|\partial_0 u_\varepsilon\| + K_3
\end{aligned}
$$

for some $K_2, K_3 \ge 0$, where we have again used (3). Hence, we get for $p := (K_1 + K_2)/c \ge 0$ and $q := K_3/c \ge 0$ that

$$\left\| \partial_0^\beta u_\varepsilon \right\|^2 \le p \left\| \partial_0^\beta u_\varepsilon \right\| + q,$$

which implies

$$\left\| \partial_0^\beta u_\varepsilon \right\| \le \frac{p}{2} + \sqrt{\frac{p^2}{4} + q}.$$

Thus, $u \in D(\partial_0^\beta)$, by Lemma 2.3. □

Another, perhaps more familiar looking, maximal regularity result can now be deduced from Theorem 2.4:

Corollary 2.5. *Assume conditions* (1),(2a) *and* (2b) *to be satisfied,* $\varrho > 1/(2r)$. *Then, for all* $f \in L^2_\varrho(\mathbb{R}, H_0)$, *there exists a unique*

$$u \in H^\beta_\varrho(\mathbb{R}, H_0) \cap D\left(C^* N_{11}(\partial_0^{-1})^{-1} \left(C + N_{10}(\partial_0^{-1})\right)\right)$$

satisfying

$$
\begin{aligned}
\partial_0 M(\partial_0^{-1})u + N_{00}\left(\partial_0^{-1}\right) u - N_{01}\left(\partial_0^{-1}\right) \left(N_{11}\left(\partial_0^{-1}\right)\right)^{-1} \left(C + N_{10}\left(\partial_0^{-1}\right)\right) u \\
+ C^* \left(N_{11}\left(\partial_0^{-1}\right)\right)^{-1} \left(C + N_{10}\left(\partial_0^{-1}\right)\right) u = f. \quad (4)
\end{aligned}
$$

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Proof. Using condition (1), by Theorem 1.1, we infer the existence of a unique $(v, w) \in L^2_\varrho(\mathbb{R}, H_0 \oplus H_1)$ such that

$$\overline{\left(\partial_0 \begin{pmatrix} M(\partial_0^{-1}) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} N_{00}(\partial_0^{-1}) & N_{01}(\partial_0^{-1}) \\ N_{10}(\partial_0^{-1}) & N_{11}(\partial_0^{-1}) \end{pmatrix} + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}\right)} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

By Theorem 2.4 (and Lemma 2.1), we get

$$\begin{aligned} \begin{pmatrix} v \\ w \end{pmatrix} &\in H^\beta_\varrho(\mathbb{R}, H_0) \oplus L^2_\varrho(\mathbb{R}, H_1) \cap D\left(\begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}\right) \\ &= D\left(\partial_0 \begin{pmatrix} M(\partial_0^{-1}) & 0 \\ 0 & 0 \end{pmatrix}\right) \cap D\left(\begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}\right) \end{aligned}$$

Hence,

$$\begin{aligned} \begin{pmatrix} f \\ 0 \end{pmatrix} &= \overline{\left(\partial_0 \begin{pmatrix} M(\partial_0^{-1}) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} N_{00}(\partial_0^{-1}) & N_{01}(\partial_0^{-1}) \\ N_{10}(\partial_0^{-1}) & N_{11}(\partial_0^{-1}) \end{pmatrix} + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}\right)} \begin{pmatrix} v \\ w \end{pmatrix} \\ &= \left(\partial_0 \begin{pmatrix} M(\partial_0^{-1}) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} N_{00}(\partial_0^{-1}) & N_{01}(\partial_0^{-1}) \\ N_{10}(\partial_0^{-1}) & N_{11}(\partial_0^{-1}) \end{pmatrix} + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}\right) \begin{pmatrix} v \\ w \end{pmatrix} \\ &= \begin{pmatrix} \partial_0 M(\partial_0^{-1})v + N_{00}(\partial_0^{-1})v + N_{01}(\partial_0^{-1})w - C^* w \\ N_{10}(\partial_0^{-1})v + N_{11}(\partial_0^{-1})w + Cv \end{pmatrix}. \end{aligned} \tag{5}$$

With Lemma 2.2, we obtain from the second line

$$w = -\left(N_{11}(\partial_0^{-1})\right)^{-1} \left(C + N_{10}(\partial_0^{-1})\right) v.$$

Substituting the latter equation into the first equation of (5), we obtain (4). On the other hand, given $u \in H^\beta_\varrho(\mathbb{R}, H_0) \cap D\left(C^* N_{11}(\partial_0^{-1})^{-1} \left(C + N_{10}(\partial_0^{-1})\right)\right)$ satisfying (4), we deduce that

$$\begin{pmatrix} u \\ -\left(N_{11}(\partial_0^{-1})\right)^{-1} \left(C + N_{10}(\partial_0^{-1})\right) u \end{pmatrix}$$

is the solution of (5). Thus, the uniqueness statement is also settled. □

3 Some Examples

Although the strength of the above result lies in the generality of the admissible "material laws", the approach is perhaps best illustrated by making a link to known results obtained by a different approach. In this spirit, our first example deals with the paradigm of maximal regularity, the heat equation, to illustrate the different perspective of our approach on this issue. We then continue with slightly more complex examples from the literature, which may not readily be seen to be covered by the general approach developed here. This includes a concluding example for a fractional-in-time evolutionary problem.

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3.1 The heat equation

As a warm-up example we consider the paradigmatic case of heat transport. Let $\Omega \subseteq \mathbb{R}^3$ be a non-empty open set where the heat transport takes place. We consider the equations of heat conduction in the body Ω, which consists of the balance of momentum law

$$\partial_0 \vartheta + \operatorname{div} q = f,$$

where $\vartheta : \mathbb{R} \times \Omega \to \mathbb{C}$ denotes the temperature density, $q : \mathbb{R} \times \Omega \to \mathbb{C}^3$ stands for the heat flux and $f : \mathbb{R} \times \Omega \to \mathbb{C}$ is an external heat source forcing term, and Fourier's law

$$q = -k \operatorname{grad} \vartheta,$$

where the heat conductivity $k \in L(L^2(\Omega)^3, L^2(\Omega)^3)$ is a bounded selfadjoint operator satisfying

$$\mathfrak{Re}\langle k\psi, \psi\rangle_{L^2(\Omega)^3} \geq c\langle \psi, \psi\rangle_{L^2(\Omega)^3} \quad (\psi \in L^2(\Omega)^3)$$

for some $c > 0$. If we impose suitable boundary conditions, say – a homogeneous Dirichlet boundary condition, on ϑ, we end up with the following system

$$\left(\partial_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & k^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix}\right) \begin{pmatrix} \vartheta \\ q \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}, \tag{6}$$

where grad_0 is defined as the distributional gradient with domain $H^1_0(\Omega)$ and $\operatorname{div} := -\operatorname{grad}_0^*$. Thus, we are indeed in the setting studied in the previous section. Since k is bounded, selfadjoint and strictly positive definite, then so is k^{-1}. Thus, conditions (2a) (with $\beta = 1$) and (1) are clearly satisfied. Moreover, for $z \in B_{\mathbb{C}}(r, r)$ we have $z^{-1} = \mathrm{i}t + \varrho$ for some $\varrho > \frac{1}{2r}, t \in \mathbb{R}$ and hence,

$$\mathfrak{Re}\langle \left(z^* k^{-1}\right)^{-1} \psi, \psi\rangle = \mathfrak{Re}(-\mathrm{i}t + \varrho)\langle k\psi, \psi\rangle \geq \varrho c \langle \psi, \psi\rangle$$

for each $\psi \in L^2(\Omega)^3$, where we have used the selfadjointness of k. This proves that condition (2b) is satisfied and thus, Theorem 2.4 yields maximal regularity of (6). In view of Corollary 2.5, we end up with the following result:

Corollary 3.1. *For all $\varrho > 0$, $f \in L^2_\varrho(\mathbb{R}, L^2(\Omega))$, there exists a unique $u \in H^1_\varrho(\mathbb{R}, L^2(\Omega)^3) \cap D(\operatorname{div} k \operatorname{grad}_0)$ such that*

$$\partial_0 u - \operatorname{div} k \operatorname{grad}_0 u = f.$$

Remark 3.2. We emphasize that each boundary condition yielding an operator matrix of the form $\begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}$ allows the application of Theorem 2.4. For several examples of such boundary condition, including mixed and non-local ones we refer to [10].

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3.2 A second order equation

Following [3, Example 6.1], where the much deeper issue of maximal regularity in certain interpolation spaces is addressed, we consider the equation

$$\partial_0^2 \vartheta + C^* \left(A + B\partial_0\right) C\vartheta = f,$$

where $C : D(C) \subseteq H_0 \to H_1$ is densely defined closed and linear between the two Hilbert spaces H_0 and H_1, and $B \in L(H_1)$ is selfadjoint, strictly positive definite and $A \in L(H_1)$. First, we note that for $\varrho > 0$ large enough the operator $(A + B\partial_0) = \partial_0 \left(A\partial_0^{-1} + B\right) = \partial_0 B \left(B^{-1}A\partial_0^{-1} + 1\right)$ is continuously invertible on $L^2_\varrho(\mathbb{R}, H_1)$, due to a Neumann series argument (for this recall that $\|\partial_0^{-1}\| \leq 1/\varrho$). Hence, setting $w := \partial_0\vartheta, q := -(A + B\partial_0)C\vartheta$, we may rewrite the above problem as a first order equation of the form

$$\left(\partial_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \left(B^{-1}A\partial_0^{-1} + 1\right)^{-1} B^{-1} \end{pmatrix} + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}\right) \begin{pmatrix} w \\ q \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

Thus, Theorem 2.4 is applicable with the choices

$$M(z) = 1, \quad N(z) = \begin{pmatrix} 0 & 0 \\ 0 & (Az + B)^{-1} \end{pmatrix}, \quad g = 0.$$

Indeed, condition (2a) (for $\beta = 1$) is obviously satisfied while condition (1) follows from

$$\begin{aligned} \mathfrak{Re}\langle N_{11}(z)\psi, \psi\rangle_{H_1} &= \mathfrak{Re}\langle B^{-1}\psi, \psi\rangle - \mathfrak{Re}\, z\langle B^{-1}AB^{-1}(AB^{-1}z + 1)^{-1}\psi, \psi\rangle_{H_1} \\ &\geq c\langle\psi, \psi\rangle - \frac{\|B^{-1}AB^{-1}\|}{\frac{1}{|z|} - \|B^{-1}A\|}\langle\psi, \psi\rangle_{H_1} \\ &\geq \left(c - \frac{\|B^{-1}AB^{-1}\|}{\frac{1}{r} - \|B^{-1}A\|}\right)\langle\psi, \psi\rangle_{H_1} \quad (\psi \in H_1) \end{aligned}$$

for $z \in B_{\mathbb{C}}(r, r)$ with $r > 0$ small enough (which corresponds to $\varrho > 0$ large enough in the argument above), where $c > 0$ is a positive definiteness constant of B^{-1}, that is, $B^{-1} \geq c$. For showing condition (2b) (for $\beta = 1$), we compute

$$\mathfrak{Re}\langle (z^*)^{-1}(Az + B)\psi, \psi\rangle_{H_1} \geq -\|A\|\langle\psi, \psi\rangle + \frac{1}{2r}c'\langle\psi, \psi\rangle,$$

for each $z \in B_{\mathbb{C}}(r, r)$, where we have used the selfadjointness of B and that $B \geq c'$ for some $c' > 0$ by assumption. The corresponding statement for the equation, we originally started out with, is as follows.

Corollary 3.3. *There exists $\varrho_0 > 0$ such that for all $\varrho \geq \varrho_0$ the following holds: For all $f \in L^2_\varrho(\mathbb{R}, H_0)$ there exists a unique $\vartheta \in H^2_\varrho(\mathbb{R}, H_0) \cap D\left(C^*\left(A + B\partial_0\right)C\right)$ satisfying*

$$\partial_0^2 \vartheta + C^* \left(A + B\partial_0\right) C\vartheta = f.$$

Proof. Again, we rely on Corollary 2.5 for $\beta = 1$. Note that in the above computations, we used the substitution $w = \partial_0\vartheta$. We infer that $w \in H^1_\varrho(\mathbb{R}, H_0)$, which yields $\vartheta \in H^2_\varrho(\mathbb{R}, H_0)$. □

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3.3 A second order integro-differential equation

Let $C : D(C) \subseteq H_0 \to H_1$ densely defined closed and linear, $k : \mathbb{R}_{\geq 0} \to L(H_1)$ weakly measurable, such that $t \mapsto \|k(t)\|$ is measurable and $|k|_{L^1_{\varrho_0}} := \int_0^\infty \|k(t)\| e^{-\varrho_0 t} \, \mathrm{d}t < \infty$ for some $\varrho_0 > 0$. Moreover, let $A, B \in L(H_1)$ with A selfadjoint and strictly positive definite. We consider the following equation

$$\left(\partial_0^2 + C^* \left(\partial_0 A + B + k*\right) C\right) u = f, \tag{7}$$

where the convolution operator $k*$ is defined by

$$\begin{aligned} k* : L^2_\varrho(\mathbb{R}, H_1) &\to L^2_\varrho(\mathbb{R}, H_1) \\ g &\mapsto \left(t \mapsto \int_0^\infty k(s) g(t-s) \, \mathrm{d}s \right) \end{aligned}$$

for $\varrho \geq \varrho_0$. By Young's inequality we have that

$$\|k * \|_{L(L^2_\varrho(\mathbb{R}, H_1))} \leq |k|_{L^1_\varrho} \leq |k|_{L^1_{\varrho_0}} < \infty,$$

so that $k*$ is a bounded linear operator on $L^2_\varrho(\mathbb{R}, H_1)$ for each $\varrho \geq \varrho_0$. Moreover, by monotone convergence, we get that $\limsup_{\varrho \to \infty} \|k * \|_{L(L^2_\varrho(\mathbb{R}, H_1))} \leq \lim_{\varrho \to \infty} |k|_{L^1_\varrho} = 0$. For a treatment of integro-differential equations within the framework of evolutionary problems we refer to [15], where this is a special case in the discussion of problems with monotone relations. We rewrite the above problem as a first order problem in the new unknowns $v := \partial_0 u$ and $q := -\left(A + \partial_0^{-1}\left(B + k*\right)\right) C v$. Thus, we arrive at

$$\left(\partial_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \left(A + \partial_0^{-1}(B + k*)\right)^{-1} \end{pmatrix} + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix} \right) \begin{pmatrix} v \\ q \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}. \tag{8}$$

We note that the operator $A + \partial_0^{-1}(B + k*)$ is indeed boundedly invertible on $L^2_\varrho(\mathbb{R}, H_1)$ for sufficiently large $\varrho > 0$, since

$$A + \partial_0^{-1}(B + k*) = A\left(1 + \partial_0^{-1} A^{-1} \left(B + k*\right)\right)$$

and

$$\|\partial_0^{-1} A^{-1} \left(B + k*\right) \|_{L(L^2_\varrho(\mathbb{R}, H_1))} \leq \frac{1}{\varrho} \|A^{-1}\| \left(\|B\| + |k|_{L^1_\varrho} \right) < 1$$

for ϱ sufficiently large. Moreover, we note that the above problem is an equation of the form discussed in Section 2 with

$$N_{11}(z) := \left(A + z \left(B + \sqrt{2\pi} \widehat{k}(-\mathrm{i}z^{-1}) \right) \right)^{-1},$$

where $\widehat{k}$ denotes the Fourier-transform of k (see [15] for more details). Condition (2a) (for $\beta = 1$) is obviously satisfied in this situation. Moreover, since

$$N_{11}(z) = A^{-1} + A^{-1} \sum_{k=1}^{\infty} (-z)^k \left(\left(B + \sqrt{2\pi}\, \widehat{k}(-\mathrm{i}z^{-1}) \right) A^{-1} \right)^k$$

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by Neumann series expansion, we infer that $\mathfrak{Re}\, N_{11}(z)$ is uniformly strictly positive definite for $z \in B_{\mathbb{C}}(\frac{1}{2\varrho}, \frac{1}{2\varrho})$ for $\varrho > 0$ large enough, since A^{-1} is strictly positive definite and

$$\begin{aligned}
&\sup_{z\in B_{\mathbb{C}}(\frac{1}{2\varrho},\frac{1}{2\varrho})} \left\| A^{-1} \sum_{k=1}^{\infty} (-z)^k \left(\left(B + \sqrt{2\pi}\, \widehat{k}(-\mathrm{i}z^{-1}) \right) A^{-1} \right)^k \right\| \\
&\leq \sup_{z\in B_{\mathbb{C}}(\frac{1}{2\varrho},\frac{1}{2\varrho})} \|A^{-1}\| \left(\frac{|z| \left(\|B\| + |k|_{L^1_{\varrho_0}} \right) \|A^{-1}\|}{1 - |z| \left(\|B\| + |k|_{L^1_{\varrho_0}} \right) \|A^{-1}\|} \right) \to 0 \quad (\varrho \to \infty).
\end{aligned}$$

This yields that condition (1) is also satisfied. Finally, using the representation $z^{-1} = \mathrm{i}t + \varrho$ for some $t \in \mathbb{R}$, $\varrho > \varrho_0$ large enough, we obtain

$$\begin{aligned}
\mathfrak{Re}\, (z^* N_{11}(z))^{-1} &= \mathfrak{Re}\, (z^*)^{-1} A + \mathfrak{Re}\, \frac{z}{z^*} \left(B + \sqrt{2\pi} \widehat{k}(-\mathrm{i}z^{-1}) \right) \\
&\geq \varrho A - \left(\|B\| + |k|_{L^1_\varrho} \right) \\
&\geq \varrho_0 c - \left(\|B\| + |k|_{L^1_{\varrho_0}} \right),
\end{aligned}$$

with $c > 0$ such that $A \geq c$. This shows condition (2b) ($\beta = 1$). Thus, Corollary 2.5 applies with $\beta = 1$ and yields the maximal regularity of (7).

Remark 3.4. The maximal regularity of a problem similar to (7) was studied in [14] in a Banach space setting, where the operators A and B were replaced by real scalars, the kernel k was assumed to be real-valued and the operator C^*C was replaced by a generator of an analytic semigroup.

3.4 A partial differential equation of fractional type

We conclude with the following example taken from [13], where the maximal regularity of the equation

$$\partial_0^\beta u - (1 + k*)Au = f$$

has been addressed in spaces of (Banach space-valued) Hölder continuous functions for some $\beta \in]0, 1[$. Here A is a sectorial operator and k is a suitable integrable, scalar-valued function, which is supported on the positive reals only. As the case of convolutions has already been addressed in the previous two subsections, we focus on the simplified equation

$$\partial_0^\beta u + C^*Cu = f, \tag{9}$$

where $C\colon D(C) \subseteq H_0 \to H_1$ is densely defined and closed in the Hilbert spaces H_0 and H_1. We show that the equation (9) admits maximal regularity in $L^2_\varrho(\mathbb{R}, H_0)$ for all $\varrho > 0$. So, let $\varrho > 0$. Setting $q := -Cu$, a corresponding 2-by-2 block operator matrix formulation reads

$$\left(\partial_0 \begin{pmatrix} \partial_0^{\beta-1} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix} \right) \begin{pmatrix} u \\ q \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

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We want to apply Theorem 2.4 (or Corollary 2.5) to

$$M(z) = z^{1-\beta}, \quad N(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

For this, note that condition (2a) is satisfied, since for all $r > 1/(2\varrho)$, we have

$$\mathfrak{Re}\langle z^{\beta-1}M(z)\phi, \phi\rangle = \mathfrak{Re}\langle z^{\beta-1}z^{1-\beta}\phi, \phi\rangle = \langle \phi, \phi\rangle \quad (z \in B_{\mathbb{C}}(r,r), \phi \in H_0)$$

Next, condition (1) follows from [11, Lemma 2.1], which states

$$\mathfrak{Re}\, \partial_0^\beta \geq \varrho^\beta.$$

For a proof of condition (2b), we observe that by [11, Lemma 2.1], we have $\mathfrak{Re}\left((z^*)^\beta\right)^{-1} = \mathfrak{Re}\left(z^\beta\right)^{-1} \geq \varrho^\beta$. Hence, we arrive at the following maximal regularity result for (9).

Corollary 3.5. *For all $\varrho > 0$, $f \in L^2_\varrho(\mathbb{R}, H_0)$, the equation* (9) *admits a unique solution $u \in H^\beta_\varrho(\mathbb{R}, H_0) \cap D(C^*C)$.*

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