J. Math. Anal. Appl. 451 (2017) 14–33

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Hypersurfaces in $\mathbb{E}_s^{n+1}$ satisfying $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$ with at most two distinct principal curvatures

Jiancheng Liu *, Chao Yang

College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, PR China

A R T I C L E I N F O

Article history:
Received 7 November 2016
Available online 2 February 2017
Submitted by R. Gornet

Keywords:
Pseudo-Euclidean space
Hypersurface
Shape operator
Mean curvature
Isoparametric

A B S T R A C T

A. Arvanitoyeorgos and G. Kaimakamis proposed in [1] the conjecture that: *any hypersurface satisfying $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$ in pseudo-Euclidean space $\mathbb{E}_s^{n+1}$ of index s has constant mean curvature.* In this paper, we prove that the conjecture is true when the hypersurfaces have at most two distinct principal curvatures. Then, we estimate that constant mean curvature, and give its explicit expression for some special cases. As a result, for that of Lorentzian type hypersurfaces which are not minimal, we prove that it must be isoparametric and give classification results.

1. Introduction

Let $x : M_r^n \to \mathbb{E}_s^{n+1}$ be an isometric immersion of a pseudo-Riemannian hypersurface M_r^n into a pseudo-Euclidean space $\mathbb{E}_s^{n+1}$. Denote by $\overrightarrow{H}$ and Δ the mean curvature vector field and the Laplace operator of M_r^n with respect to the induced metric. The equation $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$, for some real constant λ, is a natural generalization of the biharmonic equation $\Delta\overrightarrow{H} = 0$.

In 1992, Ferrández and Lucas originate in [6] the study of hypersurfaces in $\mathbb{E}_s^{n+1}$ satisfying $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$, where they proved that the hypersurfaces in $\mathbb{E}_1^3$ have constant mean curvatures and classified such hypersurfaces. Naturally, A. Arvanitoyeorgos and G. Kaimakamis made a conjecture in [1] saying that *any hypersurface satisfying $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$ in pseudo-Euclidean space $\mathbb{E}_s^{n+1}$ has constant mean curvature.*

A. Arvanitoyeorgos et al. proved in [2] that this conjecture is true for hypersurface M_r^3 of $\mathbb{E}_s^4$ whose shape operator is diagonalizable. More general, we obtained the same conclusion in [8] for M_r^n in $\mathbb{E}_s^{n+1}$ with at most three distinct principal curvatures.

Without the restriction that the shape operator is diagonalizable, there are also some papers to prove this conjecture. However, most of them are for hypersurfaces with $r = 1$ and $s = 1$, such as [3] is for hypersurface

* Corresponding author.
E-mail addresses: liujc@nwnu.edu.cn (J. Liu), yc963852@126.com (C. Yang).

http://dx.doi.org/10.1016/j.jmaa.2017.01.090

M_1^3 in $\mathbb{E}_1^4$ and [9] is for M_1^n in $\mathbb{E}_1^{n+1}$ with at most three distinct principal curvatures. As well as the situation for $r = 2$ and $s = 2$ is also proved in [1].

As a matter of course, it is interesting to study this conjecture for general indexes r and s without the restriction that the shape operator is diagonalizable. In this paper, we will show that hypersurface M_r^n satisfying $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$ in $\mathbb{E}_s^{n+1}$ with at most two distinct principal curvatures has constant mean curvature in section 3.

Once we know the mean curvature of that class of hypersurfaces is a constant, we continue in section 4 to estimate or give an explicit expression for that constant, according to the principal curvatures are all real or imaginary. In section 5, applying the results of section 4 to the special case $r = s = 1$, we classify the non-minimal Lorentzian hypersurfaces in $\mathbb{E}_1^{n+1}$ satisfying $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$ with non-diagonalizable shape operators and at most two distinct principal curvatures. For the case of diagonalizable shape operator, the problem has been studied by L. Du in [5].

2. Preliminaries

2.1. Notions and formulas of hypersurfaces in $\mathbb{E}_s^{n+1}$

Let M_r^n be a nondegenerate hypersurface in $\mathbb{E}_s^{n+1}$, $\overrightarrow{\xi}$ denote a unit normal vector field to M_r^n, then $\varepsilon = \langle\overrightarrow{\xi}, \overrightarrow{\xi}\rangle = \pm 1$. Denote by ∇ and $\widetilde{\nabla}$ the Levi-Civita connections of M_r^n and $\mathbb{E}_s^{n+1}$, respectively. For any vector fields X, Y tangent to M_r^n, the Gauss formula is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y)\overrightarrow{\xi},$$

where h is the scalar-valued second fundamental form. If we denote by A the shape operator of M_r^n associated to $\overrightarrow{\xi}$, then the Weingarten formula is given by

$$\widetilde{\nabla}_X \overrightarrow{\xi} = -A(X),$$

where $\langle A(X), Y\rangle = \varepsilon h(X, Y)$. The mean curvature vector field $\overrightarrow{H} = H\overrightarrow{\xi}$ with mean curvature $H = \frac{1}{n}\varepsilon \mathrm{tr} A$, determines a well defined normal vector field to M_r^n in $\mathbb{E}_s^{n+1}$. The Codazzi and Gauss equations are given by (cf. [12])

$$(\nabla_X A)Y = (\nabla_Y A)X,$$
$$R(X, Y)Z = \langle A(Y), Z\rangle A(X) - \langle A(X), Z\rangle A(Y),$$

where $R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$.

A non-zero vector X in $\mathbb{E}_s^{n+1}$ is called *time-like*, *space-like* or *light-like*, according to whether $\langle X, X\rangle$ is negative, positive or zero.

According to [4], the equation $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$ is equivalent to the following two equations:

$$A(\nabla H) = -\frac{n}{2}\varepsilon H(\nabla H), \tag{1}$$
$$\Delta H + \varepsilon H \mathrm{tr} A^2 = \lambda H, \tag{2}$$

where the Laplace operator Δ acting on scalar-valued function f is given by

$$\Delta f = -\sum_{i=1}^{n} \varepsilon_i (e_i e_i - \nabla_{e_i} e_i) f,$$

with $\{e_i\}_{i=1}^n$ a local orthonormal frame on M_r^n such that $\langle e_i, e_i\rangle = \varepsilon_i = \pm 1$.

A hypersurface is called *isoparametric* if the characteristic polynominal of the shape operator is a constant (cf. [7]). It is equivalent to the principal curvatures and their multiplicities being constants. In this paper, without special statement, multiplicity refers to algebraic multiplicity.

For simplicity, we call hypersurface M_r^n is a *proper hypersurface* if it is not minimal.

2.2. The shape operator of M_r^n in $\mathbb{E}_s^{n+1}$

According to [12, exercise 18, pp. 260–261], the tangent space $T_xM_r^n$ at $x \in M_r^n$ can be expressed as a direct sum of subspaces V_k, $1 \leq k \leq m$, that are mutually orthogonal and invariant under the shape operator A, and each $A|_{V_k}$ (the restriction of A on V_k) has form (a) or (b) as following.

(a) $A|_{V_k}$ has the form

$$A|_{V_k} = \begin{pmatrix} \lambda_k & & & & \\ 1 & \lambda_k & & & \\ & \ddots & \ddots & & \\ & & 1 & \lambda_k & \\ & & & 1 & \lambda_k \end{pmatrix},$$

with respect to a basis $\mathfrak{B}_k = \{u_{k_1}, u_{k_2}, \cdots, u_{k_{\alpha_k}}\}$ of V_k. The inner products of the basis elements in $\mathfrak{B}_k$ are all zero except

$$\langle u_{k_b}, u_{k_d} \rangle = \varepsilon_k = \pm 1, \quad b + d = \alpha_k + 1.$$

(b) $A|_{V_k}$ has the form

$$A|_{V_k} = \begin{pmatrix} \gamma_k & \tau_k & & & & & & & & \\ -\tau_k & \gamma_k & & & & & & & & \\ 1 & 0 & \gamma_k & \tau_k & & & & & & \\ 0 & 1 & -\tau_k & \gamma_k & & & & & & \\ & & 1 & 0 & \gamma_k & \tau_k & & & & \\ & & 0 & 1 & -\tau_k & \gamma_k & & & & \\ & & & & & & \ddots & & & \\ & & & & & & 1 & 0 & \gamma_k & \tau_k \\ & & & & & & 0 & 1 & -\tau_k & \gamma_k \end{pmatrix}, \tau_k \neq 0,$$

with respect to a basis $\overline{\mathfrak{B}}_k = \{\overline{u}_{1_1}, \overline{v}_{1_1}, \cdots, \overline{u}_{k_{\beta_k}}, \overline{v}_{k_{\beta_k}}\}$ of V_k. The inner products of the basis elements in $\overline{\mathfrak{B}}_k$ are all zero except

$$\langle \overline{u}_{k_b}, \overline{u}_{k_d} \rangle = 1 = -\langle \overline{v}_{k_b}, \overline{v}_{k_d} \rangle, \quad b + d = \beta_k + 1.$$

We denote by t the number of terms $A|_{V_k}$ having form (a). Adjusting the order of V_k, $1 \leq k \leq m$, such that $A|_{V_k}$ have form (a) for $1 \leq k \leq t$, and $A|_{V_k}$ have form (b) for $t + 1 \leq k \leq m$. Denote $A_i = A|_{V_i}$, $1 \leq i \leq t$ and $\overline{A}_j = A|_{V_j}$, $t + 1 \leq j \leq m$. Collecting all the vectors in $\mathfrak{B}_1, \cdots, \mathfrak{B}_t, \overline{\mathfrak{B}}_{t+1}, \cdots, \overline{\mathfrak{B}}_m$ in order, we get a basis $\mathfrak{B}$ of $T_xM_r^n$. With respect to this basis $\mathfrak{B}$, the shape operator A of the hypersurface M_r^n in $\mathbb{E}_s^{n+1}$ can be expressed as an almost diagonal matrix

$$A = \mathrm{diag}\{A_1, \cdots, A_t, \overline{A}_{t+1}, \cdots, \overline{A}_m\},$$

and the inner products of the elements in $\mathfrak{B}$ are all zero except

$$\langle u_{i_a}, u_{i_b}\rangle = \varepsilon_i = \pm 1, \quad a+b = \alpha_i + 1, \quad 1 \le i \le t,$$
$$\langle \overline{u}_{j_c}, \overline{u}_{j_d}\rangle = 1 = -\langle \overline{v}_{j_c}, \overline{v}_{j_d}\rangle, \quad c+d = \beta_j + 1, \quad t+1 \le j \le m,$$

where

$$\alpha_1 + \alpha_2 + \cdots + \alpha_t + 2(\beta_{t+1} + \beta_{t+2} + \cdots + \beta_m) = n.$$

Observe the forms (a) and (b), we see that $A_i, 1 \le i \le t$, has only a simple eigenvalue λ_i, and $\overline{A}_j, t+1 \le j \le m$, has eigenvalues $\gamma_j + \tau_j\sqrt{-1}$, $\gamma_j - \tau_j\sqrt{-1}$. It follows from the form of the shape operator A that M_r^n has principal curvatures

$$\lambda_1, \cdots, \lambda_t; \quad \gamma_{t+1} \pm \tau_{t+1}\sqrt{-1}, \cdots, \gamma_m \pm \tau_m\sqrt{-1}.$$

So, under the assumption that M_r^n has at most two distinct principal curvatures, the shape operator A has the following two possible forms:

(I) $t = m$, i.e. $A = \mathrm{diag}\{A_1, A_2, \cdots, A_m\}$, and there are at most two distinct values among $\{\lambda_1, \lambda_2, \cdots, \lambda_m\}$.

(II) $t = 0$, i.e. $A = \mathrm{diag}\{\overline{A}_1, \overline{A}_2, \cdots, \overline{A}_m\}$ and $\gamma_1 = \gamma_2 = \cdots = \gamma_m = \gamma$, $\tau_1 = \tau_2 = \cdots = \tau_m = \tau$, $\tau \ne 0$.

3. A partial affirmative answer to the conjecture

Theorem 3.1. *Let M_r^n be a nondegenerate hypersurface of $\mathbb{E}_s^{n+1}$ satisfying $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$ for a real constant λ. Suppose that M_r^n has at most two distinct principal curvatures, then M_r^n has constant mean curvature.*

Proof. From section 2, the shape operator A has the form (I) or (II). If A has the form (II), then its eigenvalues are not real. It follows from (1) that $\nabla H = 0$, which tells us H is a constant.

For form (I), assume that H is not a constant, then (1) implies that $-\frac{n}{2}\varepsilon H$ is an eigenvalue of the shape operator A. When $\lambda_1 = \cdots = \lambda_m$, then $\mathrm{tr}A = -\frac{n^2}{2}\varepsilon H$. On the other hand, $\mathrm{tr}A = n\varepsilon H$. These two expressions imply $H = 0$, a contradiction.

So, in the following, we need only to discuss the situation that there are two distinct values among $\{\lambda_1, \cdots, \lambda_m\}$. (1) also tells us that ∇H is an eigenvector of A with corresponding eigenvalue $-\frac{n}{2}\varepsilon H$, it may be a light-like vector or not. We will following a long discussion that each of the cases will lead to a contradiction, and complete the proof of Theorem 3.1.

First of all, we give some equations which are important and will be used frequently. In view of the form (I) and the form of A_i, $1 \le i \le m$ in section 2, we have

$$A(u_{i_a}) = \lambda_i u_{i_a} + u_{i_{a+1}}, \quad A(u_{i_{\alpha_i}}) = \lambda_i u_{i_{\alpha_i}}, \quad 1 \le i \le m, \ 1 \le a \le \alpha_i - 1. \tag{3}$$

Observe the inner products of the elements in basis $\mathfrak{B}$ given in section 2, we can express

$$\nabla H = \sum_{i=1}^{m}\sum_{a=1}^{\alpha_i} \varepsilon_i u_{i_{\alpha_i - a+1}}(H) u_{i_a}. \tag{4}$$

Let $\nabla_{u_{i_a}} u_{j_b} = \sum_{k=1}^{m}\sum_{d=1}^{\alpha_k} \Gamma_{i_a j_b}^{k_d} u_{k_d}$. Applying compatibility condition to calculate

$$\nabla_{u_D}\langle u_{i_a}, u_{i_a}\rangle, \nabla_{u_D}\langle u_{i_a}, u_{i_b}\rangle, \nabla_{u_D}\langle u_{i_a}, u_{j_d}\rangle,$$

respectively, we conclude

$$\Gamma_{Di_a}^{i_{\alpha_i-a+1}} = 0, \tag{5}$$

and

$$\Gamma_{Di_a}^{i_{\alpha_i-b+1}} = -\Gamma_{Di_b}^{i_{\alpha_i-a+1}},\ \Gamma_{Di_a}^{j_{\alpha_j-d+1}} = -\varepsilon_i\varepsilon_j\Gamma_{Dj_d}^{i_{\alpha_i-a+1}}, \tag{6}$$

for $D \in \{k_e, 1 \le k \le m, 1 \le e \le \alpha_k\}$, $1 \le i,j \le m$, $1 \le a,b \le \alpha_i$ and $1 \le d \le \alpha_j$.

In view of (3), ∇H is one of the directions $u_{i_{\alpha_i}}, 1 \le i \le m$. Without loss of generality, we suppose ∇H is in the direction of $u_{1_{\alpha_1}}$, it may be a light-like vector or not.

Case 1: ∇H is not light-like.

In this case, we will concentrate our attention to prove Lemmas 3.3 and 3.4, the two lemmas imply that H is a constant, a contradiction.

At the beginning, we give the range of indices i,j such that $2 \le i,j \le m$, and $i \ne j$.

As $u_{1_{\alpha_1}}$ is not a light-like vector, we have $\langle u_{1_{\alpha_1}}, u_{1_{\alpha_1}}\rangle \ne 0$, which means $2\alpha_1 = \alpha_1 + 1$, i.e. $\alpha_1 = 1$. It follows from (3) that $A(u_{1_1}) = \lambda_1 u_{1_1}$. Note that $\lambda_1 = -\frac{n}{2}\varepsilon H$, (5) implies that

$$\Gamma_{D1_1}^{1_1} = 0, \tag{7}$$

and (4) can be rewritten as

$$\nabla H = \sum_{i=2}^{m}\sum_{a=1}^{\alpha_i}\varepsilon_i u_{i_{\alpha_i-a+1}}(H)u_{i_a} + \varepsilon_1 u_{1_1}(H)u_{1_1}.$$

Since ∇H is in the direction of u_{1_1}, the above equation implies that

$$u_{1_1}(H) \ne 0, \quad u_{i_a}(H) = 0,\ 2 \le i \le m,\ 1 \le a \le \alpha_i. \tag{8}$$

From the expression $(\nabla_{u_B}u_C - \nabla_{u_C}u_B)(H) = [u_B, u_C](H)$, $B, C \in \{k_d, 1 \le k \le m, 1 \le d \le \alpha_k\}$ and (8), we easily get

$$\Gamma_{BC}^{1_1} = \Gamma_{CB}^{1_1}, \quad B, C \ne 1_1. \tag{9}$$

Lemma 3.2. *We have $\lambda_2 = \cdots = \lambda_m = \frac{3n\varepsilon H}{2(n-1)}$.*

Proof. Calculating the equation $\langle(\nabla_{u_{i_a}}A)u_{i_b}, u_{1_1}\rangle = \langle(\nabla_{u_{i_b}}A)u_{i_a}, u_{1_1}\rangle$ for $a = \alpha_i$, $1 \le b \le \alpha_i - 1$ and $1 \le a,b \le \alpha_i - 1$, and combining (9), we obtain

$$\Gamma_{i_{\alpha_i}i_{b+1}}^{1_1} = \Gamma_{i_{b+1}i_{\alpha_i}}^{1_1} = 0,\ 1 \le b \le \alpha_i - 1, \tag{10}$$

and

$$\Gamma_{i_a i_{b+1}}^{1_1} = \Gamma_{i_b i_{a+1}}^{1_1}, \quad 1 \le a,b \le \alpha_i - 1,$$

which together with (6) and (9), implies that

$$\Gamma_{i_a 1_1}^{i_b} = \Gamma_{i_{a+1}1_1}^{i_{b+1}},\ 1 \le a,b \le \alpha_i - 1.$$

Because of (6) and (10), it follows from the above equation that

$$\Gamma^{i_1}_{i_1 1_1} = \Gamma^{i_2}_{i_2 1_1} = \cdots = \Gamma^{i_{\alpha_i}}_{i_{\alpha_i} 1_1}, \tag{11}$$

and

$$\Gamma^{i_1}_{i_2 1_1} = \Gamma^{i_2}_{i_3 1_1} = \cdots = \Gamma^{i_{\alpha_i - 1}}_{i_{\alpha_i} 1_1} = 0. \tag{12}$$

Similarly, from the relation $\langle (\nabla_{u_{i_a}} A) u_{1_1}, u_{i_{\alpha_i - a+1}} \rangle = \langle (\nabla_{u_{1_1}} A) u_{i_a}, u_{i_{\alpha_i - a+1}} \rangle$ for $a = 1, 2, \cdots, \alpha_i - 1$ and (12), we get

$$\begin{cases} (-\frac{n}{2}\varepsilon H - \lambda_i)\Gamma^{i_1}_{i_1 1_1} = u_{1_1}(\lambda_i) + \Gamma^{i_1}_{1_1 i_2}, \\ (-\frac{n}{2}\varepsilon H - \lambda_i)\Gamma^{i_2}_{i_2 1_1} = u_{1_1}(\lambda_i) + \Gamma^{i_2}_{1_1 i_3} - \Gamma^{i_1}_{1_1 i_2}, \\ \qquad \vdots \\ (-\frac{n}{2}\varepsilon H - \lambda_i)\Gamma^{i_{\alpha_i - 1}}_{i_{\alpha_i - 1} 1_1} = u_{1_1}(\lambda_i) + \Gamma^{i_{\alpha_i - 1}}_{1_1 i_{\alpha_i}} - \Gamma^{i_{\alpha_i - 2}}_{1_1 i_{\alpha_i - 1}}, \\ (-\frac{n}{2}\varepsilon H - \lambda_i)\Gamma^{i_{\alpha_i}}_{i_{\alpha_i} 1_1} = u_{1_1}(\lambda_i) - \Gamma^{i_{\alpha_i - 1}}_{1_1 i_{\alpha_i}}. \end{cases}$$

As (11), the above equations imply that

$$\Gamma^{i_2}_{1_1 i_3} = 2\Gamma^{i_1}_{1_1 i_2}, \ \Gamma^{i_3}_{1_1 i_4} = 3\Gamma^{i_1}_{1_1 i_2}, \ \cdots, \ \Gamma^{i_{\alpha_i - 1}}_{1_1 i_{\alpha_i}} = (\alpha_i - 1)\Gamma^{i_1}_{1_1 i_2}.$$

On the other hand, $\Gamma^{i_{\alpha_i - 1}}_{1_1 i_{\alpha_i}} = -\Gamma^{i_1}_{1_1 i_2}$. So,

$$\Gamma^{i_1}_{1_1 i_2} = \Gamma^{i_2}_{1_1 i_3} = \cdots = \Gamma^{i_{\alpha_i - 1}}_{1_1 i_{\alpha_i}} = 0, \tag{13}$$

and

$$u_{1_1}(\lambda_i) = (-\frac{n}{2}\varepsilon H - \lambda_i)\Gamma^{i_a}_{i_a 1_1}, \ 1 \le a \le \alpha_i. \tag{14}$$

If $\lambda_i = -\frac{n}{2}\varepsilon H$, then $u_{1_1}(H) = 0$, which contradicts to (8). So, we conclude that $\lambda_i \neq -\frac{n}{2}\varepsilon H$. Since there are two distinct values among $\{\lambda_1, \cdots, \lambda_m\}$ and $\mathrm{tr}A = n\varepsilon H$, we have $\lambda_i = \frac{3n\varepsilon H}{2(n-1)}$. □

From Lemma 3.2 and (14), we easily find that

$$\Gamma^{i_a}_{i_a 1_1} = -\frac{3u_{1_1}(H)}{(n+2)H}, \ 2 \le i \le m, \ 1 \le a \le \alpha_i. \tag{15}$$

Denote $W = \Gamma^{i_a}_{i_a 1_1}$, we have the following Lemmas.

Lemma 3.3. *Suppose that H is not a constant. Then we have*

$$u_{1_1}(W) + W^2 = \frac{3\varepsilon_1 n^2 H^2}{4(n-1)}. \tag{16}$$

Proof. Calculate $\langle (\nabla_{u_{i_a}} A) u_{1_1}, u_{i_{\alpha_i}} \rangle = \langle (\nabla_{u_{1_1}} A) u_{i_a}, u_{i_{\alpha_i}} \rangle$, $2 \le a \le \alpha_i - 1$, and combine the results with (6), (9) and (10), we obtain

$$\Gamma^{i_1}_{1_1 i_b} = 0, \ 3 \le b \le \alpha_i.$$

From (6), (13) and the above equations, we find that

$$\Gamma^{i_b}_{1_1 i_{\alpha_i}} = 0,\ 1 \le b \le \alpha_i - 1. \tag{17}$$

Considering $\lambda_i = \frac{3n\varepsilon H}{2(n-1)}$ (see Lemma 3.2) and (8), the relation $\langle (\nabla_{u_{i_a}} A)u_{1_1},\ u_{1_1}\rangle = \langle (\nabla_{u_{1_1}} A)u_{i_a}, u_{1_1}\rangle$ for $a = 1, 2, \cdots, \alpha_i$, implies that

$$\begin{cases} \frac{n(n+2)\varepsilon H}{2(n-1)}\Gamma^{1_1}_{1_1 i_1} + \Gamma^{1_1}_{1_1 i_2} = 0, \\ \qquad\qquad \vdots \\ \frac{n(n+2)\varepsilon H}{2(n-1)}\Gamma^{1_1}_{1_1 i_{\alpha_i - 1}} + \Gamma^{1_1}_{1_1 i_{\alpha_i}} = 0, \\ \frac{n(n+2)\varepsilon H}{2(n-1)}\Gamma^{1_1}_{1_1 i_{\alpha_i}} = 0, \end{cases}$$

which tells us

$$\Gamma^{1_1}_{1_1 i_1} = \Gamma^{1_1}_{1_1 i_2} = \cdots = \Gamma^{1_1}_{1_1 i_{\alpha_i}} = 0. \tag{18}$$

Note that $\lambda_i = \lambda_j = \frac{3n\varepsilon H}{2(n-1)}$ (see Lemma 3.2), using $\langle (\nabla_{u_{i_{\alpha_i}}} A)u_{j_b}, u_{1_1}\rangle = \langle (\nabla_{u_{j_b}} A)u_{i_{\alpha_i}}, u_{1_1}\rangle$, $1 \le b \le \alpha_j - 1$ and (9), we have

$$\Gamma^{1_1}_{i_{\alpha_i} j_d} = \Gamma^{1_1}_{j_d i_{\alpha_i}} = 0,\ 2 \le d \le \alpha_j. \tag{19}$$

We know from $\langle (\nabla_{u_{i_{\alpha_i}}} A)u_{1_1}, u_{j_b}\rangle = \langle (\nabla_{u_{1_1}} A)u_{i_{\alpha_i}}, u_{j_b}\rangle$, $1 \le b \le \alpha_j - 1$ that

$$-\frac{n(n+2)\varepsilon H}{2(n-1)}\Gamma^{j_{\alpha_j - b+1}}_{i_{\alpha_i} 1_1} = \Gamma^{j_{\alpha_j - b}}_{i_{\alpha_i} 1_1} - \Gamma^{j_{\alpha_j - b}}_{1_1 i_{\alpha_i}}.$$

Applying (6) and (19) to the above equation, we get

$$\Gamma^{j_e}_{1_1 i_{\alpha_i}} = 0,\ 1 \le e \le \alpha_j - 2. \tag{20}$$

Using Gauss equation for $\langle R(u_{1_1}, u_{i_1})u_{i_{\alpha_i}}, u_{1_1}\rangle$, combining (6), (7), (10), (17), (18), (19) and (20), we have

$$\begin{aligned} u_{1_1}(\Gamma^{i_1}_{i_1 1_1}) &+ \sum_{j=2,\ j\ne i}^{m} \Gamma^{i_1}_{1_1 j_2}\Gamma^{j_2}_{i_1 1_1} + \sum_{j=2,\ j\ne i}^{m} \Gamma^{i_1}_{1_1 j_1}\Gamma^{j_1}_{i_1 1_1} - \sum_{j=2,\ j\ne i}^{m} \Gamma^{j_1}_{1_1 i_1}\Gamma^{i_1}_{j_1 1_1} \\ &+ \sum_{j=2,\ j\ne i}^{m} \Gamma^{j_1}_{i_1 1_1}\Gamma^{i_1}_{j_1 1_1} + (\Gamma^{i_1}_{i_1 1_1})^2 = \frac{3\varepsilon_1 n^2 H^2}{4(n-1)}. \end{aligned} \tag{21}$$

Calculate $\langle (\nabla_{u_{j_1}} A)u_{1_1}, u_{i_{\alpha_i}}\rangle = \langle (\nabla_{u_{1_1}} A)u_{j_1}, u_{i_{\alpha_i}}\rangle$, we get

$$\Gamma^{i_1}_{1_1 j_2} = -\frac{n(n+2)\varepsilon H}{2(n-1)}\Gamma^{i_1}_{j_1 1_1}. \tag{22}$$

The relations

$$\langle (\nabla_{u_{i_a}} A)u_{j_b}, u_{1_1}\rangle = \langle (\nabla_{u_{j_b}} A)u_{i_a}, u_{1_1}\rangle,\ 1 \le a \le \alpha_i - 1,\ 1 \le b \le \alpha_j - 1,$$

and (9) give that

$$\Gamma^{1_1}_{i_a j_{b+1}} = \Gamma^{1_1}_{j_b i_{a+1}},$$

which together with (6) and (9), implies

$$\Gamma^{j_b}_{i_a 1_1} = \Gamma^{j_{b+1}}_{i_{a+1} 1_1},\ 1 \le a \le \alpha_i - 1,\ 1 \le b \le \alpha_j - 1. \tag{23}$$

Combining (6), (19) and (23), we find

$$\Gamma^{j_b}_{i_{b+1} 1_1} = \Gamma^{j_1}_{i_2 1_1} = 0,\ 1 \le b \le \min\{\alpha_i - 1, \alpha_j\}. \tag{24}$$

If $\alpha_j < \alpha_i$, it follows from (6), (19) and (23) that $\Gamma^{i_1}_{j_1 1_1} = \Gamma^{i_{\alpha_j}}_{j_{\alpha_j} 1_1} = 0$. If $\alpha_j = \alpha_i$, from the relation $\langle (\nabla_{u_{j_b}} A) u_{1_1}, u_{i_{\alpha_i - b+1}} \rangle = \langle (\nabla_{u_{1_1}} A) u_{j_b}, u_{i_{\alpha_i - b+1}} \rangle$, and (24), we have

$$\begin{cases} -\frac{n(n+2)\varepsilon H}{2(n-1)} \Gamma^{i_1}_{j_1 1_1} = \Gamma^{i_1}_{1_1 j_2}, \\ -\frac{n(n+2)\varepsilon H}{2(n-1)} \Gamma^{i_2}_{j_2 1_1} = \Gamma^{i_2}_{1_1 j_3} - \Gamma^{i_1}_{1_1 j_2}, \\ \qquad \vdots \\ -\frac{n(n+2)\varepsilon H}{2(n-1)} \Gamma^{i_{\alpha_i - 1}}_{j_{\alpha_j - 1} 1_1} = \Gamma^{i_{\alpha_i - 1}}_{1_1 j_{\alpha_j}} - \Gamma^{i_{\alpha_i - 2}}_{1_1 j_{\alpha_j - 1}}, \\ -\frac{n(n+2)\varepsilon H}{2(n-1)} \Gamma^{i_{\alpha_i}}_{j_{\alpha_j} 1_1} = -\Gamma^{i_{\alpha_i - 1}}_{1_1 j_{\alpha_j}}, \end{cases}$$

which together with (23), tells us that

$$\Gamma^{i_1}_{1_1 j_2} = \Gamma^{i_2}_{1_1 j_3} = \cdots = \Gamma^{i_{\alpha_i - 1}}_{1_1 j_{\alpha_j}} = 0,$$

and

$$\Gamma^{i_1}_{j_1 1_1} = 0.$$

So, if $\alpha_j \le \alpha_i$, then

$$\Gamma^{i_1}_{j_1 1_1} = 0. \tag{25}$$

If $\alpha_j > \alpha_i$, calculate $\langle (\nabla_{u_{i_a}} A) u_{1_1}, u_{j_{\alpha_j - a}} \rangle = \langle (\nabla_{u_{1_1}} A) u_{i_a}, u_{j_{\alpha_j - a}} \rangle$, $1 \le a \le \alpha_i$, combining (23) and (25), we get

$$\begin{cases} -\frac{n(n+2)\varepsilon H}{2(n-1)} \Gamma^{j_2}_{i_1 1_1} = \Gamma^{j_2}_{1_1 i_2} - \Gamma^{j_1}_{1_1 i_1}, \\ \qquad \vdots \\ -\frac{n(n+2)\varepsilon H}{2(n-1)} \Gamma^{j_{\alpha_i}}_{i_{\alpha_i - 1} 1_1} = \Gamma^{j_{\alpha_i}}_{1_1 i_{\alpha_i}} - \Gamma^{j_{\alpha_i - 1}}_{1_1 i_{\alpha_i - 1}}, \\ -\frac{n(n+2)\varepsilon H}{2(n-1)} \Gamma^{j_{\alpha_i + 1}}_{i_{\alpha_i} 1_1} = -\Gamma^{j_{\alpha_i}}_{1_1 i_{\alpha_i}}. \end{cases} \tag{26}$$

If $\alpha_j > \alpha_i + 1$, then (19) and (23) give us that

$$\Gamma^{j_2}_{i_1 1_1} = \cdots = \Gamma^{j_{\alpha_i + 1}}_{i_{\alpha_i} 1_1} = 0. \tag{27}$$

As (27), it follows from (26) that if $\alpha_j > \alpha_i + 1$, then

$$\Gamma^{j_1}_{1_1 i_1} = 0. \tag{28}$$

If $\alpha_j = \alpha_i + 1$, then combining (6) and (9), we get from (23) and (26) that

$$\begin{aligned} &\Gamma_{i_1 1_1}^{j_2} = \Gamma_{i_{\alpha_i} 1_1}^{j_{\alpha_j}} = \varepsilon_i \varepsilon_j \Gamma_{j_1 1_1}^{i_1}, \\ &\Gamma_{1_1 i_1}^{j_1} = \alpha_i \Gamma_{1_1 i_{\alpha_i}}^{j_{\alpha_i}} = \varepsilon_i \varepsilon_j \frac{n(n+2)\varepsilon H}{2(n-1)} \alpha_i \Gamma_{j_1 1_1}^{i_1}. \end{aligned} \tag{29}$$

Therefore, from (22), (25), (27), (28) and (29), (21) can be simplified to

$$\begin{aligned} &\frac{n(n+2)\varepsilon H}{2(n-1)} \varepsilon_i \{ \sum_{j=2,\ \alpha_j = \alpha_i - 1}^{m} \varepsilon_j \alpha_j (\Gamma_{i_1 1_1}^{j_1})^2 - \sum_{j=2,\ \alpha_j = \alpha_i + 1}^{m} \varepsilon_j (\alpha_i + 1)(\Gamma_{j_1 1_1}^{i_1})^2 \} \\ &\qquad + u_{1_1}(\Gamma_{i_1 1_1}^{i_1}) + (\Gamma_{i_1 1_1}^{i_1})^2 = \frac{3\varepsilon_1 n^2 H^2}{4(n-1)}. \end{aligned} \tag{30}$$

Choosing $\{\alpha_{\gamma_1}, \cdots, \alpha_{\gamma_h}\}$ in $\{\alpha_2, \cdots, \alpha_m\}$, such that $\alpha_{\gamma_1} = \min\{\alpha_2, \cdots, \alpha_m\}$, $\alpha_{\gamma_{k+1}} = \alpha_{\gamma_k} + 1$, $1 \le k \le h-1$, $\alpha_{\gamma_h} + 1 \ne \alpha_l$, $2 \le l \le m$. Without loss of generality, we suppose $\alpha_{k+1} = \alpha_{\gamma_k}, 1 \le k \le h$. Taking $i = 2, 3, \cdots, h+1$ in (30), we have

$$\begin{cases} -(\alpha_2 + 1)Q_2 = P, \\ \alpha_2 Q_2 - (\alpha_3 + 1)Q_3 = P, \\ \qquad\quad \vdots \\ \alpha_{h-1} Q_{h-1} - (\alpha_h + 1)Q_h = P, \\ \alpha_h Q_h = P, \end{cases} \tag{31}$$

where,

$$P = -u_{1_1}(W) - W^2 + \frac{3\varepsilon_1 n^2 H^2}{4(n-1)}$$

and

$$Q_k = \frac{n(n+2)\varepsilon H}{2(n-1)} \varepsilon_k \varepsilon_{k+1} (\Gamma_{p_1 1_1}^{k_1})^2,\ p = k+1.$$

From (31), we can get $Q_k = 0$, $2 \le k \le h$ and (16) holds. $\square$

Lemma 3.4. *We have*

$$(n-1)W u_{1_1}(H) + u_{1_1} u_{1_1}(H) - \varepsilon_1 \varepsilon \frac{(n+8)n^2 H^3}{4(n-1)} + \varepsilon_1 \lambda H = 0. \tag{32}$$

Proof. Based upon the basis $\mathfrak{B}_k = \{u_{k_1}, u_{k_2}, \cdots, u_{k_{\alpha_k}}\}$ of V_k, $1 \le k \le m$, we can construct an orthonormal basis $\mathfrak{E}_k = \{e_{k_1}, e_{k_2}, \cdots, e_{k_{\alpha_k}}\}$ of V_k as follows.

If α_k is an even number, then we let

$$e_{k_a} = \frac{u_{k_a} - u_{k_{\alpha_k - a+1}}}{\sqrt{2}}, \quad e_{k_{\alpha_k - a+1}} = \frac{u_{k_a} + u_{k_{\alpha_k - a+1}}}{\sqrt{2}}, \quad 1 \le a \le \frac{\alpha_k}{2}, \tag{33}$$

and have

$$\langle e_{k_a}, e_{k_a} \rangle = -\varepsilon_k, \quad \langle e_{k_{\alpha_k - a+1}}, e_{k_{\alpha_k - a+1}} \rangle = \varepsilon_k.$$

If α_k is an odd number, then we give

$$e_{k_a} = \frac{u_{k_a} - u_{k_{\alpha_k - a+1}}}{\sqrt{2}}, \; e_{k_{\alpha_k - a+1}} = \frac{u_{k_a} + u_{k_{\alpha_k - a+1}}}{\sqrt{2}}, \; e_{k_{\frac{\alpha_k+1}{2}}} = u_{k_{\frac{\alpha_k+1}{2}}}, \tag{34}$$

with $1 \le a < \frac{\alpha_k+1}{2}$, and have

$$\langle e_{k_a}, e_{k_a} \rangle = -\varepsilon_k, \; \langle e_{k_{\alpha_k-a+1}}, e_{k_{\alpha_k-a+1}} \rangle = \varepsilon_k, \; \langle e_{k_{\frac{\alpha_k+1}{2}}}, e_{k_{\frac{\alpha_k+1}{2}}} \rangle = \varepsilon_k.$$

Thus, $\mathfrak{E} = \{e_{k_a}, 1 \le k \le m, 1 \le a \le \alpha_k\}$ is an orthonormal basis. Note that $e_{1_1} = u_{1_1}$, it follows from (8) that

$$e_{1_1}(H) \neq 0, \quad e_{i_a}(H) = 0, \; 2 \le i \le m, \; 1 \le a \le \alpha_i.$$

In fact, $\mathrm{tr}A^2$ is independent of the choice of basis. By calculating, we get

$$\mathrm{tr}A^2 = \frac{(n+8)n^2H^2}{4(n-1)}.$$

When α_k is an even number, then from (8) and (33), we easily obtain for $1 \le a \le \frac{\alpha_k}{2}$,

$$\begin{aligned}
\nabla_{e_{k_a}} e_{k_a}(H) &= \frac{1}{2}(\Gamma^{1_1}_{k_a k_a} + \Gamma^{1_1}_{k_{\alpha_k-a+1}k_{\alpha_k-a+1}} - \Gamma^{1_1}_{k_a k_{\alpha_k-a+1}} - \Gamma^{1_1}_{k_{\alpha_k-a+1}k_a})u_{1_1}(H),\\
\nabla_{e_{k_{\alpha_k-a+1}}} e_{k_{\alpha_k-a+1}}(H) &= \frac{1}{2}(\Gamma^{1_1}_{k_a k_a} + \Gamma^{1_1}_{k_{\alpha_k-a+1}k_{\alpha_k-a+1}} + \Gamma^{1_1}_{k_a k_{\alpha_k-a+1}} + \Gamma^{1_1}_{k_{\alpha_k-a+1}k_a})u_{1_1}(H).
\end{aligned}$$

Similarly, as α_k is an odd number, then from (8) and (34), we easily obtain for $1 \le a < \frac{\alpha_k+1}{2}$,

$$\begin{aligned}
\nabla_{e_{k_a}} e_{k_a}(H) &= \frac{1}{2}(\Gamma^{1_1}_{k_a k_a} + \Gamma^{1_1}_{k_{\alpha_k-a+1}k_{\alpha_k-a+1}} - \Gamma^{1_1}_{k_a k_{\alpha_k-a+1}} - \Gamma^{1_1}_{k_{\alpha_k-a+1}k_a})u_{1_1}(H),\\
\nabla_{e_{k_{\alpha_k-a+1}}} e_{k_{\alpha_k-a+1}}(H) &= \frac{1}{2}(\Gamma^{1_1}_{k_a k_a} + \Gamma^{1_1}_{k_{\alpha_k-a+1}k_{\alpha_k-a+1}} + \Gamma^{1_1}_{k_a k_{\alpha_k-a+1}} + \Gamma^{1_1}_{k_{\alpha_k-a+1}k_a})u_{1_1}(H),\\
\nabla_{e_{k_{\frac{\alpha_k+1}{2}}}} e_{k_{\frac{\alpha_k+1}{2}}}(H) &= \Gamma^{1_1}_{k_{\frac{\alpha_k+1}{2}} k_{\frac{\alpha_k+1}{2}}} u_{1_1}(H).
\end{aligned}$$

It follows from (2), (6) and the above that

$$\sum_{i=2}^{m}\sum_{a=1}^{\alpha_i} \Gamma^{i_a}_{i_a 1_1} u_{1_1}(H) + u_{1_1}u_{1_1}(H) - \varepsilon_1\varepsilon \frac{(n+8)n^2H^3}{4(n-1)} + \varepsilon_1\lambda H = 0. \tag{35}$$

Combining (7) and $W = \Gamma^{i_a}_{i_a 1_1}, 2 \le i \le m$, then (35) can be simplified to (32) and Lemma 3.4 follows. □

Now, we continue the proof of Theorem 3.1 for case 1.

Calculating $u_{1_1}u_{1_1}(H)$, using (15) and (16), we get

$$u_{1_1}u_{1_1}(H) = \frac{(n+2)(n+5)H}{9}W^2 - \frac{(n+2)n^2\varepsilon_1 H^3}{4(n-1)}.$$

By (15) and the above equation, (32) becomes

$$\{\frac{2(n-4)(n+2)}{9}W^2 + \frac{(n+2+\varepsilon(n+8))n^2\varepsilon_1 H^2}{4(n-1)} - \varepsilon_1\lambda\}H = 0. \tag{36}$$

Applying u_{1_1} to both sides of (36) and using (15) and (16), we obtain

$$\{\frac{2(n-4)(n+2)}{9}W^2 + \frac{(n+2+\varepsilon(n+8))n^2\varepsilon_1 H^2}{4(n-1)} - \varepsilon_1\lambda\}u_{1_1}(H)$$
$$-2(n+2)\{\frac{2(n-4)}{9}W^2 + \frac{(n-10-\varepsilon(n+8))n^2\varepsilon_1 H^2}{12(n-1)}\}HW = 0.$$

Further, using (15) and (36), the above equation implies that

$$\frac{2(n-4)}{9}W^2 + \frac{(n-10-\varepsilon(n+8))n^2\varepsilon_1 H^2}{12(n-1)} = 0,$$

which together with (36) gives that H must be a constant, and this is a contradiction.

Case 2: ∇H is light-like.

In this case, we will focus on proving Lemmas 3.5 and 3.7. The two lemmas imply that H is a constant, a contradiction.

Notice that ∇H is in the direction of $u_{1_{\alpha_1}}$, $\lambda_1 = -\frac{n}{2}\varepsilon H$, and $u_{1_{\alpha_1}}$ is light-like means $\alpha_1 \neq 1$. Since $\mathrm{tr}A = n\varepsilon H$, we can suppose $\lambda_1 = \cdots = \lambda_p = -\frac{n}{2}\varepsilon H$ and $\lambda_{p+1} = \cdots = \lambda_m = \frac{(2+\alpha_1+\cdots+\alpha_p)n\varepsilon H}{2(\alpha_{p+1}+\cdots+\alpha_m)}$. It follows from (4) that

$$u_{1_1}(H) \neq 0, \quad u_B(H) = 0,\ B \neq 1_1, \tag{37}$$

where $B \in \{h_a, 1 \le h \le m, 1 \le a \le \alpha_h\}$. Certainly,

$$u_{1_1}(\lambda_k) \neq 0,\ u_B(\lambda_k) = 0,\ B \neq 1_1,\ 1 \le k \le m. \tag{38}$$

Observe $(\nabla_{u_B}u_C - \nabla_{u_C}u_B)(H) = [u_B, u_C](H)$, $B, C \in \{h_a, 1 \le h \le m, 1 \le a \le \alpha_h\}$, using (37), we have

$$\Gamma^{1_1}_{BC} = \Gamma^{1_1}_{CB}, \quad B, C \neq 1_1. \tag{39}$$

Lemma 3.5. *We have*

$$-\varepsilon_1\sum_{i=2}^{p}\alpha_i\Gamma^{i_{\alpha_i}}_{i_{\alpha_i}1_{\alpha_1}}u_{1_1}(H) + \varepsilon H\,trA^2 = \lambda H, \tag{40}$$

where

$$trA^2 = \frac{[(n+4)(\alpha_1+\cdots+\alpha_p)+4]n^2H^2}{4[n-(\alpha_1+\cdots+\alpha_p)]}.$$

Proof. We construct an orthonormal basis $\mathfrak{E}$ the same as that in the proof of Lemma 3.4. With this orthonormal basis $\mathfrak{E}$ and (37), we have

$$\begin{cases} e_{1_1}e_{1_1}(H) = (u_{1_1}u_{1_1}(H) - u_{1_{\alpha_1}}u_{1_1}(H))/2, \\ e_{1_{\alpha_1}}e_{1_{\alpha_1}}(H) = (u_{1_1}u_{1_1}(H) + u_{1_{\alpha_1}}u_{1_1}(H))/2, \\ e_{1_b}e_{1_b}(H) = 0, \quad 2 \le b \le \alpha_1 - 1, \\ e_{k_a}e_{k_a}(H) = 0, \quad 2 \le k \le m, \quad 1 \le a \le \alpha_k. \end{cases} \tag{41}$$

Similar to the proof of Lemma 3.4, according to the orthonormal basis $\mathfrak{E}$ and (37), we obtain the expressions of $\nabla_{e_{k_a}}e_{k_a}(H)$, $1 \le k \le m$, $1 \le a \le \alpha_k$. Substitute the expressions and (41) into (2), we get

$$-\varepsilon_1 u_{1_{\alpha_1}} u_{1_1}(H) - \varepsilon_1 \sum_{k=1}^{m}\sum_{a=1}^{\alpha_k} \Gamma_{k_a 1_{\alpha_1}}^{k_a} u_{1_1}(H) + \varepsilon H \mathrm{tr} A^2 = \lambda H. \tag{42}$$

Calculating $\langle(\nabla_{u_{k_a}} A)u_{k_b}, u_{1_{\alpha_1}}\rangle = \langle(\nabla_{u_{k_b}} A)u_{k_a}, u_{1_{\alpha_1}}\rangle$, $2 \le k \le m$, for $a = \alpha_k$, $1 \le b \le \alpha_k - 1$ and $1 \le a, b \le \alpha_k - 1$, and combining (39), we obtain

$$\Gamma_{k_{\alpha_k} k_d}^{1_1} = \Gamma_{k_d k_{\alpha_k}}^{1_1} = 0,\ 2 \le k \le m,\ 2 \le d \le \alpha_k,$$

and

$$\Gamma_{k_a k_{b+1}}^{1_1} = \Gamma_{k_b k_{a+1}}^{1_1},\ 2 \le k \le m,\ 1 \le a, b \le \alpha_k - 1.$$

By using (6) and (39), it follows from the above equations that

$$\Gamma_{k_1 1_{\alpha_1}}^{k_1} = \Gamma_{k_2 1_{\alpha_1}}^{k_2} = \cdots = \Gamma_{k_{\alpha_k} 1_{\alpha_1}}^{k_{\alpha_k}},\ 2 \le k \le m, \tag{43}$$

and

$$\Gamma_{k_2 1_{\alpha_1}}^{k_1} = \Gamma_{k_3 1_{\alpha_1}}^{k_2} = \cdots = \Gamma_{k_{\alpha_k} 1_{\alpha_1}}^{k_{\alpha_k - 1}} = 0,\ 2 \le k \le m. \tag{44}$$

From the relations

$$\langle(\nabla_{u_{k_a}} A)u_{1_{\alpha_1}}, u_{k_{\alpha_k - a+1}}\rangle = \langle(\nabla_{u_{1_{\alpha_1}}} A)u_{k_a}, u_{k_{\alpha_k - a+1}}\rangle,\ 2 \le k \le m,$$

as well as (38) and (44), we know

$$\begin{cases} (-\frac{n}{2}\varepsilon H - \lambda_k)\Gamma_{k_1 1_{\alpha_1}}^{k_1} = \Gamma_{1_{\alpha_1} k_2}^{k_1}, \\ (-\frac{n}{2}\varepsilon H - \lambda_k)\Gamma_{k_2 1_{\alpha_1}}^{k_2} = \Gamma_{1_{\alpha_1} k_3}^{k_2} - \Gamma_{1_{\alpha_1} k_2}^{k_1}, \\ \qquad\vdots \\ (-\frac{n}{2}\varepsilon H - \lambda_k)\Gamma_{k_{\alpha_k - 1} 1_{\alpha_1}}^{k_{\alpha_k - 1}} = \Gamma_{1_{\alpha_1} k_{\alpha_k}}^{k_{\alpha_k - 1}} - \Gamma_{1_{\alpha_1} k_{\alpha_k - 1}}^{k_{\alpha_k - 2}}, \\ (-\frac{n}{2}\varepsilon H - \lambda_k)\Gamma_{k_{\alpha_k} 1_{\alpha_1}}^{k_{\alpha_k}} = -\Gamma_{1_{\alpha_1} k_{\alpha_k}}^{k_{\alpha_k - 1}}, \end{cases}$$

which together with (43), implies that

$$\Gamma_{1_{\alpha_1} k_2}^{k_1} = \Gamma_{1_{\alpha_1} k_3}^{k_2} = \cdots = \Gamma_{1_{\alpha_1} k_{\alpha_k}}^{k_{\alpha_k - 1}} = 0,\ 2 \le k \le m,$$

and

$$\Gamma_{h_1 1_{\alpha_1}}^{h_1} = \Gamma_{h_2 1_{\alpha_1}}^{h_2} = \cdots = \Gamma_{h_{\alpha_h} 1_{\alpha_1}}^{h_{\alpha_h}} = 0,\ p+1 \le h \le m. \tag{45}$$

The relations $\langle(\nabla_{u_{1_a}} A)u_{1_b}, u_{1_{\alpha_1}}\rangle = \langle(\nabla_{u_{1_b}} A)u_{1_a}, u_{1_{\alpha_1}}\rangle$, $1 \le a \le \alpha_1 - 1$, $2 \le b \le \alpha_1 - 1$, and (38) give that

$$\Gamma_{1_a 1_{b+1}}^{1_1} = \Gamma_{1_b 1_{a+1}}^{1_1}.$$

Applying (5), (6) and (39), we get from the above equation that

$$\Gamma_{1_{\alpha_1 - 1} 1_{\alpha_1}}^{1_{\alpha_1 - 1}} = \cdots = \Gamma_{1_2 1_{\alpha_1}}^{1_2} = \Gamma_{1_1 1_{\alpha_1}}^{1_1} = 0. \tag{46}$$

Since $[u_{1_1}, u_{1_{\alpha_1}}](H) = \nabla_{u_{1_1}} u_{1_{\alpha_1}}(H) - \nabla_{u_{1_{\alpha_1}}} u_{1_1}(H)$, combining (5) and (37), we have

$$u_{1_{\alpha_1}} u_{1_1}(H) = \Gamma^{1_1}_{1_{\alpha_1} 1_1} u_{1_1}(H). \tag{47}$$

Notice that $\alpha_1 + \cdots + \alpha_m = n$. By calculating, we easily obtain

$$\mathrm{tr} A^2 = \frac{[(n+4)(\alpha_1 + \cdots + \alpha_p) + 4] n^2 H^2}{4[n - (\alpha_1 + \cdots + \alpha_p)]}. \tag{48}$$

Combining (45), (46), (47) and (48), (42) can be simplified to (40). □

In the proofs of Lemmas 3.6 and 3.7, we give the range of indices i, j: $2 \le i, j \le p$ and $i \ne j$.

Lemma 3.6. *We have*

$$\begin{aligned} u_{1_{\alpha_1}}(\Gamma^{i_{\alpha_i}}_{i_{\alpha_i} 1_{\alpha_1}}) = {} & \Gamma^{1_{\alpha_1}}_{1_{\alpha_1} 1_{\alpha_1}} \Gamma^{i_{\alpha_i}}_{i_{\alpha_i} 1_{\alpha_1}} - (\Gamma^{i_{\alpha_i}}_{i_{\alpha_i} 1_{\alpha_1}})^2 - \sum_{c_1} (\Gamma^{j_1}_{i_1 1_{\alpha_1}} \Gamma^{i_1}_{1_{\alpha_1} j_1} \\ & - \Gamma^{i_1}_{j_1 1_{\alpha_1}} \Gamma^{j_1}_{1_{\alpha_1} i_1} + \Gamma^{j_1}_{i_1 1_{\alpha_1}} \Gamma^{i_1}_{j_1 1_{\alpha_1}}), \end{aligned} \tag{49}$$

for $2 \le i \le p$, *where* c_1 *means* $2 \le j \le p, \ j \ne i, \ \alpha_j = \alpha_i$.

Proof. It follows from (5), (6) and (39) that

$$\Gamma^{k_b}_{1_{\alpha_1} 1_{\alpha_1}} = -\varepsilon_k \varepsilon_1 \Gamma^{1_1}_{1_{\alpha_1} k_{\alpha_k - b+1}} = -\varepsilon_k \varepsilon_1 \Gamma^{1_1}_{k_{\alpha_k - b+1} 1_{\alpha_1}} = 0, \ k_b \ne 1_{\alpha_1}. \tag{50}$$

Evaluating (B, C, D) in the set

$$\{(i_{\alpha_i}, 1_a, 1_{\alpha_1}), (1_a, 1_{\alpha_1}, i_{\alpha_i}), (i_{\alpha_i}, i_b, 1_{\alpha_1}), (i_b, 1_{\alpha_1}, i_{\alpha_i})\}$$

with $1 \le a \le \alpha_1 - 1, 1 \le b \le \alpha_i - 1$, then the equation $\langle (\nabla_{u_B} A) u_C, u_D \rangle = \langle (\nabla_{u_C} A) u_B, u_D \rangle$, combining (38), implies that

$$\Gamma^{1_1}_{i_{\alpha_i} 1_{a+1}} = \Gamma^{i_1}_{1_{\alpha_1} 1_{a+1}} = \Gamma^{1_1}_{i_{\alpha_i} i_{b+1}} = \Gamma^{i_1}_{1_{\alpha_1} i_{b+1}} = 0.$$

As (6), the above is equivalent to

$$\Gamma^{1_a}_{i_{\alpha_i} 1_{\alpha_1}} = \Gamma^{1_a}_{1_{\alpha_1} i_{\alpha_i}} = \Gamma^{i_b}_{i_{\alpha_i} 1_{\alpha_1}} = \Gamma^{i_b}_{1_{\alpha_1} i_{\alpha_i}} = 0, \ 1 \le a \le \alpha_1 - 1, \ 1 \le b \le \alpha_i - 1. \tag{51}$$

From the expressions $\langle (\nabla_{u_{i_a}} A) u_{k_b}, u_{1_{\alpha_1}} \rangle = \langle (\nabla_{u_{k_b}} A) u_{i_a}, u_{1_{\alpha_1}} \rangle$, $2 \le k \le m$, $k \ne i$, we get

$$\begin{cases} (\lambda_k + \frac{n}{2}\varepsilon H) \Gamma^{1_1}_{i_{\alpha_i} k_{\alpha_k}} = 0, \\ (\lambda_k + \frac{n}{2}\varepsilon H) \Gamma^{1_1}_{i_{\alpha_i} k_b} + \Gamma^{1_1}_{i_{\alpha_i} k_{b+1}} = 0, \ 1 \le b \le \alpha_k - 1, \\ (\lambda_k + \frac{n}{2}\varepsilon H) \Gamma^{1_1}_{i_a k_{\alpha_k}} = \Gamma^{1_1}_{k_{\alpha_k} i_{a+1}}, \ 1 \le a \le \alpha_i - 1, \\ (\lambda_k + \frac{n}{2}\varepsilon H) \Gamma^{1_1}_{i_a k_b} + \Gamma^{1_1}_{i_a k_{b+1}} = \Gamma^{1_1}_{k_b i_{a+1}}, \ 1 \le a \le \alpha_i - 1, 1 \le b \le \alpha_k - 1, \end{cases}$$

which together with (39), tells us that

$$\Gamma^{1_1}_{i_a h_b} = 0, \ \text{i.e.} \ \Gamma^{h_b}_{i_a 1_{\alpha_1}} = \Gamma^{i_a}_{h_b 1_{\alpha_1}} = 0, \ p+1 \le h \le m. \tag{52}$$

The relations

$$\langle (\nabla_{u_{i_{\alpha_i}}} A) u_{j_b}, u_{1_{\alpha_1}} \rangle = \langle (\nabla_{u_{j_b}} A) u_{i_{\alpha_i}}, u_{1_{\alpha_1}} \rangle,$$
$$\langle (\nabla_{u_{j_b}} A) u_{1_{\alpha_1}}, u_{i_{\alpha_i}} \rangle = \langle (\nabla_{u_{1_{\alpha_1}}} A) u_{j_b}, u_{i_{\alpha_i}} \rangle,$$

give that

$$\Gamma^{1_1}_{i_{\alpha_i} j_{b+1}} = \Gamma^{i_1}_{1_{\alpha_1} j_{b+1}} = 0,$$

i.e.

$$\Gamma^{j_b}_{i_{\alpha_i} 1_{\alpha_1}} = \Gamma^{j_b}_{1_{\alpha_1} i_{\alpha_i}} = 0, \; 1 \le b \le \alpha_j - 1. \tag{53}$$

Using Gauss equation for $\langle R(u_{1_{\alpha_1}}, u_{i_{\alpha_i}}) u_{1_{\alpha_1}}, u_{i_1} \rangle$, combining (50), (51), (52) and (53), we have

$$\begin{aligned} u_{1_{\alpha_1}}(\Gamma^{i_{\alpha_i}}_{i_{\alpha_i} 1_{\alpha_1}}) = \Gamma^{1_{\alpha_1}}_{1_{\alpha_1} 1_{\alpha_1}} \Gamma^{i_{\alpha_i}}_{i_{\alpha_i} 1_{\alpha_1}} - (\Gamma^{i_{\alpha_i}}_{i_{\alpha_i} 1_{\alpha_1}})^2 - \sum_{j=2,\; j \ne i}^{p} (\Gamma^{j_{\alpha_j}}_{i_{\alpha_i} 1_{\alpha_1}} \Gamma^{i_{\alpha_i}}_{1_{\alpha_1} j_{\alpha_j}} \\ - \Gamma^{i_{\alpha_i}}_{j_{\alpha_j} 1_{\alpha_1}} \Gamma^{j_{\alpha_j}}_{1_{\alpha_1} i_{\alpha_i}} + \Gamma^{j_{\alpha_j}}_{i_{\alpha_i} 1_{\alpha_1}} \Gamma^{i_{\alpha_i}}_{j_{\alpha_j} 1_{\alpha_1}}). \end{aligned}$$

It follows from (6) and (39) that $\Gamma^{j_{\alpha_j}}_{i_{\alpha_i} 1_{\alpha_1}} = \varepsilon_i \varepsilon_j \Gamma^{i_1}_{j_1 1_{\alpha_1}}$. So, we can rewrite the above equation as

$$\begin{aligned} u_{1_{\alpha_1}}(\Gamma^{i_{\alpha_i}}_{i_{\alpha_i} 1_{\alpha_1}}) = \Gamma^{1_{\alpha_1}}_{1_{\alpha_1} 1_{\alpha_1}} \Gamma^{i_{\alpha_i}}_{i_{\alpha_i} 1_{\alpha_1}} - (\Gamma^{i_{\alpha_i}}_{i_{\alpha_i} 1_{\alpha_1}})^2 - \sum_{j=2,\; j \ne i}^{m} (\Gamma^{j_1}_{i_1 1_{\alpha_1}} \Gamma^{i_1}_{1_{\alpha_1} j_1} \\ - \Gamma^{i_1}_{j_1 1_{\alpha_1}} \Gamma^{j_1}_{1_{\alpha_1} i_1} + \Gamma^{j_1}_{i_1 1_{\alpha_1}} \Gamma^{i_1}_{j_1 1_{\alpha_1}}). \end{aligned} \tag{54}$$

We know from the equation

$$\langle (\nabla_{u_{i_a}} A) u_{j_b}, u_{1_{\alpha_1}} \rangle = \langle (\nabla_{u_{j_b}} A) u_{i_a}, u_{1_{\alpha_1}} \rangle, \; 1 \le a \le \alpha_i - 1, \; 1 \le b \le \alpha_j - 1,$$

that $\Gamma^{1_1}_{i_a j_{b+1}} = \Gamma^{1_1}_{j_b i_{a+1}}$, which implies that

$$\Gamma^{j_b}_{i_a 1_{\alpha_1}} = \Gamma^{j_{b+1}}_{i_{a+1} 1_{\alpha_1}}. \tag{55}$$

If $\alpha_j > \alpha_i$, then from (53) and (55), we find

$$\Gamma^{j_1}_{i_1 1_{\alpha_1}} = \cdots = \Gamma^{j_{\alpha_i}}_{i_{\alpha_i} 1_{\alpha_1}} = 0. \tag{56}$$

And from $\langle (\nabla_{u_{i_b}} A) u_{1_{\alpha_1}}, u_{j_{\alpha_j - b}} \rangle = \langle (\nabla_{u_{1_{\alpha_1}}} A) u_{i_b}, u_{j_{\alpha_j - b}} \rangle$, $1 \le b \le \alpha_i$, we get that

$$\begin{cases} -\Gamma^{j_1}_{i_1 1_{\alpha_1}} = \Gamma^{j_2}_{1_{\alpha_1} i_2} - \Gamma^{j_1}_{1_{\alpha_1} i_1}, \\ \qquad \vdots \\ -\Gamma^{j_{\alpha_i - 1}}_{i_{\alpha_i - 1} 1_{\alpha_1}} = \Gamma^{j_{\alpha_i}}_{1_{\alpha_1} i_{\alpha_i}} - \Gamma^{j_{\alpha_i - 1}}_{1_{\alpha_1} i_{\alpha_i - 1}}, \\ -\Gamma^{j_{\alpha_i}}_{i_{\alpha_i} 1_{\alpha_1}} = -\Gamma^{j_{\alpha_i}}_{1_{\alpha_1} i_{\alpha_i}}, \end{cases}$$

which together with (56), gives that if $\alpha_j > \alpha_i$, then

$$\Gamma^{j_1}_{1_{\alpha_1} i_1} = 0. \tag{57}$$

Similarly, if $\alpha_i > \alpha_j$, then

$$\Gamma^{i_1}_{j_1 1_{\alpha_1}} = \Gamma^{i_1}_{1_{\alpha_1} j_1} = 0. \tag{58}$$

Combining (56), (57) and (58), we conclude from (54) that (49) holds. □

Lemma 3.7. *We have*

$$\Gamma^{i_{\alpha_i}}_{i_{\alpha_i} 1_{\alpha_1}} = 0, \; 2 \le i \le p.$$

Proof. Differentiating (40) along the direction $u_{1_{\alpha_1}}$, combining (6), (37), (47) and (49), we obtain

$$\sum_{i=2}^{p} \alpha_i (\Gamma^{i_{\alpha_i}}_{i_{\alpha_i} 1_{\alpha_1}})^2 + \sum_{c_2} \alpha_i (\Gamma^{j_1}_{i_1 1_{\alpha_1}} \Gamma^{i_1}_{1_{\alpha_1} j_1} - \Gamma^{i_1}_{j_1 1_{\alpha_1}} \Gamma^{j_1}_{1_{\alpha_1} i_1} + \Gamma^{j_1}_{i_1 1_{\alpha_1}} \Gamma^{i_1}_{j_1 1_{\alpha_1}}) = 0,$$

where c_2 means $2 \le i, j \le p, \; i \ne j, \; \alpha_i = \alpha_j$, i.e.

$$\sum_{i=2}^{p} \alpha_i (\Gamma^{i_{\alpha_i}}_{i_{\alpha_i} 1_{\alpha_1}})^2 + \sum_{c_2} \alpha_i (\Gamma^{j_1}_{i_1 1_{\alpha_1}} \Gamma^{i_1}_{j_1 1_{\alpha_1}}) = 0.$$

In fact, if $\alpha_i = \alpha_j$, then $\Gamma^{i_1}_{j_1 1_{\alpha_1}} = \Gamma^{i_{\alpha_i}}_{j_{\alpha_j} 1_{\alpha_1}} = \varepsilon_i \varepsilon_j \Gamma^{j_1}_{i_1 1_{\alpha_1}}$, the above equation can be rewritten as

$$\sum_{i=2}^{p} \alpha_i (\Gamma^{i_{\alpha_i}}_{i_{\alpha_i} 1_{\alpha_1}})^2 + \sum_{c_2} \alpha_i \varepsilon_i \varepsilon_j (\Gamma^{j_1}_{i_1 1_{\alpha_1}})^2 = 0. \tag{59}$$

For $\alpha_i = \alpha_j$, we know from $\langle (\nabla_{u_{i_b}} A) u_{1_{\alpha_1}}, u_{j_{\alpha_j - b}} \rangle = \langle (\nabla_{u_{1_{\alpha_1}}} A) u_{i_b}, u_{j_{\alpha_j - b}} \rangle$, $1 \le b \le \alpha_i - 1$, that

$$\begin{cases} -\Gamma^{j_1}_{i_1 1_{\alpha_1}} = \Gamma^{j_2}_{1_{\alpha_1} i_2} - \Gamma^{j_1}_{1_{\alpha_1} i_1}, \\ \quad \vdots \\ -\Gamma^{j_{\alpha_j - 1}}_{i_{\alpha_i - 1} 1_{\alpha_1}} = \Gamma^{j_{\alpha_j}}_{1_{\alpha_1} i_{\alpha_i}} - \Gamma^{j_{\alpha_j - 1}}_{1_{\alpha_1} i_{\alpha_i - 1}}, \end{cases}$$

which together with (6) and (55), tells us that if $\alpha_i = \alpha_j$ and $\varepsilon_i \varepsilon_j = -1$, then

$$\Gamma^{j_1}_{i_1 1_{\alpha_1}} = \frac{\Gamma^{j_1}_{1_{\alpha_1} i_1} - \Gamma^{i_1}_{1_{\alpha_1} j_1}}{\alpha_i - 1}.$$

Putting the above equation into (59), we obtain

$$\sum_{i=2}^{p} \alpha_i (\Gamma^{i_{\alpha_i}}_{i_{\alpha_i} 1_{\alpha_1}})^2 + \sum_{c_3} \alpha_i (\Gamma^{j_1}_{i_1 1_{\alpha_1}})^2 - \sum_{c_4} \frac{\alpha_i}{(\alpha_i - 1)^2} (\Gamma^{j_1}_{1_{\alpha_1} i_1} - \Gamma^{i_1}_{1_{\alpha_1} j_1})^2 = 0,$$

where c_3 means $2 \le i, j \le p, \; i \ne j, \; \alpha_i = \alpha_j, \; \varepsilon_i \varepsilon_j = 1$ and c_4 means $2 \le i, j \le p, \; i \ne j, \; \alpha_i = \alpha_j, \; \varepsilon_i \varepsilon_j = -1$. It follows from the above equation that

$$\sum_{i=2}^{p} \alpha_i (\Gamma^{i_{\alpha_i}}_{i_{\alpha_i} 1_{\alpha_1}})^2 + \sum_{c_3} \alpha_i (\Gamma^{j_1}_{i_1 1_{\alpha_1}})^2 = 0,$$

which implies Lemma 3.7. □

Coming back to the proof of Theorem 3.1 for case 2, we have from Lemmas 3.5 and 3.7 that

$$\frac{[(n+4)(\alpha_1+\cdots+\alpha_p)+4]\varepsilon n^2H^3}{4[n-(\alpha_1+\cdots+\alpha_p)]}=\lambda H,$$

which implies H is a constant. It is a contradiction.

In view of the two cases, we complete the proof of Theorem 3.1. □

Remark. Recall the method or ideas in the proof of Theorem 3.1, we find that the principal curvatures of the hypersurfaces considered in Theorem 3.1 are all real or all imaginary. When the hypersurfaces have more than two distinct principal curvatures, the principal curvatures are possibly all real or not all real. For the later case, the shape operator can have a complicated form. For the former case, the principal curvatures are not all terms about H and it is difficult to get an algebraic equation about H.

4. Estimation of the constant mean curvature H

Based upon Theorem 3.1, we know that the hypersurface M_r^n satisfying $\Delta\overrightarrow{H}=\lambda\overrightarrow{H}$ with at most two distinct principal curvatures has constant mean curvature. In this section, we will estimate that constant according to the two possible cases: M_r^n has at most two distinct real principal curvatures or a pair of adjoint imaginary principal curvatures. As before, ε is the inner product of the normal vector field ξ with itself.

4.1. M_r^n has at most two distinct real principal curvatures

Proposition 4.1. *Let M_r^n be a nondegenerate hypersurface of $\mathbb{E}_s^{n+1}$ satisfying $\Delta\overrightarrow{H}=\lambda\overrightarrow{H}$. Suppose that M_r^n has at most two distinct principal curvatures, which are all real numbers.*

(1) *If $\varepsilon\lambda\le 0$, then M_r^n is minimal.*
(2) *If $\varepsilon\lambda>0$, then we have*
   (i) *When the principal curvatures are the same, says μ, then $\mu=0$, $H=0$, or $\mu=\varepsilon\sqrt{\frac{\varepsilon\lambda}{n}}$, $H=\sqrt{\frac{\varepsilon\lambda}{n}}$, or $\mu=-\varepsilon\sqrt{\frac{\varepsilon\lambda}{n}}$, $H=-\sqrt{\frac{\varepsilon\lambda}{n}}$.*
   (ii) *When M_r^n has two distinct principal curvatures μ and ν, then $-\sqrt{\frac{\varepsilon\lambda}{n}}<H<\sqrt{\frac{\varepsilon\lambda}{n}}$. Specially, if algebraic and geometric multiplicities of μ or ν coincide, says l, then M_r^n is minimal or $H=\frac{\sqrt{l\varepsilon\lambda}}{n}$, $\mu=\varepsilon\sqrt{\frac{\varepsilon\lambda}{l}}$, $\nu=0$ or $H=-\frac{\sqrt{l\varepsilon\lambda}}{n}$, $\mu=-\varepsilon\sqrt{\frac{\varepsilon\lambda}{l}}$, $\nu=0$.*

Proof. We know from Theorem 3.1 that H is a constant, it follows from (2) that

$$H\mathrm{tr}A^2=\varepsilon\lambda H. \tag{60}$$

(1) When $\varepsilon\lambda<0$, it is easy to see from (60) that $H=0$. When $\varepsilon\lambda=0$, then (60) implies $H=0$, or $\mathrm{tr}A^2=0$. Since $\mathrm{tr}A^2$ is equal to the sum of the squares of all principal curvatures and $\mathrm{tr}A$ is equal to the sum of all principal curvatures, so $\mathrm{tr}A^2=0$ tells us $\mathrm{tr}A=0$. Combining $\mathrm{tr}A=n\varepsilon H$, we have $H=0$, i.e. M_r^n is minimal.

(2) When $\varepsilon\lambda>0$, it follows from (60) that $H=0$ or $\mathrm{tr}A^2=\varepsilon\lambda$. For the case that the principal curvatures are the same, says μ, if $H=0$, together with $\mathrm{tr}A=n\varepsilon H$, we easily find $\mu=0$. If $\mathrm{tr}A^2=\varepsilon\lambda$, then $n\mu^2=\varepsilon\lambda$. Notice that $\mu=\varepsilon H$, we obtain $H=\sqrt{\frac{\varepsilon\lambda}{n}}$, $\mu=\varepsilon\sqrt{\frac{\varepsilon\lambda}{n}}$, or $H=-\sqrt{\frac{\varepsilon\lambda}{n}}$, $\mu=-\varepsilon\sqrt{\frac{\varepsilon\lambda}{n}}$.

For the case that M_r^n has two distinct principal curvatures μ and ν, with multiplicities l and $n-l$, respectively, if $\mathrm{tr}A^2 = \varepsilon\lambda$, by investigating $\mathrm{tr}A$ and $\mathrm{tr}A^2$, we have

$$\begin{cases} l\mu + (n-l)\nu = n\varepsilon H, \\ l\mu^2 + (n-l)\nu^2 = \varepsilon\lambda, \end{cases} \tag{61}$$

which implies that $nl\mu^2 - 2nl\varepsilon H\mu + n^2H^2 - (n-l)\varepsilon\lambda = 0$. Note that $\mu \neq \nu$, this equation has real roots if and only if

$$4n^2l^2H^2 - 4nl(n^2H^2 - (n-l)\varepsilon\lambda) > 0,$$

which tells us that $-\sqrt{\frac{\varepsilon\lambda}{n}} < H < \sqrt{\frac{\varepsilon\lambda}{n}}$.

Finally, we consider the special case that algebraic and geometric multiplicities of μ or ν coincide. If $\mathrm{tr}A^2 = \varepsilon\lambda$, then (61) holds, which implies that M_r^n is isoparametric. Using a basic identity of Cartan in [7, Theorem 2.9], we know that $\frac{l\mu\nu}{\mu-\nu} = 0$ or $\frac{(n-l)\nu\mu}{\nu-\mu} = 0$, i.e. $\mu\nu = 0$. Suppose $\mu \neq 0$ and $\nu = 0$, the Proposition 4.1 follows from (61). □

4.2. M_r^n has a pair of adjoint imaginary principal curvatures

Proposition 4.2. *Let M_r^{2n} be a nondegenerate hypersurface of $\mathbb{E}_s^{2n+1}$ satisfying $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$ with $\varepsilon\lambda \geq 0$. Suppose that M_r^{2n} has a pair of adjoint imaginary principal curvatures, then $H = 0$, or $H < -\sqrt{\frac{\varepsilon\lambda}{2n}}$ or $H > \sqrt{\frac{\varepsilon\lambda}{2n}}$. $H = 0$ if and only if the real parts of principal curvatures are zero.*

Proof. From Theorem 3.1, we know H is a constant. It follows from (2) that $H = 0$ or $\mathrm{tr}A^2 = \varepsilon\lambda$.

Denote $\gamma + \tau\sqrt{-1}$ and $\gamma - \tau\sqrt{-1}$ the two imaginary principal curvatures of M_r^{2n} with multiplicity n, then $\gamma = \varepsilon H$, and $H = 0$ if and only if $\gamma = 0$.

Following from the form (Ⅱ) of the shape operator A, we get $\mathrm{tr}A^2 = 2n(\gamma^2 - \tau^2)$. So, when we consider the possible case of $\mathrm{tr}A^2 = \varepsilon\lambda$, the two equations together with $\gamma = \varepsilon H$ imply that γ, τ are constants and $H^2 > \frac{\varepsilon\lambda}{2n}$. So, $H < -\sqrt{\frac{\varepsilon\lambda}{2n}}$ or $H > \sqrt{\frac{\varepsilon\lambda}{2n}}$. We complete the proof of Proposition 4.2. □

5. Classification for proper Lorentzian hypersurfaces

In this section, we will classify proper Lorentzian hypersurfaces (i.e. $r = s = 1$) in $\mathbb{E}_1^{n+1}$ satisfying $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$ with at most two distinct principal curvatures. In this case, $\varepsilon = \langle\xi, \xi\rangle = 1$.

Recall from [10] or [11] that, for Lorentz hypersurface M_1^n in $\mathbb{E}_1^{n+1}$, the shape operator A has four possible forms with respect to a frame at $T_xM_1^n$:

$$\text{(I)}\quad A = \mathrm{diag}\{\lambda_1, \cdots, \lambda_n\}, \qquad G = \mathrm{diag}\{1, \cdots, 1, -1\},$$

$$\text{(Ⅱ)}\quad A = \begin{pmatrix} \lambda_1 & & \\ 1 & \lambda_1 & \\ & & D_{n-2} \end{pmatrix}, \qquad G = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & I_{n-2} \end{pmatrix},$$

$$\text{(Ⅲ)}\quad A = \begin{pmatrix} \lambda_1 & & & \\ 1 & \lambda_1 & & \\ & 1 & \lambda_1 & \\ & & & D_{n-3} \end{pmatrix}, \qquad G = \begin{pmatrix} & & 1 & \\ & 1 & & \\ 1 & & & \\ & & & I_{n-3} \end{pmatrix},$$

$$\text{(IV)}\quad A=\begin{pmatrix} D_{n-2} & & \\ & \gamma & \tau \\ & -\tau & \gamma \end{pmatrix}, \qquad G=\begin{pmatrix} I_{n-1} & \\ & -1 \end{pmatrix}, \tau \neq 0,$$

where $D_{n-2} = \operatorname{diag}\{\lambda_2, \cdots, \lambda_{n-1}\}$, $D_{n-3} = \operatorname{diag}\{\lambda_2, \cdots, \lambda_{n-2}\}$ and I the identity matrix.

Our first result asserts the following non-existence property.

Proposition 5.1. *There is no proper Lorentzian hypersurface of $\mathbb{E}_1^{n+1}$ with at most two distinct principal curvatures, and satisfying $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$ for $\lambda \leq 0$.*

Proof. Assume that M_1^n is such a proper Lorentzian hypersurface, then Proposition 4.1 implies that M_1^n has a pair of imaginary principal curvatures $\gamma + \tau\sqrt{-1}$ and $\gamma - \tau\sqrt{-1}$.

From Theorem 3.1, H is a non-zero constant (since we consider the proper ones). Moreover, it follows from (2) that $\operatorname{tr}A^2 = \lambda$. On the other hand, $\gamma = H$ and $\operatorname{tr}A^2 = 2n(\gamma^2 - \tau^2)$. Those equations imply that γ and τ are all constant, and M_1^n is isoparametric. However, Magid showed in [11] that the principal curvatures of Lorentzian isoparametric hypersurface M_1^n in $\mathbb{E}_1^{n+1}$ are all real, a contradiction, and Proposition 5.1 follows. □

In view of Proposition 5.1, we need only to consider the classification problem for the case of $\lambda > 0$. When the shape operator is diagonalizable, the classification results have been obtained by L. Du in [5]. So our Theorem 5.2 only classifies such proper Lorentzian hypersurfaces with non-diagonalizable shape operators, and the results in Theorem 5.2 for $n = 2$ just coincide with that in [6].

Theorem 5.2. *Let M_1^n $(n \geq 3)$ be a nondegenerate proper Lorentzian hypersurface of $\mathbb{E}_1^{n+1}$ satisfying $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$ with $\lambda > 0$. Suppose that M_1^n has non-diagonalizable shape operator and at most two distinct principal curvatures, then one of the following holds:*

(1) *When all the principal curvatures are the same, then it must be $\sqrt{\frac{\lambda}{n}}$ or $-\sqrt{\frac{\lambda}{n}}$, in this case, M_1^n is locally congruent to a generalized umbilical hypersurface.*

(2) *When M_1^n has two distinct principal curvatures μ (with multiplicity l) and ν, then $(\mu, \nu) = (\sqrt{\frac{\lambda}{l}}, 0)$ or $(-\sqrt{\frac{\lambda}{l}}, 0)$, in this case, M_1^n is locally congruent to a generalized cylinder.*

Before proving Theorem 5.2, we remark that the generalized umbilical hypersurfaces and generalized cylinders are proper ones satisfying $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$ with at most two distinct principal curvatures. Indeed, those are certain parametrized hypersurfaces in $\mathbb{E}_1^{n+1}$, given by M. A. Magid in [11] for studying the isoparametric ones. We recall from [11] that there are two types of generalized umbilical hypersurfaces whose shape operators take the following two possible forms:

$$\begin{pmatrix} \mu & & \\ 1 & \mu & \\ & & D_{n-2}(\mu) \end{pmatrix}, \qquad \begin{pmatrix} \mu & & & \\ 1 & \mu & & \\ & 1 & \mu & \\ & & & D_{n-3}(\mu) \end{pmatrix},$$

and there are four types of generalized cylinders whose shape operators take the following four possible forms:

$$\begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & & D_{n-l-2}(0) & \\ & & & D_l(\mu) \end{pmatrix}, \quad \begin{pmatrix} \mu & & & \\ 1 & \mu & & \\ & & D_{l-2}(\mu) & \\ & & & D_{n-l}(0) \end{pmatrix},$$

$$\begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & & D_{n-l-3}(0) & \\ & & & & D_l(\mu) \end{pmatrix}, \quad \begin{pmatrix} \mu & & & & \\ 1 & \mu & & & \\ & 1 & \mu & & \\ & & & D_{l-3}(\mu) & \\ & & & & D_{n-l}(0) \end{pmatrix},$$

where μ is a nonzero constant and $D_k(x) = \mathrm{diag}\{x, \cdots, x\}$ is a k-order matrix.

Using those forms, we can get the values of $\mathrm{tr}A$ and $\mathrm{tr}A^2$. Since $\mathrm{tr}A = nH$, we also can obtain the value of H and find H is a nonzero constant, which implies $\nabla H = 0$ and $\Delta H = 0$. Then we can check that the equations (1) and (2) hold, equivalently, $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$ holds. Therefore, generalized umbilical hypersurfaces and generalized cylinders are proper Lorentzian hypersurfaces in $\mathbb{E}_1^{n+1}$ satisfying $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$ with at most two distinct principal curvatures.

Proof of Theorem 5.2. In fact, the principal curvatures of M_1^n are all real. If not, then M_1^n has two adjoint imaginary principal curvatures, it follows from the proof of Proposition 4.2 that M_1^n is isoparametric and its principal curvatures are all real, a contradiction. So the non-diagonalizable shape operator takes only forms (Ⅱ) or (Ⅲ).

If all the principal curvatures of M_1^n are the same, says μ, using the forms (Ⅱ) and (Ⅲ), it is not hard to check that the minimal polynomial of the shape operator is $(x-\mu)^2$ or $(x-\mu)^3$, where $\mu = \pm\sqrt{\frac{\lambda}{n}}$ (see Proposition 4.1). According to [11, Theorems 4.5 and 4.8], M_1^n is locally congruent to a generalized umbilical hypersurface.

If M_1^n has two distinct principal curvatures μ and ν, then by observing the forms (Ⅱ) and (Ⅲ) of A, we know that the algebraic and geometric multiplicity of μ or ν coincide. So, from Proposition 4.1, two principal curvatures are $\mu = \pm\sqrt{\frac{\lambda}{l}}$ (with multiplicity l) and 0. Again by means of the forms (Ⅱ) and (Ⅲ), the minimal polynomial of the shape operator is $(x-\mu)x^2$, $(x-\mu)^2x$, $(x-\mu)x^3$ or $(x-\mu)^3x$. Finally, we conclude from [11, Theorems 4.4, 4.6, 4.7 and 4.9] M_1^n is locally congruent to a generalized cylinder. We compete the proof of Theorem 5.2. □

Acknowledgment

This work was supported by the National Natural Science Foundation of China (No. 11261051).

References

[1] A. Arvanitoyeorgos, G. Kaimakamis, Hypersurfaces of type M_2^3 in E_2^4 with proper mean curvature vector, J. Geom. Phys. 63 (2013) 99–106.
[2] A. Arvanitoyeorgos, F. Defever, G. Kaimakamis, Hypersurfaces of E_s^4 with proper mean curvature vector, J. Math. Soc. Japan 59 (3) (2007) 797–809.
[3] A. Arvanitoyeorgos, G. Kaimakamis, M. Magid, Lorentz hypersurfaces in E_1^4 satisfying $\Delta\overrightarrow{H} = \alpha\overrightarrow{H}$, Illinois J. Math. 53 (2) (2009) 581–590.
[4] B.Y. Chen, S. Ishikawa, Biharmonic surfaces in pseudo-Euclidean spaces, Mem. Fac. Sci. Kyushu Univ. Ser. A. 45 (2) (1991) 323–347.
[5] L. Du, Submanifolds satisfying the condition $\tau_2(\phi) = \eta\tau(\phi)$ in pseudo-Riemannian space forms, PhD thesis, Northwest Normal University, China, 2016.
[6] A. Ferrández, P. Lucas, On surfaces in the 3-dimensional Lorentz–Minkowski space, Pacific J. Math. 152 (1) (1992) 93–100.

[7] J. Hahn, Isoparametric hypersurfaces in the pseudo-Riemannian space forms, Math. Z. 187 (1984) 195–208.

[8] J.-C. Liu, C. Yang, Hypersurfaces in $\mathbb{E}_s^{n+1}$ satisfying $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$ with at most three distinct principal curvatures, J. Math. Anal. Appl. 419 (2014) 562–573.

[9] J.-C. Liu, C. Yang, Lorentz hypersurfaces in $\mathbb{E}_1^{n+1}$ satisfying $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$ with at most three distinct principal curvatures, J. Math. Anal. Appl. 434 (2016) 222–240.

[10] M.A. Magid, Isometric immersions of Lorentz space with parallel second fundamental forms, Tsukuba J. Math. 8 (1) (1984) 31–54.

[11] M.A. Magid, Lorentzian isoparametric hypersurfaces, Pacific J. Math. 118 (1985) 165–197.

[12] B. O'Neill, Semi-Riemannian Geometry: with Applications to Relativity, Pure Appl. Math., vol. 103, Academic Press, New York, 1983.