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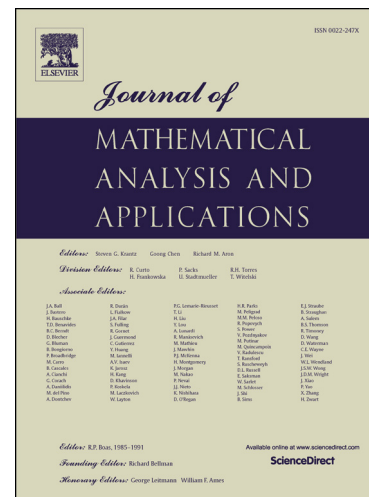
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# The stability of traveling wave solutions for a diffusive competition system of three species<sup>☆</sup>

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## Abstract

In this article, we investigate the stability of monotone traveling wave solutions for a diffusive three species competition system. By considering the initial perturbations of the traveling waves in some weighted function spaces, the monotone three-species waves become asymptotically stable. Further stability will be determined from the asymptotic behavior of the waves. This can be achieved by using the method of super- and subsolutions.

**Keywords:** Lotka-Volterra; competition-diffusion system; stability; traveling waves

**2010 MSC:** 35B35

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## 1. Introduction

In this paper we consider the Lotka-Volterra competition-diffusion system

$$u_{i,t} = D_i u_{i,xx} + r_i u_i \left( 1 - \sum_{j=1}^n b_{ij} u_j \right), \quad x \in \mathbb{R}, t > 0, \quad i = 1, \dots, n, \quad (1.1)$$

where  $D_i, r_i, b_{ij}$  ( $i, j = 1, \dots, n$ ) are positive constants. This system has attracted much attention in ecological and biological areas. Traveling wave solutions of (1.1) play an important role in the biological invasion of species. The previous works on traveling waves for two species ( $n = 2$ ) can be found in [1], [2] and the reference therein. For the three species case ( $n = 3$ ), due to the lack of maximum

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principles, many results were obtained by singular perturbation methods. For the recent works related to the model of three species, we refer the reader to [3], [4], [5], [6], [7], [8], [9], [10] and the reference therein.

In this article, we study the existence and stability of traveling wave solutions of three-species Lotka-Volterra competition-diffusion system (1.1) with  $b_{13} = 0 = b_{31}$ . The traveling wave solutions of this system can be written in the form

$$(u_1, u_2, u_3)(x, t) = (\phi, \psi, \theta)(y), \quad y := x + st. \quad (1.2)$$

By scaling the parameters, the parameters  $b_{ii}$  can be transformed such that  $b_{ii} = 1$  ( $i = 1, 2, 3$ ). By substituting (1.2) into (1.1) with  $b_{13} = 0 = b_{31}$ , we have that  $\phi, \psi, \theta$  satisfy the following system:

$$\begin{cases} s\phi' = D_1\phi'' + r_1\phi(1 - \phi - b_{12}\psi), \\ s\psi' = D_2\psi'' + r_2\psi(1 - b_{21}\phi - \psi - b_{23}\theta), \\ s\theta' = D_3\theta'' + r_3\theta(1 - b_{32}\psi - \theta), \\ (\phi, \psi, \theta)(-\infty) = (1, 0, 1), \quad (\phi, \psi, \theta)(+\infty) = (0, 1, 0), \end{cases} \quad y \in \mathbb{R}, \quad (1.3)$$

where  $p' = \frac{dp}{dy}$ . It should be noted that Guo et al. [5] have obtained the existence of solutions of (1.3).

**Proposition 1.1.** (Theorem 1 in [5]) Assume that

$$(A) \quad b_{12}, b_{32} > 1 \text{ and } b_{21} + b_{23} < 1. \quad (1.4)$$

Let  $D_2, r_2, b_{21}, b_{23}$  be given positive constants. Then for

$$s_{\min} = s_* := 2\sqrt{D_2 r_2 (1 - b_{21} - b_{23})},$$

(1.3) admits traveling wave solutions  $(\phi, \psi, \theta) = (\tilde{\phi}, \tilde{\psi}, \tilde{\theta})$  as long as

$$(D_j, r_j, b_{j2}) \in B_j^1 \cup B_j^2, \quad j = 1, 3, \quad (1.5)$$

where  $s_{\min}$  is the minimal speed for (1.3) and

$$\begin{aligned} B_j^1 &:= \{D_j \in (0, 2D_2], \quad b_{j2}(b_{21} + b_{23}) \leq 1, \quad r_j > 0\}, \\ B_j^2 &:= \left\{ \begin{array}{l} D_j \in (0, 2D_2), \quad b_{j2}(b_{21} + b_{23}) > 1, \\ 0 < r_j < \left(2 - \frac{D_j}{D_2}\right) \frac{r_2(1 - b_{21} - b_{23})}{b_{j2}(b_{21} + b_{23}) - 1}, \end{array} \right\}, \quad j = 1, 3. \end{aligned}$$

15 In [5], the existence was obtained by the theories of lattice dynamical systems. However, the results about stability of the solutions of (1.3) are not clear. Let  $L$  be defined as in (3.1), which is the linearized operator around  $(\phi, \psi, \theta)$ . A traveling wave is called linearly stable if the spectrum of  $L$  is contained in the left half plane complex plane with zero as a simple eigenvalue. It is well  
20 known that linear stability implies the stability of the nonlinear problem. That is, by the spectral results in linear stability and the estimates of the corresponding semigroups, one can prove that the traveling wave  $(\phi, \psi, \theta)$  is exponentially stable (see [11], [12], [13] for the detail proof). In the following we define some function spaces which will be considered later:

**Definition 1.2.** *Define*

$$BUC(\mathbb{R}) := \{\mathbf{p} : \mathbb{R} \rightarrow \mathbb{R}^3 \mid \mathbf{p} \text{ is bounded and uniformly continuous}\}$$

with supremum norm  $\|\cdot\|_\infty$ ;

$$BUC_\sigma(\mathbb{R}) := \{\mathbf{p} : \mathbb{R} \rightarrow \mathbb{R}^3 \mid \mathbf{p}(z), (1 + e^{\sigma z}) \mathbf{p}(z) \in BUC(\mathbb{R})\} \quad (1.6)$$

for some constant  $\sigma > 0$ .  $BUC_\sigma(\mathbb{R})$  is equipped with the weighted norm

$$\|\mathbf{p}\|_\sigma := \sup_{z \in \mathbb{R}} |(1 + e^{\sigma z}) \mathbf{p}(z)|. \quad (1.7)$$

25 It is clear that when  $\sigma = 0$ ,  $BUC_\sigma(\mathbb{R}) = BUC(\mathbb{R})$ . This means that there is no weight introduced by  $e^{\sigma z}$ .

The main results of this paper are as follows. We obtain the exponential stability by the spectral analysis of the linearized operator around  $(\phi, \psi, \theta)$  (see [11], [12], [13] and the reference therein). We obtain the existence of solutions  
30 of (1.3) by using the method of super- and subsolutions in [14]. In the existence results, the the restrictions on the parameters  $(D_j, r_j, b_{j2})$  ( $j = 1, 3$ ) are different from Proposition 1.1. However, by the method of super- and subsolutions, the asymptotic behavior of the traveling wave solutions of (1.3) would be clear, and then we can obtain the stability.

**Theorem 1.3.** Assume that (A) in Proposition 1.1 holds. Let  $D_2, r_2, b_{21}, b_{23}$  be given positive constants. Then for  $s > s_* := 2\sqrt{D_2 r_2 (1 - b_{21} - b_{23})}$ , (1.3) admits traveling wave solutions  $(\phi, \psi, \theta) = (\hat{\phi}, \hat{\psi}, \hat{\theta})$  as long as

$$(D_j, r_j, b_{j2}) \in B_j, \quad j = 1, 3, \quad (1.8)$$

where

$$B_j := \left\{ D_j \in (0, D_2], \quad 0 < r_j < \frac{D_j r_2 (1 - b_{21} - b_{23})}{D_2 (b_{j2} - 1)} \right\}, \quad j = 1, 3.$$

Moreover, we have

$$\hat{\phi}'(z) < 0, \quad \hat{\theta}'(z) < 0 \text{ and } \hat{\psi}'(z) > 0 \text{ for all } z \in \mathbb{R}. \quad (1.9)$$

- Theorem 1.4.** 1. The traveling wave solution  $(\hat{\phi}, \hat{\psi}, \hat{\theta})$  of Theorem 1.3 with  $s > s_*$  is unstable in the space  $BUC(\mathbb{R})$  with respect to  $\|\cdot\|_\infty$ .
2. The traveling wave solution  $(\hat{\phi}, \hat{\psi}, \hat{\theta})$  of Theorem 1.3 with  $s > s_*$  is exponentially stable with no shift in the space  $BUC_\sigma(\mathbb{R})$  with respect to  $\|\cdot\|_\sigma$ , where  $\sigma$  is the constant in Definition 1.2 which satisfying

$$\frac{s - \sqrt{s^2 - s_*^2}}{2D_2} < \sigma < \min \left\{ \frac{s + \sqrt{s^2 - s_*^2}}{2D_2}, \frac{s + \sqrt{s^2 + 4D_j r_j}}{2D_j} \quad (j = 1, 3) \right\}. \quad (1.10)$$

In the proof of Theorem 1.4 stated below, one can see that the asymptotic behavior of  $(\phi, \psi, \theta)$  will determine the stability in the weighted function spaces  $BUC_\sigma(\mathbb{R})$ . In  $BUC_\sigma(\mathbb{R})$ , the essential spectrum of  $L$  can be stabilized in  $BUC_\sigma(\mathbb{R})$ , but zero does not belong to the normal spectrum of  $L$ . By restrictions (1.8) on the parameters in (1.3), we can choose those heteroclinic orbits  $(\phi, \psi, \theta)$  which toward  $(1, 0, 1)$  along the eigendirection with respect to the leading eigenvalues by the method of super- and subsolutions in [14] and [15] (see Theorem 3.4). We will show that these orbits are exponentially stable with no shift. Similar approaches with the introduction of weighted function spaces can be seen in [16] and reference therein.

The paper is organized as follows. In Section 2 we give the proof of Theorem 1.3. In Section 3 we prove Theorem 1.4.

## 2. Existence

50 In this section we prove the existence results in Theorem 1.3. We obtain the existence of solutions by constructing a suitable pair of super- and subsolutions. In the choice of super- and subsolutions, we can obtain the asymptotic behavior of the solution  $\phi, \psi, \theta$  of (1.3). Then we can find the stability of  $\phi, \psi, \theta$ . Similar approaches can be found in Hsu and Yang [17], Hung [18] and Leung et al. [16].

### 55 2.1. Monotone systems

In order to analyze the problem by the theory of monotone systems, we define

$$U(z) = 1 - \phi(-z), \quad V(z) = \psi(-z), \quad W(z) = 1 - \theta(-z), \quad \text{with } z = -y = -x - st. \quad (2.1)$$

We still use  $p'$  to denote the derivative of a function  $p(z)$  with respect to  $z$ . Then one can see that (1.3) can be rewritten as

$$\begin{cases} D_1 U'' + sU' + r_1(1 - U)(-U + b_{12}V) = 0, \\ D_2 V'' + sV' + r_2V(1 - b_{21} - b_{23} + b_{21}U - V + b_{23}W) = 0, \\ D_3 W'' + sW' + r_3(1 - W)(-W + b_{32}V) = 0 \end{cases} \quad z \in \mathbb{R} \quad (2.2)$$

with the boundary condition

$$(U, V, W)(-\infty) = (1, 1, 1), \quad (U, V, W)(+\infty) = (0, 0, 0). \quad (2.3)$$

We also define

$$\Phi(z) = (U, V, W)^T(z),$$

$$\mathbf{D} = \text{diag}(D_1, D_2, D_3), \quad \mathbf{0} = (0, 0, 0)^T, \quad \mathbf{1} = (1, 1, 1)^T. \quad (2.4)$$

Then  $\Phi$  satisfies

$$\mathbf{D}\Phi'' + s\Phi' + \mathbf{F}(\Phi) = \mathbf{0}, \quad (2.5)$$

where  $\mathbf{F} = (\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3) \in \mathbb{R}^3$  denotes the nonlinear terms in (2.2):

$$\begin{aligned}\mathbf{F}_1(\Phi) &= r_1(1-U)(-U + b_{12}V); \\ \mathbf{F}_2(\Phi) &= r_2V(1 - b_{21} - b_{23} + b_{21}U - V + b_{23}W); \\ \mathbf{F}_3(\Phi) &= r_3(1-W)(-W + b_{32}V).\end{aligned}\tag{2.6}$$

We will consider the solutions in the rectangle

$$[\mathbf{0}, \mathbf{1}] := \{(u_1, u_2, u_3) | 0 \leq u_i \leq 1, i = 1, 2, 3\}.$$

It can be easily checked that (2.2) forms a monotone system in the sense from [13]:

$$\frac{\partial \mathbf{F}_i}{\partial U_j} \geq 0 \text{ for } i, j = 1, 2, 3, \text{ and } i \neq j.\tag{2.7}$$

In the following we define the ordering of vectors  $\mathbf{u} = (u_1, u_2, u_3)$  and  $\mathbf{v} = (v_1, v_2, v_3)$  as follows.  $\mathbf{u} \leq \mathbf{v}$  means that  $u_i \leq v_i$  with  $i = 1, 2, 3$ ;  $\mathbf{u} < \mathbf{v}$  means that  $\mathbf{u} \leq \mathbf{v}$  but  $\mathbf{u} \neq \mathbf{v}$ .

## 2.2. Construction of super- and subsolutions

60 In the following we apply the super- and subsolution techniques to obtain the existence of solutions of (2.2) and (2.3). The super- and subsolutions are continuous but piece-wisely smooth as defined in [14] and [15].

Let

$$C(\mathbb{R}, \mathbb{R}^3) = \{\mathbf{p} : \mathbb{R} \rightarrow \mathbb{R}^3 | \mathbf{p} \text{ is continuous}\};$$

$$C_{[0,1]}(\mathbb{R}, \mathbb{R}^3) = \{\mathbf{p} : \mathbb{R} \rightarrow \mathbb{R}^3 | \mathbf{0} \leq \mathbf{p}(z) \leq \mathbf{1} \text{ for all } z \in \mathbb{R}\}.\tag{2.8}$$

**Proposition 2.1.** (Theorem 2.2 in [14] and Theorem 3.6' in [15]) Assume that

the following hold:

- (H1)  $\mathbf{F}(\mathbf{0}) = \mathbf{F}(\mathbf{1}) = \mathbf{0}$ ,
- (H2) There exists a matrix  $\beta = \text{diag}(\beta_1, \beta_2, \beta_2)$  with  $\beta_i \geq 0$  such that  
 $\mathbf{F}(\Phi) - \mathbf{F}(\Psi) + \beta(\Phi - \Psi) \geq \mathbf{0}$   
 for  $\Phi, \Psi \in BUC(\mathbb{R})$  with  $\mathbf{0} \leq \Psi \leq \Phi \leq \mathbf{1}$ ,
- (H3) There are two constants  $a > 0$  and  $b > 0$  such that  
 $|\mathbf{F}(\Phi) - \mathbf{F}(\Psi)| \leq b \|\Phi - \Psi\|^a$   
 for  $\Phi, \Psi \in BUC(\mathbb{R})$  with  $\mathbf{0} \leq \Psi \leq \Phi \leq \mathbf{1}$ ,

where  $|\cdot|$  denotes the Euclidean norm in  $\mathbb{R}^3$ . Suppose that (2.5) has a supersolution  $\bar{\varphi} \in C_{[0,1]}(\mathbb{R}, \mathbb{R}^3)$  and a subsolution  $\underline{\varphi} \in C_{[0,1]}(\mathbb{R}, \mathbb{R}^3)$  satisfying

- (I)  $0 \leq \sup_{\zeta \leq z} \underline{\varphi}(\zeta) \leq \bar{\varphi}(z) \leq 1$ , for  $z \in \mathbb{R}$ ;
- (II)  $\mathbf{F}(\mathbf{p}) \neq \mathbf{0}$  for  $\mathbf{p} \in (\mathbf{0}, \inf_{z \in \mathbb{R}} \bar{\varphi}(z)] \cup [\sup_{z \in \mathbb{R}} \underline{\varphi}(z), \mathbf{1})$ ;
- (III)  $\bar{\varphi}'(z^+) \leq \bar{\varphi}'(z^-)$ , for  $z \in \mathbb{R}$ ;
- (IV)  $\underline{\varphi}'(z^+) \geq \underline{\varphi}'(z^-)$ , for  $z \in \mathbb{R}$ .

Then (2.5) and (2.3) have a monotone solution.

**Lemma 2.2.** Let  $s > s_*$  and

$$\mu_V^\pm = \frac{s \pm \sqrt{s^2 - s_*^2}}{2D_2}. \quad (2.9)$$

Assume that the parameters in (1.3) satisfy (1.4) and (1.8). Define

$$\bar{U}(z) = \min \{1, e^{-\mu_V^+ z}\}, \quad \bar{V}(z) = \min \{1, e^{-\mu_V^- z}\}, \quad \bar{W}(z) = \min \{1, e^{-\mu_V^+ z}\} \quad (2.10)$$

and

$$\underline{U}(z) = 0, \quad \underline{V}(z) = \max \left\{ (1 - b_{21} - b_{23}) (1 - M e^{-\varepsilon z}) e^{-\mu_V^- z}, 0 \right\}, \quad \underline{W}(z) = 0, \quad (2.11)$$

where  $M$  and  $\varepsilon$  are positive constants. Then  $\bar{\varphi}(z) = (\bar{U}, \bar{V}, \bar{W})(z)$  is an supersolution of (2.2) and if  $M > 0$  is sufficiently large and  $\varepsilon > 0$  is sufficiently small, then  $\underline{\varphi}(z) = (\underline{U}, \underline{V}, \underline{W})(z)$  is a subsolution of (2.2).  $\bar{\varphi}$  and  $\underline{\varphi}$  satisfy conditions (I), (III) and (IV) in Proposition 2.1. Moreover, we have that

$$\mathbf{0} \leq \underline{\varphi}(z) \leq \bar{\varphi}(z) \leq \mathbf{1}, \quad \text{for all } z \in \mathbb{R}. \quad (2.12)$$



*Proof.* From (2.10) and (2.11) one can easily find that  $\bar{\varphi}$  and  $\underline{\varphi}$  satisfy conditions (I), (III) and (IV) and (2.12). By (2.9) we have that

$$-\mu_V^- \text{ and } -\mu_V^+ \text{ are the two zeros of } D_2 \mu^2 + s\mu + r_2(1 - b_{21} - b_{23}) = 0. \quad (2.13)$$

We prove that  $\bar{\varphi}(z)$  is a supersolution of (2.2) first. By (2.13) we have that

$$\begin{aligned} & D_1 (-\mu_V^-)^2 + s(-\mu_V^-) + \sup_{z \geq 0} e^{\mu_V^+ z} r_1 (1 - \bar{U}) (-\bar{U} + b_{12} \bar{V}) \\ &= D_1 \left[ \frac{-r_2(1 - b_{21} - b_{23})}{D_2} - s \frac{-\mu_V^-}{D_2} \right] + s(-\mu_V^-) + r_1(b_{12} - 1) \\ &= s\mu_V^- \left( \frac{D_1}{D_2} - 1 \right) - \left[ \frac{D_1}{D_2} r_2(1 - b_{21} - b_{23}) - r_1(b_{12} - 1) \right] \leq 0. \end{aligned}$$

The above last inequality holds since  $D_1$ ,  $D_2$  and  $r_1$  satisfy the constraints in (1.8). The proof about  $\bar{V}$  and  $\bar{W}$  can be obtained in a similar manner by using (1.8). By Theorem 2.3 in [14], we can find that  $\bar{\varphi} = (\bar{U}, \bar{V}, \bar{W})$  is a supersolution of (2.2). We then prove that  $(U, V, W)$  is a subsolution of (2.2). Recall that (2.13) holds. Then we can choose  $\varepsilon > 0$  small enough such that

$$0 < \varepsilon < \mu_V^-, \quad D_2 (-\mu_V^- - \varepsilon)^2 + s(-\mu_V^- - \varepsilon) + r_2(1 - b_{21} - b_{23}) < 0.$$

Let  $M > 1$  be large enough such that

$$-M \left[ D_2 (-\mu_V^- - \varepsilon)^2 + s(-\mu_V^- - \varepsilon) + r_2(1 - b_{21} - b_{23}) \right] \geq r_2(1 - b_{21} - b_{23}).$$

By the computation similar to [14], it can be seen that  $(U, V, W)$  is a subsolution of (2.2). The proof is completed.  $\square$

From Lemma 2.2, we can prove Theorem 1.3 as follows:

*Proof of Theorem 1.3.* We would apply Proposition 2.1 to obtain the existence of traveling waves of (2.2). From the expression of the nonlinear terms  $\mathbf{F}$  in (2.6), one can also obtain that  $\mathbf{F}$  satisfies (H1), (H2) and (H3) in Proposition 2.1. Under Lemma 2.2, we have the existence of a pair of supersolution and subsolution in the form (2.10) and (2.11). Moreover, conditions (I), (III) and (IV) can be achieved. We still need to verify Condition (II), i.e.,  $\mathbf{F}(\mathbf{p}) \neq \mathbf{0}$  for  $\mathbf{p} \in (\mathbf{0}, \inf_{z \in \mathbb{R}} \bar{\varphi}(z)) \cup [\sup_{z \in \mathbb{R}} \underline{\varphi}(z), \mathbf{1})$ . By (2.10) and (2.11), we

have that  $(\mathbf{0}, \inf_{z \in \mathbb{R}} \bar{\varphi}(z))$  is an empty set and  $\sup_{z \in \mathbb{R}} \underline{\varphi}(z) = (0, M_V, 0)$ , where  $M_V := \sup_{z \in \mathbb{R}} V(z) > 0$ . Therefore Condition (II) is equivalent to the following statement:  $\mathbf{F}(U, V, W) \neq \mathbf{0}$  for

$$0 \leq U < 1, M_V \leq V < 1, 0 \leq W < 1. \quad (2.14)$$

One can find that there are eight equilibria of the system (1.3):

$$\begin{aligned} E_1 &:= (0, 0, 0); & E_2 &:= (0, 0, 1); \\ E_3 &:= (1, 0, 0); & E_4 &:= (1, 0, 1); \\ E_5 &:= (1, 1, 1); & E_6 &:= \left( \frac{b_{12}(b_{21}-1)}{b_{12}b_{21}-1}, \frac{b_{21}-1}{b_{12}b_{21}-1}, 1 \right); \\ E_7 &:= \left( 1, \frac{b_{23}-1}{b_{23}b_{32}-1}, \frac{b_{32}(b_{23}-1)}{b_{23}b_{32}-1} \right); & E_8 &:= (b_{12}V^*, V^*, b_{32}V^*), \end{aligned}$$

where

$$V^* = \frac{b_{21} + b_{23} - 1}{b_{12}b_{21} + b_{23}b_{32} - 1}.$$

One can see that all the equilibria  $E_1, \dots, E_7$  are not in the range (2.14). Hence we only need to consider  $E_8$ . If  $b_{12}b_{21} + b_{23}b_{32} - 1 > 0$ , then by (1.4) we have that  $V^* < 0$ . Hence  $E_8$  is not in (2.14). If  $b_{12}b_{21} + b_{23}b_{32} - 1 < 0$ , by direct observation and (1.4), it follows that

$$V^* - 1 = \frac{b_{21}(b_{12}-1) + b_{23}(b_{32}-1)}{1 - b_{12}b_{21} - b_{23}b_{32}} > 0$$

and  $E_8$  is still not in (2.14). Therefore Condition (II) is proved and we have (2.3). Therefore we can obtain the existence of a traveling wave solution  $\Phi = \hat{\Phi}(z) = (\hat{U}, \hat{V}, \hat{W})(z)$  of (2.2) and (2.3). One can see that  $\hat{\Phi}$  satisfies (2.5). From [14] and [15], the obtained solution  $\hat{\Phi}$  belongs to the set

$$\Gamma := \left\{ \Phi \in C_{[0,1]}(\mathbb{R}, \mathbb{R}^3) \left| \begin{array}{l} \text{(i) } \Phi \text{ is nonincreasing in } \mathbb{R}; \\ \text{(ii) } \underline{\varphi}(z) \leq \Phi(z) \leq \bar{\varphi}(z) \text{ for all } z \in \mathbb{R}; \\ \text{(iii) } |\Phi(z_1) - \Phi(z_2)| \leq C|z_1 - z_2| \\ \text{for some } C > 0 \text{ for all } z_1, z_2 \in \mathbb{R}. \end{array} \right. \right\}$$

Therefore we have that

$$\underline{U}(z) \leq \hat{U}(z) \leq \bar{U}(z), \quad \underline{V}(z) \leq \hat{V}(z) \leq \bar{V}(z), \quad \text{and} \quad \underline{W}(z) \leq \hat{W}(z) \leq \bar{W}(z). \quad (2.15)$$

From (2.1) it follows that that (1.3) has a traveling wave solution  $(\hat{\phi}, \hat{\psi}, \hat{\theta})(y)$  with

$$\hat{\phi}(y) := 1 - \hat{U}(-y), \quad \hat{\psi}(y) := \hat{V}(-y), \quad \hat{\theta}(y) := 1 - \hat{W}(-y).$$

Finally, by standard maximum principle arguments one can prove that  $\hat{U}, \hat{V}$ , and  $\hat{W}$  satisfy  $\hat{U}'(z), \hat{V}'(z), \hat{W}'(z) < 0$ . From (2.1), we can obtain (1.9). The  
 70 proof is finished.  $\square$

### 3. Stability

#### 3.1. Linearized operators

Let  $I_3$  denote the  $3 \times 3$  identity matrix. Moreover, we let  $L$  be the linearized operator around  $\hat{\Phi}(z)$ :

$$L\mathbf{p} = \mathbf{D}\mathbf{p}'' + M_L(z)\mathbf{p}' + N_L(z)\mathbf{p}, \quad (3.1)$$

where

$$M_L(z) = sI_3, \quad N_L(z) = \mathbf{F}'(\hat{\Phi}(z)).$$

If we want to consider the exponential stability of  $\hat{\Phi}(z)$  in Theorem 1.3 in the space  $BUC(\mathbb{R})$ , we need to investigate the spectra of the linearized operator  $L$   
 75 defined in (3.1).

In some situations, we need to obtain more detailed results about stability. To achieve this, one can consider the weighted function spaces  $BUC_\sigma(\mathbb{R})$  (where  $\sigma > 0$ ) which are smaller than  $BUC(\mathbb{R})$ . We introduce the operator

$$T\mathbf{p} := (1 + e^{\sigma z})\mathbf{p}(z)$$

from  $BUC_\sigma(\mathbb{R})$  to  $BUC(\mathbb{R})$ , where  $\mathbf{p} \in BUC_\sigma(\mathbb{R})$  for some constant  $\sigma > 0$ . Define

$$\tilde{L} := TLT^{-1}. \quad (3.2)$$

If  $L : BUC_\sigma(\mathbb{R}) \rightarrow BUC_\sigma(\mathbb{R})$  is considered in  $BUC_\sigma(\mathbb{R})$ , then  $\tilde{L} : BUC(\mathbb{R}) \rightarrow BUC(\mathbb{R})$  can be seen as the operator in  $BUC(\mathbb{R})$ . Moreover, it is easily seen that  $\sigma(L) = \sigma(\tilde{L})$ . Therefore, in the study of the spectral problems in  $BUC_\sigma(\mathbb{R})$ ,

we will always consider  $\tilde{L}\mathbf{p} = \lambda\mathbf{p}$  with  $\mathbf{p} \in BUC(\mathbb{R})$  instead of  $L$ . As in [13],  $\tilde{L}$  can be expressed as

$$\tilde{L}\mathbf{p} = \mathbf{D}\mathbf{p}'' + M_{\tilde{L}}(z)\mathbf{p}' + N_{\tilde{L}}(z)\mathbf{p},$$

where  $\mathbf{p} = (p, q, r)^T$ ,

$$M_{\tilde{L}}(z) = sI_3 - 2g_1(z)\mathbf{D}, \quad N_{\tilde{L}}(z) = g_2(z)\mathbf{D} - sg_1(z)I_3 + J(\hat{U}, \hat{V}, \hat{W})(z), \quad (3.3)$$

and

$$g_1(z) = \frac{\sigma e^{\sigma z}}{1 + e^{\sigma z}}, \quad g_2(z) = \frac{\sigma^2 e^{\sigma z} (e^{\sigma z} - 1)}{(1 + e^{\sigma z})^2}. \quad (3.4)$$

### 3.2. Essential spectrum

For a linear operator  $\mathcal{L}$  from  $BUC(\mathbb{R})$  to  $BUC(\mathbb{R})$ , we give the following definition.

**Definition 3.1.** Let  $\sigma(\mathcal{L})$  be the spectrum of  $\mathcal{L}$ . We denote  $\sigma_n(\mathcal{L})$  as the normal spectrum of  $\mathcal{L}$  which consists of isolated eigenvalues with finite multiplicity.  $\sigma_e(\mathcal{L}) := \sigma(\mathcal{L}) \setminus \sigma_n(\mathcal{L})$  is defined as the essential spectrum of  $\mathcal{L}$ .

#### 3.2.1. Essential spectrum of $L$

We prove the first result of Theorem 1.4 in the following lemma:

**Lemma 3.2.**  $(\hat{\phi}, \hat{\psi}, \hat{\theta})$  is unstable in  $BUC(\mathbb{R})$ .

*Proof.* Define

$$S_L^\pm := \{ \lambda \in \mathbb{C} \mid \det(-\tau^2 \mathbf{D} + i\tau M_L^\pm + N_L^\pm - \lambda I_3) = 0 \},$$

where  $M_L^\pm := M_L(\pm\infty) = sI_3$ ,  $N_L^\pm := N_L(\pm\infty) = \mathbf{F}'(\hat{\Phi}(\pm\infty))$ . One can find that

$$S_L^+ = \left\{ \lambda \left| \begin{array}{l} Re\lambda = -D_1 \left( \frac{Im\lambda}{s} \right)^2 - r_1, \\ Re\lambda = -D_2 \left( \frac{Im\lambda}{s} \right)^2 + r_2 (1 - b_{21} - b_{23}), \\ Re\lambda = -D_3 \left( \frac{Im\lambda}{s} \right)^2 - r_3, \end{array} \right. \right\};$$

$$S_L^- = \left\{ \lambda \left| \begin{array}{l} Re\lambda = -D_1 \left( \frac{Im\lambda}{s} \right)^2 + r_1 (1 - b_{12}), \\ Re\lambda = -D_2 \left( \frac{Im\lambda}{s} \right)^2 - r_2, \\ Re\lambda = -D_3 \left( \frac{Im\lambda}{s} \right)^2 + r_3 (1 - b_{32}), \end{array} \right. \right\}.$$

By applying the spectral theories in [12] or [13], the boundary of the essential spectra of  $L$  are described by  $S_L^\pm$ . Hence the essential spectrum  $\sigma_e(L)$  is contained in the six parabolas  $S_L^\pm$  and the regions inside the union of them. One can find that

$$\sup_{\lambda \in \sigma_e(L)} Re\lambda = \max \{ -r_1, r_2 (1 - b_{21} - b_{23}), -r_3, r_1 (1 - b_{12}), -r_2, r_3 (1 - b_{32}) \} > 0.$$

85 Therefore by Theorem 3.1 of Chapter 5 in [13],  $(\hat{U}, \hat{V}, \hat{W})$  (i.e.,  $(\hat{\phi}, \hat{\psi}, \hat{\theta})$ ) is unstable in the space  $BUC(\mathbb{R})$  in the Lyapunov sense. The proof is finished.  $\square$

### 3.2.2. Essential spectrum of $\tilde{L}$

In this subsection, we establish the following lemma.

**Lemma 3.3.** *If  $\sigma$  satisfies (1.10), it follows that  $\sup_{\lambda \in \sigma_e(\tilde{L})} Re\lambda < 0$ .*

*Proof.* Define

$$S_L^\pm := \left\{ \lambda \in \mathbb{C} \mid \det \left( -\tau^2 \mathbf{D} + i\tau M_L^\pm + N_L^\pm - \lambda I_3 \right) = 0 \right\},$$

with respect to the operator  $\tilde{L}$ , where  $M_L^\pm := M_L(\pm\infty)$ ,  $N_L^\pm := N_L(\pm\infty)$ . From (3.4), it follows that

$$g_1(z) \rightarrow \sigma, g_2(z) \rightarrow \sigma^2 \text{ as } z \rightarrow \infty;$$

$$g_1(z) \rightarrow 0, g_2(z) \rightarrow 0 \text{ as } z \rightarrow -\infty.$$

That is, there is no contribution from the weight as  $z \rightarrow -\infty$ . Therefore

$$M_L^- = sI_3, N_L^- = \mathbf{F}'(\hat{\Phi}(-\infty)),$$

$$M_L^+ = \text{diag}(s - 2D_1\sigma, s - 2D_2\sigma, s - 2D_3\sigma),$$

and

$$N_L^+ = \begin{pmatrix} D_1\sigma^2 - s\sigma - r_1 & r_1b_{12} & 0 \\ 0 & D_2\sigma^2 - s\sigma & 0 \\ 0 & +r_2(1 - b_{21} - b_{23}) & D_3\sigma^2 - s\sigma - r_3 \end{pmatrix}.$$

The following calculations give us

$$\begin{aligned} & \left( -\tau^2 \mathbf{D} + i\tau M_L^+ + N_L^+ - \lambda I_3 \right) = \\ & \begin{pmatrix} -\tau^2 D_1 + i\tau(s - 2D_1\sigma) & & \\ +D_1\sigma^2 - s\sigma & r_1b_{12} & 0 \\ -r_1 - \lambda & & \\ & -\tau^2 D_2 + i\tau(s - 2D_2\sigma) & \\ 0 & +D_2\sigma^2 - s\sigma & 0 \\ & +r_2(1 - b_{21} - b_{23}) - \lambda & \\ & & -\tau^2 D_3 + i\tau(s - 2D_3\sigma) \\ 0 & r_3b_{32} & +D_3\sigma^2 - s\sigma \\ & & -r_3 - \lambda \end{pmatrix}, \\ & \left( -\tau^2 \mathbf{D} + i\tau M_L^- + N_L^- - \lambda I_3 \right) = \\ & \begin{pmatrix} -\tau^2 D_1 + i\tau s & & \\ +r_1(1 - b_{12}) - \lambda & 0 & 0 \\ & -\tau^2 D_2 + i\tau s & \\ r_2b_{21} & -r_2 - \lambda & r_2b_{23} \\ & & -\tau^2 D_3 + i\tau s \\ 0 & 0 & +r_3(1 - b_{32}) - \lambda \end{pmatrix}. \end{aligned}$$

Hence we can find that

$$S_L^+ = \left\{ \lambda \left| \begin{aligned} Re\lambda &= -D_1 \left( \frac{Im\lambda}{s-2D_1\sigma} \right)^2 + D_1\sigma^2 - s\sigma - r_1, \\ Re\lambda &= -D_2 \left( \frac{Im\lambda}{s-2D_2\sigma} \right)^2 + D_2\sigma^2 - s\sigma + r_2(1 - b_{21} - b_{23}), \\ Re\lambda &= -D_3 \left( \frac{Im\lambda}{s-2D_3\sigma} \right)^2 + D_3\sigma^2 - s\sigma - r_3, \end{aligned} \right. \right\},$$

and  $S_L^- = S_L^-$ . Hence the essential spectrum  $\sigma_e(\tilde{L})$  is contained in the six parabolas  $S_L^\pm$  and the regions inside the union of them by the essential spectral theories in [12] again. In order to obtain  $\sup_{\lambda \in \sigma_e(\tilde{L})} \operatorname{Re} \lambda < 0$ , we need to choose  $\sigma$  such that the vertices of the parabolas of  $S_L^+$  are less than zero. That is,

$$\max\{D_1\sigma^2 - s\sigma - r_1, D_2\sigma^2 - s\sigma + r_2(1 - b_{21} - b_{23}), D_3\sigma^2 - s\sigma - r_3\} < 0.$$

90 The above inequality is equivalent to (1.10). (Recall that  $s_* = 2\sqrt{D_2 r_2(1 - b_{21} - b_{23})}$ .) Hence the lemma is proved.  $\square$

### 3.3. The normal spectrum and asymptotic behavior

In this subsection we prove the second result of Theorem 1.4. After the analysis of  $\sigma_e(\tilde{L})$ , we still need to investigate the location of the normal spectrum  $\sigma_n(\tilde{L})$ . We can study this problem by the spectral property of monotone systems (see, for example, [13]).

**Theorem 3.4.** *If  $\sigma$  satisfies (1.10), the normal spectrum  $\sigma_n(\tilde{L})$  satisfies the following:*

1. There is no  $\lambda \in \sigma_n(\tilde{L})$  with  $\operatorname{Re} \lambda > 0$ .
- 100 2.  $0 \notin \sigma_n(\tilde{L})$ .

*Proof.* We prove (1) first. By the monotone property of the traveling wave solution  $(\hat{U}, \hat{V}, \hat{W})$  in (1.9), we have that  $\hat{U}' < 0, \hat{V}' < 0, \hat{W}' < 0$ . Hence  $-(1 + e^{\sigma z})(\hat{U}', \hat{V}', \hat{W}')^T(z)$  is a positive solution of  $\tilde{L}\mathbf{p} = \mathbf{0}$ . By Theorem 5.1 in Chapter 4 of [13], there is no  $\lambda \in \sigma_n(\tilde{L})$  with  $\operatorname{Re} \lambda > 0$ .

105 In the following we prove result (2). Before the proof we find the asymptotic behavior of  $\hat{V}(z)$  and  $\hat{V}'(z)$  first. From the second equation of (2.2), it follows that

$$\begin{aligned} 0 &= D_2 \hat{V}'' + s \hat{V}' + r_2 \hat{V} (1 - b_{21} - b_{23} + b_{21} \hat{U} - \hat{V} + b_{23} \hat{W}) \\ &= D_2 \hat{V}'' + s \hat{V}' + r_2 (1 - b_{21} - b_{23}) \hat{V} + r_2 \hat{V} (b_{21} \hat{U} - \hat{V} + b_{23} \hat{W}) \\ &= D_2 \hat{V}'' + s \hat{V}' + r_2 (1 - b_{21} - b_{23}) \hat{V} + o(\hat{V}) \end{aligned}$$

as  $z \rightarrow \infty$ . Therefore

$$\hat{V}(z) = (c_4 e^{-\mu_V^- z} + c_5 e^{-\mu_V^+ z}) (1 + o(1)) \quad (3.5)$$

for some constants  $c_4, c_5 \in \mathbb{R}$ , as  $z \rightarrow \infty$ .

From (3.1) and the limits of  $(\hat{U}, \hat{V}, \hat{W})$  as  $z \rightarrow \infty$ , one can see that the second equation of  $L(p, q, r)^T = \mathbf{0}$  is asymptotically like

$$D_2 q'' + s q' + r_2 (1 - b_{21} - b_{23}) q = 0$$

as  $z \rightarrow \infty$ . Since  $(\hat{U}', \hat{V}', \hat{W}')^T$  is the bounded solution of  $L(p, q, r)^T = 0$ , we also have that

$$\hat{V}'(z) = (c_4 (-\mu_V^-) e^{-\mu_V^- z} + c_5 (-\mu_V^+) e^{-\mu_V^+ z}) (1 + o(1)) \quad (3.6)$$

as  $z \rightarrow \infty$ . In the following we prove that (3.5) and (3.6) can be controlled such that  $c_4 \neq 0$ . Suppose that  $c_4 = 0$ . Then by (3.6) we must have that  $c_5 \neq 0$ . By (2.15) it follows that

$$\lim_{z \rightarrow \infty} \frac{V(z)}{\hat{V}(z)} \leq 1. \quad (3.7)$$

On the other hand, by (2.11) and (3.5),

$$\begin{aligned} \lim_{z \rightarrow \infty} \frac{V(z)}{\hat{V}(z)} &= \lim_{z \rightarrow \infty} \frac{(1 - b_{21} - b_{23})(1 - M e^{-\varepsilon z}) e^{-\mu_V^- z}}{c_4 e^{-\mu_V^- z} + c_5 e^{-\mu_V^+ z}} \\ &= \lim_{z \rightarrow \infty} \frac{(1 - b_{21} - b_{23})(1 - M e^{-\varepsilon z})}{c_5 e^{-(\mu_V^+ - \mu_V^-)z}} = \infty \end{aligned}$$

since  $\mu_V^+ - \mu_V^- > 0$ . Thus we have a contradiction and therefore  $c_4 \neq 0$ . By (1.10) we have that

$$\mu_V^- < \sigma. \quad (3.8)$$

Then from (3.6), (3.8) and the fact that  $c_4 \neq 0$ , we have that

$$\lim_{z \rightarrow \infty} (1 + e^{\sigma z}) V'(z) = \lim_{z \rightarrow \infty} e^{\sigma z} (c_4 (-\mu_V^-) e^{-\mu_V^- z} + c_5 (-\mu_V^+) e^{-\mu_V^+ z}) = \infty.$$

By Theorem 5.1 in Chapter 4 of [13], we have that  $0 \notin \sigma_n(\tilde{L})$ . Therefore the proof is finished.  $\square$

In the following we give the proof of Theorem 1.4.



*Proof of Theorem 1.4.* Under Theorem 1.3, we have that if  $s > s_*$  and the parameters  $(D_j, r_j, b_{j2})$  ( $j = 1, 3$ ) satisfy (1.4) and (1.8), there exists traveling wave solutions  $(\hat{U}, \hat{V}, \hat{W})$ . By the resolvent estimates for  $(\lambda I - \tilde{L})^{-1}$  like Lemma 2.1 in Chapter 5 of [13], we have that  $\tilde{L}$  generates an analytic semigroup  $e^{t\tilde{L}}$ . Since Theorem 3.4 are satisfied, by the standard theory about linear stability implying asymptotic stability in Theorem 1.2 or Theorem 2.1 of Chapter 5 of [13], one has that  $(\hat{U}, \hat{V}, \hat{W})$  (i.e.,  $(\hat{\phi}, \hat{\psi}, \hat{\theta})$ ) is exponentially stable with no shift in  $BUC_\sigma(\mathbb{R})$ . The proof is completed.  $\square$

### 3.4. Remarks

In the existence of traveling waves of (1.3), the restrictions on the parameters in (2.2) is simpler in Theorem 1.3 than that in Proposition 1.1 since we do not consider the conditions to guarantee the minimal speed of (2.2). However, we cannot use the super- subsolution techniques in [14] to obtain the existence of traveling waves when  $s = s_*$ . Moreover, even if we have the existence for  $s = s_*$ , one can see that (1.10) will not hold. Therefore it is impossible to choose the weight so that the perturbation of the waves in the weighted space decay exponentially in time. In order to find further results of stability when  $s = s_*$ , we need to consider other kinds of perturbation of the waves. The techniques for dealing with the stability of Fisher-KPP waves such as [19], [20], [21] and the reference therein can provide us some information in the future work.

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