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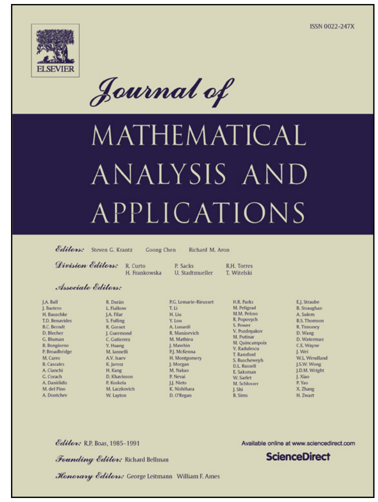

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Maitere Aguerrea, Carlos Gómez

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On existence of semi-wavefronts for a non-local reaction-diffusion equation with distributed delay

Maitere Aguerrea, Carlos Gómez

Facultad de Ciencias Básicas, Universidad Católica del Maule,Casilla 617, Talca, Chile.

Abstract

We study the problem of existence of semi-wavefront solutions for a non-local delayed reaction-diffusion equation with monostable nonlinearity. In difference with previous works, we consider non-local interaction which can be asymmetric in space. As a consequence of this asymmetry, we must analyze the existence of expansion waves for both positive and negative speeds. In the paper, we use a framework of the general theory recently developed for a certain nonlinear convolution equation. This approach allows us to prove the wave existence for the range of admissible speeds $c \in \mathbb{R} \setminus (c_\star^-, c_\star^+)$, where the critical speeds $c_\star^-$ and $c_\star^+$ can be calculated explicitly from some associated equations. The main result is then applied to several non-local reaction-diffusion epidemic and population models.

Keywords: reaction-diffusion equation; traveling wave; non-local interaction; delay; existence.

1. Introduction.

The main object of study in this paper is the following monostable non-local reaction-diffusion equation

$$u_t(t,x) = u_{xx}(t,x) - f(u(t,x)) + \int_0^\infty \int_{\mathbb{R}} K(s,w) g(u(t-s,x-w)) dw ds. \quad (1.1)$$

Email address: maguerrea@ucm.cl, cgomez@ucm.cl (Maitere Aguerrea, Carlos Gómez)

With appropriate f, g (where $f(0) = g(0) = 0$) and K, equation (1.1) is often used to model ecological and biological processes where the typical interpretation of $u(t,x)$ is the density of population. Thus we will be interested only in non-negative solutions of the above equation. Now, it is well known that the key elements determining the dynamics of solutions of equation (1.1) are the semi-wavefronts, i.e. bounded positive classical non-constant solutions $u(t,x) = \phi(x+ct)$ satisfying one of the boundary conditions $\phi(-\infty) = 0$, $\phi(+\infty) = 0$. The parameter c is called the speed of propagation. An important special case of semi-wavefront is a wavefront, i.e. semi-wavefront whose profile ϕ converges at both $+\infty$ and $-\infty$.

Over the last decade, the existence and uniqueness of semi-wavefronts and wavefronts for the general non-local equation (1.1) have been investigated in a series of papers where different geometric and smoothness conditions on f, K and g were assumed (see e.g. [1, 7, 23, 29]). One of the main goals of this paper is to weaken two major (at least, on our opinion) geometric restrictions imposed on the functions K, g. The first one is the evenness condition $K(s,x) = K(s,-x),\ x \in \mathbb{R}$, assumed in [7, 23, 29]. However, several recent studies indicate that asymmetric kernels might appear in the population modeling in a natural way, cf. [11]. Importantly, the asymmetry of interaction can produce interesting ecological effects [11, 15]. So, first, we get rid of the above mentioned restrictive symmetry assumption. The second geometric restriction to be weaken in this paper is the sub-tangency inequality

$$g(s) \le g'(0)s, \quad s \ge 0. \tag{1.2}$$

Indeed, we show that the bulk of existence results still holds even if we do not use (1.2). Finally, it is worth to mention that our approach also allows to assume less restrictive smoothness conditions on f, g. In order to be more precise, let us list our main hypotheses:

H_0**:** $K \in L^1(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$ and $\int_0^\infty \int_{\mathbb{R}} K(s,w)dwds = 1$. Moreover, for any $c \in \mathbb{R}$, there exist $\gamma_1^\#(c) < 0 < \gamma_2^\#(c)$ such that for each $z \in (\gamma_1^\#(c), \gamma_2^\#(c))$ the integral $\int_0^\infty \int_{\mathbb{R}} K(s,w)e^{-z(cs+w)}dwds$ is finite and it diverges, if $z > \gamma_2^{\#}(c)$ or $z < \gamma_1^{\#}(c)$.

H_1**:** Function $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ is bounded and $g(0) = 0$, $g(s) > 0$ for all $s > 0$. In addition, the right-hand Dini derivates $g'_-(0^+) > 0$ and $g'_+(0^+)$ are finite.

H_2**:** Locally Lipschitzian function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing with $f(0) = 0$ and $f(\bar{\xi}_2) > \sup_{s\geq 0} g(s)$ for some $\bar{\xi}_2 > 0$. Moreover, $0 < f'_-(0^+) < g'_-(0^+)$ and $f(s) \geq f'_-(0^+)s$ for $s \in [0, +\infty)$.

Our main results are given below:

Theorem 1.1. *Assume that* H_0*-*H_2 *hold. Then there exist* $c_\star^-, c_\star^+ \in \mathbb{R}$ *such that for every* $c \in \mathbb{R}\backslash(c_\star^-, c_\star^+)$ *the equation (1.1) has a semi-wavefront solution* $u(x,t) = \phi(x+ct)$ *propagating with speed c. If* $c \geq c_\star^+$*, then* $\phi(-\infty) = 0$ *and* $\liminf_{t\to+\infty} \phi(t) > 0$*. If* $c \leq c_\star^-$*, then* $\phi(+\infty) = 0$ *and* $\liminf_{t\to-\infty} \phi(t) > 0$*. Furthermore, if equation* $f(s) = g(s)$ *has only two solutions:* 0 *and* κ*, with* κ *being globally attracting with respect to* $f^{-1} \circ g : (0, \xi_2] \to (0, \xi_2]$*, then each of these semi-wavefront is a wavefront.*

Theorem 1.2. *Assume that* H_0*-*H_2 *hold. Then there exist* $c_*^-, c_*^+ \in \mathbb{R}$ *such that* $c_\star^- \leq c_*^- < c_*^+ \leq c_\star^+$*, and for any* $c \in (c_*^-, c_*^+)$*, the equation (3.1) has no semi-wavefront solution* $u(t,x) = \phi(x+ct)$ *propagating with speed c.*

Remark 1.3. We observe that if g satisfies condition $g(s) \leq Ls, s > 0$ with $L = g'_-(0)$, then $c_\star^+ = c_*^+$ and $c_\star^- = c_*^-$. Moreover, Theorems 1.1 and 1.2 imply that $c_\star^+$ and $c_\star^-$ are the minimal speeds of propagation. Now, the situation when $L > g'_-(0)$ and g is non-monotone is clearly more complicated: in particular, the existence of the minimal speed of semi-wavefronts propagation with the usual properties is not yet proved in such a case. In other words, if we take a general non-monotone reaction term g satisfying $L > g'_-(0)$, then the question about the structure of the set of all admissible speeds from the interval $[c_*^+, c_\star^+]$ is largely open at this moment (even for the particular case of local interactions with discrete delay).

Remark 1.4. By considering waves in the form $\psi(x+ct) := \phi(-(x+ct)$, we find that $c_\star^+ = -c_\star^-$, $c_*^+ = -c_*^-$ for spatially symmetric kernels. However, as Example 3.5 below shows, if kernel $K(s,x)$ is not symmetric in the second variable then may happen that $c_\star^+ \neq -c_\star^-$(again, profiles propagating with speed $c \geq c_\star^+$ will satisfy $\phi(-\infty) = 0$ while profiles propagating with the speed $c \leq c_\star^-$ will satisfy $\phi(+\infty) = 0$). Models with spatially asymmetric kernels have been studied by means of the dynamical system methods in [18, 34, 31]. The existence of the left and right minimal speeds was proved for some subclasses of monotone and non-monotone semiflows in [18, 34]. The existence and non-existence results of [34] were applied to equation (1.1) considered with $K(s,w) = \delta(s-\tau)k(w), \tau > 0$ and with g satisfying inequality (1.2). Similar results were also obtained in [11], by using sub-supersolutions method. In this way, Theorems 1.1 and 1.2 extend studies of [11, 34], where the existence and non-existence theorems were established for g satisfying the sub-tangency condition (1.2).

The existence and non-existence results are established by applying the general wave's existence and uniqueness theory developed in [2, 11]. These works deal with the scalar integral equation

$$\phi(t) = \int_X d\rho(\tau) \int_{\mathbb{R}} \mathcal{N}(s,\tau) g(\phi(t-s),\tau) ds, \quad t \in \mathbb{R}, \tag{1.3}$$

where (X,ρ) denotes a space with finite measure ρ, $\mathcal{N}(s,\tau) \geq 0$ is integrable on $\mathbb{R} \times X$ with $\int_{\mathbb{R}} \mathcal{N}(s,\tau)ds > 0$, $\tau \in X$, while measurable $g : \mathbb{R}_+ \times X \to \mathbb{R}_+$, $g(0,\tau) \equiv 0$, is continuous in ϕ for every fixed $\tau \in X$. In order to apply the techniques of [11], we will transform equation (1.1) into the form (1.3).

We conclude the introduction by saying several words about the organization of the paper. The next section contains some preliminary results. In Section 3, we describe geometric properties of the bounded solutions of equation (1.1), the associated characteristic equations are studied and our main existence results are proved. In the final section, we apply Theorems 1.1 to some non-local reaction-diffusion epidemic and population models with distributed time delay (these models were also previously studied in [4, 8, 11, 12, 21, 23, 25, 27, 28, 30]).

2. Preliminaries.

In this section, we first extend various abstract results proved in [11]. Then we show how to transform equation (1.1) into the convolution form (1.3). We assume that the functions $N(s,\tau)$, $g(v,\tau)$ and $\rho(\tau)$ satisfy all assumptions mentioned in the introduction.

We begin by stating a general result obtained in [11, Theorem 7] (Proposition 2.1 below). This result ensures the existence of semi-wavefront solutions of the equation (1.3) under the following conditions

(N) There exists $\tau_0 \in X$, $\rho(\tau_0) = 1$, such that $g(v,\tau)$ is increasing in $v \in \mathbb{R}_+$ for each fixed $\tau \neq \tau_0$ and $g(v,\tau) > 0, v > 0$. Furthermore, there exists $\xi_2 > 0$ such that $\theta(v) := v - \tilde{g}(v)$ is strictly increasing on $[0, \xi_2]$, where

$$\tilde{g}(v) := \int_{\mathbb{R}} \int_{X \setminus \{\tau_0\}} g(v,\tau)\mathcal{N}(s,\tau)d\rho(\tau)ds,$$

and $\theta(\xi_2) > \max_{v \geq 0} g(v,\tau_0) \int_{\mathbb{R}} \mathcal{N}(s,\tau_0)ds$.

(P) Bounded continuous solution $\phi(t) \geq 0$ of (1.3) vanishes at some point only if $\phi(t) \equiv 0$.

We also need the following characteristic function χ associated with the variational equation along the trivial steady state of (1.3):

$$\chi(z) := 1 - \int_{\mathbb{R}} \int_X \mathcal{N}(s,\tau)g'(0,\tau)d\rho(\tau)e^{-sz}ds,$$

as well as the function $G(v) := \theta^{-1}(Cg(v,\tau_0))$, where $C = \int_{\mathbb{R}} \mathcal{N}(s,\tau_0)ds$.

We have the following general result:

Proposition 2.1. *(See [11, Theorema 7]) Assume (N) and (P) and let $g'(0,\tau) > 0$, $G'(0)$ be finite and*

$$g(s,\tau) \leq g'(0,\tau)s \quad \textit{for all } s \geq 0, \tau \in X. \tag{2.1}$$

If $\chi(z), \chi(0) < 0$, is well defined and changes its sign on some open interval $(0, \bar{w})$ [respectively, on $(-\bar{w}, 0)$], then equation (1.3) has at least one semi-wavefront ϕ with $\sup_{s\in\mathbb{R}} \phi(s) \le \xi_2, \phi(-\infty) = 0$, and $\liminf_{t\to+\infty} \phi(t) > \xi_1 > 0$ [respectively, $\phi(+\infty) = 0$, and $\liminf_{t\to-\infty} \phi(t) > \xi_1 > 0$]. Moreover, if the equation $G(s) = s$ has exactly two solutions 0 and κ on $\mathbb{R}_+$, and the point κ is globally attracting for the map $G : (0, \xi_2] \to (0, \xi_2]$, then $\phi(+\infty) = \kappa$.

Remark 2.2. Note that ξ_1 can be found explicitly, see [11, Lemma 5]. We also observe that the proof of Proposition 2.1 uses the sublinearity assumption (2.1) and $G'(0) < \infty$ in an essential way.

Nevertheless, it is possible to show that the Proposition 2.1 remains valid if we assume (N) and (P) and the following weak conditions (L), (G_1) instead of the restrictive (2.1) and $G'(0) < \infty$.

(L) $g(s, \tau) \le l(\tau)s$ for all $s \ge 0, \tau \in X$ and some a measurable map $l(\tau) : X \to \mathbb{R}_+$ such that $\int_X l(\tau)d\rho(\tau) \int_{\mathbb{R}} \mathcal{N}(s, \tau)ds < \infty$.

(M) $\tilde{g}_L := \int_{\mathbb{R}} \int_{X\setminus\{\tau_0\}} \mathcal{N}(s, \tau)l(\tau)d\rho(\tau)ds < 1$, where τ_0 is as in (N).

We also will consider the following modified characteristic functions

$$\chi_L(z) := 1 - \int_{\mathbb{R}} \int_X \mathcal{N}(s, \tau)l(\tau)d\rho(\tau)e^{-sz}ds.$$

$$\chi_-(z) := 1 - \int_{\mathbb{R}} \int_X \mathcal{N}(s, \tau)g'_-(0^+, \tau)d\rho(\tau)e^{-sz}ds,$$

where we suppose that the right-hand Dini derivate $g'_-(0^+, \tau) > 0$ for each $\tau \in X$.

Lemma 2.3. *Assume that (N), (M) and (L) hold, $\chi_-(0) < 0$ and $g'_-(0^+, \tau) > 0, \tau \in X$. Then, for some $\xi_1 \in (0, \xi_2)$, $G([\xi_1, \xi_2]) \subset [\xi_1, \xi_2]$ and $\min_{s\in[\xi_1,\xi_2]} G(s) = G(\xi_1)$, while $G(s) > s$ for $s \in (0, \xi_1]$, where ξ_2 is as in (N). Moreover, the right-hand Dini derivate $G'_+(0^+)$ is finite.*

Proof. First, note that $G(0) = 0$ and $0 < G(v) < \xi_2, v > 0$. Since $\chi_-(0) < 0$ we have

$$1 - \int_{\mathbb{R}} \int_{X\setminus\{\tau_0\}} \mathcal{N}(s, \tau)g'_-(0^+, \tau)d\rho(\tau)ds < Cg'_-(0^+, \tau_0).$$

Observe that, by the general version of the Fatou lemma given in [5], we get that

$$\int_{\mathbb{R}} \int_{X\setminus\{\tau_0\}} \mathcal{N}(s,\tau) g'_-(0^+,\tau) ds d\rho(\tau) \leq \tilde{g}'_-(0).$$

Hence, $1 - \tilde{g}'_-(0) < Cg'_-(0^+,\tau_0)$. Since $g'_+(0^+,\tau) \leq l(\tau)$ we get from (M) that $\theta'_-(0)$ is finite and

$$0 < 1 - \tilde{g}_L \leq \theta'_-(0) \leq 1 - \tilde{g}'_-(0).$$

Thus $\tilde{g}'_-(0) < 1$ and

$$1 < \frac{Cg'_-(0^+,\tau_0)}{1 - \tilde{g}'_-(0)} < \infty.$$

On the other hand,

$$\begin{aligned}
\liminf_{v\to 0+} \frac{G(v)}{v} &= \liminf_{v\to 0+} \left(\left(\frac{\theta^{-1}(Cg(v,\tau_0))}{\theta(\theta^{-1}(Cg(v,\tau_0))} \right) \frac{Cg(v,\tau_0)}{v} \right) \\
&\geq \liminf_{v\to 0+} \left(\frac{\theta^{-1}(Cg(v,\tau_0))}{\theta(\theta^{-1}(Cg(v,\tau_0))} \right) Cg'_-(0^+,\tau_0) = \frac{Cg'_-(0^+,\tau_0)}{1 - \tilde{g}'_-(0)} > 1.
\end{aligned}$$

so that $G'_-(0+) > 1$. Thus $G(s) > s$ for each $s \in (0,\xi_1]$ for some $\xi_1 \in (0,\xi_2)$. Since $G(0) = 0$ and $0 < G(v) < \xi_2, v \in (0,\xi_2]$, we can choose ξ_1 sufficiently small such that $G([\xi_1,\xi_2]) \subset [\xi_1,\xi_2]$ and $\min_{s\in[\xi_1,\xi_2]} G(s) = G(\xi_1)$. Finally, $G'_+(0+)$ is finite because of

$$\limsup_{v\to 0+} \frac{G(v)}{v} \leq \limsup_{v\to 0+} \left(\frac{\theta^{-1}(Cg(v,\tau_0))}{\theta(\theta^{-1}(Cg(v,\tau_0))} \right) Cg'_+(0^+,\tau_0) \leq \frac{Cg'_+(0^+,\tau_0)}{1 - \tilde{g}_L}.$$

□

Now, we are ready to state the following useful extension of Proposition 2.1.

Theorem 2.4. *Assume (N), (P), (L), (M) and $g'_-(0^+,\tau) > 0$, $\tau \in X$. Suppose also that $\chi_L(z)$ is well defined on some open interval $(-\bar{w},\bar{w})$ and $\chi_-(0) < 0$. If $\chi_L(z)$ changes its sign on the open interval $(0,\bar{w})$ [respectively, on $(-\bar{w},0)$], then equation (1.3) has at least one semi-wavefront ϕ satisfying $\sup_{s\in\mathbb{R}} \phi(s) \leq \xi_2$, $\phi(-\infty) = 0$, and $\liminf_{t\to+\infty} \phi(t) > \xi_1$ [respectively, $\sup_{s\in\mathbb{R}} \phi(s) \leq \xi_2$, $\phi(+\infty) = 0$, and $\liminf_{t\to-\infty} \phi(t) > \xi_1$]. Moreover, if the equation $G(s) = s$ has exactly two solutions*

0 and κ on $\mathbb{R}_+$, and the point κ is globally attracting for the map $G : (0, \xi_2] \to (0, \xi_2]$, then $\phi(+\infty) = \kappa$ [respectively, $\phi(-\infty) = \kappa$].

To prove Theorem 2.4, we will need Lemma 2.5 and Theorem 2.6 below:

Lemma 2.5. *Suppose that (L) holds and assume that $g'_-(0^+, \tau) > 0$, $\tau \in X$, and $\chi_-(0) \in (-\infty, 0)$. Let $\phi : \mathbb{R} \to [0, +\infty)$ be a bounded solution to equation (1.3). If $\phi(-\infty) = 0$ and, for each fixed $t' \in \mathbb{R}$, it holds that $\phi(t) \not\equiv 0$ for all $t \leq t'$, then χ_- is well defined and has a zero on some non-degenerate interval $(0, \gamma_\phi]$. If $\phi(+\infty) = 0$, then χ_- has a zero on some non-degenerate interval $[\gamma_\phi, 0)$.*

Proof. The proof (where we follow the approach of [11]) will be divided into the four steps.

Step I. First we consider the bilateral Laplace transforms

$$\Phi(z) := \int_{\mathbb{R}} e^{-zs}\phi(s)ds, \ \mathcal{L}(z) := \int_{\mathbb{R}}\int_X \mathcal{N}(s,\tau)g'_-(0^+,\tau)d\rho(\tau)e^{-sz}ds,$$

and, for $\delta > 0$, the measurable function

$$\lambda_\delta^-(\tau) := \inf_{u\in(0,\delta)} \frac{g(u,\tau)}{u} \geq 0.$$

Observe that, by the monotone convergence theorem,

$$\begin{aligned}\lim_{\delta\to 0^+}\int_X\int_{\mathbb{R}} \mathcal{N}(s,\tau)\lambda_\delta^-(\tau)dsd\rho(\tau) &= \int_X\int_{\mathbb{R}} \mathcal{N}(s,\tau)g'_-(0^+,\tau)dsd\rho(\tau)\\ &= 1-\chi_-(0) > 1.\end{aligned}$$

Therefore,

$$\int_X\int_{\mathbb{R}} \mathcal{N}(s,\tau)\lambda_\delta^-(\tau)dsd\rho(\tau) \in (1,\infty),$$

for all $0 < \delta < \delta'$, being δ' sufficiently small. In this way, since ϕ satisfies (1.3) and $g(s,\tau) \geq \lambda_\delta^-(\tau)s$ for $s \in (0,\delta) \subset (0,\delta')$, $\tau \in X$, [2, Theorem 1] assures that there exist $\bar{x} > 0$ such that

$$\int_{-\infty}^0 \phi(s)e^{-s\bar{x}}ds \text{ and } \int_{\mathbb{R}}\int_X \mathcal{N}(s,\tau)\lambda_\delta^-(\tau)d\rho(\tau)e^{-s\bar{x}}ds$$

are convergent. Consequently, since $0 < g'_-(0^+,\tau) \leq 2\lambda_\delta^-(\tau)$ for all $\delta > 0$ sufficiently small, we have

$$\int_{\mathbb{R}}\int_X \mathcal{N}(s,\tau)g'_-(0,\tau)d\rho(\tau)e^{-s\bar{x}}ds < \infty. \tag{2.2}$$

Thus $\Phi(z)$, $\mathcal{L}(z)$ are finite for all $0 \le \Re z \le \bar{x}$. Now, we denote the maximal open vertical strips of convergence for these two integrals as $\sigma_\phi < \Re z < \gamma_\phi$ and $\sigma_K < \Re z < \gamma_K$, respectively. Note that, from [2, Lemma 1] we get that $\sigma_K \le \sigma_\phi < \gamma_\phi \le \gamma_K$ and $\mathcal{L}(\gamma_\phi)$ is always a finite number, so that $\chi_-(\gamma_\phi) = 1 - \mathcal{L}(\gamma_\phi) \in \mathbb{R}$.

Step II. For real $z \in (0, \gamma_\phi)$ we consider the integrals

$$\mathcal{G}(z,\tau) := \int_{\mathbb{R}} e^{-zs} g(\phi(s),\tau) ds, \;\; \mathcal{K}(z,\tau) := \int_{\mathbb{R}} e^{-zs} \mathcal{N}(s,\tau) ds.$$

Since ϕ is non-negative and bounded, and since $g'_-(0+,\tau) > 0$ exists and g satisfies the condition (L), the convergence of $\mathcal{G}(z,\tau)$ is equivalent to the convergence of $\Phi(z)$ for $z > 0$. Note that applying the bilateral Laplace transform to equation (1.3), we obtain that

$$\Phi(z) = \int_X \mathcal{K}(z,\tau)\mathcal{G}(z,\tau) d\rho(\tau). \tag{2.3}$$

Moreover, observe that $\mathcal{L}, \mathcal{G}, \Phi$ are positive at each real point of the convergence, and

$$\int_X \mathcal{K}(z,\tau) \frac{\mathcal{G}(z,\tau)}{\Phi(z)} d\rho(\tau) = 1. \tag{2.4}$$

Step III. Now, we will prove that $\chi(z)$ has a zero on $(0, \gamma_\phi]$. First, we suppose that $\Phi(\gamma_\phi) = \lim_{z \to \gamma_\phi -} \Phi(z) = \infty$. Let T_δ be the rightmost non-positive number such that $\phi(s) \le \delta$ for $s \le T_\delta$. Then

$$\lambda_\delta^- \int_{-\infty}^{T_\delta} e^{-zs}\phi(s) ds \le \int_{-\infty}^{+\infty} e^{-zs} g(\phi(s),\tau) ds \le l(\tau) \int_{-\infty}^{+\infty} e^{-zs}\phi(s) ds.$$

As a consequence, for each positive $\delta > 0$,

$$\lambda_\delta^- \le \liminf_{z \to \gamma_\phi -} \frac{\mathcal{G}(z,\tau)}{\Phi(z)} \le l(\tau),$$

and

$$g'_-(0^+,\tau) \le \liminf_{z \to \gamma_\phi -} \frac{\mathcal{G}(z,\tau)}{\Phi(z)}, \;\; \text{for each } \tau \in X.$$

Now, the non-negative function $\mathcal{F}(\tau) := \liminf_{z \to \gamma_\phi -} \frac{\mathcal{G}(z,\tau)}{\Phi(z)}$ is well defined for each $\tau \in X$ and is measurable on X. By using Fatou Lemma as $z \to \gamma_\phi -$ in

(2.4) we obtain

$$1-\chi_-(\gamma_\phi)=\int_X \mathcal{K}(\gamma_\phi,\tau)g'_-(0^+,\tau)d\rho(\tau)\le \int_X \mathcal{K}(\gamma_\phi,\tau)\mathcal{F}(\tau)d\rho(\tau)$$
$$\le \liminf_{z\to\gamma_\phi-}\int_X \mathcal{K}(z,\tau)\frac{\mathcal{G}(z,\tau)}{\Phi(z)}d\rho(\tau)=1.$$

Therefore $\chi_-(\gamma_\phi)\ge 0$, and since $\chi_-(0)<0$ we get the required assertion.

Step IV. Let us prove that $\chi_-(z)=0$ has a root in $(0,\gamma_\phi]$ even if $\Phi(\gamma_\phi)=\lim_{z\to\gamma_\phi-}\Phi(z)>0$ is finite. Since $\phi(t)\not\equiv 0,\ t\le t'$ for each fixed t', in such a case $\gamma_\phi<\infty$.

Suppose now on the contrary that the characteristic equation

$$\chi_-(z)=1-\int_{\mathbb{R}}\int_X \mathcal{N}(s,\tau)g'_-(0+,\tau)d\rho(\tau)e^{-sz}ds=0$$

has not real roots on $[0,\gamma_\phi]$. Then $\chi_-(0)<0$ implies $\chi_-(\gamma_\phi)<0$. Set

$$\zeta(t):=\phi(t)e^{-\gamma_\phi t},\ \mathcal{N}_1(s,\tau):=e^{-\gamma_\phi s}\mathcal{N}(s,\tau).$$

Then, for $t<T_\delta-N$, $N>0$, we have from (1.3) that

$$\int_{-\infty}^t\zeta(v)dv=\int_{-\infty}^t\phi(v)e^{-\gamma_\phi v}dv$$
$$\ge\int_X d\rho(\tau)\int_{-N}^N\mathcal{N}_1(s,\tau)\int_{-\infty}^t g(\phi(v-s),\tau)e^{-\gamma_\phi(v-s)}dvds$$
$$\ge\int_X d\rho(\tau)\int_{-N}^N\lambda_\delta^-(\tau)\mathcal{N}_1(s,\tau)\int_{-\infty}^t\zeta(v-s)dvds$$
$$\ge\left(\int_X d\rho(\tau)\int_{-N}^N\lambda_\delta^-(\tau)\mathcal{N}_1(s,\tau)ds\right)\int_{-\infty}^{t-N}\zeta(v)dv=:\kappa_\delta\int_{-\infty}^{t-N}\zeta(v)dv.$$

On the other hand, in virtue of the monotone convergence theorem, we have

$$\lim_{\delta\to0+}\lim_{N\to+\infty}\int_X d\mu(\tau)\int_{-N}^N\lambda_\delta^-(\tau)\mathcal{N}_1(s,\tau)ds=1-\chi_-(\gamma_\phi)>1.$$

Hence, for some appropriate $\delta,N>0$, the increasing function $\xi(t)=\int_{-\infty}^t\zeta(s)ds$ satisfies $\xi(t)\ge\kappa_\delta\xi(t-N)$, $t<T_\delta-N$ with $\kappa_\delta>1$. We now consider the function $h(t)=\xi(t)e^{-\gamma t}$ with $\gamma=N^{-1}\ln\kappa_\delta>0$. For all $t<T_\delta-N$ we have

$$h(t-N)=\xi(t-N)e^{-\gamma(t-N)}\le\frac{1}{\kappa_\delta}\xi(t)e^{-\gamma t}e^{\gamma N}=h(t).$$

Hence $\sup_{t\le 0} h(t) < \infty$ and $\xi(t) = O(e^{\gamma t})$, $t \to -\infty$. After taking $\bar{x} \in (0,\gamma)$ and integrating by parts, we obtain

$$\int_{-\infty}^{t} \zeta(s)e^{-\bar{x}s}ds = \xi(t)e^{-\bar{x}t} + \bar{x}\int_{-\infty}^{t} \xi(s)e^{-\bar{x}s}ds < +\infty.$$

This implies that the integral $\int_{-\infty}^{t} \phi(s)e^{-(\gamma_\phi+\bar{x})s}ds$ converges, contradicting to the definition of γ_ϕ, which completes Step IV.

Finally, note that if $\phi(+\infty) = 0$, then $\psi(t) := \phi(-t)$ satisfies the equation

$$\psi(t) = \int_X d\rho(\tau)\int_{\mathbb{R}} \mathcal{N}(-s,\tau)g(\phi(t-s),\tau)ds, \quad t \in \mathbb{R},$$

and $\psi(-\infty) = 0$. Moreover, the associated characteristic equation

$$\hat{\chi}_-(z) := 1 - \int_{\mathbb{R}}\int_X \mathcal{N}(-s,\tau)g'_-(0+,\tau)d\rho(\tau)e^{-sz}ds = \chi_-(-z), \tag{2.5}$$

and thus $\hat{\chi}_-(0) = \chi_-(0) < 0$. In consequence, $\hat{\chi}_-(z)$ has at least one positive root, so that $\chi_-(z)$ has at least one negative root. The lemma is proved. □

Theorem 2.6. *Assume all conditions of Theorem 2.4. Let ϕ be a positive bounded solution to equation (1.3). If $\inf_{s\in\mathbb{R}}\phi(s) < \xi_1$, then $\lim_{t\to\omega}\phi(t) = 0$ and $\liminf_{t\to-\omega}\phi(t) > \xi_1$ for some $\omega \in \{-\infty,\infty\}$.*

Proof. The proof will be divided into the tree steps.

Step I. Here we prove the following property: if ϕ is a positive bounded solution to equation (1.3), then either $\liminf_{t\to+\infty}\phi(t) > 0$ or $\phi(+\infty) = 0$ (a similar statement is also true at $-\infty$). This result may be proved in much the same way as [11, Theorem 3, p.5]. First, observe that $\phi(t)$ is uniformly continuous on $\mathbb{R}$:

$$\begin{aligned}|\phi(t+h) - \phi(t)| &\le \int_X d\rho(\tau)\int_{\mathbb{R}} |\mathcal{N}(s+h,\tau) - \mathcal{N}(s,\tau)|g(\phi(t-s),\tau)ds \\ &\le |\phi|_\infty \int_X l(\tau)d\rho(\tau)\int_{\mathbb{R}} |\mathcal{N}(s+h,\tau) - \mathcal{N}(s,\tau)|ds =: |\phi|_\infty\sigma(h).\end{aligned}$$

Next, by condition (L), we have

$$l(\tau)\int_{\mathbb{R}} \mathcal{N}(s,\tau)ds \in L_1(X),$$

so that $\lim_{h\to 0}\sigma(h) = 0$, in view of the continuity of translation in $L_1(\mathbb{R})$ and the Lebesgue's dominated convergence theorem.

Now, let suppose that $\limsup_{t\to+\infty}\phi(t) = S > 0$ and $\liminf_{t\to+\infty}\phi(t) = 0$. Then Lemma 2.5 allows to repeat Step 3 of the proof of Theorem 3 in [11] (where χ is replaced with χ_-) and, for each fixed $j > S^{-1}$, to find a positive solution $\zeta_j : \mathbb{R} \to (0, 1/j]$ of (1.3) such that

$$0 < \max_{t\in\mathbb{R}} \zeta_j(t) = \zeta_j(0) \le 1/j.$$

Now, let us consider $y_j(t) = \zeta_j(t)/\zeta_j(0)$. Each y_j satisfies

$$y_j(t) = \int_X d\rho(\tau) \int_{\mathbb{R}} \mathcal{N}(s,\tau)a_j(t-s,\tau)y_j(t-s)ds, \tag{2.6}$$

where $a_j(t,\tau) = g(\zeta_j(t),\tau)/\zeta_j(t)$. In addition, note that the sequence $\{y_j(t)\}_{j=1}^{+\infty}$ is equicontinuous. In fact, since $a_j(t,\tau) \le l(\tau)$ for all $\tau \in X$, we get that

$$\begin{aligned}
|y_j(t+h) - y_j(t)| &\le \int_X d\rho(\tau) \int_{\mathbb{R}} a_j(t-s)y_j(t-s)|\mathcal{N}(s+h,\tau) - \mathcal{N}(s,\tau)|ds \\
&\le \int_X d\rho(\tau) \int_{\mathbb{R}} a_j(t-s)|\mathcal{N}(s+h,\tau) - \mathcal{N}(s,\tau)|ds \\
&\le \int_X d\rho(\tau) \int_{\mathbb{R}} l(\tau)|\mathcal{N}(s+h,\tau) - \mathcal{N}(s,\tau)|ds = \sigma(h)
\end{aligned}$$

where $\sigma(h)$ was defined on step 1. In consequence, $\{y_j(t)\}$ has a subsequence converging to a continuous function $y_* : \mathbb{R} \to [0,1],\ y_*(0) = 1$.

On the other hand,

$$\left|\int_{\mathbb{R}} \mathcal{N}(s,\tau)a_j(t-s,\tau)y_j(t-s)ds\right| \le l(\tau)\int_{\mathbb{R}} \mathcal{N}(s,\tau)ds \in L_1(X).$$

Thus, by the Fatou lemma,

$$\begin{aligned}
y_*(t) &= \liminf_{j\to\infty} \int_X d\rho(\tau) \int_{\mathbb{R}} \mathcal{N}(s,\tau)a_j(t-s,\tau)y_j(t-s)ds \\
&\ge \int_X g'_-(0,\tau)d\rho(\tau) \int_{\mathbb{R}} \mathcal{N}(s,\tau)y_*(t-s)ds \ge 0.
\end{aligned} \tag{2.7}$$

To finish the proof, note that cannot exist any nontrivial continuous function $y_* \ge 0$ satisfying (2.7). Indeed, since $\chi_-(0) < 0$ there exists $N > 0$ such that

$$\int_X g'_-(0,\tau)d\rho(\tau)\int_{-N}^{N} \mathcal{N}(s,\tau)ds > 1.$$

From (2.7) for $t > t'$, we obtain

$$\int_{t'}^{t} y_*(v)dv \geq \int_X g'_-(0,\tau)d\rho(\tau) \int_{-N}^{N} \mathcal{N}(s,\tau) \int_{t'}^{t} y_*(v-s)dvds$$
$$= \int_X g'_-(0,\tau)d\rho(\tau) \int_{-N}^{N} \mathcal{N}(s,\tau)\Big(\int_{t'-s}^{t'} + \int_{t'}^{t} + \int_{t}^{t-s}\Big)y_*(v)dvds,$$

from which

$$\int_{t'}^{t} y_*(v)dv \leq \frac{2\int_X \int_{-N}^{N} |s|\mathcal{N}(s,\tau)g'_-(0,\tau)dsd\rho(\tau)}{\int_X \int_{-N}^{N} \mathcal{N}(s,\tau)g'_-(0,\tau)dsd\rho(\tau) - 1}, \quad t' < t.$$

Therefore $y_* \in L_1(\mathbb{R})$. Now by integrating (2.7) over the real line, we get that

$$\int_{\mathbb{R}} y_*(v)dv \geq \left[\int_X g'_-(0,\tau)d\rho(\tau) \int_{\mathbb{R}} \mathcal{N}(s,\tau)ds\right] \int_{\mathbb{R}} y_*(v)dv,$$

a contradiction. Hence, the dichotomy principle is established at $+\infty$. The other case can be reduced to the previous one by doing the change of variables $\psi(t) := \phi(-t)$ and considering $\hat{\chi}_-$ (defined in (2.5)) instead of χ_-.

Step II. We now observe that $\chi_-(0) < 0$ and χ_- concave on its maximal domain of definition, so that all real zeros of χ should be of the same sign (if they exist). Thus, if $\chi_-(z)$ does not have any real positive [negative] zero and ϕ is a positive bounded solution of (1.3), then $\liminf_{t\to-\infty} \phi(t) > 0$ [respectively, $\liminf_{t\to+\infty} \phi(t) > 0$], in view of Lemma 2.5. As a consequence, equation (1.3) can not have positive pulse like solutions (i.e. solutions satisfying $\phi(-\infty) = \phi(+\infty) = 0$).

Step III. Now we will prove the following uniform persistence property: if $m = \inf_{s\in\mathbb{R}} \phi(s) < \xi_1$, then $\lim_{t\to\omega} \phi(t) = 0$ and $\liminf_{t\to-\omega} \phi(t) > \xi_1$. First note that, since ϕ is a bounded positive solution of equation (1.3), we have

$$0 \leq m := \inf_{t\in\mathbb{R}} \phi(t) \leq \sup_{t\in\mathbb{R}} \phi(t) =: M < +\infty.$$

Repeating the proof of Lemma 10 in [11], we get

$$[m, M] \subseteq G([m, M]). \tag{2.8}$$

Now, since $G(s) > s, s \in (0, \xi_1)$ and $m < \xi_1$, using Lemma 2.3 we obtain $m = 0$. Hence, due to the positivity of $\phi(t)$, there exists $\omega \in \{-\infty, +\infty\}$ such that $\liminf_{t\to\omega}\phi(t)=0$. Then, applying the step I and II, we find that $\mu:=\liminf_{t\to-\omega}\phi(t)>0$ and $\phi(\omega)=0$. Making use of standard limiting solution argument, we see that, for some $t_j\to-\omega$, the sequence $\phi(t+t_j)$ is converging in the compact-open topology of $C(\mathbb{R})$ to some function $\phi_1(t)$, $\mu=\inf_{t\in\mathbb{R}}\phi_1(t)\le\sup_{t\in\mathbb{R}}\phi_1(t)\le M$ solving equation (1.3). By (2.8), we have $[\mu,M]\subseteq G([\mu,M])$ which implies $\mu>\xi_1$. □

Proof Theorem 2.4. We follow the approach presented in [11, Theorem 7]. The proof will be divided into two steps.

Step I. First, let ξ_2 and ξ_1 be as in the hypothesis (N) and Lemma 2.3, respectively. Consider the sequence of measurable functions

$$g_n(s,\tau)=\begin{cases} l(\tau)s, & s\in[0,1/n],\\ \max\{\frac{l(\tau)}{n},g(s,\tau)\}, & s>1/n.\end{cases} \tag{2.9}$$

Clearly, $g_n(s,\tau)$ are continuous in s for each fixed τ. We also claim that for all sufficiently large n, each $g_n(s,\tau)$ satisfy the hypotheses of Proposition 2.1, where ξ_1 and ξ_2 do not depend on n.

Proof the claim: since $g(s,\tau)$ satisfies (N), we have that $g_n(s,\tau)$ is increasing in $s\in\mathbb{R}_+$ for all $n\in\mathbb{N}$ and each fixed $\tau\neq\tau_0$, and $g_n(s,\tau_0)>0, s>0$. In addition, the functions $\theta_n(v)=v-\tilde g_n(v)$, where

$$\tilde g_n(v)=\int_{\mathbb{R}}\int_{X\setminus\{\tau_0\}}g_n(v,\tau)\mathcal{N}(s,\tau)d\rho(\tau)ds,$$

are strictly increasing on $[0,\xi_2]$ for each $n\in\mathbb{N}$ such that $n\ge 1/\xi_2$. Indeed, for $0\le v_1<v_2\le\frac{1}{n}\le\xi_2$,

$$\begin{aligned}\theta_n(v_2)-\theta_n(v_1)&=(v_2-v_1)-(\tilde g_n(v_2)-\tilde g_n(v_1))\\ &=(v_2-v_1)(1-\tilde g_L)>0.\end{aligned}$$

Now, if $\frac{1}{n}\le v_1<v_2\le\xi_2$, then using strict monotonicity of θ, we have that

$$\theta(v_2)-\theta(v_1)=(v_2-v_1)-(\tilde g(v_2)-\tilde g(v_1))>0.$$

Thus $\tilde g(v_2)-\tilde g(v_1)<(v_2-v_1)$. On the other hand, we claim that

$$\tilde g_n(v_2)-\tilde g_n(v_1)\le\tilde g(v_2)-\tilde g(v_1).$$

Indeed, let us consider the measurable subsets of $X' := X \setminus \{\tau_0\}$:

$$A_j := \{\tau \in X' : g(v_j, \tau) \leq l(\tau)/n\}, \quad B_j := \{\tau \in X' : g(v_j, \tau) > l(\tau)/n\}.$$

Note that $B_j = X' \setminus A_j$. Since for each $\tau \in X'$, $g(v, \tau)$ is increasing in $v > 0$, we have $A_2 \subset A_1$. Thus $B_1 \subset B_2$ and $B_2 \setminus B_1 = A_1 \setminus A_2$. Consequently, $X' = B_1 \cup (B_2 \setminus B_1) \cup A_2$ is a disjoint union of three sets, and since

$$\int_{\mathbb{R}} \int_{A_2} (g_n(v_2, \tau) - g_n(v_1, \tau))\mathcal{N}(s, \tau) d\rho(\tau) ds = 0,$$

we get that

$$\begin{aligned}
\tilde{g}_n(v_2) - \tilde{g}_n(v_1) &= \int_{\mathbb{R}} \int_{B_2 \setminus B_1} (g(v_2, \tau) - l(\tau)/n)\mathcal{N}(s, \tau) d\rho(\tau) ds \\
&\quad + \int_{\mathbb{R}} \int_{B_1} (g(v_2, \tau) - g(v_1, \tau))\mathcal{N}(s, \tau) d\rho(\tau) ds \\
&\leq \int_{\mathbb{R}} \int_{B_2} (g(v_2, \tau) - g(v_1, \tau))\mathcal{N}(s, \tau) d\rho(\tau) ds \\
&\leq \int_{\mathbb{R}} \int_{X'} (g(v_2, \tau) - g(v_1, \tau))\mathcal{N}(s, \tau) d\rho(\tau) ds \\
&= \tilde{g}(v_2) - \tilde{g}(v_1).
\end{aligned}$$

Hence $\theta_n(v_2) - \theta_n(v_1) \geq (v_2 - v_1) - (\tilde{g}(v_2) - \tilde{g}(v_1)) > 0$.

Finally, if $v_1 \leq \frac{1}{n} < v_2 \leq \xi_2$, then

$$\theta_n(v_2) - \theta_n(v_1) = (\theta_n(v_2) - \theta_n(1/n)) + (\theta_n(1/n) - \theta_n(v_1)) > 0.$$

by the above. Hence θ_n are strictly increasing on $[0, \xi_2]$ for all $n > 1/\xi_2$.

On the other hand, note that $\lim_{n\to\infty} g_n(s, \tau) = g(s, \tau)$ uniformly on $\mathbb{R}_+$ for every fixed $\tau \in X$. Since $g_{n+1}(s, \tau) \leq g_n(s, \tau), n \in \mathbb{N}$, $\{\tilde{g}_n\}$ is a decreasing sequence of measurable nonnegative functions. Now, for each fixed $v \geq 0$ we have $\lim_{n\to\infty} \tilde{g}_n(v) = \tilde{g}(v)$, and, by virtue of the Dini's Theorem, the convergence is uniformly on compact sets. We thus get $\lim_{n\to\infty} \theta_n(\xi_2) = \theta(\xi_2)$, and

$$\theta_n(\xi_2) > \max_{v\geq 0} g(v, \tau_0) \int_{\mathbb{R}} \mathcal{N}(s, \tau_0) ds = C \max_{v\geq 0} g(v, \tau_0),$$

for n is sufficiently large, see the hypothesis (N). Finally, the uniform convergence of $g_n(s, \tau_0)$ to $g(s, \tau_0)$ on $\mathbb{R}_+$, allows conclude that $\theta_n(\xi_2) > C \max_{v\geq 0} g_n(v, \tau_0)$, and consequently $g_n(s, \tau)$ satisfy the hypothesis (N) for each large n.

In particular, for each n sufficiently large, we can define the functions $G_n(v) := \theta_n^{-1}(Cg_n(v,\tau_0))$, $v \in [0,\xi_2]$. Note that $G_n(0) = 0$, $0 < G_n(v) < \xi_2$, $v > 0$. Moreover, from the uniform convergence of the sequences $g_n(s,\tau)$ and $\tilde{g}_n(v)$, we have that $G_n(v)$ converge to $G(v)$ uniformly on $[0,\xi_2]$, and finally, since $\tilde{g}_n'(0) = \tilde{g}_L < 1$, we get that

$$G_n'(0) = \frac{Cg_n'(0,\tau_0)}{1-\tilde{g}_L} = \frac{Cl(\tau_0)}{1-\tilde{g}_L} < \infty.$$

On the other hand, for each $n \in \mathbb{N}$ the characteristic function $\chi_n(z)$ satisfies

$$\begin{aligned}
\chi_n(z) &= 1 - \int_{\mathbb{R}}\int_X \mathcal{N}(s,\tau)g_n'(0,\tau)d\rho(\tau)e^{-sz}ds \\
&= 1 - \int_{\mathbb{R}}\int_X \mathcal{N}(s,\tau)l(\tau)d\rho(\tau)e^{-sz}ds \\
&= \chi_L(z).
\end{aligned}$$

Since $\chi_-(0) < 0$, we have $\chi_n(0) = \chi_L(0) < 0$, so that

$$1 - \int_{\mathbb{R}}\int_{X\setminus\{\tau_0\}} l(\tau)\mathcal{N}(s,\tau)d\rho(\tau)ds - Cl(\tau_0) = 1 - \tilde{g}_L - Cl(\tau_0) < 0,$$

and $G_n'(0) > 1$. Hence $G_n(v) = G_n'(0)v$ for $v \in [0,\delta_n)$ for some $\delta_n > 0$.

Now, suppose that $G_n(a) = a$ for some $a \in [0,\xi_2]$. Since $g_n(v,\tau) \geq g(v,\tau)$, $v \geq 0$, we have

$$a = Cg_n(a,\tau_0) + \tilde{g}_n(a) \geq Cg(a,\tau_0) + \tilde{g}(a).$$

We thus get $\theta(a) \geq Cg(a,\tau_0)$ and $G(a) \leq a$. From Lemma 2.3 we have $G(v) > v$ for each $v \in (0,\xi_1]$, and hence $a > \xi_1$. Since $G_n(v) > v$ for $v \in [0,\delta_n)$, then $G_n(v) > v$ for each $v \in (0,\xi_1]$.

Now, by Lemma 2.3, we have that $\theta^{-1}(Cg(s,\tau_0)) \geq \theta^{-1}(Cg(\xi_1,\tau_0))$, $s \in [\xi_1,\xi_2]$. Thus $g(s,\tau_0) \geq g(\xi_1,\tau_0)$ for each $s \in [\xi_1,\xi_2]$. In addition, the condition $g_n(v,\tau_0) \geq g(v,\tau_0)$, $v \geq 0$, yields

$$\theta_n^{-1}(Cg_n(v,\tau_0)) \geq \theta_n^{-1}(Cg(v,\tau_0)), v \in [0,\xi_2].$$

Thus for $v \in [\xi_1,\xi_2]$ we have that $\theta_n^{-1}(Cg_n(v,\tau_0)) \geq \theta_n^{-1}(Cg(\xi_1,\tau_0))$. On the other hand, note that for each n sufficiently large $g_n(\xi_1,\tau_0) = g(\xi_1,\tau_0)$, so that

$$G_n(v) = \theta_n^{-1}(Cg_n(v,\tau_0)) \geq \theta_n^{-1}(Cg_n(\xi_1,\tau_0)) = G_n(\xi_1) = G(\xi_1),\ v \in [\xi_1,\xi_2].$$

Hence $\min_{s\in[\xi_1,\xi_2]} G_n(s) = G_n(\xi_1) = G(\xi_1)$ for each n large. This ends the proof of the claim.

Consequently, since $g_n(s,\tau) \le g_n'(0,\tau)s$ for all $s \ge 0$ and g_n satisfy all hypotheses of Preposition 2.1 for all n sufficiently large, with ξ_1 and ξ_2 as in Lemma 2.3, which do not depend of n, there exist positive continuos function ϕ_n such that

$$\phi_n(t) = \int_X d\rho(\tau) \int_{\mathbb{R}} \mathcal{N}(s,\tau) g_n(\phi_n(t-s),\tau) ds, \tag{2.10}$$

such that $\phi_n(t) \le \xi_2$, $t \in \mathbb{R}$, and if $\chi_n(z) = \chi_L(z)$ changes sign on some open interval $(0,\bar{w})$, the functions ϕ_n satisfy the boundary conditions $\phi_n(-\infty) = 0$ and $\liminf_{t\to+\infty} \phi_n(t) > \xi_1$, and if $\chi_n(z)$ changes sign on some open interval $(-\bar{w},0)$, then $\phi_n(+\infty) = 0$ and $\liminf_{t\to-\infty} \phi_n(t) > \xi_1$.

Step II. Now we will study the convergence of the sequence ϕ_n on compact sets. First, note that the functions $\phi_n(t+t_0)$ also satisfy equation (2.10), hence we can assume that $\phi_n(0) = \xi_1/2$. On the other hand, since $\{\phi_n\}$ is uniformly bounded, Lebesgue's dominates convergence theorem, continuity of the translation in $L_1(\mathbb{R})$ and the estimation

$$\begin{aligned}
|\phi_n(t+h) - \phi_n(t)| &\le \int_X l(\tau) d\rho(\tau) \int_{\mathbb{R}} |\mathcal{N}(t+h-u,\tau) - \mathcal{N}(t-u,\tau)| \phi_n(u) du \\
&= \int_X l(\tau) d\rho(\tau) \int_{\mathbb{R}} |\mathcal{N}(h+s,\tau) - \mathcal{N}(s,\tau)| \phi_n(t-s) ds \\
&\le \xi_2 \int_X l(\tau) d\rho(\tau) \int_{\mathbb{R}} |\mathcal{N}(s+h,\tau) - \mathcal{N}(s,\tau)| ds \to 0,\ h \to 0,
\end{aligned}$$

imply that the sequence $\{\phi_n\}$ is equicontinuous on $\mathbb{R}$. Therefore there exists a subsequence $\{\phi_{n_j}\}$ which converges uniformly on compact sets to some bounded function $\phi \in C(\mathbb{R},\mathbb{R})$, by the Ascoli-Arzelà Theorem. Note that Lebesgue's dominated convergence theorem implies that ϕ satisfies equation (1.3). Consequently, $0 \le \phi(t) \le \xi_2$, $t \in \mathbb{R}$ and $\phi(0) = \xi_1/2$. Thus $\inf_{t\in\mathbb{R}} \phi(t) < \xi_1$, and Theorem 2.6 shows that $\phi(-\infty) = 0$ and $\liminf_{t\to+\infty} \phi(t) > \xi_1$, if $\chi_n(z)$ changes sign on some open interval $(0,\bar{w})$, and if $\chi_n(z)$ changes sign on some open interval $(-\bar{w},0)$, then $\phi(+\infty) = 0$ and $\liminf_{t\to-\infty} \phi(t) > \xi_1$.

Finally, set $m' := \liminf_{t\to w} \phi(t) \le \limsup_{t\to w} \phi(t) =: M'$, $w \in \{-\infty,+\infty\}$

and suppose that $m' > \xi_1$. Then repeating the proof of Lemma 10 in [11], we get $[m', M'] \subseteq G([m', M'])$, and hence $m', M' \in (\xi_1, \xi_2]$. In addition, since the equation $G(s) = s$ has exactly two solutions 0 and κ on $\mathbb{R}_+$, we obtain that $\kappa \in (\xi_1, \xi_2]$. Moreover, since κ is globally attracting for the map $G : (0, \xi_2] \to (0, \xi_2]$, and

$$[m', M'] \subseteq G([m', M']) \subseteq G^2([m', M']) \subseteq \cdots \subseteq G^n([m', M']) \subseteq \cdots,$$

where $G^n := G \circ \cdots \circ G$ (n times), then $\lim_{n\to\infty} G^n([m', M']) = \kappa$ and thus, $m' = M' = \kappa$, and finally, we have $\phi(w) = \kappa$. This completes the proof. $\square$

3. Semi-wavefront solutions for non-local delayed reaction-diffusion equation (1.1)

In this section, we study the problem of existence and non-existence of semi-wavefront solutions for (1.1) using the framework developed in the Section 2. We will rewrite equation (1.1) in the form (1.3) in order to apply Theorem 2.4 and prove the existence of solutions $u(t, x) = \phi(x + ct)$ for some range of admissible speeds.

Everywhere in this section, we are assuming the hypotheses H_0-H_2.

3.1. Modification of the equation (1.1)

Note that the profile ϕ must satisfy the equation

$$y''(t) - cy'(t) - f(y(t)) + \int_0^\infty \int_{\mathbb{R}} K(s, w) g\left(y(t - cs - w)\right) dw ds = 0 \tag{3.1}$$

for all $t \in \mathbb{R}$. This equation can be written as

$$y''(t) - cy'(t) - \beta y(t) + f_\beta(y(t)) + \int_0^\infty \int_{\mathbb{R}} K(s, w) g(y(t - cs - w)) dw ds = 0, \tag{3.2}$$

where $f_\beta(s) = \beta s - f(s)$ and $\beta > 0$. Clearly, now we have to prove the existence of positive bounded solution ϕ of equation (3.1), satisfying $\phi(-\infty) = 0$ or $\phi(+\infty) = 0$.

Next, if ϕ is a positive bounded solution to (3.1), it should satisfy the integral equation

$$\begin{aligned}\phi(t) &= \frac{1}{\sigma(c)}\left(\int_{-\infty}^{t} e^{\nu(c)(t-s)}(\mathcal{G}\phi)(s)ds + \int_{t}^{+\infty} e^{\mu(c)(t-s)}(\mathcal{G}\phi)(s)ds\right) \\ &= \int_{\mathbb{R}} k_1(t-s)(\mathcal{G}\phi)(s)ds,\ t\in\mathbb{R},\end{aligned} \tag{3.3}$$

where

$$k_1(s) := (\sigma(c))^{-1}\begin{cases} e^{\nu(c)s}, & s\geq 0 \\ e^{\mu(c)s}, & s<0\end{cases},$$

$\sigma(c) = \sqrt{c^2+4\beta}$, $\nu(c) < 0 < \mu(c)$ are the roots of $z^2 - cz - \beta = 0$ and the operator $\mathcal{G}$ is defined as

$$(\mathcal{G}\phi)(t) := \int_0^\infty \int_{\mathbb{R}} K(s,w)g(\phi(t-cs-w))dwds + f_\beta(\phi(t)).$$

Note that $(\mathcal{G}\phi)(t)$ can be rewritten as

$$\begin{aligned}(\mathcal{G}\phi)(t) &= \int_{\mathbb{R}} g(\phi(t-r))\int_0^\infty K(s,r-cs)dsdr + f_\beta(\phi(t)) \\ &= \int_{\mathbb{R}} g(\phi(t-r))k_2(r)dr + f_\beta(\phi(t)),\end{aligned} \tag{3.4}$$

where

$$k_2(r) := \int_0^\infty K(s,r-cs)ds$$

is well defined a.e. on $\mathbb{R}$. Finally, from (3.4) we get that ϕ also must satisfy the equation

$$\phi(t) = (k_1 * k_2) * g(\phi)(t) + k_1 * f_\beta(\phi)(t),\ t\in\mathbb{R}, \tag{3.5}$$

where $*$ denotes the convolution $(f*g)(t) = \int_{\mathbb{R}} f(t-s)g(s)ds$.

Equation (3.5) can be rewritten as

$$\phi(t) = \int_X d\rho(\tau)\int_{\mathbb{R}} \mathcal{N}(s,\tau)g(\phi(t-s),\tau)ds,\ t\in\mathbb{R}, \tag{3.6}$$

with

$$\mathcal{N}(s,\tau) = \begin{cases} (k_1*k_2)(s), & \tau=\tau_0, \\ k_1(s), & \tau=\tau_1,\end{cases} \qquad g(s,\tau) = \begin{cases} g(s), & \tau=\tau_0, \\ f_\beta(s), & \tau=\tau_1,\end{cases}, \tag{3.7}$$

$X = \{\tau_0, \tau_1\}$ and $\rho(\tau_0) = \rho(\tau_1) = 1$. Note that the function $g(\cdot, \cdot)$ is continuous on $\mathbb{R} \times X$ with $g(0, \tau) \equiv 0$ and $\mathcal{N}(s, \tau)$ is integrable on $\mathbb{R} \times X$ with

$$\int_{\mathbb{R}} \mathcal{N}(s, \tau) ds = \frac{1}{\beta} > 0, \, \tau \in X.$$

Set now

$$\bar{f}_\beta(s) = \max\{f_\beta(t) : 0 \le t \le s\}, s \ge 0,$$

$$\bar{g}(s, \tau) = \begin{cases} g(s), & \tau = \tau_0, \\ \bar{f}_\beta(s), & \tau = \tau_1, \end{cases} \tag{3.8}$$

and consider the modified convolution equation

$$\phi(t) = \int_X d\rho(\tau) \int_{\mathbb{R}} \mathcal{N}(s, \tau) \bar{g}(\phi(t - s), \tau) ds, \, t \in \mathbb{R}. \tag{3.9}$$

Observe that for every $M > 0$ we can choose $\beta > 0$ sufficiently large such that

$$\bar{f}_\beta(s) = f_\beta(s) \ge 0, \, s \in [0, M].$$

Thus a solutions family $\{\phi_\beta\}$ of (3.9), uniformly bounded by some constant $M > 0$, where $\beta(M)$, is also a family of bounded solutions of (3.6).

Lemma 3.1. *If f satisfies the condition H_2, then for each $M > 0$ there exists $\beta > 0$ such that $\bar{f}_\beta(s) = \beta s - f(s)$, $s \in [0, M]$. Furthermore, the function $\bar{f}_\beta$ is monotone increasing on $(0, +\infty)$, $\bar{f}_\beta(0) = 0$ and $\bar{f}_\beta(s) > 0, s > 0$. Finally, the Dini derivate $\bar{f}'_{\beta+}(0) = \beta - f'_-(0) > 0$ is finite and $\bar{f}_\beta(s) \le \bar{f}'_{\beta+}(0)s, s \ge 0$.*

Proof. Let $u \in [0, M]$. Since f is increasing and locally Lipschitzian function, there exists $\beta = \beta(M) > 0$ such that $f'_+(0) < \beta$ and

$$f(u) - f(s) \le \beta(u - s), \quad 0 \le s \le u.$$

In particular, $\bar{f}_\beta(u) = \beta u - f(u)$ for each $u \in [0, M]$, $\bar{f}_\beta(0) = 0$. Now, if $\bar{f}_\beta(s_0) = 0$ for some $s_0 > 0$, then $\bar{f}_\beta(t) = 0$ for all $0 \le t \le s_0$, which is impossible in view of $f'_+(0) = \beta - f'_-(0) > 0$. Thus, $\bar{f}_\beta(s) > 0$ for each $s > 0$. Finally, by hypothesis H_2, $f(s) \ge f'_-(0)s$ for all $s \ge 0$. This implies that

$$f_\beta(s) = \beta s - f(s) \le (\beta - f'_-(0))s \le (\beta - f'_-(0))u, \quad 0 \le s \le u.$$

In consequence, $\bar{f}_\beta(u) = \max\{f_\beta(t) : 0 \le t \le u\} \le (\beta - f'_-(0))u$ for each $u \ge 0$, which completes the proof. □

Lemma 3.2. *Assume that H_0-H_2 hold. If $\mathcal{N}(s,\tau)$ and $\bar{g}(v,\tau)$ are defined as in (3.7) and (3.8), respectively, then all the conditions of hypotheses (N), (L) and (M) are true with $\xi_2 = \bar{\xi}_2$. Moreover, $\bar{g}'_-(0+,\tau) > 0$ for each τ.*

Proof. First, consider $\beta = \beta(\bar{\xi}_2) > 0$ sufficiently large such that $\beta > f'_+(0)$. Then the function $\bar{g}(\cdot,\cdot)$ is continuous on $\mathbb{R} \times X$ with $\bar{g}(0,\tau) \equiv 0$ and $\bar{g}(s,\tau) > 0$ for each $(s,\tau) \in \mathbb{R}_+ \times X$ (see Lemma 3.1). Note that $\rho(\tau_0) = 1$. In addition, since $\bar{g}(v,\tau_1) = \bar{f}_\beta(v)$, we have

$$\tilde{g}(v) = \int_{\mathbb{R}} g(v,\tau_1)\mathcal{N}(s,\tau_1)ds = \bar{f}_\beta(v)\int_{\mathbb{R}} k_1(s)ds = \frac{\bar{f}_\beta(v)}{\beta},$$

and hence $\bar{g}(v,\tau_1)$ and $\tilde{g}(v)$ are monotone increasing on $\mathbb{R}_+$. Moreover, the function $\theta(v) = v - \frac{\bar{f}_\beta(v)}{\beta} = \frac{f(v)}{\beta}$ is strictly increasing on $[0,\bar{\xi}_2]$, where $f(\bar{\xi}_2) > \sup_{s\ge 0} g(s)$, by H_2. Note also that

$$\theta(\bar{\xi}_2) = \frac{f(\bar{\xi}_2)}{\beta} > \frac{1}{\beta}\sup_{v\ge 0} g(v) = \sup_{v\ge 0}\bar{g}(v,\tau_0)\int_{\mathbb{R}}\mathcal{N}(s,\tau_0)ds.$$

Hence, hypothesis (N) is satisfied with $\xi_2 = \bar{\xi}_2$.

Next, since g is bounded and $g'_+(0) < \infty$, we can find some constant $L > g'_+(0)$ such that

$$g(s) \le Ls,\ \ s \ge 0. \tag{3.10}$$

Thus, from Lemma 3.1 and (3.10) we conclude that

$$\bar{g}(s,\tau) \le l(\tau)s, s \ge 0, \tag{3.11}$$

where

$$l(\tau) = \begin{cases} L, & \tau = \tau_0, \\ \beta - f'_-(0), & \tau = \tau_1. \end{cases} \tag{3.12}$$

Note that

$$\int_X l(\tau)d\rho(\tau)\int_{\mathbb{R}}\mathcal{N}(s,\tau)ds = 1 + \frac{L - f'_-(0)}{\beta} < \infty$$

and

$$\tilde{g}_L = \int_{\mathbb{R}} \int_{X\setminus\{\tau_0\}} \mathcal{N}(s,\tau) l(\tau) d\rho(\tau) ds = \frac{(\beta - f'_-(0))}{\beta} < 1.$$

Thus (L) and (M) are also satisfied. Finally, since $g'_-(0), \bar{f}'_{\beta-}(0) > 0$, it is clear that $\bar{g}'_-(0+,\tau) > 0$ for each $\tau \in \{\tau_0, \tau_1\}$.

□

3.2. The characteristic equations

First, we consider the equation

$$\mathcal{R}(z,c) := z^2 - cz - q + p\int_0^\infty \int_{\mathbb{R}} K(s,w) e^{-z(cs+w)} dwds = 0,$$

where $p > q > 0$.

Lemma 3.3. *Suppose that for each $c \in \mathbb{R}$, the function $\mathcal{R}(z,c)$ is defined for all z from some maximal interval $(\delta_1(c), \delta_2(c)) \ni 0$. Then there exist $c^-_{\#} = c^-_{\#}(\mathcal{R}), c^+_{\#} = c^+_{\#}(\mathcal{R}) \in \mathbb{R}$ such that $c^-_{\#} < c^+_{\#}$ and the following statements are true:*

- (i) *If $c > c^+_{\#}$, then $\mathcal{R}(z,c)$ has at least one positive zero $z = \lambda_1(c) \in (0, \delta_2(c))$, it may have at most two positive zeros on $(0, \delta_2(c))$ and it does not have any negative zero. If $c < c^+_{\#}$, then $\mathcal{R}(z,c)$ does not have any positive zero on $(0, \delta_2(c))$. Furthermore, if $c = c^+_{\#}$ and $\lim_{z\uparrow\delta_2(c^+_{\#})} \mathcal{R}(z, c^+_{\#}) \neq 0$, then $\mathcal{R}(z, c^+_{\#})$ has a unique double zero on $(0, \delta_2(c^+_{\#}))$, denoted by $z = \lambda_1(c^+_{\#})$, and $\mathcal{R}(z, c^+_{\#}) > 0$ for all $z \neq \lambda_1(c^+_{\#}) \in [0, \delta_2(c^+_{\#}))$.*
- (ii) *If $c < c^-_{\#}$, the function $\mathcal{R}(z,c)$ has at least one negative zero $z = \lambda_1(c) \in (\delta_1(c), 0)$, it may have at most two negative zeros on $(\delta_1(c), 0)$ and it does not have any positive zero. If $c > c^-_{\#}$, then $\mathcal{R}(z,c)$ does not have any negative zero on $(0, \delta_1(c))$. Furthermore, if $c = c^-_{\#}$ and $\lim_{z\downarrow\delta_1(c^-_{\#})} \mathcal{R}(z, c^-_{\#}) \neq 0$, then $\mathcal{R}(z, c^-_{\#})$ has a unique double zero on $(0, \delta_1(c^-_{\#}))$, denoted by $z = \lambda_1(c^-_{\#})$, and $\mathcal{R}(z, c^-_{\#}) > 0$ for all $z \neq \lambda_1(c^-_{\#}) \in (\delta_1(c^-_{\#}), 0]$.*

Proof. First, we observe that the existence of $c^+_{\#}$ is given in [1, Lemma 3.1]. Now, to prove the existence of $c^-_{\#}$ we define the function $\mathcal{W}(z,c) := \mathcal{R}(-z,c)$.

Observe that λ is a root of $\mathcal{W}(z,c)=0$ if and only if $-\lambda$ is a root of $\mathcal{R}(z,c)=0$. Since

$$\mathcal{W}(z,c) = z^2 + cz - q + p\int_0^\infty \int_{\mathbb{R}} K(s,-w)e^{-z(-cs+w)}dwds, \tag{3.13}$$

[1, Lemma 3.1] implies that there exists the minimal speed $c^+_\#(\mathcal{W})$. Therefore taking $c^-_\#(\mathcal{R}) := -c^+_\#(\mathcal{W})$ we establishes the statements given in (ii) of the lemma. Finally, it is clear that $c^-_\# < c^+_\#$, and this complete the proof. □

Set now

$$\chi_0(z,c) := z^2 - cz - f'_-(0) + g'_-(0)\int_0^\infty \int_{\mathbb{R}} K(s,w)e^{-z(cs+w)}dwds,$$

and

$$\chi_L(z,c) := z^2 - cz - f'_-(0) + L\int_0^\infty \int_{\mathbb{R}} K(s,w)e^{-z(cs+w)}dwds.$$

Since hypothesis H_2 implies that $g'_-(0) > f'_-(0) > 0$, Lemma 3.3 guarantees the existence of c^+_* [respectively, $c^+_\star$] which is the minimal value of c for which $\chi_0(z,c)=0$ [respectively, $\chi_L(z,c)=0$] has at least one positive root. Similarly, there exist c^-_* [respectively, $c^-_\star$] which is the maximal value of c for which $\chi_0(z,c)=0$ [respectively, $\chi_L(z,c)=0$] has at least one negative root. Note that $c^+_\star \geq c^+_*$, $c^-_\star \leq c^-_*$, because of $L \geq g'_-(0)$.

On the other hand, for some $\beta > 0$ we have

$$\begin{aligned}
\chi_-(z) &= 1 - g'_-(0)\int_{\mathbb{R}} \mathcal{N}(s,\tau_1)e^{-zs}ds - (\beta - f'_-(0))\int_{\mathbb{R}} \mathcal{N}(s,\tau_2)e^{-zs}ds \\
&= 1 - \frac{\beta - f'_-(0)}{\beta + cz - z^2} - \frac{g'_-(0)}{\beta + cz - z^2}\int_0^\infty \int_{\mathbb{R}} K(r,w)e^{-z(rc+w)}dwdr \\
&= -\frac{\chi_0(z,c)}{\beta + cz - z^2}.
\end{aligned} \tag{3.14}$$

Thus the zeros of function $\chi_-(z)$ are determined by the roots of characteristic equation $\chi_0(z,c)=0$. Note also that

$$\chi_-(0) = -\frac{\chi_0(0,c)}{\beta} = \frac{f'_-(0) - g'_-(0)}{\beta} < 0. \tag{3.15}$$

Similarly, we obtain that

$$\chi_L(z) = -\frac{\chi_L(z,c)}{\beta + cz - z^2},\ \chi_L(0) = \frac{f'_-(0) - L}{\beta} < 0. \tag{3.16}$$

Our next result shows some estimates for admissible wave speeds. Set

$$E := -\frac{g'_-(0)\int_0^\infty \int_{\mathbb{R}} K(s,w)w\,dwds}{1 + g'_-(0)\int_0^\infty \int_{\mathbb{R}} K(s,w)s\,dwds}.$$

Note that if $K(s,w) = K(s,-w),\ w \in \mathbb{R}$, then $E = 0$.

Lemma 3.4. *Suppose that* $H_0 - H_2$ *hold. Then* $c_*^+ > E > c_*^-$. *Moreover, if*

$$\int_0^\infty \int_{\mathbb{R}} K(s,w)w\,dwds \leq 0, \tag{3.17}$$

then $c_*^+ > 0$. *If* K *does not satisfy the condition (3.17), then* $c_*^- < 0$.

Proof. Let $c \geq c_*^+$. Then Lemma 3.3 and the convexity of χ_0 with respect to z guarantee that $\chi_0'(0,c) < 0$, and therefore

$$c\Big(1 + g'_-(0)\int_0^\infty \int_{\mathbb{R}} K(s,w)s\,dwds\Big) > -g'_-(0)\int_0^\infty \int_{\mathbb{R}} K(s,w)w\,dwds,$$

which gives $c > E$. Thus $c_*^+ > E$. Similarly, if $c \leq c_*^-$, then $\chi_0'(0,c) > 0$. From this it follows that $c < E$, and so $c_*^- < E$. Now, if condition (3.17) is valid, then $c_*^+ > E \geq 0$. Finally, if (3.17) is not true, then $c_*^- < E < 0$. □

Example 3.5. We consider a space structured population with maturation effects described by the delays, for example marine species, where the juveniles move by advection as well as diffusion, but the adults move by diffusion only. If $u(t,x)$ denotes the density of the adult population at $x \in \mathbb{R}$ and time t, then the evolution of $u(t,x)$ is described following model:

$$u_t(t,x) = d_a u_{xx}(t,x) - \mu_a u(t,x) + \int_0^\infty \int_{\mathbb{R}} g(u(t-s,x-w))\frac{\mu_j e^{\frac{-(w+v_js)^2}{4d_js} - \mu_j s}}{2\sqrt{\pi d_j s}}dwds, \tag{3.18}$$

where g is the birth function, d_j, v_j, μ_j are respectively the diffusion rate, the advection velocity and the death rate for juveniles and d_a, μ_a are respectively the diffusion rate and the death rate for adults (see [11] for more details). Note that spatial asymmetry occurs in this model with

$$K(s,w) = \frac{\mu_j e^{\frac{-(w+v_j s)^2}{4d_j s} - \mu_j s}}{2\sqrt{\pi d_j s}}.$$

By scaling the variables, we can suppose that $d_a = 1$. Thus the characteristic function $\chi(z,c)$ takes the following form

$$\chi(z,c) = z^2 - cz - \mu_a + p\int_0^{\infty}\int_{\mathbb{R}} \frac{\mu_j e^{\frac{-(w+v_j s)^2}{4d_j s} - \mu_j s}}{2\sqrt{\pi d_j s}} e^{-z(cs+w)} dwds,$$

where $p := g'_-(0) > \mu_a$. A straightforward calculation of the integral gives

$$\chi(z,c) = z^2 - cz - \mu_a + \frac{p\mu_j}{\mu_j + (c - v_j)z - d_j z^2}.$$

Note that $\chi(0,c) = p - \mu_a > 0$ and $\lim_{c\downarrow -\infty} \chi(z,c) = +\infty$ for $z \in (0,+\infty)$. In addition,

$$\frac{\partial^2 \chi}{\partial z^2}(z,c) > 0, \quad z \in [0,+\infty),$$

and hence it has at most two real zeros for each c. Moreover, the kernel of equation (3.18) satisfies condition (3.17). In consequence, Lemma 3.4 implies that $c_*^+ > 0$. For example, in the particular case when $v_j = 0.02$, $d_j = 100$, $\mu_j = 0.001$, $\mu_a = 0.05$ and $p = 2$, we obtain the critical speeds $c_*^- = -2.82483\cdots$ and $c_*^+ = 2.85797\cdots$ (see Figures 1 - 4). Observe that $|c_*^-| \neq |c_*^+|$.

We now show possible situations between $c_*^{\pm}$ and $c_\star^{\pm}$. Furthermore, if we assume also that g satisfies the condition (3.10) with $L = 2.7 > g'_+(0)$, we obtain the critical speeds $c_\star^- = -3.12959\cdots$ and $c_\star^+ = 3.1623\cdots$. In consequence,

$$c_\star^- < c_*^- < 0 < c_*^+ < c_\star^+,$$

and hence there are not semi-wavefronts to equation (3.18) propagating at the velocity $c \in (-2.82483\cdots, 2.85797\cdots)$ (see proof of Theorem 1.2). In particular, equation (3.18) does not have stationary semi-wavefronts. Now, under the same conditions stated above, but replacing v_j by $v_j = 4.7$, we obtain $c_*^- = 0.376044\cdots$, $c_*^+ = 6.99332\cdots$, $c_\star^- = 0.0332958\cdots$ and $c_\star^+ = 7.25469\cdots$.

Thus

$$0 < c_\star^- < c_*^- < c_*^+ < c_\star^+,$$

and hence, in this case, there is at least one stationary semi-wavefront to equation (3.18) (see Theorem 3.10). Moreover, if $0 < c \leq c_\star^-$, then $\phi(+\infty) = 0$ and, in consequence, the extinction of semi-wavefront $u(x,t) = \phi(x+ct)$ occurs.

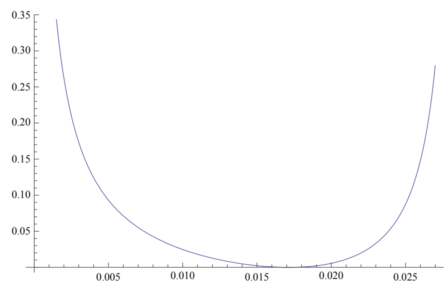

Figure 1: $\chi(z,c)$ for $c = 2.85797\cdots$.

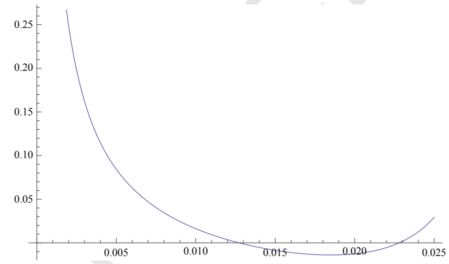

Figure 2: $\chi(z,c)$ for $c = 3$.

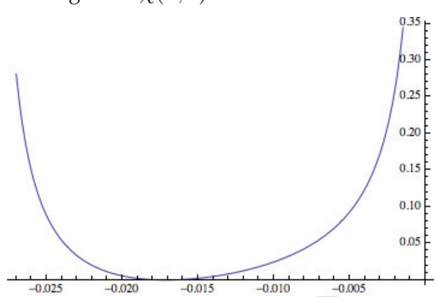

Figure 3: $\chi(z,c)$ for $c = -2.82483\cdots$.

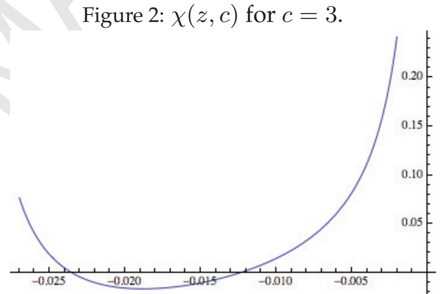

Figure 4: $\chi(z,c)$ for $c = -3$.

3.3. Separation properties and non-existence of wavefronts

First, we establish the following simple fact:

Lemma 3.6. *Suppose that H_0-H_2 hold. If $u(t,x) = \phi(x+ct) \geq 0$ is a bounded solution of equation (3.1) such that ϕ vanishes at some point, then $\phi \equiv 0$.*

Proof. Let $M = \sup_{t\in\mathbb{R}} \phi(t)$. Then by Lemma 3.1 there exists $\beta = \beta(M)$ such that $\bar{f}_\beta(s) = f_\beta(s) \geq 0$ for all $s \in [0, M]$.

Now, suppose that there exists $t_0 \in \mathbb{R}$ such that $\phi(t_0) = 0$. From (3.3) we get that

$$\phi(t_0) = \int_{\mathbb{R}} k_1(t_0 - s)(\mathcal{G}\phi)(s)ds = 0.$$

Since $k_1(t) > 0$ and $(\mathcal{G}\phi)(t) \geq 0$ for all $t \in \mathbb{R}$, we necessarily have $(\mathcal{G}\phi)(t) = 0$ for all $t \in \mathbb{R}$. According to $g(0) = 0$, $g(t) > 0, t > 0$, and $K \geq 0$, we get that

$$\int_{\mathbb{R}} g(\phi(t-r))k_2(r)dr \geq 0,\ t \in \mathbb{R}.$$

In addition, since $f_\beta(s) \geq 0$ for all $s \in [0, M]$, we deduce from (3.4) that

$$\int_{\mathbb{R}} g(\phi(t-r))k_2(r)dr = f_\beta(\phi(t)) = 0,\ t \in \mathbb{R}.$$

Thus $\phi(t) = 0$ for all $t \in \mathbb{R}$, and the lemma follows. □

We can now prove the non-existence assertion of Theorem 1.2.

Proof Theorem 1.2. Take c_*^+, c_*^- as defined in Section 3.2.

Next, suppose that for some $c \in (c_*^-, c_*^+)$ there exists a semi-wavefront solution ϕ of (3.1) propagating with the speed c. Let $M = \sup_{t\in\mathbb{R}} \phi(t)$. Then by Lemma 3.1 there exists $\beta = \beta(M) > f'_+(0)$ such that $\bar{g}(v,\tau) = g(v,\tau)$, $v \in [0, M]$, $\tau \in X$. Moreover, Lemma 3.2 implies that hypothesis (L) holds and that $g'_-(0+,\tau) > 0$ for each $\tau \in X$. We also observe that $\chi_-(0) < 0$, by (3.15). Finally, ϕ is a bounded solution of equation (3.6) with β defined as above, and hence all the hypotheses of Lemma 2.5 hold.

Now, if $\phi(-\infty) = 0$, then Lemmas 3.6 and 2.5 imply that $\chi_-(z)$ is well defined on some $(0, \gamma_1]$ and $\chi_-(z') = 0$ for some $z' \in (0, \gamma_1]$. In view of (3.14), this yields $\chi_0(z', c) = 0$, which contradicts the minimality of c_*^+. If $\phi(+\infty) = 0$, then from Lemmas 3.6 and 2.5 we also obtain that $\chi_-(z)$ is well defined on some $[\gamma_2, 0)$ and $\chi_-(z'') = 0$ for some $z'' \in [\gamma_2, 0)$. Thus $\chi_0(z'', c) = 0$, which contradicts the maximality of c_*^-. □

Theorem 3.7. *Assume that* H_0-H_2 *hold. Let* $u(t,x) = \phi(x+ct) > 0$ *be a bounded solution of equation (3.1).*

1. *If* $\liminf_{t\to-\infty}\phi(t) = 0$, *then* $\phi(-\infty) = 0$, *the critical speed* c_*^+ *is finite and* $c \geq c_*^+$. *Moreover,* $\liminf_{t\to+\infty}\phi(t) \geq \delta_1(\phi) > 0$ *for some* $\delta_1(\phi) > 0$.
2. *If* $\liminf_{t\to+\infty}\phi(t) = 0$, *then* $\phi(+\infty) = 0$, *the critical speed* c_*^- *is finite and* $c \leq c_*^-$. *Moreover,* $\liminf_{t\to-\infty}\phi(t) \geq \delta_2(\phi) > 0$ *for some* $\delta_2(\phi) > 0$.
3. *Equation (3.1) can not have positive pulse solutions, i.e. solutions satisfying* $\phi(-\infty) = \phi(+\infty) = 0$.

Proof. For $M = \max\{\bar{\xi}_2, \sup_{t\in\mathbb{R}}\phi(t)\}$ consider $\beta = \beta(M) > f'_-(0)$ given in Lemma 3.1. Then $\bar{g}(v,\tau) = g(v,\tau)$ on $[0, M]$ for each $\tau \in \{\tau_0, \tau_1\}$. In addition, by Lemma 3.2, conditions (N) and (L) hold, and $g'_-(0+,\tau) > 0$ for each τ. Note also that $\chi_-(0) < 0$.

On the other hand, since ϕ is a bounded solution of (3.6), then steps I and II of the proof in Theorem 2.6 imply that for some $w \in \{-\infty, +\infty\}$, either $\phi(w) = 0$ or $\liminf_{t\to w}\phi(t) > \delta(\phi) > 0$. If $\liminf_{t\to-\infty}\phi(t) = 0$, then $\phi(-\infty) = 0$ and $\chi_-(z)$ has a positive root, by Lemma 2.5. Thus c_*^+ is finite, $c \geq c_*^+$ and all real zeros of $\chi_-(z)$ are positive. By Lemma 2.5, $\phi(+\infty) \neq 0$, and hence $\liminf_{t\to+\infty}\phi(t) \geq \delta(\phi) > 0$. Now, if $\liminf_{t\to+\infty}\phi(t) = 0$, then $\phi(+\infty) = 0$ and $\chi_-(z)$ has a negative root, by Lemma 2.5. Thus c_*^- is finite, $c \leq c_*^-$ and all real zeros of $\chi_-(z) = 0$ are negative. From the same lemma we get that $\phi(-\infty) \neq 0$, and hence $\liminf_{t\to-\infty}\phi(t) \geq \delta(\phi) > 0$. In consequence, ϕ can not satisfy the boundary condition $\phi(-\infty) = \phi(+\infty) = 0$.

□

3.4. The existence problem

Theorem 3.8. *(Existence of semi-wavefronts) Let assumptions* H_0*-*H_2 *hold. Then the equation (3.1) has at least one semi-wavefront* $u(x,t) = \phi(x+ct)$ *propagating with speed* $c \geq c_\star^+$ *such that* $\phi(-\infty) = 0$ *and* $\liminf_{t\to+\infty}\phi(t) > 0$. *Furthermore, if equation* $f(s) = g(s)$ *has only two solutions:* 0 *and* κ, *with* κ *being globally attracting with respect to* $f^{-1}\circ g : (0, \bar{\xi}_2] \to (0, \bar{\xi}_2]$, *then* $\phi(+\infty) = \kappa$.

Proof. The proof will be divided into 3 steps.

Step I. Consider $\beta = \beta(\bar{\xi}_2) > f'_+(0)$. Then the assumptions H_0-H_2 and Lemma 3.2 ensure that $\bar{g}(v,\tau), \tau \in \{\tau_0, \tau_1\}$ satisfies all the conditions of the hypotheses (N), (L) and (M). Moreover, $\bar{g}'_-(0+,\tau) > 0$ for each τ. In addition, since Lemma 3.6 implies that hypothesis (P) is valid, all the assumptions of Theorem 2.4 are satisfied.

On the other hand, hypothesis H_0 implies that for each $c \in \mathbb{R}$, $\chi_L(z,c)$, and hence $\chi_L(z)$ and $\chi_-(z)$ are well defined on the maximal interval $(\gamma_1^{\#}(c), \gamma_2^{\#}(c))$ with $\gamma_1^{\#}(c) < 0 < \gamma_2^{\#}(c)$. Moreover, $\chi_L(0,c) < 0$, $\chi_L(0) < 0$ and $\chi_-(0) < 0$, by (3.15) and (3.16). Now, if $c > c_\star$, then $\chi_L(z,c)$ changes its sign on $(0, \gamma_2^{\#}(c))$, and hence $\chi_L(z)$ also changes its sign on $(0, \gamma_2^{\#}(c))$, by (3.16). Thus Theorem 2.4 implies that the equation (3.9) has at least one semi-wavefront ϕ satisfying $\sup_{s\in\mathbb{R}} \phi(s) \leq \bar{\xi}_2$, $\phi(-\infty) = 0$, and $\liminf_{t\to+\infty} \phi(t) > \xi_1$. Since $g(v,\tau) = \bar{g}(v,\tau)$ if $v \in [0, \bar{\xi}_2]$, then ϕ is also a semi-wavefronts solution to (3.6), and hence it is a semi-wavefronts solution to (3.1) propagating with speed $c > c_\star^+$.

Step II. For the case $c = c_\star^+$, we define $c_n := \frac{nc_\star^+ + 1}{n}$. Since $c_n > c_\star^+$, the previous result assures the existence of positive bounded solutions ψ_n to (3.1) such that $\sup_{s\in\mathbb{R}} \psi_n(s) \leq \bar{\xi}_2$, $\psi_n(-\infty) = 0$ and $\liminf_{t\to+\infty} \psi_n(t) > \xi_1$, for each $n \in \mathbb{N}$. Since $\psi_n(t+t_0)$ also satisfy the equation (3.1), we can assume that $\psi_n(0) = \xi_1/2$. In addition, from (3.3) we have

$$\psi_n(t) = \int_{\mathbb{R}} (k_1)_n(t-s)(\mathcal{G}_n\psi_n)(s)ds,\ t \in \mathbb{R}, \tag{3.19}$$

where

$$(k_1)_n(s) = (\sigma(c_n))^{-1} \begin{cases} e^{\nu(c_n)s}, & s \geq 0 \\ e^{\mu(c_n)s}, & s < 0 \end{cases}, \quad \sigma(c_n) = \sqrt{c_n^2 + 4\beta},$$

$\nu(c_n) < 0 < \mu(c_n)$ are the roots of $z^2 - c_n z - \beta = 0$ and the operators $\mathcal{G}_n$ are defined as

$$(\mathcal{G}_n\psi)(t) := \int_0^\infty \int_{\mathbb{R}} K(s,w)g(\psi(t - c_n s - w))dwds + f_\beta(\psi(t)).$$

Now, since $\sigma(c_n) > \sigma(c_\star^+)$, differentiating (3.19) we find that

$$|\psi_n'(t)| \leq \frac{1}{\sigma(c_\star^+)} \left(\sup_{u\geq 0} g(u) + (\beta - f'_-(0))\xi_2 \right).$$

In consequence, $\{\psi_n\}$ is pre-compact in the compact open topology of $C(\mathbb{R},\mathbb{R})$ and we can find a subsequence $\{\psi_{n_j}\}$ which converges, uniformly on compact subsets of $\mathbb{R}$, to some bounded function $\psi \in C(\mathbb{R},\mathbb{R})$. In addition, note that

$$\lim_{j\to+\infty}(\mathcal{G}_{n_j}\psi_{n_j})(t) = \int_0^\infty \int_{\mathbb{R}} K(s,w)g(\psi(t-c_\star^+ s - w))dwds + f_\beta(\psi(t)),$$

for every $t \in \mathbb{R}$. Thus Lebesgue's dominated convergence theorem implies that ψ is a solution of (3.1) propagating with the speed $c = c_\star^+$. Clearly, $\sup_{s\in\mathbb{R}}\psi(s) \leq \bar{\xi}_2$ and $\psi(0) = \xi_1/2$. Thus $\inf_{s\in\mathbb{R}}\psi(s) < \xi_1$, and from Theorem 2.6 we have $\psi(-\infty) = 0$ and $\liminf_{t\to+\infty}\psi(t) > \xi_1$.

Step III. If equation $f(s) = g(s)$ has only two solutions: 0 and κ, then only 0 and κ satisfy the equation $G(s) = f^{-1}(g(s)) = s$. Being κ globally attracting with respect to $G : (0,\bar{\xi}_2] \to (0,\bar{\xi}_2]$, from Theorem 2.4, with $\xi_2 = \bar{\xi}_2$, we have $\phi(+\infty) = \kappa$. $\square$

Remark 3.9. (a) Sufficient conditions to ensure the global stability of $f^{-1}\circ g$ are given in [24]. (b) If $g'_-(0) = L$, then $l(\tau) = g'_-(0,\tau)$ and $\chi_-(z) = \chi_L(z)$. Thus $c_\star^+ = c_*^+$, and the existence holds for each $c \geq c_*^+$.

Small changes in the previous proof allow to prove the follow theorem:

Theorem 3.10. *(Existence of semi-wavefronts) Let assumptions H_0-H_2 hold. Then the equation (3.1) has at least one semi-wavefront $u(x,t) = \phi(x+ct)$ propagating with speed $c \leq c_\star^-$ such that $\phi(+\infty) = 0$ and $\liminf_{t\to-\infty}\phi(t) > 0$. Furthermore, if equation $f(s) = g(s)$ has only two solutions: 0 and κ, with κ being globally attracting with respect to $f^{-1}\circ g : (0,\bar{\xi}_2] \to (0,\bar{\xi}_2]$, then $\phi(-\infty) = \kappa$.*

4. Applications.

In this section, we apply Theorem 1.1 to some non-local reaction-diffusion epidemic and population models with distributed time delay, previously studied in [4, 8, 12, 21, 23, 25, 27, 28, 30].

Epidemic dynamics model: Developing some ideas from [23, 27, 28, 30], we first consider here the following non-local system with distributed delay

$$\begin{cases} u_t(t,x) = du_{xx}(t,x) - f(u(t,x)) + \int_{\mathbb{R}} \mathcal{K}(x-y)v(t,y)dy, \\ \\ v_t(t,x) = -\alpha v(t,x) + \int_0^\infty g(u(t-s,x))P(ds), \end{cases} \tag{4.1}$$

where $\alpha, d > 0$, $x \in \mathbb{R}, t \geq 0$, and P is a probability measure on $\mathbb{R}_+$. The functions $u(t,x)$ and $v(t,x)$ denote the densities of the infectious agent and the infective human population at a point x and time t. The nonnegative kernel K can be asymmetric and $\int_{\mathbb{R}} \mathcal{K}(w)dw =: A > 0$. The function g can be non-monotone. By scaling the variables, we can suppose that $d = 1$.

Next, suppose that $(u(t,x), v(t,x)) = (\phi(x+ct), \psi(x+ct))$ is a semi-wavefront solution of system (4.1), i.e. the continuous non-constant uniformly bounded functions $u(t,x) = \phi(x+ct)$ and $v(t,x) = \psi(x+ct)$ are positive and satisfy of the boundary conditions $\phi(-\infty) = \psi(-\infty) = 0$ or $\phi(+\infty) = \psi(+\infty) = 0$. Then the wave profiles ϕ and ψ must satisfy the following system:

$$\phi''(t) - c\phi'(t) - f(\phi(t)) + \int_{\mathbb{R}} \mathcal{K}(u)\psi(t-u)du = 0, \tag{4.2}$$

$$c\psi'(t) + \alpha\psi(t) - \int_0^\infty g(\phi(t-cs))P(ds) = 0. \tag{4.3}$$

Suppose, for example, that $\phi(-\infty) = \psi(-\infty) = 0$ and $c > 0$. Then, integrating (4.3) between $-\infty$ and t, we find that ψ satisfies

$$\begin{aligned} \psi(t) &= \frac{1}{c}\int_0^\infty \int_0^\infty e^{-\frac{\alpha}{c}u} g(\phi(t-u-cr))P(dr)du \\ &= \int_0^\infty \int_r^\infty e^{-\alpha(w-r)} g(\phi(t-cw))dwP(dr) \\ &= \int_0^\infty \int_0^w e^{-\alpha(w-r)} g(\phi(t-cw))P(dr)dw \\ &= \int_0^\infty \mathcal{K}_1(w) g(\phi(t-cw))dw, \end{aligned}$$

where

$$\mathcal{K}_1(w) = \int_0^w e^{-\alpha(w-r)} P(dr).$$

In the case $c = 0$, we have $\alpha\psi(t) = g(\phi(t))$. Now, if we suppose that $\phi(+\infty) = \psi(+\infty) = 0$ and $c < 0$, then we obtain a similar result. Hence

$$\psi(t) = \begin{cases} \int_0^\infty \mathcal{K}_1(w) g(\phi(t - cw)) dw, & c \neq 0, \\ \psi(t) = g(\phi(t))/\alpha, & c = 0. \end{cases} \tag{4.4}$$

Substituting these results in equation (4.2), we get

$$\phi''(t) - c\phi'(t) - f(\phi(t)) + \int_0^{+\infty} \int_{\mathbb{R}} K(s,w) g(\phi(t - s - cw) ds dw = 0, \tag{4.5}$$

where

$$K(s,w) := \begin{cases} \mathcal{K}(s)\mathcal{K}_1(w), & c \neq 0, \\ \mathcal{K}(s)\frac{\delta(w)}{\alpha}, & c = 0. \end{cases}$$

Note that

$$\int_0^{+\infty} \int_{\mathbb{R}} K(s,w) dw ds = \frac{A}{\alpha} > 0,$$

hence we can suppose, without loss of generality, that the non-negative kernel $K(s,w)$ is normalized by $\int_0^{+\infty} \int_{\mathbb{R}} K(s,w) dw ds = 1$.

On the other hand, observe that the characteristic functions χ_0 and χ_L associated to (4.5) have the following form:

$$\chi_0(z,c) = z^2 - cz - f'_-(0) + \frac{g'_-(0)}{cz + \alpha} \int_0^\infty e^{-zcr} P(dr) \int_{\mathbb{R}} \mathcal{K}(w) e^{-zw} dw, \tag{4.6}$$

and

$$\chi_L(z,c) = z^2 - cz - f'_-(0) + \frac{L}{cz + \alpha} \int_0^\infty e^{-zcr} P(dr) \int_{\mathbb{R}} \mathcal{K}(w) e^{-zw} dw, \tag{4.7}$$

for $\Re(cz + \alpha) > 0$. In this way, if $g'_-(0) - f'_-(0) > 0$ then Lemma 3.3 ensures the existence of c_*^+ [respectively, $c_\star^+$] which is the minimal value of c for which (4.6) [respectively, (4.7)] has at least one positive zero.

We can now formulate the following existence result:

Theorem 4.1. *Let assumptions H_0-H_2 hold. Then for each wave speed $c \notin (c_\star^-, c_\star^+)$, system (4.1) admits at least one wavefront solution*

$$(u(t,x), v(t,x)) = (\phi(x + ct), \psi(x + ct)),$$

satisfying $\phi(-\infty) = \psi(-\infty) = 0$ and $\liminf_{t\to+\infty}(\phi(t),\psi(t)) > (0,0)$ [respectively, $\phi(+\infty) = \psi(+\infty) = 0$ and $\liminf_{t\to-\infty}(\phi(t),\psi(t)) > (0,0)$], if $c \geq c_\star^+$ [respectively, if $c \leq c_\star^-$]. Moreover, system has no semi-wavefront solution propagating with speed $c \in (c_*^-, c_*^+)$. Finally system (4.1) does not have positive pulse solutions.

Proof. Note that hypothesis H_0 implies that for each $c \in \mathbb{R}$, $\chi_L(z,c)$ is well defined on the maximal interval $(\gamma_1(c), \gamma_2(c))$ with $\gamma_1(c) < 0 < \gamma_2(c)$. Thus for each $c \geq c_\star^+$ Theorem 3.8 implies the existence of a semi-wavefront solution ϕ of (4.5) propagating with speed c, such that $\phi(-\infty) = 0$ and $\liminf_{t\to+\infty}\phi(t) > 0$. Moreover, we also have the non-existence of semi-wavefront of (4.5), if $c \in (c_*^-, c_*^+)$. In addition, Theorem 3.7 ensure that equation (4.5) can not have positive pulses.

Now, since ϕ is a positive bounded function, ψ defined in (4.4) is also bounded:

$$|\psi(t)| \leq \frac{\sup_{u\geq 0} g(u)}{\alpha}, t \in \mathbb{R}.$$

In addition, applying the Lebesgue's dominated convergence theorem [Fatou lemma, respectively] we get that $\psi(-\infty) = 0$ [$\liminf_{t\to+\infty}\psi(t) > 0$, respectively]. Thus $(\phi(t),\psi(t))$ is a semi-wavefront of (4.2) and (4.3) propagating with speed $c \geq c_\star^+$, and, in consequence, $(u(t,x), v(t,x)) = (\phi(x+ct), \psi(x+ct))$ is a semi-wavefront solution of (4.1) satisfying the bounded condition $\phi(-\infty) = \psi(-\infty) = 0$. A similar argument applies if $c \leq c_*^-$. □

Remark 4.2. Observe that if equation $f(s) = g(s)$ has only two solutions: 0 and κ, with κ being globally attracting with respect to $f^{-1}\circ g : (0,\bar{\xi}_2] \to (0,\bar{\xi}_2]$, where $\bar{\xi}_2$ is defined in H_0, then the semi-wavefront $(u(t,x), v(t,x)) = (\phi(x+ct), \psi(x+ct))$ is in fact a wavefront.

Remark 4.3. Theorem 4.1 completes or improves some results of [23, 27, 28, 30].

A population dynamics model: Let u and v denote the numbers of mature and immature population of a single species at time $t \geq 0$, respectively. We will study the system

$$\begin{cases} u_t(t,x) = du_{xx}(t,x) - f(u(t,x)) + \int_0^\infty \int_{\mathbb{R}} K(s,w)g(u(t-s,x-w))dwds, \\ \\ v_t(t,x) = Dv_{xx}(t,x) - \gamma v(t,x) + g(u(t,x)) - \int_0^\infty \int_{\mathbb{R}} K(s,w)g(u(t-s,x-w))dwds, \end{cases} \tag{4.8}$$

where $\gamma, D, d > 0$, the nonnegative kernel K can be asymmetric and satisfies $0 < \int_0^\infty \int_{\mathbb{R}} K(s,w)dwds < 1$. Note that by scaling the variables, we can suppose that $d = 1$. Now, observe that in the system (4.8) the first equation can be solved independently of the second. In this way, if the system (4.8) admits a semi-wavefront solution $(u(t,x), v(t,x)) = (\phi(x+ct), \psi(x+ct))$, then $v(t,x) = \psi(x+ct)$ must satisfy the equation

$$D\psi''(t) - c\psi'(t) - \gamma\psi(t) + (\mathcal{H}\phi)(t) = 0,$$

where the operator $\mathcal{H}$ is defined by

$$(\mathcal{H}\phi)(t) = g(\phi(t)) - \int_0^\infty \int_{\mathbb{R}} K(s,w)g(\phi(t-cs-w))dwds.$$

Note that $\mathcal{H}\phi \in C(\mathbb{R}, \mathbb{R})$, $\mathcal{H}0 = 0$ and

$$|(\mathcal{H}\phi)(t)| \leq \sup_{t\geq 0} g(t) \left(1 + \int_0^\infty \int_{\mathbb{R}} K(s,w)dwds\right).$$

Thus $\mathcal{H}\phi$ is a bounded function and ψ can be represented by

$$\psi(t) = \int_{\mathbb{R}} k_1(t-s)(\mathcal{H}\phi)(s)ds = \int_{\mathbb{R}} k_1(s)(\mathcal{H}\phi)(t-s)ds, \tag{4.9}$$

where

$$k_1(s) = \left(\sqrt{c^2 + 4D\gamma}\right)^{-1} \begin{cases} e^{\tilde{\nu}(c)s}, & s \geq 0, \\ e^{\tilde{\mu}(c)s}, & s < 0, \end{cases}$$

and $\tilde{\nu}(c) < 0 < \tilde{\mu}(c)$ are the roots of $Dz^2 - cz - \gamma = 0$. Moreover,

$$|\psi(t)| \leq \frac{\sup_{t\geq 0} g(t)}{\gamma} \left(1 + \int_0^\infty \int_{\mathbb{R}} K(s,w)dwds\right).$$

On the other hand, if $\phi(-\infty) = 0$, then we have $\mathcal{H}(\phi(-\infty)) = 0$ and $\psi(-\infty) = 0$, by Lebesgue's theorem of dominated convergence. In addition, if $\liminf_{t\to+\infty} \phi(t) > 0$, then the Fatou lemma implies that $\liminf_{t\to+\infty} \psi(t) > 0$. Similar argument applies at $+\infty$. In consequence, we obtain the following lemma:

Lemma 4.4. *Assume that H_0-H_2 are true. If $u(t,x)=\phi(x+ct)$ is a semi-wavefront of the first equation of the system (4.8), satisfying $\phi(-\infty)=0$ and $\liminf_{t\to+\infty}\phi(t)>0$, then the second equation of (4.8) has a semi-wavefront $v(t,x)=\psi(x+ct)$ with $\psi(-\infty)=0$ and $\liminf_{t\to+\infty}\psi(t)>0$. If $\phi(+\infty)=0$ and $\liminf_{t\to-\infty}\phi(t)>0$, then $\psi(+\infty)=0$ and $\liminf_{t\to-\infty}\psi(t)>0$.*

Finally, consider the characteristic functions $\chi_0(z,c)$ and $\chi_L(z,c)$ associated with the first equation of system (4.8). Note that Lemma 3.3 ensure the existence of $c_*^+, c_\star^+, c_*^-, c_\star^-$, defined as in Section 3.2. In addition, we can suppose, without loss of generality, that the non-negative kernel $K(s,w)$ is normalized by $\int_0^{+\infty}\int_{\mathbb{R}} K(s,w)dwds = 1$. Then the following theorem is a direct consequence of Theorem 3.8.

Theorem 4.5. *Let assumptions H_0-H_2 hold. Then for each wave speed $c\notin(c_\star^-, c_\star^+)$, the system (4.8) admits at least one semi-wavefront solution*

$$(u(t,x),v(t,x))=(\phi(x+ct),\psi(x+ct)),$$

satisfying $\phi(-\infty)=\psi(-\infty)=0$ and $\liminf_{t\to+\infty}(\phi(t),\psi(t))>(0,0)$, if $c\geq c_\star^+$ or $\phi(+\infty)=\psi(+\infty)=0$ and $\liminf_{t\to-\infty}(\phi(t),\psi(t))>(0,0)$, if $c\leq c_\star^-$. Furthermore, the system (4.8) has no semi-wavefront solution propagating with speed $c\in(c_*^-, c_*^+)$. Finally system (4.8) does not have positive pulse solutions.

Proof. First, for each $c\geq c_\star^+$ Theorem 1.1 implies the existence of a semi-wavefront solution $\phi(x+ct), \phi(-\infty)=0$, $\liminf_{t\to+\infty}\phi(t)>0$ of the first equation of system (4.8). If $c\leq c_\star^-$, then a similar result holds with $\phi(+\infty)=0$ and $\liminf_{t\to-\infty}\phi(t)>0$. Moreover, Theorem 3.7 ensures that this equation can not have pulse solutions.

Now, we define the function ψ as in (4.9). Then Lemma 4.4 implies that the second equation of the system (4.8) has a semi-wavefront $v(t,x))=\psi(x+ct)$ satisfying $\psi(-\infty)=0$ and $\liminf_{t\to+\infty}\psi(t)>0$, if $c\geq c_\star^+$, and $\psi(+\infty)=0$ and $\liminf_{t\to-\infty}\psi(t)>0$, if $c\leq c_\star^-$. Thus $(u(t,x),v(t,x))=(\phi(x+ct),\psi(x+ct))$ is a semi-wavefront solution to system (4.8) satisfying $\phi(-\infty)=\psi(-\infty)=0$

and $\liminf_{t\to+\infty}(\phi(t),\psi(t)) > (0,0)$, if $c \geq c_\star^+$ or $\phi(+\infty) = \psi(+\infty) = 0$ and $\liminf_{t\to-\infty}(\phi(t),\psi(t)) > (0,0)$, if $c \leq c_\star^-$.

Next, Theorem 1.2 implies the non-existence of semi-wavefront propagating with speed $c \in (c_*^-, c_*^+)$, which completes the proof. $\square$

Remark 4.6. Observe that if equation $f(s) = g(s)$ has only two solutions: 0 and κ, with κ being globally attracting with respect to $f^{-1} \circ g : (0, \bar{\xi}_2] \to (0, \bar{\xi}_2]$, then we also have that the semi-wavefront $(u(t,x), v(t,x)) = (\phi(x+ct), \psi(x+ct))$ is a wavefront.

Remark 4.7. We note that Theorem 4.5 completes or improves some results of [8, 12, 23, 25], where the non-existence or the uniqueness was established under stronger assumptions (K is Gaussian or symmetric kernel, and g monotone). In [12, 25] only the particular cases $f(s) = \beta s^2$ and $g(s) = s$, were studied, and in [23], the assumptions were either $f(s) = f'(0)s$ or $g(s) = g'(0)s$.

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