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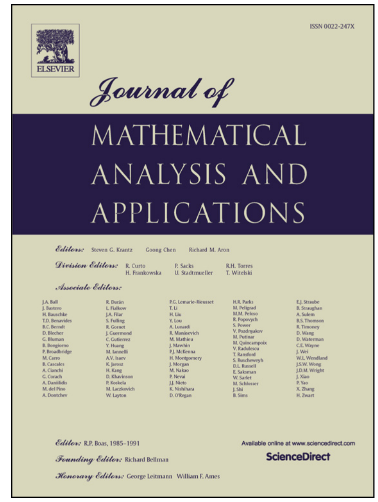

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MEASURE OF NONCOMPACTNESS, SURJECTIVITY OF GRADIENT OPERATORS AND AN APPLICATION TO THE $p-$LAPLACIAN

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Abstract. It is shown that if X is a real Banach space with dual X^* and $F : X \to X^*$ is a continuous gradient operator that is coercive in a certain sense and proper on closed bounded sets, then it is surjective. Use of the notion of measure of noncompactness enables sufficient conditions for properness to be given. These give rise to a surjectivity theorem for compact perturbations of strongly monotone maps and also facilitate discussion of a Dirichlet boundary-value problem involving the $p-$Laplacian.

1. Introduction

This paper is a direct development of the ideas in [6]. In that paper, the following result was proved: let F be a bounded continuous gradient operator acting in a real, infinite-dimensional Hilbert space with scalar product denoted by $\langle \cdot, \cdot \rangle$. Suppose that F is *strongly coercive* in the sense that

$$\langle F(x), x \rangle \geq c\|x\|^2 \tag{1.1}$$

for some $c > 0$ and for all $x \in H$, and suppose moreover that it is α-*coercive* [4] in the sense that

$$\omega(F) \equiv \inf \left\{ \frac{\alpha(F(A))}{\alpha(A)} : A \subset H, A \, \text{bounded} \, , \alpha(A) > 0 \right\} > 0, \tag{1.2}$$

$\alpha(A)$ denoting the measure of noncompactness (see, e.g., Chapter 1 of [1]) of the set A. Under the above conditions, F is surjective. This result was proved in [6] as a simple consequence of the Ekeland variational principle (see, e.g., Chapter 4 of [7]), after converting the existence problem into a minimization problem for an appropriately defined functional on H.

The purpose of the present paper is both to extend in various directions this surjectivity theorem, and to show that the resulting abstract result can be usefully applied to boundary-value problems for nonlinear partial differential equations. (In [6], the theorem was only applied within the context of operator theory itself, and precisely to the spectral theory of nonlinear operators, see in particular Chapters 6 and 7 of [1]).

We generalize the result in [6] considering gradient maps F that act from a Banach space X to its dual space X^*, we weaken considerably the coercivity condition (1.1), and finally we replace the "quantitative" technical assumption $\omega(F) > 0$ with

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the qualitative and more general condition on F of being locally proper (see below for the precise definitions).

Theorems in the literature concerning the surjectivity of nonlinear maps $F : X \to X^*$ are plentiful and typically impose the assumptions of monotonicity and coercivity on F. By monotonicity is meant that all $x, y \in X$,

$$\langle F(x) - F(y), x - y\rangle \geq 0,$$

while coercivity requires that

$$\langle F(x), x\rangle \,/\, \|x\| \to \infty \text{ as } \|x\| \to \infty.$$

Here $\langle \cdot, \cdot\rangle$ stands for the duality pairing between X and X^*. Thus Theorem 26.11 of [12] shows that if F is a gradient operator that is bounded, monotone and coercive, then it is surjective. The celebrated Minty-Browder theorem (see [5], Theorem 5.16 and [9], p. 323) requires neither that F be a gradient operator nor that it be bounded, but some continuity condition on F is imposed. Later work of Leray and Lions (see [9], p. 323) led to a weakening of the monotonicity assumption and the introduction of pseudo-monotone maps (see [16, Chapter 2, Section 2.4]). In contrast, Theorem 2.1 of the present paper (Section 2) makes no monotonicity requirement and shows that F is surjective if it is a continuous gradient map that satisfies a strong coercivity condition and is proper on closed bounded sets. A sufficient condition for properness is that a refinement of $\omega(F)$ - as defined in (1.2) - be positive, and from this theorems concerning compact perturbations of monotone maps can be recovered, in the setting of gradient operators: see Corollary 3.3.

In Section 3 we discuss in some detail the refinement of $\omega(F)$ useful to ensure local properness. In fact, we start with a similar work for the companion number $\alpha(F)$ to $\omega(F)$; recall (see, for instance, [13]) that in general, given Banach spaces X, Y and a bounded continuous map $F : X \to Y$, $\alpha(F)$ is defined by

$$\alpha(F) = \inf\left\{k : \alpha_Y\left(F(A)\right) \leq k\alpha_X(A) \text{ for all bounded } A \subset X\right\},$$

where α_X, α_Y correspond to the measures of noncompactness of subsets of X, Y respectively. Evidently F is compact if and only if $\alpha(F) = 0$; hence $\alpha(F)$ is often referred to as the measure of noncompactness of F. When $F = T$, a bounded linear map from X to itself, there are various important theorems that are now classical. For example,

$$\lim_{n\to\infty} \left(\alpha\left(T^n\right)\right)^{1/n}$$

exists and equals the radius of the essential spectrum of T. We refer to Chapter 1 of [11] for further information, proofs and references for these classical results concerning linear maps; for nonlinear maps the reader can consult the papers [13], [4] and Chapter 2 of [1].

Finally in Section 4, we apply the abstract results to the following Dirichlet problem for the $p-$Laplacian Δ_p:

$$-\Delta_p u - \lambda_1 |u|^{p-2} u \; + f(x,u) = h \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{1.3}$$

Here $2 \leq p < \infty$, $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$, Ω is a bounded open subset of $\mathbb{R}^n$, $h \in L_{p'}(\Omega)\,(p' = p/(p-1))$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies adequate assumptions.

Moreover, λ_1 is the first eigenvalue for the Dirichlet $p-$Laplacian in Ω [14]. We see (1.3) as a perturbation of the same problem where $f = 0$, namely

$$-\Delta_p u - \lambda_1 |u|^{p-2} u \ = h \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{1.4}$$

If $p = 2$, we know by the Fredholm theory for linear elliptic problems that (1.4) is not solvable for any $h \in L_2(\Omega)$, but only for those h satisfying the orthogonality condition

$$\int_\Omega h\phi_1 \, dx = 0 \tag{1.5}$$

where ϕ_1 is a normalized eigenfunction corresponding to λ_1. In fact, restrictions on $h \in L_{p'}(\Omega)$ for the solvability of (1.4) hold for any p, as follows in particular by the sharp results of Takáč [18]. Our question is: do there exist nonzero f's such that (1.3) has a solution for *any* $h \in L_{p'}(\Omega)$? Or in other words: can we perturb the problem (1.4) with an appropriate additional term f so as to restore surjectivity? Indeed, we use the results of Section 3 to prove solvability of (1.3) for any $h \in L_{p'}(\Omega)$ if f satisfies - in addition to the standard regularity and growth assumptions - a definiteness condition of the form $sf(x,s) \geq m|s|^p$, for some $m > 0$ and all $(x,s) \in \Omega \times \mathbb{R}$.

2. A surjectivity theorem for gradient operators

Let X be a real Banach space with norm $\|.\|$ and with dual X^*. We denote with $\langle x, y\rangle$ the pairing between $x \in X^*$ and $y \in X$. Recall (see e.g. [2], Definition 2.5.1) that a map $F : X \to X^*$ is said to be a *gradient* (or *potential)* operator if there exists a differentiable functional $f : X \to \mathbb{R}$ such that

$$F(x) = f'(x) \qquad \text{for all} \quad x \in X \tag{2.1}$$

where $f'(x) \in X^*$ denotes the (Fréchet) derivative of f at the point $x \in X$. When it is so, the functional f - the *potential* of F - is defined up to an additive constant; assuming for convenience that $f(0) = 0$ and assuming in addition that F is continuous, f is explicitly related to F via the equation

$$f(x) = \int_0^1 \langle F(tx), x\rangle \, dt. \tag{2.2}$$

We recall moreover that given a differentiable functional $f : X \to \mathbb{R}$, a point $x \in X$ is said to be a *critical point* of f if $f'(x) = 0$. Therefore, the zeroes of a gradient operator are precisely the critical points of its potential.

Finally, we recall that a map $F : X \to Y$ (X, Y metric spaces) is said to be *proper* if the preimage $F^{-1}(K)$ is a compact subset of X whenever $K \subset Y$ is compact, and is said to be *proper on closed bounded sets* if given any closed bounded set M of X, the set $M \cap F^{-1}(K)$ is compact whenever $K \subset Y$ is compact.

Theorem 2.1. *Let X be a real Banach space with dual X^*, and let $F : X \to X^*$ be a continuous gradient operator. Suppose that F satisfies the following assumptions:*
i) $\langle F(x), x\rangle \geq c\|x\|^p$ for some $c > 0$, some $p > 1$ and all $x \in X$;
ii) F is proper on closed bounded sets.
Then F is surjective.

Proof. Let f be the potential of F. Using i) we have, for $x \in X$ and $t \in \mathbb{R}, t > 0$,

$$\langle F(tx), x\rangle = \frac{\langle F(tx), tx\rangle}{t} \geq \frac{c\|tx\|^p}{t} = ct^{p-1}\|x\|^p$$

whence

$$f(x) = \int_0^1 \langle F(tx), x\rangle \, dt \geq c'\|x\|^p \tag{2.3}$$

with $c' = c/p > 0$. To prove that F is surjective, we take $y \in X'$ and look for an $x \in X$ such that $F(x) = y$; however, since $F = f'$ this equation is equivalent to to the search of a critical point x for the functional f_1 defined on X putting

$$f_1(x) = f(x) - \langle y, x\rangle, \qquad x \in X \tag{2.4}$$

for we have evidently

$$f_1'(x) = f'(x) - y = F(x) - y. \tag{2.5}$$

Now using (2.3), and writing simply $\|y\|$ rather than $\|y\|_{X^*}$, we get

$$f_1(x) \geq c'\|x\|^p - \|y\|\|x\| \tag{2.6}$$

which shows in particular that f_1 is coercive on X in the sense that

$$f_1(x) \to +\infty \text{ as } \|x\| \to \infty. \tag{2.7}$$

We claim that f_1 is bounded below on X. For suppose on the contrary that $\inf_{x\in X} f_1(x) = -\infty$, and let $(x_n) \subset X$ be such that

$$f_1(x_n) \to -\infty. \tag{2.8}$$

The sequence (x_n) is necessarily bounded (otherwise there would exists a subsequence (x_{n_k}) with $\|x_{n_k}\| \to \infty$, and thus $f_1(x_{n_k}) \to +\infty$ by (2.7), contradicting (2.8)). But then by (2.6), $f_1(x_n)$ must be bounded below, contradicting again (2.8).

Therefore f_1 is bounded below on X. As f_1 is of class C^1, the Ekeland Variational Principle (see, in particular, Theorem 4.4 of [7]) ensures the existence of a minimizing sequence along which the derivative of f_1 tends to 0, that is, a sequence $(x_n) \subset X$ such that

$$f_1(x_n) \to c_1 \equiv \inf_{x\in X} f_1(x) \qquad \text{and} \qquad f_1'(x_n) \to 0.$$

Using the expression (2.5) of f_1', we see that the latter condition is equivalent to

$$F(x_n) \to y.$$

The sequence (x_n) is bounded by virtue of (2.7), and since F is proper on closed bounded sets by assumption, the convergence of $F(x_n)$ implies that (x_n) contains a convergent subsequence. Letting (x_{n_k}) denoting this subsequence and putting $x = \lim_{k\to\infty} x_{n_k}$, we then see immediately by the continuity of f_1 and F that $f_1(x) = c_1$ and $F(x) = y$. □

It may be useful to single out a slightly more general version of the argument used in the proof of Theorem 2.1 in order to show that the relevant functional is bounded below.

Proposition 2.1. *Let X be a Banach space and let $f : X \to \mathbb{R}$ be such that*

$$f(x) \geq \phi(\|x\|) \tag{2.9}$$

where $\phi : [0, +\infty[\to [0, +\infty[$ is bounded on bounded sets and such that $\phi(t) \to +\infty$ as $t \to +\infty$. Then f is bounded below on X. Moreover, any minimizing sequence is necessarily bounded.

Proof. Identical to that given in Theorem 2.1 for $f = f_1$ and for the special case

$$\phi(t) = c' t^p - \|y\| t.$$

□

Let us now consider in more detail the assumptions stated in Theorem 2.1. We first ask whether the coercivity condition i) can be weakened. Note that the hypothesis that $p > 1$ is used in an essential way: the proof of Theorem 2.1 breaks down if $p = 1$. Nevertheless it is possible to replace the term $\|x\|^p$ in assumption i) by one of the form $\|x\| \, h\left(\|x\|\right)$, where h is a suitable function, such as one of logarithmic type, that grows more slowly than any power. A convenient way of explaining this is by means of the so-called slowly varying functions, a detailed account of which is given in [10, Chapter 3, Subsection 3.4.3].

A measurable function $b : [1, \infty) \to (0, \infty)$ is said to be *slowly varying* (sv) if given any $\varepsilon > 0$, the map $t \longmapsto t^{\varepsilon} b(t)$ is equivalent to (i.e. bounded above and below by constant multiples of) a non-decreasing function and $t \longmapsto t^{-\varepsilon} b(t)$ is equivalent to a non-increasing function on $[1, \infty)$.

Functions of power type are plainly not sv. To illustrate the sv class, define functions l_i on $[1, \infty)$ by

$$l_0(t) = t, l_i(t) = 1 + \log l_{i-1}(t) \text{ for } i \in \mathbb{N}.$$

Examples of sv functions b are given by

(i) $b(t) = \prod_{i=1}^{m} l_i^{\alpha_i}(t)$, $m \in \mathbb{N}, \alpha \in \mathbb{R}^m$;
(ii) $b(t) = \exp\left(\log^{\alpha} t\right), 0 < a < 1$;
(iii) $b(t) = \exp\left(l_m^a t\right), \, 0 < a < 1, m \in \mathbb{N}$;
(iv) $b(t) = l_m(t), m \in \mathbb{N}$.

Given any sv function b, define by γ_b the function defined on $(0, \infty)$ by

$$\gamma_b(t) = b\left(\max\left\{t, 1/t\right\}\right), \; t > 0.$$

It turns out (see [10], p.109) that if b is sv, then

$$\int_0^t \gamma_b(s) ds \asymp t\gamma_b(t), \; t > 0, \tag{2.10}$$

where the symbol $\asymp$ stands for "equivalent to" in the sense explained above.

Proposition 2.2. *Let all the assumptions in Theorem 2.1 be satisfied except that i) is replaced by the weaker assumption*

i') $\langle F(x), x \rangle \geq \|x\| \gamma_b(\|x\|)$ for all $x \in X$ and for some sl funtion b with $b(t) \to \infty$ as $t \to \infty$.

Then F is surjective.

Proof. Using i), we first have for $t > 0$ and $x \in X$

$$\langle F(tx), x\rangle = t^{-1}\langle F(tx), tx\rangle \geq \|x\|\gamma_b(t\|x\|)$$

whence, using the expression (2.2) for the potential f and the equivalence (2.10), we obtain

$$f(x) \geq \|x\| \int_0^1 \gamma_b(t\|x\|)\,dt = \int_0^{\|x\|} \gamma_b(s)ds \geq c\,\|x\|\,b(\|x\|) \tag{2.11}$$

for some $c > 0$. It follows that the functional f_1 defined in (2.4) satisfies the lower bound

$$f_1(x) \geq \|x\|(cb(\|x\|) - \|y\|) \tag{2.12}$$

and therefore satisfies the condition (2.9) of Proposition 2.1 with

$$\phi(t) = t(cb(t) - \|y\|).$$

Using this Proposition, and arguing as in the proof of Theorem 2.1 via the Ekeland's principle, the conclusion follows. □

More space is needed for an adequate discussion of condition ii) in Theorem 2.1, and we place this separately in the next Section.

3. Measure of noncompactness and α-coercivity

The aim of this Section is to discuss conditions on a map $F : X \to X^*$ that ensure its local properness, as required by condition ii) of Theorem 2.1. We do this using the definition and properties of the measure of noncompactness of sets and operators (see, for instance, [1, Chapter 2], [4] or [13]). If A is a bounded subset of a Banach space X, we let $\alpha(A)$ denote its (Kuratowski) *measure of noncompactness*, defined by

$$\alpha(A) = \inf\{\epsilon > 0 : A \text{ can be covered by finitely many subsets of diameter} \leq \epsilon\}.$$

Thus, $\alpha(A) = 0$ if and only if A is totally bounded, or equivalently if the closure $\overline{A}$ of A is compact. Consider now a mapping F of a Banach space X into a Banach space Y. The measure of noncompactness of bounded sets in either space will be denoted with the same letter α. Here and henceforth, we shall only consider maps $F : X \to Y$ that are bounded on bounded sets, so that $\alpha(F(A))$ is defined whenever $A \subset X$ is bounded; when we need to stress that F satisfies this condition, we merely say that F is *bounded*. We assume moreover that $dimX = \infty$, so that there exist bounded sets $A \subset X$ with $\alpha(A) > 0$. We first generalize the known definitions (see, e.g., [1, Chapter 2], [4] or [13]) as follows.

Definition 3.1. A map $F : X \to Y$ is said to be *α-Hölder of exponent* γ $(0 < \gamma < \infty)$ if $\alpha(F(A)) \leq k[\alpha(A)]^\gamma$ for some $k \geq 0$ and all bounded subsets A of X; in this case we put

$$\alpha_\gamma(F) = \inf\{k \geq 0 : \alpha(F(A)) \leq k[\alpha(A)]^\gamma \text{ for all bounded } A \subset X\}, \tag{3.1}$$

that is,

$$\alpha_\gamma(F) = \sup\left\{\frac{\alpha(F(A))}{[\alpha(A)]^\gamma} : A \subset X, A \text{ bounded}, \alpha(A) > 0\right\}. \tag{3.2}$$

Thus if F is α-Hölder of exponent γ, it follows that for all bounded $A \subset X$,

$$\alpha(F(A)) \leq \alpha_\gamma(F)[\alpha(A)]^\gamma. \tag{3.3}$$

In particular when $\gamma = 1$, F is $\alpha-$Lipschitz; in this case we continue to write - as in [13] and in [4], for instance - $\alpha(F)$ rather than $\alpha_1(F)$. We note moreover that the following statements are equivalent:

- F is *compact*, that is, the closure $\overline{F(A)}$ of $F(A)$ is compact for any bounded $A \subset E_1$;
- $\forall \gamma > 0$, F is α-Hölder of exponent γ and $\alpha_\gamma(F) = 0$;
- $\exists \gamma > 0 : F$ is α-Hölder of exponent γ and $\alpha_\gamma(F) = 0$.

Example 3.1. *Let $F : X \to Y$ be Hölder continuous of exponent γ, i.e., such that*

$$\|F(x) - F(y)\| \leq k\|x - y\|^\gamma \tag{3.4}$$

for some $k \geq 0$ and for all $x, y \in X$. Then F is α-Hölder of the same exponent γ and we have

$$\alpha_\gamma(F) \leq k. \tag{3.5}$$

The following two Propositions yield elementary properties of $\alpha_\gamma(F)$ that are stated for completeness and that can be proved immediately.

Proposition 3.1. *Let $F, G : X \to Y$ be α-Hölder of exponent γ. Then so are $F + G$ and λF ($\lambda \in \mathbb{R}$), and moreover*

- $\alpha_\gamma(\lambda F) = |\lambda|\alpha_\gamma(F)$
- $\alpha_\gamma(F + G) \leq \alpha_\gamma(F) + \alpha_\gamma(G)$.

We simply mention for further use the inequalities

$$\alpha(F + G)(A) \leq \alpha(F(A) + G(A)) \leq \alpha(F(A)) + \alpha(G(A)) \tag{3.6}$$

that give rise to the second property stated in Proposition 3.1.

Proposition 3.2. *Let X, Y, Z be Banach spaces, let $F : X \to Y$ be α-Hölder of exponent γ and let $G : Y \to Z$ be $\alpha-$Lipschitz. Then $G \circ F : X \to Z$ is α-Hölder of exponent γ, and moreover*

$$\alpha_\gamma(G \circ F) \leq \alpha(G)\alpha_\gamma(F).$$

Next, given $F : X \to Y$ and given γ with $0 < \gamma < \infty$, let $\omega_\gamma(F)$ be defined as follows:

$$\omega_\gamma(F) = \inf\left\{\frac{[\alpha(F(A))]^\gamma}{\alpha(A)} : A \subset X, A\,\text{bounded}\,, \alpha(A) > 0\right\}. \tag{3.7}$$

It follows by (3.7) that, for all bounded $A \subset X$,

$$[\alpha(F(A))]^\gamma \geq \omega_\gamma(F)\alpha(A). \tag{3.8}$$

We stress the fact that unlike $\alpha_\gamma(F)$ (which is defined as a real nonnegative number only for α-Hölder maps), $\omega_\gamma(F)$ is defined as a real nonnegative number for any F. One basic property of the number $\omega_\gamma(F)$ is the fact that - as in the case $\gamma = 1$ [13] - its strict positivity guarantees the local properness of F.

Proposition 3.3. *Let $F : X \to Y$ be continuous and such that $\omega_\gamma(F) > 0$ for some $\gamma \in]0, \infty[$. Then F is proper on closed bounded sets, that is, given any closed bounded set M of X, the set $M \cap F^{-1}(K)$ is compact whenever $K \subset Y$ is compact.*

Proof. Let M and K be as in the statement. We have

$$K \supset K \cap F(M) \supset F(F^{-1}(K)) \cap F(M) \supset F(M \cap F^{-1}(K))$$

and therefore

$$[\alpha(K)]^\gamma \geq [\alpha(F(M \cap F^{-1}(K)))]^\gamma$$

whence, using (3.8), it follows that

$$[\alpha(K)]^\gamma \geq \omega_\gamma(F)\alpha(M \cap F^{-1}(K)). \tag{3.9}$$

As K is compact by assumption, the left-hand side of (3.9) is zero, whence the result follows since $M \cap F^{-1}(K)$ is a closed subset of X. □

The behaviour of ω_γ under addition and composition is somewhat more involved than that of α_γ.

Proposition 3.4. *Let $F, G : X \to Y$, fix $\gamma > 0$ and suppose that G is α-Hölder of exponent $1/\gamma$. Then if $\gamma \leq 1$,*

$$\omega_\gamma(F+G) \leq \omega_\gamma(F) + \alpha^\gamma_{1/\gamma}(G) \tag{3.10}$$

while if $\gamma > 1$,

$$\omega_\gamma(F+G) \leq 2^{\gamma-1}(\omega_\gamma(F) + \alpha^\gamma_{1/\gamma}(G)). \tag{3.11}$$

Proof. The case $\gamma = 1$ was dealt with in [13]. If $\gamma < 1$, consider the inequality (3.6) and divide each term by $\alpha(A) > 0$:

$$\frac{[\alpha(F+G)(A)]^\gamma}{\alpha(A)} \leq \frac{[\alpha(F(A)+G(A))]^\gamma}{\alpha(A)} \leq \frac{[\alpha(F(A))]^\gamma}{\alpha(A)} + \frac{[\alpha(G(A))]^\gamma}{\alpha(A)}. \tag{3.12}$$

Now observe that for any A,

$$\frac{[\alpha(G(A))]^\gamma}{\alpha(A)} = \left(\frac{[\alpha(G(A))]}{[\alpha(A)]^{1/\gamma}}\right)^\gamma \leq \alpha^\gamma_{1/\gamma}(G).$$

Use this bound for the last term of (3.12) and then take the infimum of the first and second member of the resulting inequality: the definition (3.7) then yields (3.10).

The case $\gamma > 1$ is dealt with in the same way, save that one must use here the inequality $(a+b)^\gamma \leq 2^{\gamma-1}(a^\gamma + b^\gamma)$. □

The inequalities (3.10) and (3.11) give rise in turn to useful lower bounds for $\omega_\gamma(F+G)$. Indeed if $0 < \gamma \leq 1$, we have from (3.10) and Proposition 3.1

$$\omega_\gamma(F) = \omega_\gamma(F+G-G) \leq \omega_\gamma(F+G) + \alpha^\gamma_{1/\gamma}(G)$$

which yields

$$\omega_\gamma(F+G) \geq \omega_\gamma(F) - \alpha^\gamma_{1/\gamma}(G) \quad (0 < \gamma \leq 1), \tag{3.13}$$

while using in a similar way (3.11) we obtain

$$\omega_\gamma(F+G) \geq \frac{1}{2^{p-1}}\omega_\gamma(F) - \alpha^\gamma_{1/\gamma}(G) \quad (\gamma > 1). \tag{3.14}$$

The estimates (3.13) and (3.14) are important as they show that the property $\omega_\gamma(F) > 0$ is stable under $\alpha-$Hölder additive perturbations G of F of sufficiently small constant $\alpha_\gamma(G)$. In view of this remarkable property, maps satisfying the condition $\omega_\gamma(F) > 0$ have been named (in case $\gamma = 1$) *α-coercive* [4], and allow among others for the construction of the *degree for quasi-Fredholm maps* (see for instance [3] and [4]), a topological invariant having properties similar to those of the classical Leray-Schauder degree. For our present needs, the following precise statement will be sufficient.

Corollary 3.1. *Let $F : X \to Y$ be such that $\omega_\gamma(F) > 0$ for some $\gamma \in]0, \infty[$, and let $G : X \to Y$ be compact. Then $\omega_\gamma(F + G) > 0$. In particular, $F + G$ is proper on closed bounded sets.*

Remark 3.1. *It follows in particular from* (3.10) *and* (3.13) *that if G is compact and $0 < \gamma \leq 1$, then*

$$\omega_\gamma(F + G) = \omega_\gamma(F). \tag{3.15}$$

The next two statements deal with the behaviour of ω_γ with respect to composition.

Proposition 3.5. *Let X, Y, Z be Banach spaces, let $F : X \to Y$, let $G : Y \to Z$ and suppose that G is α-Hölder of exponent γ. Then*

$$\omega(G \circ F) \leq \alpha_\gamma(G)\omega_\gamma(F). \tag{3.16}$$

Proof. Using (3.3) we have, for all bounded $A \subset X$,

$$\alpha(G(F(A)) \leq \alpha_\gamma(G)[\alpha(F(A))]^\gamma. \tag{3.17}$$

Now divide both members of (3.17) by $\alpha(A) > 0$, take the infimum of the respective ratios over all bounded $A \subset X$ with $\alpha(A) > 0$, and finally use the definition (3.7) to obtain (3.16). □

Corollary 3.2. *Let F be a bijective mapping of a Banach space X onto itself, and suppose that the inverse map $F^{-1} : X \to X$ is α-Hölder of exponent γ. Then both $\alpha_\gamma(F^{-1})$ and $\omega_\gamma(F)$ are strictly positive, and precisely $\omega_\gamma(F) \geq [\alpha_\gamma(F^{-1})]^{-1}$.*

Proof. By the assumptions and by Proposition 3.5 we have, letting I denote the identity map in X,

$$1 = \omega(I) = \omega(F^{-1} \circ F) \leq \alpha_\gamma(F^{-1})\omega_\gamma(F),$$

whence the result follows. □

Our next result gives in an important case an explicit lower bound for ω_γ.

Theorem 3.1. *Let X be a real, reflexive Banach space with dual X^*, and let $F : X \to X^*$ be bounded, continuous and* strongly monotone *in the sense that*

$$\langle F(x) - F(y), x - y\rangle \geq k\|x - y\|^p \tag{3.18}$$

for some $k > 0$, some $p \in [2, \infty[$ and all $x, y \in X$. Put $\gamma = 1/(p - 1)$; then

$$\omega_\gamma(F) \geq k^\gamma. \tag{3.19}$$

Proof. By the Minty-Browder Theorem (see e.g. [5], Theorem 5.16), F is bijective. Moreover, it follows from (3.18) that

$$\|F(x) - F(y)\| \geq k\|x - y\|^{p-1} \tag{3.20}$$

which shows that, for all $u, v \in F(X) = X^*$,

$$\|F^{-1}(u) - F^{-1}(v)\| \leq k^{-1/(p-1)}\|u - v\|^{1/(p-1)}. \tag{3.21}$$

Thus $F^{-1} : X^* \to X$ is Hölder continuous of exponent γ. It follows by Example 3.1 that F^{-1} is α-Hölder of the same exponent, and (3.5) shows in particular that

$$\alpha_\gamma(F) \leq k^{-1/(p-1)}. \tag{3.22}$$

The conclusion now follows from Corollary 3.2. □

The restriction of p to the interval $[2, \infty[$ stems from the fact that when $1 < p < 2$ there are no maps F sasfying (3.18). For if there were, then by (3.21), F^{-1} would be Hölder-continuous with exponent $1/(p-1) > 1$; so that the Fréchet derivative of F^{-1} would exist and be zero; hence F^{-1} would be constant, which is impossible.

Finally, the results of this Section can be gathered in order to give a concrete form of the general surjectivity Theorem 2.1.

Corollary 3.3. *Let X be a real, reflexive Banach space with dual X^*, and let $F, G : X \to X^*$ be bounded continuous gradient operators. Suppose that the following assumptions are satisfied:*

i) $\langle F(x) + G(x), x\rangle \geq c\|x\|^p$ for some $c > 0$, some $p \in [2, \infty[$ and all $x \in X$;

ii) F is strongly monotone in the sense of (3.18);

iii) G is compact.

Then $F + G$ is surjective.

Proof. Apply Theorem 2.1 observing that $F + G$ is proper on closed bounded sets, as follows by Theorem 3.1 and Corollary 3.1. □

4. Application: the $p-$Laplacian

Let $p \in [2, \infty)$, let Ω be a bounded open subset of $\mathbb{R}^n$, denote by $\|\cdot\|_p$ the usual norm on the Lebesgue space $L_p(\Omega)$ and put $X = \overset{0}{W}{}^1_p(\Omega)$, the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_X$ given by

$$\|u\|_X := \left(\int_\Omega |\nabla u|^p \, dx\right)^{1/p}.$$

Endowed with this norm X is a reflexive Banach space that is compactly embedded in $L_p(\Omega)$. The $p-$Laplacian Δ_p is defined on appropriate functions u by

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right). \tag{4.1}$$

It is naturally associated with a map $T : X \to X^*$ given by

$$\langle T(u), v\rangle_X := \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v dx \quad (u, v \in X), \tag{4.2}$$

where $\langle w, v\rangle_X$ denotes the value of $w \in X^*$ at $v \in X$. This map has various interesting properties. First, we have

$$\langle T(u), u\rangle_X = \int_\Omega |\nabla u|^p \, dx = \|u\|_X^p \quad (u \in X). \tag{4.3}$$

Moreover, in view of the inequality

$$C_p |a-b|^p \leq \left\{|a|^{p-2} a - |b|^{p-2} b\right\} \cdot (a-b), \; C_p = \frac{2}{p(2^{p-1}-1)},$$

valid for all $a, b \in \mathbb{R}^n$ (see [14], Appendix 4), we have

$$\begin{aligned}\langle T(u) - T(v), u-v\rangle_X &= \int_\Omega \left\{|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v\right\} \cdot (\nabla u - \nabla v) \, dx \\ &\geq C_p \|u-v\|_X^p.\end{aligned} \tag{4.4}$$

Thus T is strongly monotone. Moreover, use of Hölder's inequality shows that

$$\|T(u)\|_{X^*} = \sup_{\|v\|_X \le 1} |\langle T(u), v\rangle_X| \le \|u\|_X^{p-1}, \tag{4.5}$$

which implies that T is bounded. If $\|w\|_X = 1$, then

$$\|T(w)\|_{X^*} \ge |\langle T(w), w\rangle_X| = \|w\|_X^p = 1,$$

and so given any $u \in X\backslash\{0\}$,

$$\|T(u)\|_{X^*} = \|u\|_X^{p-1} \left\| T\left(\frac{u}{\|u\|_X}\right)\right\|_{X^*} \ge \|u\|_X^{p-1}.$$

Together with (4.5) this shows that

$$\|T(u)\|_{X^*} = \|u\|_X^{p-1} \text{ for all } u \in X. \tag{4.6}$$

In fact, T is continuous. To check this the following inequalities, valid for all $w, z \in \mathbb{R}^n$, will be useful:

$$\left| |z|^{(p-2)/2} z - |w|^{(p-2)/2} w \right|^2 \le \frac{p^2}{4} \left(|z|^{p-2} z - |w|^{p-2} w \right) \cdot (z - w) \tag{4.7}$$

and

$$\left| |z|^{p-2} z - |w|^{p-2} w \right| \le (p-1) \left(|z|^{(p-2)/2} + |w|^{(p-2)/2} \right) \left| |z|^{(p-2)/2} z - |w|^{(p-2)/2} w \right|. \tag{4.8}$$

For a proof of these see [15, Section 10]. Combination of (4.7) and (4.8) shows that

$$\begin{aligned}\left| |z|^{p-2} z - |w|^{p-2} w \right| \le{}& \frac{p(p-1)}{2} \left(|z|^{(p-2)/2} + |w|^{(p-2)/2} \right) \\ &\times \left| |z|^{p-2} z - |w|^{p-2} w \right|^{1/2} |z - w|^{1/2},\end{aligned}$$

so that

$$\begin{aligned}\left| |z|^{p-2} z - |w|^{p-2} w \right| &\le \frac{p^2(p-1)^2}{4} \left(|z|^{(p-2)/2} + |w|^{(p-2)/2} \right)^2 |z - w| \\ &\le \frac{p^2(p-1)^2}{2} \left(|z|^{p-2} + |w|^{p-2} \right) |z - w|.\end{aligned} \tag{4.9}$$

It follows that for all $u, v \in X$,

$$\begin{aligned}\|T(u) - T(v)\|_{X^*} &\le \sup_{\|w\|_X = 1} \left| \int_\Omega \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \cdot w dx \right| \\ &\le \left| \int_\Omega \left| |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right|^{p'} dx \right|^{1/p'} \\ &\le \frac{p^2(p-1)^2}{2} \left| \int_\Omega \left(|\nabla u|^{p-2} + |\nabla v|^{p-2} \right)^{p'} |\nabla u - \nabla v|^{p'} dx \right|^{1/p'} \\ &\le \frac{p^2(p-1)^2}{2} \left(\int_\Omega \left(|\nabla u|^{p-2} + |\nabla v|^{p-2} \right)^{p/(p-2)} dx \right)^{(p-2)/p} \|u - v\|_X \\ &\le p^2(p-1)^2 \left(\int_\Omega \left(|\nabla u|^p + |\nabla v|^p \right) dx \right)^{(p-2)/p} \|u - v\|_X \\ &= p^2(p-1)^2 \left(\|u\|_X^p + \|u\|_X^p \right)^{(p-2)/p} \|u - v\|_X.\end{aligned}$$

Standard procedures now show that T is continuous.

To complete this catalogue of properties of T we note that T is a gradient operator since $\langle T(u), v\rangle_X = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v dx$ is the directional derivative in the direction $v \in X$ of the C^1-functional

$$E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \, dx \qquad (u \in X).$$

We summarise these results in the following

Proposition 4.1. *The map $T : X \to X^*$ defined by (4.2) is a bounded, continuous, strongly monotone gradient operator.*

By an *eigenvalue* of the (Dirichlet) $p-$Laplacian is meant a real number λ such that there is a function $u \neq 0$ (an *eigenfunction*) such that

$$-\Delta_p u = \lambda |u|^{p-2} u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{4.10}$$

This eigenvalue problem is interpreted in the weak sense, so that one asks whether there exist $\lambda \in \mathbb{R}$ and $u \in X\backslash\{0\}$ such that for all $v \in X$,

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v dx = \lambda \int_\Omega |u|^{p-2} uv dx. \tag{4.11}$$

In term of the map T this means that

$$\langle T(u), v\rangle_X = \lambda \int_\Omega |u|^{p-2} uv dx. \tag{4.12}$$

It is well known (see [14]) that there is a principal eigenvalue, that is, a least such eigenvalue, denoted by λ_1, and that it is positive, simple, isolated and characterised variationally by

$$\lambda_1 = \inf_{u \in X\backslash\{0\}} \|u\|_X^p \, / \, \|u\|_p^p \, . \tag{4.13}$$

This characterization of λ_1 and (4.3) show in particular that, for all $u \in X$,

$$\langle T(u), u\rangle_X = \|u\|_X^p \geq \lambda_1 \|u\|_p^p \, . \tag{4.14}$$

We consider the problem

$$-\Delta_p u - \lambda_1 |u|^{p-2} u \; + f(x, u) = h \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{4.15}$$

where $h \in L_{p'}(\Omega)\,(p' = p/(p-1))$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the conditions

$$|f(x, s)| \leq a |s|^{p-1} + b \tag{4.16}$$

and

$$sf(x, s) \geq m |s|^p \tag{4.17}$$

for all $(x, s) \in \Omega \times \mathbb{R}$. Here a and b are non-negative constants and m is a positive constant; all of these are independent of (x, s). Problem (4.15) too is interpreted in the weak sense, so that we ask whether there exists $u \in X$ such that for all $v \in X$,

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v dx - \lambda_1 \int_\Omega |u|^{p-2} uv dx + \int_\Omega f(x, u) v dx = \int_\Omega hv dx. \tag{4.18}$$

Such a problem, with $f = 0$, was discussed by Drábek and Holubová in [8]. When $1 < p < 2$ they gave conditions on h under which (4.18) has no solutions, and other conditions on h leading to the existence of at least one solution; they expect that similar results can be obtained when $2 < p < \infty$. Here we show that when $p \in [2, \infty)$, there are general conditions on a non-zero f sufficient to ensure that there is a solution of (4.18) for all $h \in L_{p'}(\Omega)$.

As

$$\left|\int_{\Omega}|u|^{p-2}\,uvdx\right|\leq\|v\|_{p}\,\|u\|_{p}^{p/p'}=\|v\|_{p}\,\|u\|_{p}^{p-1}\,, \tag{4.19}$$

it follows that the map $v\longmapsto\int_{\Omega}|u|^{p-2}\,uvdx$ belongs to $(L_{p}(\Omega))^{*}\hookrightarrow\hookrightarrow X^{*}$; thus we have a compact map $K:X\to X^{*}$ defined by

$$\langle K(u),v\rangle_{X}=\int_{\Omega}|u|^{p-2}\,uvdx. \tag{4.20}$$

By (4.19) and by Poincaré's inequality we have, for some $k>0$,

$$\|K(u)\|_{X^{*}}\leq k\,\|u\|_{X}^{p-1}\quad(u\in X)\,,$$

so that K is bounded. That it is also continuous follows from an argument similar to that used to prove continuity of T and involving (4.9). Consideration of the functional

$$u\longmapsto\frac{1}{p}\int_{\Omega}|u|^{p}\,dx$$

shows that K is a gradient operator.

Next we consider the term $\int_{\Omega}f(x,u)vdx$ in (4.18). Using (4.16) we see that

$$\left|\int_{\Omega}f(x,u)vdx\right|\leq\|v\|_{p}\,\|f(\cdot,u)\|_{p'}\leq\|v\|_{p}\left\{a\,\|u\|_{p}^{p-1}+\|b\|_{p'}\right\},$$

and so a bounded, compact map $N:X\to X^{*}$ may be defined by

$$\langle N(u),v\rangle_{X}=\int_{\Omega}f(x,u)vdx. \tag{4.21}$$

In view of (4.17),

$$\langle N(u),u\rangle_{X}\geq m\,\|u\|_{p}^{p}\,. \tag{4.22}$$

The map N is of Nemytskii type and from the familiar properties of such operators (see, for example, [9], p. 127) we see that N is continuous. Moreover, it follows by arguments entirely similar to those used for the case $p=2$ in [17], Appendix B, that N is a gradient operator with potential

$$I(u)=\int_{\Omega}P(x,u)\,dx,\quad u\in X$$

where $P:\Omega\times\mathbb{R}\to\mathbb{R}$ is defined by

$$P(x,z)=\int_{0}^{z}f(x,t)\,dt.$$

We summarise these facts on the following

Proposition 4.2. *The maps $K,N:X\to X^{*}$ defined by (4.20), (4.21) respectively are bounded, continuous, compact gradient operators.*

The problem (4.18) is equivalent to the operator equation

$$T(u)-\lambda_{1}K(u)+N(u)=\widehat{h}, \tag{4.23}$$

where $\widehat{h}\in X^{*}$ is defined by

$$\left\langle\widehat{h},v\right\rangle_{X}=\int_{\Omega}hvdx\qquad(v\in X)\,.$$

Put $G = -\lambda_1 K + N$. Then use of (4.3), (4.20) and (4.22) shows that, for all $u \in X$,

$$\langle T(u) + G(u), u \rangle_X \geq \|u\|_X^p + (m - \lambda_1)\|u\|_p^p . \qquad (4.24)$$

Thus if $m \geq \lambda_1$,

$$\langle T(u) + G(u), u \rangle_X \geq \|u\|_X^p \quad (u \in X). \qquad (4.25)$$

If $0 < m < \lambda_1$, using (4.14) we see that

$$(m - \lambda_1)\|u\|_p^p \geq \left(\frac{m}{\lambda_1} - 1 \right) \|u\|_X^p ,$$

and using this in (4.24) yields

$$\langle F(u) + G(u), u \rangle_X \geq \frac{m}{\lambda_1} \|u\|_X^p \quad (u \in X).$$

Corollary 3.3 can now be applied to give the following result.

Theorem 4.1. *Suppose that f is continuous and satisfies (4.16) and (4.17) with $m > 0$. Then given any $h \in L_{p'}(\Omega)$ $(p' = p/(p-1))$, problem (4.18) has a solution $u \in X$.*

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MEASURE OF NONCOMPACTNESS, SURJECTIVITY OF GRADIENT OPERATORS AND AN APPLICATION TO THE $p-$LAPL

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