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Global boundedness to a parabolic-parabolic chemotaxis model with nonlinear diffusion and singular sensitivity ☆

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ABSTRACT

This article deals with the parabolic-parabolic chemotaxis system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla\varphi(v)) + f(u), & x \in \Omega,\ t > 0, \\ v_t = \triangle v - v + u, & x \in \Omega,\ t > 0 \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{R}^n (n \geq 1)$ with smooth boundary conditions, $D, S \in C^2([0, +\infty))$ nonnegative, with $D(u) = a_0(u + 1)^{-\alpha}$ for $a_0 > 0$ and $\alpha < 0$, $0 \leq S(u) \leq b_0(u + 1)^{\beta}$ for $b_0 > 0, \beta \in \mathbb{R}$, and where the singular sensitivity satisfies $0 < \varphi'(v) \leq \frac{\chi}{v^k}$ for $\chi > 0, k \geq 1$. In addition, $f : \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying $f(s) \equiv 0$ or generalizing the logistic source $f(s) = rs - \mu s^m$ for all $s \geq 0$ with $r \in \mathbb{R}, \mu > 0$, and $m > 1$. It is shown that for the case without a growth source, if $2\beta - \alpha < 2$, the corresponding system possesses a globally bounded classical solution. For the case with a logistic source, if $2\beta + \alpha < 2$ and $n = 1$ or $n \geq 2$ with $m > 2\beta + 1$, the corresponding system has a globally classical solution.

1. Introduction

This article is concerned with the existence and boundedness of globally classical solutions to the following parabolic-parabolic chemotaxis system:

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla\varphi(v)) + f(u), & x \in \Omega,\ t > 0, \\ v_t = \triangle v - v + u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x),\ \ v(x, 0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded domain in $\mathbb{R}^n (n \geq 1)$ with smooth boundary $\partial\Omega$. The nonlinear nonnegative functions $D(s)$, the density-dependent sensitivity $S(s)$, and the signal-dependent sensitivity $\varphi(s)$ satisfy

$$D(s), S(s) \in C^2([0,\infty)) \text{ and } \varphi(s) \in C^{2+w}_{loc}((0,\infty))(0 < w < 1), \tag{1.2}$$

$$D(s) = a_0(s+1)^{-\alpha} \quad \text{for all } s \geq 0, \tag{1.3}$$

$$0 \leq S(s) \leq b_0(s+1)^{\beta} \text{ for all } s \geq 0 \text{ and } S(0) = 0, \tag{1.4}$$

$$0 < \varphi'(s) \leq \tfrac{\chi}{s^k} \quad \text{for all } s \geq 0, \tag{1.5}$$

where $a_0, b_0, \chi > 0$, $k \geq 1$, and $\alpha, \beta \in \mathbb{R}$ are constants. The function $f : \mathbb{R} \to \mathbb{R}$ is smooth and satisfies

$$f(s) \equiv 0 \text{ for any } s > 0 \tag{1.6}$$

or generalizes the logistic source

$$f(s) = rs - \mu s^m \text{ for all } s > 0 \text{ and } f(0) \geq 0, \tag{1.7}$$

where $r \in \mathbb{R}, \mu > 0$, and $m > 1$. In addition, $\frac{\partial}{\partial\nu}$ represents the outer normal derivative on $\partial\Omega$, and the initial data satisfy

$$\begin{cases} u_0(x) \in C(\overline{\Omega}),\ u_0(x) \geq 0, \text{ and } u_0(x) \not\equiv 0,\ x \in \overline{\Omega}, \\ v_0(x) \in W^{1,\infty}(\Omega) \text{ and } v_0(x) > 0,\ x \in \overline{\Omega}. \end{cases} \tag{1.8}$$

In the model (1.1), u and v represent the cell density and the concentration of an attractive signal produced by the cells themselves, respectively. A source of logistic type $f(u)$ is included in (1.1) to represent an unlimited growth of the cell density.

We briefly recall related results obtained in previous literature in the field.

(I) The case without a growth source (viz., $f \equiv 0$)

The system (1.1) is a generalized version of the parabolic-parabolic chemotaxis system

$$\begin{cases} u_t = \triangle u - \nabla \cdot (u \nabla \varphi(v)), & x \in \Omega,\ t > 0, \\ v_t = \triangle v - u + v, & x \in \Omega,\ t > 0, \end{cases} \tag{1.9}$$

which has been studied often [3,5,7,8,12,15,18–20,27,32,33,36,35]. For the classical case $\varphi(v) = v$, it has been shown that if Ω is a ball, then the radial solution may blow up in finite time when $n = 2$ [8] or $n \geq 3$ [36]. Moreover, if $\|u_0\|_{L^{\frac{n}{2}}(\Omega)}$ and $\|\nabla v_0\|_{L^n(\Omega)}$ are small, then the solution is global and bounded [33]. For the case that $\varphi'(v) = \frac{\chi}{v}$ with $\chi > 0$, all solutions are global in time when $n = 1$ [19]. For $n \geq 2$, Winkler [35] proved the existence of global classical solutions if $\chi < \sqrt{\frac{2}{n}}$ and global existence of weak solutions to (1.9) if $\chi < \sqrt{\frac{n+2}{3n-4}}$. Later, Fujie [3] obtained the uniform-in-time boundedness of solutions to (1.9) for $\chi < \sqrt{\frac{2}{n}}$. For the two-dimensional case, Lankeit [12] studied the existence of a bounded solution for $\sqrt{\frac{2}{N}} \leq \chi < \chi_0$ with $\chi_0 > 1$. Furthermore, Lankeit and Winkler [15] proved the problem (1.9) possesses at least a globally defined generalized solution for $N = 2$ or $N = 3$ and $\chi < \sqrt{8}$ or $N \geq 4$ and $\chi < \frac{N}{N-2}$. In addition, for the case that $0 < \varphi'(v) \leq \frac{\chi}{v^k}$ with $k > 1, \chi > 0$, Fujie and Yokata [27] proved the global existence and boundedness of classical solutions to (1.9).

For (1.1), the case of $\varphi(v) = v$ has also been studied [10,22,31]. Tao and Winkler [22] proved that if Ω is convex and $\frac{S(u)}{D(u)} \leq cu^{\frac{2}{n}-\varepsilon}$ for large u, then the solutions are globally bounded. Later, Ishida et al. [10]

extended the result to the case without the assumption of convexity. Recently, for the case $0 < \varphi'(v) \le \frac{\chi}{v^k}$ with $k \ge 1$, Ding [2] proved that (1.1) possesses a globally bounded classical solution if $\alpha+\beta < 1$ and $\alpha \ge 0$ under the assumptions (1.2)–(1.5). However, the case $\alpha < 0$ was left as an open problem.

(II) The case with f being a logistic source (viz., $f = ru - \mu u^m$)

Generally, a logistic source is beneficial for the global existence and boundedness of a solution. First, consider the following chemotaxis system:

$$\begin{cases} u_t = \triangle u - \nabla \cdot (u \nabla \varphi(v)) + ru - \mu u^m, & x \in \Omega,\ t > 0, \\ \tau v_t = \triangle v - u + v, & x \in \Omega,\ t > 0, \end{cases} \tag{1.10}$$

where $\Omega \subset \mathbb{R}^n (n \ge 1)$ is a bounded smooth domain. We recall the related result with $\varphi(v) = \chi v$ with $\chi > 0$ in (1.10). Winkler [30] showed that if μ is sufficiently large, then the problem (1.10) possesses a unique global-in-time classical solution that is bounded in $\Omega \times (0, \infty)$ for $n \ge 1$, $m = 2$, and $\tau = 1$. For $\tau = 0$ and if the growth source $f(u)$ satisfies $f(u) \le r - \mu u^m$, it has been proved that for $m = 2$ there exist weak solutions for arbitrary $\mu > 0$ and globally classical solutions for $\mu > \frac{n-2}{n}\chi$ [24]. Under radially symmetric assumptions, the corresponding solution blows up if $n \ge 5$ and $m < \frac{3}{2} + \frac{1}{2n-2}$ [34]. With source term $f(u)$ controlled by $-c_0(u + u^m)$ and $r - \mu u^m$ with $c_0, m, \mu > 0$ and $r \ge 0$, the global existence of very weak solutions under specific assumptions on the initial data, μ and m, has been proven for $\tau = 0$ or 1 [26,29]. In addition, for the case that $\varphi(v) = \log v$, $m = 2$, and $n = 2$, the parabolic-elliptic system (1.10) with $\tau = 0$ was considered in [6], where it was shown that there exists a unique global bounded classical solution whenever $r > \frac{\chi^2}{4}$ for $0 < \chi \le 2$ or $r > \chi - 1$ for $\chi > 2$. Later, Zhao and Zheng [37] considered the case $\tau = 1$ and obtained the existence and boundedness of solutions to (1.10) under the same conditions as in [6]. Moreover, Zheng et al. [40] showed the long-time behavior of solutions under the additional condition $\mu > \max\{\frac{1}{2}, \frac{\chi^2 r}{2\delta_0^2}\}$. Recently, Fujie et al. [4] studied the asymptotic behavior in a chemotaxis model with nonlinear general diffusion for tumor invasion. There is an essential estimate $\int_0^\infty \int_\Omega |\nabla v|^2 < \infty$, which does not hold in this article. Up to know, the long-time behavior of (1.1) has not been solved.

Before stating our main results for the model (1.1), we mention the following chemotaxis-consumption model [1,11,13,14,16,17,21,23,38,39]:

$$\begin{cases} u_t = \nabla \cdot (D(u) \nabla u) - \nabla \cdot (S(u) \nabla \varphi(v)) + ru - \mu u^m, & x \in \Omega,\ t > 0, \\ v_t = \triangle v - uv, & x \in \Omega,\ t > 0, \end{cases} \tag{1.11}$$

where Ω is a bounded and smooth domain of $\mathbb{R}^{\mathbb{N}} (N \ge 1)$. Marras and Viglaloro [17] proved that if $2\beta+\alpha < 2$, the corresponding problem (1.11) possesses a unique global bounded solution provided μ is large enough under the assumption that $D(u) = (u+1)^{-\alpha}$, $S(u) = (u+1)^{\beta}$, and $\varphi'(v) = \frac{\chi}{(1+av)^2}$, where $a \ge 0$, $\chi > 0$, and $\alpha, \beta \in \mathbb{R}$. Later, Zhao and Zheng [38] proved that there exists a global classical solution if $m > 1$ for $n = 1$ or $m > 1 + \frac{n}{2}$ for $n \ge 2$ under the case that $D(u) = 1$ and $\varphi'(v) \in C^1(0, \infty)$ satisfies $\varphi'(v) \to \infty$ as $v \to 0$. Specially, when $\varphi'(v) = \frac{1}{v}$, $n = 2$, and $m > 2$, they showed the asymptotic behavior of solutions provided μ is large enough. For $r = \mu = 0$, as $D(u) \equiv 1$, $S(u) = u$, and $\varphi(v) = v$, Tao [21] proved that the global classical solution of (1.11) is uniformly bounded provided that the initial data are small enough. For $n = 3$, Tao and Winkler [23] showed that the problem (1.11) has a globally bounded weak solution in $\Omega \times (T, +\infty)$ under the assumption that Ω is a bounded convex domain. Recently, Lankeit [13] studied the existence of locally bounded global solutions of (1.11) where $D(u)$ satisfies $D(u) \ge \delta u^{-\alpha}$, $S(u) = u$, and $\varphi'(v) = \frac{1}{v}$. He showed the system (1.11) has a globally classical (or weak) solution when $N \ge 2$ and $\alpha < -\frac{n}{4}$. Liu [16] studied the global classical solution of (1.11) with $D(u) = 1$, $\varphi'(v) = \frac{1}{v}$, and $S(u)$ satisfying $0 < S(u) < K(u+1)^{\beta}$, and showed the system (1.11) admits a global classical solution when either $n = 1$ and $\beta < 2$ or $n \ge 2$ and $\beta < 1 - \frac{n}{4}$.

In the present article, we study the existence and boundedness of a globally classical solution for the chemotaxis system with nonlinear diffusion and singular sensitivity. For convenience, we use C_i and c_i to represent positive constants that can be different in different cases but remain independent of the relevant quantities. Our main results in this article are as follows.

Theorem 1.1. *Let $f(u) \equiv 0$ in (1.1) and let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with a smooth boundary. Assume the initial data satisfy (1.8) and D, S, and φ satisfy (1.2)–(1.5) with $\alpha < 0$, $k \geq 1$, and*

$$2\beta - \alpha < 2. \tag{1.12}$$

Then the problem (1.1) possesses a globally bounded and classical solution (u, v).

Theorem 1.2. *Let $f(u) = ru - \mu u^m$ with $r \in \mathbb{R}, \mu > 0$, and $m > 1$ in (1.1) and let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with a smooth boundary. Assume the initial data satisfy (1.8) and D, S, and φ satisfy (1.2)–(1.5) with $\alpha < 0$, $k \geq 1$, $2\beta + \alpha < 2$, and*

$$\begin{cases} m > 1 & \text{if } n = 1, \\ m > 2\beta + 1 & \text{if } n \geq 2. \end{cases} \tag{1.13}$$

Then the problem (1.1) possesses a globally and classical solution (u, v).

Remark 1. Theorem 1.1 solves the boundedness result of solutions that was left as an open problem for the case $\alpha < 0$ in [2]. In addition, compared with Theorem 1.1, the condition $2\beta + \alpha < 2$ in Theorem 1.2 is better than $2\beta - \alpha < 2$ in Theorem 1.1, which shows that the logistic source is beneficial for the existence of global solutions.

2. Preliminaries

Firstly, we state a result concerning the local existence of classical solutions, which can be proved by the standard contraction argument [2,4,37].

Lemma 2.1. *Suppose that $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with a smooth boundary. Assume D, S, and φ satisfy (1.2)–(1.5) with $\alpha < 0, k \geq 1$, and $2\beta - \alpha < 2$ and the initial data satisfy (1.8). If $f \equiv 0$ or $f(u) = ru - \mu u^m$ with $r \in R, \mu > 0$, and $m > 1$, then there exists $T_{\max} \in (0, \infty]$ and a local-in-time nonnegative classical solution (u, v) in $\Omega \times (0, T_{\max})$ satisfies*

$$(u, v) \in (C^0(\overline{\Omega} \times (0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})))^2. \tag{2.1}$$

Moreover, if $T_{\max} < \infty$, then

$$\limsup_{t \to T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{2.2}$$

Furthermore, the solution (u, v) of (1.1) satisfies the following properties [2,37]:

$$\begin{cases} \|u(\cdot, t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} & \text{if } f \equiv 0, \\ \|u(\cdot, t)\|_{L^1(\Omega)} \leq C & \text{if } f(u) = ru - \mu u^m \end{cases} \tag{2.3}$$

for all $t \in (0, T_{\max})$ and C is a positive constant.

In addition, according to Lemma 2.2 in [3] and (2.7) in [37], we obtain the lower bound of v.

Lemma 2.2. *Let $f(u) \equiv 0$ in (1.1). Assume (u, v) is the solution of the problem (1.1). Then there exists a constant $\eta > 0$ such that*

$$\inf_{x\in\Omega} v(x,t) \geq \eta \ \ \textit{for all}\ t \in (0, T_{\max}). \tag{2.4}$$

Let $f(u) = ru - \mu u^m$ with $r \in R, \mu > 0$, and $m > 1$ in (1.1). Assume (u, v) is the solution of the problem (1.1). Then

$$v(x,t) \geq \delta(t) := \inf_{y\in\Omega} v_0(y)e^{-t} \ \ \textit{for all}\ (x,t) \in \Omega \times (0, T_{\max}). \tag{2.5}$$

Lemma 2.3. *[25]. Assume $y(t) \geq 0$ satisfies*

$$\begin{cases} y'(t) + c_1 y^p \leq c_2, & t > 0, \\ y(0) = y_0 \end{cases}$$

with constants c_1, $p > 0$, and $c_2 \geq 0$. Then there exists a constant $c_3 > 0$ such that $y(t) \leq c_3$ for all $t > 0$.

Lemma 2.4. *[3,9] Let $1 \leq p, q \leq \infty$.*

(i) If $\frac{n}{2}(\frac{1}{p} - \frac{1}{q}) < 1$, then there exists $C_1 > 0$ such that

$$\|v(\cdot,t)\|_{L^q(\Omega)} \leq C_1(1 + \sup_{s\in(0,\infty)} \|u(\cdot,s)\|_{L^p(\Omega)}) \ \ \textit{for all}\ t > 0. \tag{2.6}$$

(ii) If $\frac{1}{2} + \frac{n}{2}(\frac{1}{p} - \frac{1}{q}) < 1$, then there exists $C_2 > 0$ such that

$$\|\nabla v(\cdot,t)\|_{L^q(\Omega)} \leq C_2(1 + \sup_{s\in(0,\infty)} \|u(\cdot,s)\|_{L^p(\Omega)}) \ \ \textit{for all}\ t > 0. \tag{2.7}$$

The following special case of the Gagliardo-Nirenberg inequality will be frequently used in the later analysis [28].

Lemma 2.5. *Assume $0 < q \leq p \leq \frac{2n}{n-2}$ (or $0 < q \leq p < \infty$ if $n = 1, 2$) and $r > 0$. Then for any $\psi \in W^{1,2}(\Omega) \cap L^q(\Omega) \cap L^r(\Omega)$, there exists a constant $C_3 > 0$ such that*

$$\|\psi\|_{L^p(\Omega)} \leq C_3(\|\nabla\psi\|_{L^2(\Omega)}^{\lambda^*} \|\psi\|_{L^q(\Omega)}^{1-\lambda^*} + \|\psi\|_{L^r(\Omega)}), \tag{2.8}$$

where $\lambda^* \in (0, 1)$ satisfies

$$\lambda^* = \frac{\frac{1}{q} - \frac{1}{p}}{\frac{1}{q} + \frac{1}{n} - \frac{1}{2}}. \tag{2.9}$$

The following lemma gives the boundedness criterion of solutions for the model (1.1), and we refer readers to [2,35] for details.

Lemma 2.6. *Let (u, v) be a solution of (1.1) defined on $[0, T_{\max})$. If there exist $C_4 > 0$ and $p > n$ such that*

$$\|u(\cdot,t)\|_{L^p(\Omega)} \leq C_4 \ \ \textit{for all}\ t \in [0, T_{\max}), \tag{2.10}$$

then we have

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \le C_5 \quad \textit{for all}\, t \in [0, T_{\max}) \tag{2.11}$$

with $C_5 > 0$ *independent of* t.

3. Proof of Theorem 1.1

Before proving our main results, we give the following a priori estimate for solutions to (1.1). For convenience, we denote $T = T_{\max}$.

Lemma 3.1. *Assume* D, S, φ, *and* f *satisfy (1.2)–(1.6) with* $k \ge 1$ *and* $2\beta+\alpha < 2$. *Then for any* $p > 2-\alpha-2\beta$ *and* $\varepsilon_1 > 0$, *there exists* $C_1(\varepsilon_1) > 0$ *such that*

$$\begin{aligned}&\tfrac{d}{dt}\textstyle\int_\Omega (u+1)^p + \frac{a_0 p(p-1)}{2}\int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2 \\ &\le \varepsilon_1 \textstyle\int_\Omega \frac{|\nabla v|^2}{v^{2k+\gamma}}(u+1)^{p+\frac{\alpha}{2}+\beta-1} + C_1(\varepsilon_1)\int_\Omega u^2, \quad t \in (0,T),\end{aligned} \tag{3.1}$$

where $\gamma = \frac{2k(1-\frac{\alpha}{2}-\beta)}{p+\alpha+2\beta-2} > 0$.

Proof. Multiplying the first equation in (1.1) by $p(u+1)^{p-1}$ and integrating the result with respect to x over Ω, we have

$$\begin{aligned}&\tfrac{d}{dt}\textstyle\int_\Omega (u+1)^p \\ &= -p(p-1)\textstyle\int_\Omega (u+1)^{p-2}D(u)|\nabla u|^2 + p(p-1)\int_\Omega (u+1)^{p-2}S(u)\nabla\varphi(v)\cdot\nabla u\end{aligned} \tag{3.2}$$

for all $t \in (0,T)$. Combining (1.4), (1.5), and Young's inequality, we have

$$\begin{aligned}&p(p-1)\textstyle\int_\Omega (u+1)^{p-2}S(u)\nabla\varphi(v)\cdot\nabla u \\ &\le \tfrac{a_0 p(p-1)}{2}\textstyle\int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2 + \frac{p(p-1)}{2a_0}\int_\Omega (u+1)^{p-2+\alpha}S^2(u)|\nabla\varphi(v)|^2 \\ &\le \tfrac{a_0 p(p-1)}{2}\textstyle\int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2 + \frac{\chi^2 b_0^2 p(p-1)}{2a_0}\int_\Omega \frac{(u+1)^{p+\alpha+2\beta-2}}{v^{2k}}|\nabla v|^2\end{aligned} \tag{3.3}$$

for all $t \in (0,T)$. Since $p > 2-\alpha-2\beta$, $k \ge 1$, and $2\beta+\alpha < 2$, we have from Young's inequality that

$$\tfrac{\chi^2 b_0^2 p(p-1)}{2a_0}\textstyle\int_\Omega \frac{(u+1)^{p+\alpha+2\beta-2}}{v^{2k}}|\nabla v|^2 \le \varepsilon_1\int_\Omega \frac{|\nabla v|^2}{v^{2k+\gamma}}(u+1)^{p+\frac{\alpha}{2}+\beta-1} + c_1(\varepsilon_1)\int_\Omega |\nabla v|^2, \tag{3.4}$$

where $\gamma = \frac{2k(1-\frac{\alpha}{2}-\beta)}{p+\alpha+2\beta-2} > 0$. By Lemma 2.4, there exists a constant $c_2 > 0$ such that

$$\textstyle\int_\Omega |\nabla v|^2 \le c_2 \int_\Omega u^2. \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3), we have

$$\begin{aligned}&p(p-1)\textstyle\int_\Omega (u+1)^{p-2}S(u)\nabla\varphi(v)\nabla u \\ &\le \tfrac{a_0 p(p-1)}{2}\textstyle\int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2 + \varepsilon_1\int_\Omega \frac{|\nabla v|^2}{v^{2k+\gamma}}(u+1)^{p+\frac{\alpha}{2}+\beta-1} + c_3(\varepsilon_1)\int_\Omega u^2\end{aligned} \tag{3.6}$$

for all $t \in (0,T)$. Inserting (3.6) into (3.2), we obtain (3.1). □

Lemma 3.2. *Assume* D, S, φ, *and* f *satisfy (1.2)–(1.6) with* $k \ge 1$, $\alpha < 0$, *and* $2\beta - \alpha < 2$. *Then for any* $p > \max\{2-\frac{\alpha}{2}-\beta, 4-2\alpha-4\beta\}$, *there exists* $C_2 > 0$ *such that*

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma-2}} + \frac{(2k+\gamma-2)(2k+\gamma-1)}{4}\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma}}|\nabla v|^2 \\
&\le \frac{a_0p(p-1)}{4}\int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2 + C_2\int_\Omega u^2 + C_2\int_\Omega (u+1)^{p+\frac{\alpha}{2}+\beta-1},\ t\in(0,T),
\end{aligned}$$

where $\gamma = \frac{2k(1-\frac{\alpha}{2}-\beta)}{p+\alpha+2\beta-2}$.

Proof. By comparison with the second equation in (1.1) and combination with (1.8), we have $v > 0$ in $\Omega\times(0,T)$. By direct computation of $\frac{d}{dt}\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma-2}}$ and using the first two equations in (1.1) and considering (1.3)–(1.5), we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma-2}} &= (p+\tfrac{\alpha}{2}+\beta-1)\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-2}}{v^{2k+\gamma-2}}\nabla\cdot(D(u)\nabla u - S(u)\nabla\varphi(v))dx \\
&\quad -(2k+\gamma-2)\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma-1}}(\triangle v - v + u)dx \\
&\le -a_0(p+\tfrac{\alpha}{2}+\beta-1)(p+\tfrac{\alpha}{2}+\beta-2)\int_\Omega \frac{(u+1)^{p-\frac{\alpha}{2}+\beta-3}}{v^{2k+\gamma-2}}|\nabla u|^2 \\
&\quad +a_0(p+\tfrac{\alpha}{2}+\beta-1)(2k+\gamma-2)\int_\Omega \frac{(u+1)^{p-\frac{\alpha}{2}+\beta-2}}{v^{2k+\gamma-1}}|\nabla u||\nabla v| \\
&\quad -\chi(p+\tfrac{\alpha}{2}+\beta-1)(2k+\gamma-2)\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-2}}{v^{3k+\gamma-1}}S(u)|\nabla v|^2 \\
&\quad +\chi b_0(p+\tfrac{\alpha}{2}+\beta-1)(p+\tfrac{\alpha}{2}+\beta-2)\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+2\beta-3}}{v^{3k+\gamma-2}}|\nabla u||\nabla v| \\
&\quad -(2k+\gamma-2)(2k+\gamma-1)\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma}}|\nabla v|^2 \\
&\quad +(2k+\gamma-2)(p+\tfrac{\alpha}{2}+\beta-1)\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-2}}{v^{2k+\gamma-1}}|\nabla u||\nabla v| \\
&\quad +(2k+\gamma-2)\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma-2}} \\
&\quad -(2k+\gamma-2)\int_\Omega \frac{u(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma-1}}
\end{aligned} \tag{3.7}$$

for all $t\in(0,T)$. Because $p > 2-\frac{\alpha}{2}-\beta$ and $k\ge 1$, it is easy to see that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma-2}} + (2k+\gamma-2)(2k+\gamma-1)\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma}}|\nabla v|^2 \\
&\le a_0(p+\tfrac{\alpha}{2}+\beta-1)(2k+\gamma-2)\int_\Omega \frac{(u+1)^{p-\frac{\alpha}{2}+\beta-2}}{v^{2k+\gamma-1}}|\nabla u||\nabla v| \\
&\quad +\chi b_0(p+\tfrac{\alpha}{2}+\beta-1)(p+\tfrac{\alpha}{2}+\beta-2)\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+2\beta-3}}{v^{3k+\gamma-1}}|\nabla u||\nabla v| \\
&\quad +(2k+\gamma-2)(p+\tfrac{\alpha}{2}+\beta-1)\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-2}}{v^{2k+\gamma-1}}|\nabla u||\nabla v| \\
&\quad +(2k+\gamma-2)\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma-2}}.
\end{aligned} \tag{3.8}$$

Now we deal with the terms on the right-hand side of (3.8). We know from Cauchy's inequality, $k\ge 1$, and $v\ge\eta$ that

$$\begin{aligned}
A_1 &:= a_0(p+\tfrac{\alpha}{2}+\beta-1)(2k+\gamma-2)\int_\Omega \frac{(u+1)^{p-\frac{\alpha}{2}+\beta-2}}{v^{2k+\gamma-1}}|\nabla u||\nabla v| \\
&\le \varepsilon_2\int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2 + c_1(\varepsilon_2)\int_\Omega \frac{|\nabla v|^2}{v^{4k+2\gamma-2}}(u+1)^{p+2\beta-2} \\
&\le \varepsilon_2\int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2 + \frac{c_1(\varepsilon_2)}{\eta^{2k+\gamma-2}}\int_\Omega \frac{|\nabla v|^2}{v^{2k+\gamma}}(u+1)^{p+2\beta-2}.
\end{aligned} \tag{3.9}$$

Using Young's inequality and $2\beta-\alpha<2$, we infer that

$$\frac{c_1(\varepsilon_2)}{\eta^{2k+\gamma-2}}\int_\Omega \frac{|\nabla v|^2}{v^{2k+\gamma}}(u+1)^{p+2\beta-2} \le \frac{(2k+\gamma-1)(2k+\gamma-2)}{4}\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma}}|\nabla v|^2 + c_2(\varepsilon_2)\int_\Omega \frac{|\nabla v|^2}{v^{2k+\gamma}}. \tag{3.10}$$

By Lemma 2.4 and $v\ge\eta$, we have

$$c_2(\varepsilon_2)\int_\Omega \frac{|\nabla v|^2}{v^{2k+\gamma}} \le c_3(\varepsilon_2)\int_\Omega u^2. \tag{3.11}$$

Combining (3.9)–(3.11), we have

$$A_1 \le \varepsilon_2 \int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2 + \frac{(2k+\gamma-1)(2k+\gamma-2)}{4}\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma}}|\nabla v|^2 + c_3(\varepsilon_2)\int_\Omega u^2. \tag{3.12}$$

Next we estimate the second term on the right-hand side of (3.8). We know from $\alpha < 0$ and $2\beta - \alpha < 2$ that $2\beta + \alpha < 2$, which together with $p > 4 - 2\alpha - 4\beta$, $v \ge \eta$ and Young's inequality yields

$$\begin{aligned}
A_2 &:= \chi b_0 (p+\tfrac{\alpha}{2}+\beta-1)(p+\tfrac{\alpha}{2}+\beta-2)\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+2\beta-3}}{v^{3k+\gamma-1}}|\nabla u||\nabla v| \\
&\le \frac{\chi b_0(p+\frac{\alpha}{2}+\beta-1)(p+\frac{\alpha}{2}+\beta-2)}{\eta^k}\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+2\beta-3}}{v^{2k+\gamma-1}}|\nabla u||\nabla v| \\
&\le \varepsilon_2\int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2 + \frac{c_4(\varepsilon_2)}{\eta^{2k+\gamma-2}}\int_\Omega \frac{(u+1)^{p+4\beta+2\alpha-4}}{v^{2k+\gamma}}|\nabla v|^2 \\
&\le \varepsilon_2\int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2 + \frac{(2k+\gamma-1)(2k+\gamma-2)}{4}\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma}}|\nabla v|^2 + c_5(\varepsilon_2)\int_\Omega u^2.
\end{aligned} \tag{3.13}$$

Similarly,

$$\begin{aligned}
A_3 &:= (2k+\gamma-2)(p+\tfrac{\alpha}{2}+\beta-1)\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-2}}{v^{2k+\gamma-1}}|\nabla u||\nabla v| \\
&\le \varepsilon_2\int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2 + \frac{c_6(\varepsilon_2)}{\eta^{2k+\gamma-2}}\int_\Omega \frac{(u+1)^{p+2\beta+2\alpha-2}}{v^{2k+\gamma}}|\nabla v|^2 \\
&\le \varepsilon_2\int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2 + \frac{(2k+\gamma-1)(2k+\gamma-2)}{4}\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma}}|\nabla v|^2 + c_7(\varepsilon_2)\int_\Omega u^2.
\end{aligned} \tag{3.14}$$

In addition, for $k \ge 1$ it is easy to see

$$A_4 := (2k+\gamma-2)\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma-2}} \le c_8\int_\Omega (u+1)^{p+\frac{\alpha}{2}+\beta-1}. \tag{3.15}$$

Collecting (3.8) and (3.12)–(3.15) and setting $\varepsilon_2 = \frac{a_0 p(p-1)}{12}$ yields Lemma 3.2. □

Lemma 3.3. *Suppose that (1.2)–(1.6) hold with $\alpha < 0$, $k \ge 1$, and $2\beta - \alpha < 2$. Let $p > \max\{n, 2 - \frac{\alpha}{2} - \beta, 4 - 2\alpha - 4\beta\}$. Then there exists a constant $C_3 > 0$ such that*

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le C_3 \tag{3.16}$$

for all $t \in (0,T)$.

Proof. Since $2\beta - \alpha < 2$ and $\alpha < 0$, it is easy to obtain $2\beta + \alpha < 2$. We set $\varepsilon_1 = \frac{(2k+\gamma-2)(2k+\gamma-1)}{4}$ in Lemma 3.1. Combining Lemmas 3.1 and 3.2, we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_\Omega (u+1)^p + \int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma-2}}\Big) + \frac{a_0 p(p-1)}{4}\int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2 \\
&\le c_1\int_\Omega u^2 + c_1\int_\Omega (u+1)^{p+\frac{\alpha}{2}+\beta-1}
\end{aligned} \tag{3.17}$$

for all $t \in (0,T)$. We have from the Gagliardo-Nirenberg inequality that

$$\begin{aligned}
\int_\Omega (u+1)^p &= \|(u+1)^{\frac{p-\alpha}{2}}\|_{L^{\frac{2p}{p-\alpha}}}^{\frac{2p}{p-\alpha}} \\
&\le c_2\|\nabla (u+1)^{\frac{p-\alpha}{2}}\|_{L^2}^{\frac{2p}{p-\alpha}\theta_1}\|(u+1)^{\frac{p-\alpha}{2}}\|_{L^{\frac{2}{p-\alpha}}}^{\frac{2p}{p-\alpha}(1-\theta_1)} + c_2\|(u+1)^{\frac{p-\alpha}{2}}\|_{L^{\frac{2}{p-\alpha}}}^{\frac{2p}{p-\alpha}},
\end{aligned} \tag{3.18}$$

where $\theta_1 = \frac{\frac{p-\alpha}{2} - \frac{p-\alpha}{2p}}{\frac{p-\alpha}{2} + \frac{1}{n} - \frac{1}{2}} \in (0,1)$ because $p > 1$ and $\alpha < 0$. Applying (2.3) to (3.18), we infer that

$$\int_\Omega (u+1)^p \le c_3\left(\int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2\right)^{\frac{\theta_1 p}{p-\alpha}} + c_3, \tag{3.19}$$

where

$$\frac{\theta_1 p}{p-\alpha} = \frac{(p-1)n}{n(p-\alpha)+2-n}. \tag{3.20}$$

It follows that $\frac{\theta_1 p}{p-\alpha} < 1$ because $p > 1$ and $\alpha < 0$. Applying Young's inequality to (3.19), we have

$$\int_\Omega (u+1)^p \le \frac{a_0 p(p-1)}{16} \int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2 + c_4. \tag{3.21}$$

In addition, it follows from Young's inequality, $v \ge \eta$, and $2\beta + \alpha < 2$ that

$$\begin{aligned} \int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma-2}} &\le \frac{1}{\eta^{2k+\gamma-2}} \int_\Omega (u+1)^{p+\frac{\alpha}{2}+\beta-1} \\ &\le \int_\Omega (u+1)^p dx + c_5, \end{aligned} \tag{3.22}$$

which together with (3.21) implies

$$\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma-2}} \le \frac{a_0 p(p-1)}{16} \int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2 + c_6. \tag{3.23}$$

Similarly, we can obtain

$$c_1 \int_\Omega u^2 \le \frac{a_0 p(p-1)}{16} \int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2 + c_7 \tag{3.24}$$

and

$$c_1 \int_\Omega (u+1)^{p+\frac{\alpha}{2}+\beta-1} \le \frac{a_0 p(p-1)}{16} \int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2 + c_8. \tag{3.25}$$

Thus, we define $\phi(t) := \int_\Omega (u+1)^p + \int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma-2}}$. Combining (3.17), (3.21), and (3.23)–(3.25), we see that

$$\phi'(t) + \phi(t) \le c_9, \ t \in (0, T) \tag{3.26}$$

with $c_9 > 0$. So by Lemma 2.3, there exists a constant $c_{10} > 0$ such that

$$\|u(\cdot, t)\|_{L^p(\Omega)} \le c_{10} \quad \text{for all } t \in (0, T).$$

The proof of Lemma 3.3 is complete. □

The proof of Theorem 1.1. In view of Lemmas 2.1 and 2.6, Theorem 1.1 is a direct consequence of Lemma 3.3. □

4. Proof of Theorem 1.2

Lemma 4.1. *Assume D, S, and φ satisfy (1.2)–(1.5) with $k \ge 1$ and $2\beta + \alpha < 2$ and f satisfies (1.7) with $r \in R, \mu > 0$, and $m > 1$. Then for any $\varepsilon_1 > 0$ and $p > \max\{2 - m, 2 - \alpha - 2\beta\}$, there exist $C_1(\varepsilon_1)$ and $C_2 > 0$ such that*

$$\begin{aligned} &\frac{d}{dt} \int_\Omega (u+1)^p + \frac{a_0 p(p-1)}{2} \int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2 \\ &\le \varepsilon_1 \int_\Omega \frac{|\nabla v|^2}{v^{2k+\gamma}} (u+1)^{p+\frac{\alpha}{2}+\beta-1} + C_1(\varepsilon_1) \int_\Omega u^2 - \frac{\mu p}{2} \int_\Omega (u+1)^{m+p-1} + C_2, \ t \in (0, T), \end{aligned} \tag{4.1}$$

where $\gamma = \frac{2k(1-\frac{\alpha}{2}-\beta)}{p+\alpha+2\beta-2}$.

Proof. Multiplying the first equation in (1.1) by $p(u+1)^{p-1}$ and integrating the result with respect to x over Ω, we have

$$\begin{aligned}&\tfrac{d}{dt}\int_\Omega (u+1)^p\\ &= -p(p-1)\int_\Omega (u+1)^{p-2}D(u)|\nabla u|^2 + p(p-1)\int_\Omega (u+1)^{p-2}S(u)\nabla\varphi(v)\nabla u\\ &\quad + rp\int_\Omega u(u+1)^{p-1} - \mu p\int_\Omega u^m(u+1)^{p-1}.\end{aligned} \tag{4.2}$$

Since $-u^m \le -(u+1)^m + m(u+1)^{m-1}$ for $m>1$, we have

$$\begin{aligned}&rp\int_\Omega u(u+1)^{p-1} - \mu p\int_\Omega u^m(u+1)^{p-1}\\ &\le |r|p\int_\Omega (u+1)^p - \mu p\int_\Omega (u+1)^{m+p-1} + m\mu p\int_\Omega (u+1)^{m+p-2}.\end{aligned} \tag{4.3}$$

We know from Young's inequality and $p > 2-m$ that

$$|r|p\int_\Omega (u+1)^p \le \tfrac{\mu p}{4}\int_\Omega (u+1)^{m+p-1} + c_1 \tag{4.4}$$

and

$$m\mu p\int_\Omega (u+1)^{m+p-2} \le \tfrac{\mu p}{4}\int_\Omega (u+1)^{m+p-1} + c_2. \tag{4.5}$$

Collecting (4.3)–(4.5), we have

$$rp\int_\Omega u(u+1)^{p-1} - \mu p\int_\Omega u^m(u+1)^{p-1} \le -\tfrac{\mu p}{2}\int_\Omega (u+1)^{m+p-1} + c_3. \tag{4.6}$$

Inserting (4.6) and (3.6) into (4.2), we obtain (4.1). The proof of Lemma 4.1 is complete. □

Lemma 4.2. *Assume $m>1$, $\alpha<0$, and $2\beta+\alpha<2$. Then there exists a constant $p_0>1$ such that for any $p>p_0$ the following inequalities hold:*

$$0 < \frac{p-\alpha-\frac{(p-\alpha)(m+p-3)}{(p+2\beta-2)(m+p-1)}}{p-\alpha+1} < 1 \tag{4.7}$$

and

$$0 < \frac{(p+2\beta-2)(m+p-1)-m-p+3}{(p-\alpha+1)(m+p-3)} < 1. \tag{4.8}$$

Proof. Let $p > \max\{1-\alpha, 3-2\beta, 3-m\}$. Then we have

$$\frac{p-\alpha-\frac{(p-\alpha)(m+p-3)}{(p+2\beta-2)(m+p-1)}}{p-\alpha+1} > 0 \tag{4.9}$$

and

$$\frac{(p+2\beta-2)(m+p-1)-m-p+3}{(p-\alpha+1)(m+p-3)} > 0. \tag{4.10}$$

Define the function $g:\mathbb{R}^+\to\mathbb{R}$ satisfies

$$g(s) = 2s^2 + (2m-\alpha+2\beta-6)s - \alpha(m-3) + 2(\beta-1)(m-1),\ s>0.$$

It is easy to see there exists a positive constant $p_1>1$ such that $g(s)>0$ for any $s>p_1$. This means

$$\frac{p-\alpha-\frac{(p-\alpha)(m+p-3)}{(p+2\beta-2)(m+p-1)}}{p-\alpha+1}<1 \quad \text{for } p>p_1. \tag{4.11}$$

In addition, since $2\beta+\alpha<2$, if we take $p_2=\frac{2(\beta-1)(m-1)-(2-\alpha)(m-3)}{2-\alpha-2\beta}$, then for any $p>p_2$, we obtain

$$\frac{(p+2\beta-2)(m+p-1)-m-p+3}{(p-\alpha+1)(m+p-3)}<1. \tag{4.12}$$

Combining (4.9)–(4.12) and setting

$$p_0=\max\{1-\alpha,3-2\beta,3-m,p_1,p_2\},$$

we obtain (4.7) and (4.8). The proof of Lemma 4.2 is complete. □

Lemma 4.3. *Assume D, S, and φ satisfy (1.2)–(1.5) with $\alpha<0$, $k\geq 1$, and $2\beta+\alpha<2$ and f satisfies (1.7) with $\mu>0$ and*

$$\begin{cases} m>1 & \text{if } n=1,\\ m>2\beta+1 & \text{if } n\geq 2.\end{cases}$$

If p_0 is given by Lemma 4.2, then for any $p>\max\{p_0,2-\frac{\alpha}{2}-\beta,4-2\alpha-4\beta\}$, there exists $C_3(T)>0$ such that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma-2}}+\frac{(2k+\gamma-2)(2k+\gamma-1)}{3}\int_\Omega\frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma}}|\nabla v|^2\\&\leq\frac{a_0p(p-1)}{4}\int_\Omega(u+1)^{p-\alpha-2}|\nabla u|^2+\frac{\mu p}{2}\int_\Omega(u+1)^{m+p-1}+C_3(T)\int_\Omega u^2\\&\quad+C_3(T)\int_\Omega(u+1)^{p+\frac{\alpha}{2}+\beta-1}+C_3(T)\end{aligned}$$

for all $t\in(0,T)$.

Proof. Similarly to the proof of Lemma 3.2, by direct computation of $\frac{d}{dt}\int_\Omega\frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma-2}}$, we have

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma-2}}\\&=(p+\frac{\alpha}{2}+\beta-1)\int_\Omega\frac{(u+1)^{p+\frac{\alpha}{2}+\beta-2}}{v^{2k+\gamma-2}}[\nabla\cdot(D(u)\nabla u-S(u)\nabla\varphi(v))+f(u)]dx\\&\quad-(2k+\gamma-2)\int_\Omega\frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma-1}}(\triangle v-v+u)dx.\end{aligned}\tag{4.13}$$

Compared with (3.7) in Lemma 3.2, we need only to estimate $\int_\Omega\frac{(u+1)^{p+\frac{\alpha}{2}+\beta-2}}{v^{2k+\gamma-2}}f(u)$. Since $f(u)=ru-\mu u^m$ and $\mu>0$, we have

$$\begin{aligned}\int_\Omega\frac{(u+1)^{p+\frac{\alpha}{2}+\beta-2}}{v^{2k+\gamma-2}}f(u)&\leq|r|\int_\Omega\frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma-2}}-\mu\int_\Omega\frac{(u+1)^{p+\frac{\alpha}{2}+\beta-2}u^m}{v^{2k+\gamma-2}}\\&\leq|r|\int_\Omega\frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma-2}}.\end{aligned}\tag{4.14}$$

Collecting (3.7), (3.8), (4.13), and (4.14), we get

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma-2}}+(2k+\gamma-2)(2k+\gamma-1)\int_\Omega\frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma}}|\nabla v|^2\\&\leq A_1+A_2+A_3+A_4+A_5,\end{aligned}\tag{4.15}$$

where

$$
\begin{aligned}
A_1 &:= a_0(p+\tfrac{\alpha}{2}+\beta-1)(2k+\gamma-2)\int_\Omega \tfrac{(u+1)^{p-\frac{\alpha}{2}+\beta-2}}{v^{2k+\gamma-1}}|\nabla u||\nabla v|,\\
A_2 &:= \chi b_0(p+\tfrac{\alpha}{2}+\beta-1)(p+\tfrac{\alpha}{2}+\beta-2)\int_\Omega \tfrac{(u+1)^{p+\frac{\alpha}{2}+2\beta-3}}{v^{3k+\gamma-1}}|\nabla u||\nabla v|,\\
A_3 &:= (2k+\gamma-2)(p+\tfrac{\alpha}{2}+\beta-1)\int_\Omega \tfrac{(u+1)^{p+\frac{\alpha}{2}+\beta-2}}{v^{2k+\gamma-1}}|\nabla u||\nabla v|,\\
A_4 &:= (2k+\gamma-2)\int_\Omega \tfrac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma-2}},\\
A_5 &:= |r|(p+\tfrac{\alpha}{2}+\beta-1)\int_\Omega \tfrac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma-2}}.
\end{aligned}
$$

Next we estimate A_1–A_5. Applying Cauchy's inequality to A_1 and using $v \geq \delta(T)$, we have

$$A_1 \leq \varepsilon_2 \int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2 + c_1(T)\int_\Omega (u+1)^{p+2\beta-2}|\nabla v|^2. \tag{4.16}$$

We know from Young's inequality that

$$
\begin{aligned}
&c_1(T)\int_\Omega (u+1)^{p+2\beta-2}|\nabla v|^2\\
&\leq \varepsilon_3 \int_\Omega |\nabla v|^{m+p-1} + c_2(T)\int_\Omega (u+1)^{\frac{(m+p-1)(p+2\beta-2)}{m+p-3}}
\end{aligned}
\tag{4.17}
$$

for all $t \in (0,T)$.

Firstly, we estimate the first term on right-hand side of the inequality (4.17). For any $n \geq 1$, by Lemma 2.4, there exists a constant $c_3 > 0$ such that

$$\int_\Omega |\nabla v|^{m+p-1} \leq c_3 \int_\Omega (u+1)^{m+p-1}. \tag{4.18}$$

Next we estimate the last term in (4.17). We firstly consider the case $n = 1$. Applying the Gagliardo-Nirenberg inequality, we have

$$
\begin{aligned}
&c_2(T)\int_\Omega (u+1)^{\frac{(m+p-1)(p+2\beta-2)}{m+p-3}}\\
&= c_2(T)\|(u+1)^{\frac{p-\alpha}{2}}\|_{L^{\frac{2(m+p-1)(p+2\beta-2)}{(m+p-3)(p-\alpha)}}}^{\frac{2(m+p-1)(p+2\beta-2)}{(p-\alpha)(m+p-3)}}\\
&\leq c_3(T)\|\nabla(u+1)^{\frac{p-\alpha}{2}}\|_{L^2}^{\frac{2(m+p-1)(p+2\beta-2)}{(p-\alpha)(m+p-3)}\theta_2}\|(u+1)^{\frac{p-\alpha}{2}}\|_{L^{\frac{2}{p-\alpha}}}^{\frac{2(m+p-1)(p+2\beta-2)}{(p-\alpha)(m+p-3)}(1-\theta_2)}\\
&\quad + c_3(T)\|(u+1)^{\frac{p-\alpha}{2}}\|_{L^{\frac{2}{p-\alpha}}}^{\frac{2(m+p-1)(p+2\beta-2)}{(p-\alpha)(m+p-3)}},
\end{aligned}
$$

where $\theta_2 = \frac{p-\alpha-\frac{(p-\alpha)(m+p-3)}{(p+2\beta-2)(m+p-1)}}{p-\alpha+1}$. It follows that $\theta_2 \in (0,1)$ because of (4.7). So using $\int_\Omega u dx \leq M$, we have

$$c_2(T)\int_\Omega (u+1)^{\frac{(m+p-1)(p+2\beta-2)}{m+p-3}} \leq c_4(T)\left(\int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2\right)^{\frac{(p+2\beta-2)(m+p-1)-m-p+3}{(p-\alpha+1)(m+p-3)}} + c_4(T).$$

We have from Lemma 4.2 and $2\beta+\alpha < 2$ that for any $m > 1$ the inequality

$$0 < \frac{(p+2\beta-2)(m+p-1)-m-p+3}{(p-\alpha+1)(m+p-3)} < 1$$

holds for $p > p_0$. Thus, Young's inequality yields

$$c_2(T)\int_\Omega (u+1)^{\frac{(m+p-1)(p+2\beta-2)}{m+p-3}} \leq \frac{a_0 p(p-1)}{24}\int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2 + c_5(T). \tag{4.19}$$

Next we consider the case $n \geq 2$. Since $m > 2\beta+1$ and $p > p_0$, applying Young's inequality to the last term in (4.17), we have

$$c_2(T)\int_\Omega (u+1)^{\frac{(m+p-1)(p+2\beta-2)}{m+p-3}} \le \frac{\mu p}{4}\int_\Omega (u+1)^{m+p-1} + c_6(T). \tag{4.20}$$

For the case $n = 1$, setting $\varepsilon_2 = \frac{a_0 p(p-1)}{24}$ in (4.16) and $\varepsilon_3 = \frac{\mu p}{2c_3}$ in (4.17) and inserting the estimate (4.17)–(4.19) into (4.16), we have

$$A_1 \le \frac{a_0 p(p-1)}{12}\int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2 + \frac{\mu p}{2}\int_\Omega (u+1)^{m+p-1} + c_7(T). \tag{4.21}$$

For the case $n \ge 2$, setting $\varepsilon_2 = \frac{a_0 p(p-1)}{12}$ in (4.16) and $\varepsilon_3 = \frac{\mu p}{4c_3}$ in (4.17) and inserting the estimate (4.17), (4.18), and (4.20) into (4.16), we have

$$A_1 \le \frac{a_0 p(p-1)}{12}\int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2 + \frac{\mu p}{2}\int_\Omega (u+1)^{m+p-1} + c_8(T). \tag{4.22}$$

Finally, we consider the estimates of A_2–A_4. Similarly to Lemma 3.2, for any $n \ge 1$, since $\alpha < 0$, $2\beta + \alpha < 2$, and $v \ge \delta(T)$, we also have

$$A_2 \le \frac{a_0 p(p-1)}{12}\int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2 + \frac{(2k+\gamma-1)(2k+\gamma-2)}{3}\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma}}|\nabla v|^2 + c_9(T)\int_\Omega u^2 \tag{4.23}$$

and

$$A_3 \le \frac{a_0 p(p-1)}{12}\int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2 + \frac{(2k+\gamma-1)(2k+\gamma-2)}{3}\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma}}|\nabla v|^2 + c_{10}(T)\int_\Omega u^2, \tag{4.24}$$

which used $2\beta + 3\alpha < 2$. In addition, since $v \ge \delta(T)$, we have

$$A_4 + A_5 \le c_{11}(T)\int_\Omega (u+1)^{p+\frac{\alpha}{2}+\beta-1}. \tag{4.25}$$

Inserting (4.21)–(4.25) into (4.15), we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma-2}} + \frac{(2k+\gamma-2)(2k+\gamma-1)}{3}\int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma}}|\nabla v|^2\\
&\le \frac{a_0 p(p-1)}{4}\int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2 + \frac{\mu p}{2}\int_\Omega (u+1)^{m+p-1} + c_{12}(T)\int_\Omega u^2\\
&\quad + c_{12}(T)\int_\Omega (u+1)^{p+\frac{\alpha}{2}+\beta-1} + c_{12}(T)
\end{aligned}$$

for all $t \in (0, T)$. The proof of Lemma 4.3 is complete. □

Lemma 4.4. *Assume D, S, and φ satisfy (1.2)–(1.5) with $\alpha < 0$, $k \ge 1$, and $2\beta + \alpha < 2$ and f satisfies (1.7) with $\mu > 0$ and*

$$\begin{cases} m > 1 & \text{if } n = 1,\\ m > 2\beta + 1 & \text{if } n \ge 2.\end{cases}$$

If p_0 is given by Lemma 4.2, then for any $p > \max\{p_0, 2 - \frac{\alpha}{2} - \beta, 4 - 2\alpha - 4\beta, n\}$ there exists $C_4(T) > 0$ such that

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le C_4(T) \quad \text{for all } t \in (0, T). \tag{4.26}$$

Proof. Set $\varepsilon_1 = \frac{(2k+\gamma-2)(2k+\gamma-1)}{3}$ in (4.1). We know from Lemmas 4.1 and 4.3 that

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_\Omega (u+1)^p + \int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma-2}}\Big) + \frac{a_0 p(p-1)}{4}\int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2\\
&\le c_1(T)\int_\Omega u^2 + c_2(T)\int_\Omega (u+1)^{p+\frac{\alpha}{2}+\beta-1} + c_3(T),\ t \in (0, T).
\end{aligned} \tag{4.27}$$

Similarly to the proof of Lemma 3.3, using Young's inequality, we have

$$c_1(T)\int_\Omega u^2 \le \frac{a_0p(p-1)}{8}\int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2 + c_4(T) \tag{4.28}$$

and

$$c_2(T)\int_\Omega (u+1)^{p+\frac{\alpha}{2}+\beta-1} \le \frac{a_0p(p-1)}{8}\int_\Omega (u+1)^{p-\alpha-2}|\nabla u|^2 + c_5(T) \tag{4.29}$$

for $t\in(0,T)$. Inserting (4.28) and (4.29) into (4.27), we infer that

$$\frac{d}{dt}\Big(\int_\Omega (u+1)^p + \int_\Omega \frac{(u+1)^{p+\frac{\alpha}{2}+\beta-1}}{v^{2k+\gamma-2}}\Big) \le c_6(T),\ t\in(0,T), \tag{4.30}$$

which yields (4.26). The proof of Lemma 4.4 is complete. □

The proof of Theorem 1.2. In view of Lemmas 2.1 and 2.6, Theorem 1.2 is a direct consequence of Lemma 4.4. □

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