

# Accepted Manuscript

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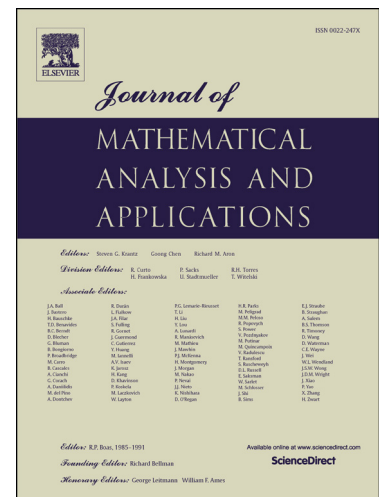
PII: S0022-247X(19)30214-8  
DOI: <https://doi.org/10.1016/j.jmaa.2019.03.005>  
Reference: YJMAA 23015

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 1 October 2018

Please cite this article in press as: G. Xu et al., On potential wells to a semilinear hyperbolic equation with damping and conical singularity, *J. Math. Anal. Appl.* (2019), <https://doi.org/10.1016/j.jmaa.2019.03.005>

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# On potential wells to a semilinear hyperbolic equation with damping and conical singularity

Guangyu Xu\* Chunlai Mu† Hong Yi‡

College of Mathematics and Statistics, Chongqing University,  
Chongqing, 401331, P.R.China

## Abstract

This paper deals with a semilinear hyperbolic equation with damping and conical singularity, which was built in [J. Math. Anal. Appl. 455(2017) 569-591], where the weak solution with low initial energy ( $I(0) < d$ ,  $d$  is potential well depth) was considered. We extend the previous results on following three aspects. Firstly, we consider vacuum isolating phenomenon of solution under initial energy  $I(0) \leq 0$  and  $0 < I(0) < d$  respectively. We find that there are two explicit vacuum regions which are annulus and ball respectively. Moreover, we get the asymptotic behavior of energy functional as  $t$  tends to the maximal existence time, and then two necessary and sufficient conditions for the weak solution existing globally and blowing up in finite time is obtained. Secondly, we discuss the weak solution with critical initial energy and establish the global existence and non-existence results. Finally, the weak solution with arbitrary positive initial energy is studied. In this case, the initial conditions such that the weak solution exists globally and blows up in finite time are given, respectively.

**Keywords:** Cone Sobolev space; Vacuum isolating phenomena; Global existence; Finite time blow-up; High initial energy

**2010 MSC:** 35A01, 35B06, 35B44, 35D30, 35L20, 35R01.

## 1 Introduction

Let  $X$  be a  $(n-1)$ -dimensional closed compact  $C^\infty$ -smooth manifold, which is regarded as the local model near the conical points.  $\mathbb{B} = [0, 1) \times X$ ,  $\partial\mathbb{B} = \{0\} \times X$ , near  $\partial\mathbb{B}$  we use the coordinates  $(x_1, x') = (x_1, x_2, \dots, x_n)$  for  $0 \leq x_1 < 1, x' \in X$ . We denote by  $\mathbb{B}_0$  the interior of  $\mathbb{B}$ .

In this paper, we consider the initial boundary value problem with following hyperbolic equation:

$$(1.1) \quad \begin{cases} u_{tt} - \Delta_{\mathbb{B}} u + V(x)u + \gamma u_t = g_t(x)|u|^{p-1}u, & t > 0, x \in \mathbb{B}_0, \\ u(x, t) = 0, & t \geq 0, x \in \partial\mathbb{B}, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & x \in \mathbb{B}_0, \end{cases}$$

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\*Corresponding author, e-mail: guangyuswu@126.com

†e-mail: clmu2005@163.com

‡e-mail: honghongyi@126.com

where the initial values are nontrivial functions and belong to the weighted Sobolev space, we will introduce the weighted Sobolev space in the next section.  $2 < p + 1 < \frac{2n}{n-2}$ , the Fuchsian type Laplace operator is defined as

$$\Delta_{\mathbb{B}} := \nabla_{\mathbb{B}}^2 = (x_1 \partial_{x_1})^2 + \partial_{x_2}^2 + \cdots + \partial_{x_n}^2,$$

which is an elliptic operator with conical degeneration on the boundary  $x_1 = 0$ , and the corresponding gradient operator is

$$\nabla_{\mathbb{B}} := (x_1 \partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n}).$$

The potential function  $V(x) \in L^{\frac{n}{4}}(\mathbb{B}_0) \cap C(\mathbb{B}_0)$  is positive such that  $\inf_{x \in \mathbb{B}} V(x) > 0$ . We suppose that  $g_t(x) : \mathbb{B} \rightarrow \mathbb{R}$  is a nonnegative function which  $g_t(x) := g(x, t)$  for every  $x \in \mathbb{B}_0$  and  $g(x, t) \in L^\infty(\mathbb{B}_0) \cap C^1(\mathbb{B}_0)$ .

In recent years, evolution equations with conical degeneration has been attracting considerable attention in the research field of analysis of nonlinear PDEs. In [12, 13, 19, 33], the authors studied the differential operators subject to homogeneous boundary conditions on weighted  $L^p$ -Sobolev spaces over a manifold with conical singularities. Later on, a class of weighted Sobolev spaces and the corresponding cone Sobolev inequality and Poincaré inequality (see Section 2) was introduced in [7, 8]. Based on those precursory results, many academicians investigated PDEs with conical degeneration and their analogue, for example, one can see [1, 2, 3] for wave equation and see [6, 9, 16, 17, 39] for edge-degenerate equation with singular potentials. For more works about degenerate differential operator and equation on singular manifolds, we would like to mention the papers [10, 11, 14, 18, 31, 34, 35, 36].

In [4], Chen et al. considered the following problem with degenerate parabolic equation

$$(1.2) \quad \begin{cases} u_t - \Delta_{\mathbb{B}} u = |u|^{p-1} u, & t > 0, x \in \mathbb{B}_0, \\ u(x, t) = 0, & t \geq 0, x \in \partial \mathbb{B}, \\ u(x, 0) = u_0(x), & x \in \mathbb{B}_0. \end{cases}$$

By the means of potential well method, which was established in [29, 32], the authors studied the solution with low and critical initial energy. They obtained existence theorems of global solution with exponential decay and showed the blow-up in finite time of solution. Moreover, the vacuum isolating behavior of solution was got. And then, the authors in [38] also discussed the solution of problem (1.2). They showed the explicit vacuum region, the global existence region and the blow-up region. Moreover, the upper bounds of the blow-up time and blow-up rate was estimated. More important, they further obtained a new initial condition such that the solution blow up in finite time for arbitrary initial energy, and in this case, they also estimated the upper bound of blow-up time.

As far as the wave equations is concerned, it is well known that most of the works only pay attention to the solution with low initial energy, and relatively few is known about the high initial energy case. The authors in [20] discussed the Cauchy problem of the following Boussinesq type equation

$$u_{tt} - u_{xx} - \beta_1 u_{xxtt} + \beta_2 u_{xxxx} = [f(u)]_{xx},$$

where  $f(u) = \alpha|u|^p$ ,  $1 < p < +\infty$ . They established a global existence result for weak solution with arbitrarily positive initial energy. Gazzola and Squassina in [15] studied the following damped semilinear wave equation

$$u_{tt} - \Delta u - \omega \Delta u_t + \mu u_t = |u|^{p-2} u.$$

They got a blow-up in finite time result with high initial energy under the case  $\omega = 0$ .

Recently, Wang and Su in [37] considered the initial boundary value problem of the dissipative Boussinesq equation

$$u_{tt} - \Delta u + \Delta^2 u - \alpha \Delta u_t + \gamma \Delta^2 u_t + \Delta f(u) = 0,$$

where  $f(u) = \beta|u|^{p-1}u$ ,  $0 \neq \beta \in \mathbb{R}$ ,  $1 < p < +\infty$ . They showed the threshold result of global existence and non-existence of solution with low initial energy, the solution with critical initial energy was also studied. By convex method and some analysis techniques, they further got the global existence and finite time blow-up results under arbitrarily positive initial energy.

Back to problem (1.1), under low initial energy, the authors in [1] used the cone Sobolev inequality, Poincaré inequality and potential wells method to discuss the invariance of solution set and present the existence and non-existence of global solution. In order to introduce their main results specifically, we give some necessary notations and definitions, which was introduced in [1] and also be used in this paper.

Throughout the paper, the norm in the space  $\mathcal{L}_p^{\frac{n}{p}}(\mathbb{B}) = \mathcal{H}_{p,0}^{0,\frac{n}{p}}(\mathbb{B})$  is defined by

$$\|u\|_{\mathcal{L}_p^{\frac{n}{p}}(\mathbb{B})} := \left( \int_{\mathbb{B}} |u|^p \frac{dx_1}{x_1} dx' \right)^{\frac{1}{p}}, \quad \forall p \in (1, +\infty).$$

Let

$$(1.3) \quad C_* := \inf \left\{ \frac{\|\sqrt{V(x)}u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}}{\|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}} \right\}, \quad C_{**} := \sup \left\{ \frac{\|g_t^{\frac{1}{p+1}}(x)u\|_{\mathcal{L}_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}}{\|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}} \right\}.$$

Since  $2 < p+1 < \frac{2n}{n-2}$ , by [4, Propositions 2.2], we know that the embedding

$$\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) \hookrightarrow \mathcal{L}_{p+1}^{\frac{n}{p+1}}(\mathbb{B})$$

is continuous. Then, we define the energy functional by

$$(1.4) \quad I(t) := \frac{1}{2} \|u_t(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \frac{1}{p+1} \int_0^t \left\| \left( \frac{d}{d\tau} g_\tau(x) \right)^{\frac{1}{p+1}} u(\tau) \right\|^{p+1} d\tau + J(u(t)),$$

where

$$J(u) := \frac{1}{2} \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \frac{1}{2} \|\sqrt{V(x)}u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 - \frac{1}{p+1} \|g_t^{\frac{1}{p+1}}(x)u\|_{\mathcal{L}_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}^{p+1}.$$

The corresponding Nehari functional and Nehari manifold can be denoted by

$$K(u) := \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \|\sqrt{V(x)}u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 - \|g_t^{\frac{1}{p+1}}(x)u\|_{\mathcal{L}_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}^{p+1},$$

$$\mathcal{N} := \left\{ u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) : K(u) = 0, \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1} dx' \neq 0 \right\},$$

then the depth of potential well is

$$(1.5) \quad d := \inf \left\{ \sup_{\lambda \geq 0} J(\lambda u), u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}), \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1} dx' \neq 0 \right\} = \inf_{u \in \mathcal{N}} J(u).$$

So we introduce the following potential well

$$(1.6) \quad W = \left\{ u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) : K(u) > 0, \quad J(u) < d \right\} \cup \{0\}.$$

It follows from the definitions of  $J(u), I(u)$  that

$$(1.7) \quad J(u) = \frac{p-1}{2(p+1)} \left( \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \|\sqrt{V(x)}u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 \right) + \frac{1}{p+1} K(u).$$

We denote  $\lambda_1$  be the first nonzero eigenvalue of following Dirichlet problem

$$\begin{cases} -\Delta_{\mathbb{B}} \phi(x) = \lambda \phi(x), & x \in \mathbb{B}_0, \\ \phi(x) = 0, & x \in \partial \mathbb{B}, \end{cases}$$

then by [1, Proposition 2.7] we know that  $\lambda_1 > 0$  and satisfies following inequality:

$$(1.8) \quad \lambda_1 \|u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 \leq \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2.$$

For any  $\delta > 0$ , we set

$$(1.9) \quad \begin{aligned} K_{\delta}(u) &:= \delta \left( \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \|\sqrt{V(x)}u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 \right) - \|g_t^{\frac{1}{p+1}}(x)u\|_{\mathcal{L}_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}^{p+1}, \\ \mathcal{N}_{\delta} &:= \left\{ u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) : K_{\delta}(u) = 0, \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1} dx' \neq 0 \right\}, \\ d(\delta) &:= \inf_{u \in \mathcal{N}_{\delta}} J(u), \end{aligned}$$

For  $\delta > 0, u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ , we let

$$(1.10) \quad \begin{aligned} \mathcal{B}_{\delta} &:= \left\{ \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})} < (1 + C_*^2)^{\frac{1}{p-1}} C_{**}^{-\frac{p+1}{p-1}} \delta^{\frac{1}{p-1}} \right\}, \\ \overline{\mathcal{B}}_{\delta} &:= \mathcal{B}_{\delta} \cup \partial \mathcal{B}_{\delta} = \left\{ \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})} \leq (1 + C_*^2)^{\frac{1}{p-1}} C_{**}^{-\frac{p+1}{p-1}} \delta^{\frac{1}{p-1}} \right\}, \\ \mathcal{B}_{\delta}^c &:= \left\{ \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})} > (1 + C_*^2)^{\frac{1}{p-1}} C_{**}^{-\frac{p+1}{p-1}} \delta^{\frac{1}{p-1}} \right\}, \\ \overline{\mathcal{B}}_{\delta}^c &:= \mathcal{B}_{\delta}^c \cup \partial \mathcal{B}_{\delta}^c = \left\{ \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})} \geq (1 + C_*^2)^{\frac{1}{p-1}} C_{**}^{-\frac{p+1}{p-1}} \delta^{\frac{1}{p-1}} \right\}. \end{aligned}$$

Next, we give the definition of weak solution to problem (1.1).

**Definition 1.1.** ([1, Definition 3.7]) *Function  $u = u(x, t) \in L^{\infty}(0, T; \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}))$  with  $u_t \in L^2(0, T; \mathcal{L}_2^{\frac{n}{2}}(\mathbb{B}))$  is called a weak solution of problem (1.1) on  $[0, T) \times \mathbb{B}_0$ , if, for all  $v \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ ,*

$$\int_{\mathbb{B}} (\gamma u + u_t - \gamma u_0 - u_1) \cdot v \frac{dx_1}{x_1} dx' + \int_0^t \int_{\mathbb{B}} (\nabla_{\mathbb{B}} u \cdot \nabla_{\mathbb{B}} v + V(x)u \cdot v - g_t(x)|u|^{p-1}u \cdot v) \frac{dx_1}{x_1} dx' d\tau = 0,$$

and  $u(x, 0) = u_0(x)$  in  $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ ,  $u_t(x, 0) = u_1(x)$  in  $\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})$ . Moreover, there hold the following energy inequality

$$(1.11) \quad I(t) + \gamma \int_0^t \|u_{\tau}(\tau)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 d\tau \leq I(0), \quad \forall t \in (0, T),$$

where  $0 \leq T \leq +\infty$  and  $I(t)$  defined in (1.4).

We define the maximal existence time  $T$  of the weak solution to problem (1.1) as follow.

**Definition 1.2.** Let  $u(x, t)$  be a weak solution of problem (1.1).

- (i). If  $u(x, t)$  exists for all  $t \in [0, +\infty)$ , then  $T = +\infty$ ;
- (ii). If there exists a  $t_0 \in (0, +\infty)$  such that  $u(x, t)$  exists for  $t \in [0, t_0)$  but doesn't exists at  $t = t_0$ , then  $T = t_0$ .

We summarize some results obtained in [1], which are relevant to the work in this paper.

Combining the conclusions of [1, Remark 3.10] and [1, Proposition 3.11] we can get following theorem.

**Theorem 1.3.** Let  $u(t) = u(x, t)$  be the weak solution of problem (1.1) and  $d(\delta)$  be defined in (1.9). We further denote  $\delta_1 < \delta_2$  be the two roots of equation  $d(\delta) = e$ ,  $0 < e \leq d$ .

- (i). If  $0 < J(u_0) \leq e$ , then for any  $t \in [0, +\infty)$ , we have

$$(1.12) \quad u(t) \notin U_e := \bigcup_{\delta_1 < \delta < \delta_2} \mathcal{N}_\delta,$$

i.e., any solution is isolated by  $U_e$ .

- (ii). If  $I(0) = 0$  and  $\|\nabla_{\mathbb{B}} u_0\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})} \neq 0$ , then for any  $t \in [0, T)$ ,

$$\|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^{p-1} \geq \frac{(p+1)(1+C_*^2)}{2C_{**}^{p+1}}.$$

The results of [1, Theorem 4.1] and [1, Theorem 4.4] discussed the weak solution of problem (1.1) under  $I(0) < d$ , and the relation between global existence and finite time blow-up is derived as a sharp condition, which can be stated as follow.

**Theorem 1.4.** Let  $u(t) = u(x, t)$  be the weak solution of problem (1.1) with  $I(0) < d$ .

- (i). If  $K(u_0) > 0$  or  $\|\nabla_{\mathbb{B}} u_0\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})} = 0$ , then  $u(t)$  exists globally and  $u(t) \in W$  for all  $t \in [0, +\infty)$ , where  $W$  is defined in (1.6);
- (ii). If  $\gamma \leq (p-1)\sqrt{1+C_*^2}\lambda_1^{\frac{1}{2}}$ ,  $K(u_0) < 0$ , then  $u(t)$  blows up in finite time with  $\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})$ -norm.

**Corollary 1.5.** For  $\gamma \leq (p-1)\sqrt{1+C_*^2}\lambda_1^{\frac{1}{2}}$ , if there exists some  $t_0 \geq 0$  such that  $u(t_0) \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ ,  $u_t(t_0) \in \mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})$  and  $I(t_0) \leq 0$ , then the corresponding weak solution of problem (1.1) blows up in finite time.

*Proof.* Without loss of generality, assume  $t_0 = 0$ , so by the proof of [1, Theorem 4.4] we know that the weak solution blow up in finite time if  $I(0) \leq 0$ .  $\square$

By Theorem 1.3(i), we know that there is a vacuum region  $U_e$  for the solution with  $0 < I(0) < d$ . However, from the perspective of set, it is easy to see that the region is so abstract and we can not get the explicit information about it. Furthermore, it is natural to ask that whether the vacuum region exists when  $I(0) \leq 0$ .

From above Theorem 1.4, we also note that the results of global existence and non-existence in [1] holds only under the solution with  $I(0) < d$ , for the critical and high initial energy case, i.e.,  $I(0) \geq d$ , it is still open that whether the global solution exists or the solution blows up in finite time.

The main purpose of this paper is to extend the partial results got in [1]. Firstly, we study the vacuum isolating phenomenon of solution for problem (1.1). According to what we know, this is the first attempt to tackle the vacuum isolating phenomenon for hyperbolic equation with damping and conical singularity. It should be point out that the studies of vacuum isolating phenomena has developed widely, there are many works about this aspect for evolution equations. One can see [4, 5, 24, 25, 26, 27, 28, 30] for the nonlinearly wave equation, see [23] for the nonlocal parabolic equation and see [38] for semilinear parabolic equation with conical degeneration.

As for problem (1.1), we prove that the vacuum region  $U_e$  defined in (1.12) is an annulus, that is to say, all solution will isolate by this annulus. Moreover,  $U_e$  splits the space  $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$  into a inner ball and a corresponding outer region. We further prove that solution exists globally if initial value belongs to the inner ball, while solution blows up in finite time if initial data belongs to the annulus' exterior (see Theorem 3.1).

The key of the proof for this result is to calculate the concrete value of  $d$ , which will be given by Lemma 3.5 below. Then, by the known conclusion in [1, Proposition 3.5] we can get the clear expression of  $d(\delta)$  (see (3.7)). We further obtain the following three equivalent propositions under assumption  $J(u) < d(\delta)$  in Lemma 3.6 below:

$$\begin{aligned} K_\delta(u) &> 0 \Leftrightarrow u \in \mathcal{B}_\delta; \\ K_\delta(u) &< 0 \Leftrightarrow u \in \mathcal{B}_\delta^c; \\ K_\delta(u) &= 0 \text{ and } u \neq 0 \Leftrightarrow u \in \partial\mathcal{B}_\delta, \end{aligned}$$

where  $\mathcal{B}_\delta, \mathcal{B}_\delta^c, \partial\mathcal{B}_\delta$  defined in (1.10). Hence, we can deduce our result by the definition of  $U_e$  given by (1.12) and the definition of  $\mathcal{N}_\delta$  in (1.9).

When  $I(0) \leq 0$ , we find another vacuum region  $U_{r^*}$ , which is an ball (see Theorem 3.2). It is interesting to note that the ball expands as  $I(0)$  decreasing and

$$\lim_{I(0) \rightarrow -\infty} U_{r^*} = \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}).$$

In this case, according to Corollary 1.5 we know that all solution will blow up in finite time.

For the blow-up weak solution with  $I(0) < d$ , we also consider the asymptotic behaviors of the energy functional as  $t \rightarrow T^-$ . Based on this results, we further obtain two necessary and sufficient conditions for solution existing globally and blowing up in finite time, respectively (see Theorem 3.3).

Secondly, we discuss the global existence and non-existence of weak solution to problem (1.1) with  $I(0) = d$  (see Theorem 4.1). Under some assumptions and make some detailed analyses, the main idea of the proof is turning the case  $I(0) = d$  into the case  $I(0) < d$ , so we can deduce our result with the help of Theorem 1.4.

Finally, we study the weak solution under arbitrary positive initial energy, i.e.,  $I(0) > 0$ . Inspired by the method in papers [20, 37], we obtain the initial conditions such that the weak solution of problem (1.1) exists globally and blows up in finite time, respectively (see Theorem 5.1 and 5.3).

The rest of this paper is organized as follows.



- For reader's convenience, in Section 2, we first introduce some definitions and properties of cone Sobolev spaces.
- In Section 3, we consider the weak solution of problem (1.1) under low initial energy. We will give some preliminary lemmas in Subsection 3.1, which will be used in the proofs and we give the complete proofs of Theorem 3.1–3.3 in Subsection 3.2.
- In Section 4, the weak solution with critical initial energy is considered, and we give the proofs of Theorem 4.1.
- We prove Theorem 5.1 and Theorem 5.3 in Section 5.

## 2 Preliminaries

The detail research of manifold with conical singularities and the corresponding cone Sobolev spaces can be found in [7, 8, 13, 33, 34]. In this subsection, we shall introduce some definitions and properties of cone Sobolev spaces briefly, which is enough to make our paper readable.

Let  $X$  be a closed, compact,  $C^\infty$  manifold, we set  $X^\Delta = (\bar{\mathbb{R}}_+ \times X)/(\{0\} \times X)$  as a local model interpreted as a cone with the base  $X$ . We denote  $X^\nabla = \mathbb{R}_+ \times X$  as the corresponding open stretched cone with  $X$ . An  $n$ -dimensional manifold  $B$  with conical singularities is a topological space with a finite subset  $B_0 = \{b_1, \dots, b_M\} \subset B$  of conical singularities. For simplicity, we assume that the manifold  $B$  has only one conical point on the boundary. Thus, near the conical point, we have a stretched manifold  $\mathbb{B}$ , associated with  $B$ .

**Definition 2.1.** Let  $\mathbb{B} = [0, 1) \times X$  be the stretched manifold of the manifold  $B$  with conical singularity, then for any cut-off function  $\omega$ , supported by a collar neighborhood of  $(0, 1) \times \partial\mathbb{B}$ , the cone Sobolev space  $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$ , for  $m \in \mathbb{N}, \gamma \in \mathbb{R}$  and  $1 < p < +\infty$ , is defined as follow

$$\mathcal{H}_p^{m,\gamma}(\mathbb{B}) = \{u \in W_{loc}^{m,p}(\mathbb{B}_0) \mid \omega u \in \mathcal{H}_p^{m,\gamma}(X^\nabla)\}.$$

Moreover, the subspace  $\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{B})$  of  $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$  is defined by

$$\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{B}) = \omega \mathcal{H}_p^{m,\gamma}(X^\nabla) + (1 - \omega)W_0^{m,p}(\mathbb{B}_0),$$

where  $W_0^{m,p}(\mathbb{B}_0)$  denotes the closure of  $C_0^\infty(\mathbb{B}_0)$  in Sobolev spaces  $W^{m,p}(\tilde{X})$ , here  $\tilde{X}$  is a closed compact  $C^\infty$  manifold of dimension  $n$  that containing  $\mathbb{B}$  as a sub-manifold with boundary.

**Definition 2.2.** We say  $u(x) \in \mathcal{L}_p^\gamma(\mathbb{B})$  with  $1 < p < +\infty$  and  $\gamma \in \mathbb{R}$  if

$$\|u\|_{\mathcal{L}_p^\gamma(\mathbb{B})}^p = \int_{\mathbb{B}} x_1^n |x_1^{-\gamma} u(x)|^p \frac{dx_1}{x_1} dx' < +\infty.$$

Observe that if  $u(x) \in \mathcal{L}_p^{\frac{n}{p}}(\mathbb{B}), v(x) \in \mathcal{L}_q^{\frac{n}{q}}(\mathbb{B})$  with  $p, q \in (1, +\infty)$  and  $\frac{1}{p} + \frac{1}{q} = 1$ , then we have the following Hölder's inequality

$$\int_{\mathbb{B}} |u(x)v(x)| \frac{dx_1}{x_1} dx' \leq \|u\|_{\mathcal{L}_p^{\frac{n}{p}}(\mathbb{B})} \|v\|_{\mathcal{L}_q^{\frac{n}{q}}(\mathbb{B})}.$$

**Lemma 2.3.** [Poincaré inequality] Let  $\mathbb{B} = [0, 1) \times X$  be a bounded subspace in  $\mathbb{R}_+^n$  with  $X \in \mathbb{R}^{n-1}$ , and  $1 < p < +\infty, \theta \in \mathbb{R}$ . If  $u(x) \in \mathcal{H}_{p,0}^{1,\theta}(\mathbb{B})$ , then

$$\|u(x)\|_{\mathcal{L}_p^\theta(\mathbb{B})} \leq \mu \|\nabla_{\mathbb{B}} u(x)\|_{\mathcal{L}_p^\theta(\mathbb{B})},$$

where  $\mu$  is a positive constant depending only on  $\mathbb{B}$ .



### 3 Low initial energy case

As mentioned above, in this section we consider the weak solution of problem (1.1) under  $I(0) < d$ . We first give the result about vacuum isolating phenomenon with  $0 < I(0) < d$ .

**Theorem 3.1.** *Let  $e \in (0, d)$ ,  $\delta_1 < \delta_2$  are the two positive roots of equation  $d(\delta) = e$ , then for all weak solution  $u(t) = u(x, t)$  of problem (1.1) with  $0 < I(0) \leq e$ , the vacuum region  $U_e$  defined in (1.12) can be denoted as follow:*

$$U_e = \left\{ u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) : (1 + C_*^2)^{\frac{1}{p-1}} C_{**}^{-\frac{p+1}{p-1}} \delta_1^{\frac{1}{p-1}} < \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})} < (1 + C_*^2)^{\frac{1}{p-1}} C_{**}^{-\frac{p+1}{p-1}} \delta_2^{\frac{1}{p-1}} \right\}.$$

Moreover,  $\overline{\mathcal{B}_{\delta_1}}$  and  $\overline{\mathcal{B}_{\delta_2}^c}$  are two invariant sets, and if  $u_0 \in \overline{\mathcal{B}_{\delta_1}}$ , then  $u(t)$  exists globally. While the weak solution  $u(t)$  blows up in finite time provided  $u_0 \in \overline{\mathcal{B}_{\delta_2}^c}$  and  $\gamma \leq (p-1)\sqrt{1 + C_*^2}\lambda_1^{\frac{1}{2}}$ .

The result on vacuum isolating phenomenon with the case  $I(0) \leq 0$  can be described as following theorem.

**Theorem 3.2.** *Suppose  $I(0) \leq 0$ ,  $r^* = r^*(I(0))$  is the unique positive root of equation  $G(r) = 0$ , where  $G(r)$  is a function given by (3.13). Let  $u(t) = u(x, t)$  be the weak solution of problem (1.1), then there exists a bounded region*

$$(3.1) \quad U_{r^*} = \left\{ u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) : \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})} < r^* \right\}$$

such that  $U_{r^*}$  is a vacuum region, i.e.,  $u(t) \notin U_{r^*}$  for all  $t \in [0, T)$ , and  $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) \setminus U_{r^*}$  is an invariant region.

The following is our final result with regard to the weak solution under  $I(0) < d$ .

**Theorem 3.3.** *Let  $u(t) = u(x, t)$  be the weak solution of problem (1.1) with  $I(0) < d$  and  $T \in (0, +\infty]$  be the maximal existence time. If  $\gamma \leq (p-1)\sqrt{1 + C_*^2}\lambda_1^{\frac{1}{2}}$ ,  $K(u_0) < 0$ , then the maximal existence time  $T < +\infty$  and*

$$(3.2) \quad \lim_{t \rightarrow T^-} I(t) = -\infty.$$

Moreover, we have following conclusions:

- (i).  $\gamma \leq (p-1)\sqrt{1 + C_*^2}\lambda_1^{\frac{1}{2}}$ ,  $K(u_0) < 0 \Leftrightarrow T < +\infty \Leftrightarrow (3.2)$  holds;
- (ii).  $K(u_0) > 0 \Leftrightarrow T = +\infty \Leftrightarrow I(t) > 0$  for all  $t \in [0, T)$ .

#### 3.1 Some auxiliary lemmas

In order to prove above theorems, we need some preparations. By [1, Lemma 3.4] we have following lemma.

**Lemma 3.4.** *Assume that  $u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ , then we have following conclusions:*

(i). The functional  $J(\lambda u)$  admits its maximum at  $\lambda = \lambda_*$ , where

$$(3.3) \quad \lambda_* = \left( \frac{\|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 - \|\sqrt{V(x)}u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2}{\|g_t^{\frac{1}{p+1}}(x)u\|_{\mathcal{L}_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}^{p+1}} \right)^{\frac{1}{p-1}}.$$

For  $0 \leq \lambda < \lambda_*$ ,  $J(\lambda u)$  is strictly increasing and for  $\lambda_* < \lambda$ , it is strictly decreasing.

(ii).  $K(\lambda_* u) = 0$  and  $K(\lambda u) > 0$  if  $0 < \lambda < \lambda_*$ . For  $\lambda_* < \lambda$ ,  $K(\lambda u) < 0$ .

(iii). The depth of potential well  $d$  defined in (1.5) can be denoted as follow

$$(3.4) \quad d = \frac{p-1}{2(p+1)} (1 + C_*^2)^{\frac{p+1}{p-1}} C_{**}^{-2\frac{p+1}{p-1}} \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2.$$

We infer from Lemma 3.4(iii) that the authors in [1] get a expression for  $d$  like (3.4), which is dependent on  $\|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2$ . Now, we modify the proof in [1, Lemma 3.4](iii) and give the precise value of  $d$ .

**Lemma 3.5.** Let  $d$  defined in (1.5), we have

$$(3.5) \quad d = \frac{p-1}{2(p+1)} (1 + C_*^2)^{\frac{p+1}{p-1}} C_{**}^{-2\frac{p+1}{p-1}}.$$

*Proof.* By Lemma 3.4 we know there exists a positive constant  $\lambda_*$  defined in (3.3) such that  $\lambda_* u \in \mathcal{N}$  and  $J(\lambda u)$  attains its maximum at  $\lambda = \lambda_*$ . Hence,

$$(3.6) \quad d = \inf_{u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) \setminus \{0\}} \sup_{\lambda \geq 0} J(\lambda u) = \inf_{u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) \setminus \{0\}} J(\lambda_* u).$$

By the definition of  $J(u)$  and the value of  $\lambda_*$  in (3.3), we can see

$$\begin{aligned} J(\lambda_* u) &= \frac{\lambda_*^2}{2} \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \frac{\lambda_*^2}{2} \|\sqrt{V(x)}u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 - \frac{\lambda_*^{p+1}}{p+1} \|g_t^{\frac{1}{p+1}}(x)u\|_{\mathcal{L}_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}^{p+1} \\ &= \frac{p-1}{2(p+1)} \left( \frac{\|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \|\sqrt{V(x)}u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2}{\|g_t^{\frac{1}{p+1}}(x)u\|_{\mathcal{L}_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}^{p+1}} \right)^{\frac{p+1}{p-1}} \\ &= \frac{p-1}{2(p+1)} \left[ \left( 1 + \frac{\|\sqrt{V(x)}u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2}{\|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2} \right) \times \frac{\|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2}{\|g_t^{\frac{1}{p+1}}(x)u\|_{\mathcal{L}_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}^2} \right]^{\frac{p+1}{p-1}}. \end{aligned}$$

Then it follows from the definitions of  $C_*$ ,  $C_{**}$  in (1.3) that

$$\begin{aligned} \inf_{u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) \setminus \{0\}} J(\lambda_* u) &= \inf_{u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) \setminus \{0\}} \frac{p-1}{2(p+1)} \left[ \left( 1 + \frac{\|\sqrt{V}u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2}{\|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2} \right) \times \frac{\|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2}{\|g_t^{\frac{1}{p+1}}u\|_{\mathcal{L}_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}^2} \right]^{\frac{p+1}{p-1}} \\ &= \frac{p-1}{2(p+1)} (1 + C_*^2)^{\frac{p+1}{p-1}} C_{**}^{-2\frac{p+1}{p-1}}. \end{aligned}$$

□

According to the conclusion of [1, Proposition 3.5] we know that

$$d(\delta) = \delta^{\frac{2}{p-1}} \frac{p+1-2\delta}{p-1} d,$$

which together with the value of  $d$  in (3.5), we get

$$(3.7) \quad d(\delta) = \frac{p+1-2\delta}{2(p+1)} \delta^{\frac{2}{p-1}} (1 + C_*^2)^{\frac{p+1}{p-1}} C_{**}^{-2\frac{p+1}{p-1}}.$$

**Lemma 3.6.** *Assume  $J(u) < d(\delta)$ ,  $0 < \delta < \frac{p+1}{2}$ , then we have*

- (i).  $K_\delta(u) > 0$  if and only if  $u \in \mathcal{B}_\delta$ ;
- (ii).  $K_\delta(u) < 0$  if and only if  $u \in \mathcal{B}_\delta^c$ ;
- (iii).  $K_\delta(u) = 0$ ,  $u \neq 0$  if and only if  $u \in \partial\mathcal{B}_\delta$

*Proof.* The method in the following proof is similar to [38, Lemma 2.6], for reader's convenience, we give a complete proof here.

(i). For any  $u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ , it follows from the definitions of  $C_*$ ,  $C_{**}$  that

$$(3.8) \quad \|\sqrt{V(x)}u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 \geq C_*^2 \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 \geq \frac{C_*^2}{C_{**}^2} \|g_t^{\frac{1}{p+1}}(x)u\|_{\mathcal{L}_{p+1}^{\frac{n}{2}}(\mathbb{B})}^2.$$

If  $u \in \mathcal{B}_\delta$ , by the definition of  $\mathcal{B}_\delta$  we know

$$(3.9) \quad \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})} < (1 + C_*^2)^{\frac{1}{p-1}} C_{**}^{-\frac{p+1}{p-1}} \delta^{\frac{1}{p-1}}$$

then by the second inequality in (3.8) we have

$$\|g_t^{\frac{1}{p+1}}(x)u\|_{\mathcal{L}_{p+1}^{\frac{n}{2}}(\mathbb{B})}^{p+1} \leq C_{**}^{p+1} \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^{p-1} \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 < \delta (1 + C_*^2) \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2.$$

By the first inequality in (3.8) we can obtain

$$\delta \left( \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \|\sqrt{V(x)}u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 \right) \geq \delta (1 + C_*^2) \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2,$$

together with above two inequalities and the definition of  $K_\delta(u)$ , gives  $K_\delta(u) > 0$ .

On the other hand, we assume  $K_\delta(u) < 0$ , then together with the definition of  $K_\delta(u)$  and (3.8) we know  $\|\sqrt{V(x)}u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 > 0$ . It follows from the definition of  $J(u)$  and  $K_\delta(u)$  that

$$(3.10) \quad J(u) = \left( \frac{1}{2} - \frac{\delta}{p+1} \right) \left( \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \|\sqrt{V(x)}u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 \right) + \frac{1}{p+1} K_\delta(u).$$

Since  $0 < \delta < \frac{p+1}{2}$ , using the assumption  $J(u) < d(\delta)$  and the value of  $d(\delta)$  given in (3.7), we know

$$\frac{p+1-2\delta}{2(p+1)} \delta^{\frac{2}{p-1}} (1 + C_*^2)^{\frac{p+1}{p-1}} C_{**}^{-2\frac{p+1}{p-1}} > J(u),$$

which together  $K_\delta(u) > 0$ , (3.10) and the first inequality in (3.8), leads to

$$\begin{aligned} & \frac{p+1-2\delta}{2(p+1)} \delta^{\frac{2}{p-1}} (1+C_*^2)^{\frac{p+1}{p-1}} C_{**}^{-2\frac{p+1}{p-1}} \\ & > \left( \frac{1}{2} - \frac{\delta}{p+1} \right) \left( \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \|\sqrt{V(x)}u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 \right) \\ & \geq \left( \frac{1}{2} - \frac{\delta}{p+1} \right) (1+C_*^2) \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 \end{aligned}$$

then we get (3.9), i.e.,  $u \in \mathcal{B}_\delta$ .

(ii). If  $u \in \mathcal{B}_\delta^c$ , we can infer from the definition of  $\mathcal{B}_\delta^c$  that

$$(3.11) \quad \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})} > (1+C_*^2)^{\frac{1}{p-1}} C_{**}^{-\frac{p+1}{p-1}} \delta^{\frac{1}{p-1}}.$$

Using the first inequality in (3.8) we can see

$$\begin{aligned} & \left( \frac{1}{2} - \frac{\delta}{p+1} \right) \left( \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \|\sqrt{V(x)}u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 \right) \\ & \geq \left( \frac{1}{2} - \frac{\delta}{p+1} \right) (1+C_*^2) \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 \\ & > \frac{p+1-2\delta}{2(p+1)} \delta^{\frac{2}{p-1}} (1+C_*^2)^{\frac{p+1}{p-1}} C_{**}^{-\frac{2(p+1)}{p-1}} \\ & = d(\delta), \end{aligned}$$

then combine (3.10) and  $J(u) < d(\delta)$ , we get  $K_\delta(u) < 0$ .

On the other hand, we assume  $K_\delta(u) < 0$ . Then together with the inequalities in (3.8) and the definition of  $K_\delta(u)$  we can deduce that

$$\begin{aligned} \delta (1+C_*^2) \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 & \leq \delta \left( \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \|\sqrt{V(x)}u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 \right) \\ & < \|g_t^{\frac{1}{p+1}}(x)u\|_{\mathcal{L}_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}^{p+1} \\ & \leq C_{**}^{p+1} \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^{p+1}, \end{aligned}$$

which implies (3.11). Namely,  $u \in \mathcal{B}_\delta^c$ .

(iii). By (i) and (ii), we get (iii) immediately.  $\square$

### 3.2 Proofs of main results under $I(0) < d$

*Proof of Theorem 3.1.* Let  $U_e$  be the set defined in (1.12), then by the definitions of  $U_e, \mathcal{N}_\delta$  and Lemma 3.6, we have

$$\begin{aligned} U_e &= \bigcup_{\delta_1 < \delta < \delta_2} \mathcal{N}_\delta \\ &= \bigcup_{\delta_1 < \delta < \delta_2} \left\{ u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) : \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})} = (1+C_*^2)^{\frac{1}{p-1}} C_{**}^{-\frac{p+1}{p-1}} \delta^{\frac{1}{p-1}} \right\} \\ &= \left\{ u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) : (1+C_*^2)^{\frac{1}{p-1}} C_{**}^{-\frac{p+1}{p-1}} \delta_1^{\frac{1}{p-1}} < \|\nabla_{\mathbb{B}} u\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})} < (1+C_*^2)^{\frac{1}{p-1}} C_{**}^{-\frac{p+1}{p-1}} \delta_2^{\frac{1}{p-1}} \right\}. \end{aligned}$$

Namely, there is no solution of problem (1.1) in  $U_e$  and all solutions are isolated by  $U_e$ , then  $\overline{\mathcal{B}_{\delta_1}}$  and  $\overline{\mathcal{B}_{\delta_2}^c}$  are both invariant regions.

If  $u_0 \in \overline{\mathcal{B}_{\delta_1}}$ , then the corresponding solution  $u(t) \in \overline{\mathcal{B}_{\delta_1}}$  for all  $t \in [0, T)$  since  $\overline{\mathcal{B}_{\delta_1}}$  is an invariant region. By the definition of  $\overline{\mathcal{B}_{\delta_1}}$  we have

$$\|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}_{\frac{n}{2}}(\mathbb{B})} \leq (1 + C_*^2)^{\frac{1}{p-1}} C_{**}^{-\frac{p+1}{p-1}} \delta_1^{\frac{1}{p-1}},$$

which means that  $u(t)$  exist globally.

If  $u_0 \in \overline{\mathcal{B}_{\delta_2}^c}$ , since  $\overline{\mathcal{B}_{\delta_2}^c}$  is an invariance set, we get  $u(t) \in \overline{\mathcal{B}_{\delta_2}^c}$  for all  $t \in [0, T)$ , i.e.,

$$\|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}_{\frac{n}{2}}(\mathbb{B})} \geq (1 + C_*^2)^{\frac{1}{p-1}} C_{**}^{-\frac{p+1}{p-1}} \delta_2^{\frac{1}{p-1}}.$$

Combining Lemma 3.6(ii)(iii) we know that

$$K_{\delta_2}(u(t)) \leq 0, \quad \forall t \in [0, T).$$

Since  $\delta_2 > 1$ , then by the definition of  $K_{\delta_2}(u)$  and the above inequality, we have  $K(u(t)) < 0$  for all  $t \in [0, T)$ , especially,  $K(u_0) < 0$ , so the assumptions of Theorem 1.4(ii) holds and we get  $u(t)$  blows up in finite time. Hence, Theorem 3.1 is proved.  $\square$

*Proof of Theorem 3.2.* We can infer from (1.11) that the energy functional  $I(t)$  is nonincreasing with respect to  $t$ . Since  $I(0) \leq 0$ , we obtain  $I(t) \leq 0$  for all  $t \in [0, T)$ . By the definition of  $I(t)$  we can see

$$I(0) \geq I(t) \geq \frac{1}{2} \|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}_{\frac{n}{2}}(\mathbb{B})}^2 + \frac{1}{2} \|\sqrt{V(x)} u(t)\|_{\mathcal{L}_{\frac{n}{2}}(\mathbb{B})}^2 - \frac{1}{p+1} \|g_t^{\frac{1}{p+1}} u(t)\|_{\mathcal{L}_{\frac{n}{p+1}}(\mathbb{B})}^{p+1},$$

then

$$\frac{1}{p+1} \|g_t^{\frac{1}{p+1}} u(t)\|_{\mathcal{L}_{\frac{n}{p+1}}(\mathbb{B})}^{p+1} \geq \frac{1}{2} \|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}_{\frac{n}{2}}(\mathbb{B})}^2 + \frac{1}{2} \|\sqrt{V(x)} u(t)\|_{\mathcal{L}_{\frac{n}{2}}(\mathbb{B})}^2 - I(u_0).$$

Combining the definitions of  $C_*$ ,  $C_{**}$ , we can deduce from above inequality that

$$(3.12) \quad \frac{1}{p+1} C_{**}^{p+1} \|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}_{\frac{n}{2}}(\mathbb{B})}^{p+1} \geq \frac{1}{2} \|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}_{\frac{n}{2}}(\mathbb{B})}^2 + \frac{1}{2} C_*^2 \|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}_{\frac{n}{2}}(\mathbb{B})}^2 - I(0).$$

In order to complete our proof, we consider a function  $G(r)$  which is defined by

$$(3.13) \quad G(r) := \frac{1}{p+1} C_{**}^{p+1} r^{p+1} - \frac{1}{2} (1 + C_*^2) r^2 + I(0).$$

Obviously, the equation  $G(r) = 0$  admits a unique positive root  $r^* = r^*(I(0))$  due to  $p > 1$ , and  $r^*(I(0))$  proposes the following properties:

- (i).  $G(r) \geq 0$  if and only if  $r \geq r^*(I(0))$ ;
- (ii).  $r^*(0) = \left( \frac{(p+1)(1+C_*^2)}{2C_{**}^{p+1}} \right)^{\frac{1}{p-1}}$ ;
- (iii).  $r^*(I(0))$  is increasing as  $I(0)$  decreasing and  $\lim_{I(0) \rightarrow -\infty} r^*(I(0)) = +\infty$ .

From the inequality (3.12) and the definition of  $G(r)$  we can see that

$$G(\|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}) \geq 0.$$

Then by the properties of  $G$  we get

$$\|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})} \geq r^*,$$

where  $r^* = r^*(I(0))$  is the unique positive root of equation  $G(r) = 0$ . So the set  $U_{r^*}$  defined in (3.1) is a vacuum region such that all solution are isolated by  $U_{r^*}$ , and then  $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) \setminus U_{r^*}$  is a invariant set.  $\square$

**Remark 3.7.** By the properties of  $r^*(I(0))$  (ii) and (3.12) we know that

$$\|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})} \geq \left( \frac{(p+1)(1+C_*^2)}{2C_{**}^{p+1}} \right)^{\frac{1}{p-1}}, \quad \text{when } I(0) = 0,$$

which is consistent with the conclusion in Theorem 1.3(ii). Moreover, we can infer from the properties of  $r^*(I(0))$  (iii) that  $U_{r^*}$  expands as  $J(u_0)$  decreasing, and

$$\lim_{I(0) \rightarrow -\infty} U_{r^*} = \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}).$$

*Proof of Theorem 3.3.* Let  $u(t) = u(x, t)$  be a weak solution of problem (1.1). Firstly, by Theorem 1.4 (ii) we know the solution blows up at some finite time  $T$  with  $\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})$ -norm, that is,

$$(3.14) \quad \lim_{t \rightarrow T^-} \|u(\cdot, t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})} = +\infty.$$

Using Hölder's inequality we get

$$(3.15) \quad \int_0^t \|u_\tau(\tau)\|_2^2 d\tau \geq \frac{1}{t} \left( \int_0^t \|u_\tau(\tau)\|_2 d\tau \right)^2.$$

By [40, page 75, Proposition 3.3], we have

$$\begin{aligned} \int_0^t \|u_\tau(\tau)\|_2 d\tau &\geq \left\| \int_0^t u_\tau(\tau) d\tau \right\|_2 \\ &= \|u(t) - u_0\|_2 \\ &\geq | \|u(t)\|_2 - \|u_0\|_2 |. \end{aligned}$$

Then together above inequality with (3.15), gives

$$\int_0^t \|u_\tau(\tau)\|_2^2 d\tau \geq \frac{1}{t} (\|u(t)\|_2 - \|u_0\|_2)^2.$$

which combines (1.11), leads to

$$\begin{aligned} I(t) &\leq I(0) - \gamma \int_0^t \|u_\tau(\tau)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 d\tau \\ &\leq I(0) - \frac{1}{t} (\|u(t)\|_2 - \|u_0\|_2)^2. \end{aligned}$$

Let  $t \rightarrow T^-$  in above inequality and use (3.14), we get (3.2).

Next, we prove the two equivalent conclusions.

(i). If  $K(u_0) < 0$ , together with  $I(0) < d$  and Theorem 1.4(ii) we know the solution blows up in finite time, i.e.,  $T < +\infty$ . Moreover, by above proof we know (3.2) holds. Hence, in order to complete the proof, we now only need to show

$$(3.2) \text{ holds} \Rightarrow K(u_0) < 0.$$

Since (3.2) holds, by the continuity of  $I(t)$  with respect to  $t$  we know there exists a  $t_0$  such that  $I(t_0) < 0$ , which implies that  $u(t)$  blow up in finite time with the help of Corollary 1.5. On the other hand, we claim that  $K(u_0) \neq 0$ . If the claim is not true, then for any nontrivial initial value  $u_0 \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ , it follows from the definition of  $\mathcal{N}$  that  $u_0 \in \mathcal{N}$ . By the definition of  $d$  we get  $J(u_0) \geq d$ , which contradicts  $J(u_0) \leq I(0) < d$ . Finally, we show that  $K(u_0) > 0$  can not happen by contradiction. If  $K(u_0) > 0$ , then by Theorem 1.4(i) we know that  $u(t)$  exists globally, which contradicts the fact that  $u(t)$  blow up in finite time, so there must be  $K(u_0) < 0$ .

(ii). If  $K(u_0) > 0$ , together with  $I(0) < d$  and Theorem 1.4(i) we get  $T = +\infty$ . Then it follows from Corollary 1.5 that  $I(t) > 0$  for all  $t \in [0, +\infty)$ . So we only need to prove

$$I(t) > 0 \text{ for all } t \in [0, T) \Rightarrow K(u_0) > 0.$$

Similar to the proof of (i), we can get  $K(u_0) \neq 0$ . If  $K(u_0) < 0$ , together with  $I(0) < d$  we know (3.2) holds, i.e.,  $\lim_{t \rightarrow T^-} I(t) = -\infty$ . By the continuity of  $I(t)$  with respect to  $t$  we know there exists a  $t_0$  such that  $I(t_0) < 0$ , which contradicts  $I(t) > 0$  for all  $t \in [0, T)$ , so we get  $I(u_0) > 0$  and our proof is complete.  $\square$

## 4 Critical initial energy case

**Theorem 4.1.** *Let  $u(t) = u(x, t)$  be a weak solution of problem (1.1) with  $I(0) = d$ .*

(i). *If  $K(u_0) \geq 0$ , then  $u(t)$  exists globally;*

(ii). *If  $\gamma \leq (p-1)\sqrt{1+C_*^2\lambda_1^{\frac{1}{2}}}$ ,  $K(u_0) < 0$ , then the solution but not steady-state solution blows up in finite time.*

*Proof of Theorem 4.1.* (i). We prove this conclusion for the following two cases.

(a). If  $\|\nabla_{\mathbb{B}} u_0\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})} \neq 0$ . By  $I(0) = d$ , we get  $u_0 \neq 0$ . Let sequence  $\{\lambda_n\}_{n=1}^{+\infty}$  satisfy  $0 < \lambda_n < 1, n = 1, 2, \dots, \lambda_n \rightarrow 1$  as  $n \rightarrow +\infty$  and  $u_{0n}(x) = \lambda_n u_0(x)$ . Considering the weak solution of problem (1.1) corresponding to the initial condition

$$(4.1) \quad u(x, 0) = u_{0n}(x), \quad u_t(x, 0) = u_1(x), \quad x \in \mathbb{B}_0.$$

From  $K(u_0) \geq 0$  and Lemma 3.4, we know the constant defined in (3.3) satisfies  $\lambda_* = \lambda_*(u_0) \geq 1$ . Thus, it follows from Lemma 3.4(ii) and  $0 < \lambda_n < 1$  that  $K(u_{0n}) = K(\lambda_n u_0) > 0$ . Similarly, by Lemma 3.4(i) we can see  $I(u_{0n}) = I(\lambda_n u_0) < I(0) = d$ . On the other hand, combining  $K(u_{0n}) > 0$  and (1.7) we know  $J(u_{0n}) > 0$ , so we further have  $I(u_{0n}) > 0$  by the definition of  $I(u_{0n})$ . So we transformed the case  $I(0) = d, K(u_0) \geq 0$  into  $0 < I(u_{0n}) <$



$d, K(u_{0n}) > 0$  under the initial condition (4.1). Then similar to the proof of Theorem 1.4(i) (see [1, Theorem 4.1]), for each  $n$ , problem (1.1) with initial condition (4.1) gets a global weak solution  $u_n(t)$  for  $0 < t < +\infty$ , and

$$\begin{aligned} \|\nabla_{\mathbb{B}} u_n(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 &< \frac{2(p+1)d}{(p-1)(1+C_*^2)}, \\ \frac{1}{2} \int_0^t \|\partial_\tau u_n(\tau)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 d\tau + \frac{1}{p+1} \int_0^t \left\| \left( \frac{d}{d\tau} g_\tau(x) \right)^{\frac{1}{p+1}} u_n(t) \right\|_{\mathcal{L}_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}^{p+1} d\tau &< d, \\ \|g_t^{\frac{1}{p+1}}(x) u_n(t)\|_{\mathcal{L}_{p+1}^{\frac{n}{p+1}}(\mathbb{B})} &< C_{**}^{p+1} \left( \frac{2(p+1)d}{(p-1)(1+C_*^2)} \right)^{\frac{p+1}{2}}, \\ \|\sqrt{V(x)} u_n(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})} &< C_*^2 \left( \frac{2(p+1)d}{(p-1)(1+C_*^2)} \right)^2, \end{aligned}$$

which implies that there exists a subsequence  $\{u_n\}_{n \geq 1}$  still denotes  $\{u_n\}$  and a function  $u$  satisfy:

$$\begin{cases} u_m(t) \rightarrow u(t), & \text{weakly star in } L^\infty\left(0, +\infty; \mathcal{H}_{2,0}^{1, \frac{n}{2}}(\mathbb{B})\right) \text{ and a.e. in } \mathbb{B}_0 \times [0, +\infty), \\ \partial_t u_n(t) \rightarrow \partial_t u(t), & \text{weakly star in } \left(0, +\infty; \mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})\right), \\ V(x)|u_n(t)|^2 \rightarrow V(x)|u(t)|^2, & \text{weakly star in } L^\infty\left(0, +\infty; \mathcal{H}_{2,0}^{1, \frac{n}{2}}(\mathbb{B})\right) \text{ and a.e. in } \mathbb{B}_0 \times [0, +\infty), \\ g_t(x)|u_n(t)|^p \rightarrow g_t(x)|u(t)|^p, & \text{weakly star in } L^\infty\left(0, +\infty; \mathcal{H}_{2,0}^{1, \frac{n}{2}}(\mathbb{B})\right) \text{ and a.e. in } \mathbb{B}_0 \times [0, +\infty). \end{cases}$$

So we take the limit to derive that the weak solution of problem (1.1) exists globally.

(b). If  $\|\nabla_{\mathbb{B}} u_0\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})} = 0$ . By Lemma 2.3, i.e., the Poincaré inequality, we know  $\|u_0\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})} = 0$ , which implies that  $u_0 = 0$  almost everywhere for  $x \in \mathbb{B}_0$ , so we can infer from the definition of  $J(u)$  that  $J(u_0) = 0$ . Thus, it follows from the definition of  $I(t)$  that

$$d = I(0) = \frac{1}{2} \|u_1\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2.$$

Let  $\lambda_m = 1 - \frac{1}{m}$ ,  $u_{1m}(x) = \lambda_m u_1(x)$ ,  $m = 2, 3, \dots$ . Considering the weak solution of problem (1.1) corresponding to the following initial conditions

$$(4.2) \quad u(x, 0) = u_0(x), \quad u_t(x, 0) = u_{1m}(x), \quad x \in \mathbb{B}_0.$$

We denote  $I_m(0)$  as the initial energy for problem (1.1) corresponding to the initial conditions (4.2). Then we get

$$I_m(0) := \frac{1}{2} \|u_{1m}\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + J(u_0) = \frac{1}{2} \|\lambda_m u_1\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 < I(0) = d.$$

Hence, similar to the proof of the case (a), we also transformed the case  $I(0) = d, K(u_0) > 0$  into  $0 < I_m(0) < d, K(u_0) > 0$  under the initial condition (4.2), the remainder of proof is the same as that in the part (a).

(ii). Under the assumption of  $I(0) = d, K(u_0) < 0$ , we first prove that

$$(4.3) \quad K(u(t)) < 0, \quad \forall t \in [0, T).$$

Arguing by contradiction, assuming that there exists a  $t_0 \in (0, T)$  such that  $K(u(t_0)) = 0$  and  $K(u(t_0)) < 0$  for all  $0 < t < t_0$ . Then for any  $0 < t < t_0$ , taking  $\delta = 1$  in Lemma 3.6(ii) we know

$$\|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})} > (1 + C_*^2)^{\frac{1}{p-1}} C_{**}^{-\frac{p+1}{p-1}}.$$

Taking  $\delta = 1$  in Lemma 3.6(iii) we get

$$\|\nabla_{\mathbb{B}} u(t_0)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})} = (1 + C_*^2)^{\frac{1}{p-1}} C_{**}^{-\frac{p+1}{p-1}},$$

which implies that  $u(t_0) \in \mathcal{N}$ . Thus, we infer from the definition of  $d$  that  $J(u(t_0)) \geq d$ . By (1.11) and the definition of  $I(t)$ , for any  $t \in [0, T)$ , it is easy to see that

$$J(u(t)) + \gamma \int_0^t \|u_{\tau}(\tau)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 d\tau \leq I(0).$$

Choosing  $t = t_0$  in above inequality and using the assumption  $I(0) = d$  we obtain

$$\int_0^{t_0} \|u_{\tau}(\tau)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 d\tau = 0,$$

then

$$\|u_t(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})} = 0, \quad \forall t \in [0, t_0],$$

which infers that  $u_t(t) = 0$  almost everywhere for  $x \in \mathbb{B}$  and  $t \in [0, t_0]$ . Hence, we deduce  $u(t_0) = u_0$  and  $K(u(t_0)) = K(u_0) < 0$ , this contradicts  $K(u(t_0)) = 0$ , so we get (4.3).

Secondly, for any solution but not steady-state solution of problem (1.1), we claim there exists a  $t_1 \in (0, T)$  such that

$$(4.4) \quad \int_0^{t_1} \|u_{\tau}(\tau)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 d\tau > 0.$$

If it is false, then for any  $t \in [0, T)$ , there hold

$$\int_0^t \|u_{\tau}(\tau)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 d\tau = 0.$$

Similar to the deduction above, we can get  $u_t(t) = 0$  almost everywhere for  $x \in \mathbb{B}$  and  $t \in [0, T)$ . Then  $u(x, t) = u_0$ , i.e.,  $u(x, t)$  is a steady-state solution. Hence, our claim (4.4) is true.

It follows from (1.11),  $I(0) = d$  and (4.4) that

$$I(t_1) \leq I(0) - \gamma \int_0^{t_1} \|u_{\tau}(\tau)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 d\tau < d.$$

Furthermore, by (4.3) we know  $K(u(t_1)) < 0$ , so taking  $t_1$  as initial time, we can infer from Theorem 1.4(ii) that  $u(t)$  blows up in finite time, then our proof is complete.  $\square$

## 5 Arbitrary positive initial energy case

**Theorem 5.1.** *Let  $u(t) = u(x, t)$  be the weak solution of problem (1.1) with  $I(0) > 0$ . If the initial value satisfies:*

$$(5.1) \quad I(0) + \sqrt{\frac{\lambda_1(p-1)}{4(p+1)}} \left( 2 \int_{\mathbb{B}} u_0 u_1 \frac{dx_1}{x_1} dx' + \gamma \|u_0\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 \right) < 0$$

and  $K(u_0) > \|u_1\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2$ , then  $u(t)$  exists globally.

*Proof.* We define

$$\varphi(t) := \|u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \gamma \int_0^t \|u(\tau)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 d\tau,$$

then

$$(5.2) \quad \varphi'(t) = 2 \int_{\mathbb{B}} u(t) u_t(t) \frac{dx_1}{x_1} dx' + \gamma \|u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2.$$

Moreover, by the formula [1, (4.13)] and the definitions of  $\varphi(t)$ ,  $K(u(t))$  we can get

$$(5.3) \quad \varphi''(t) = -2 \left( K(u(t)) - \|u_t(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 \right).$$

Set  $F(t) := K(u(t)) - \|u_t(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2$ , by the assumption (5.1) we know  $F(0) > 0$ .

Next, we claim that  $F(t) > 0$  for all  $t \in [0, T)$ . Arguing by contradiction, we assume that there exists a  $t_0 \in (0, T)$  such that  $F(t_0) = 0$ ,  $F(t) > 0$  for  $t \in (0, t_0)$  and  $F(t) < 0$  for  $t \in (t_0, T)$ . It follows from (1.11) and the definition of  $I(t)$  that

$$(5.4) \quad I(0) \geq \frac{1}{2} \|u_t(t_0)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + J(u(t_0)) + \gamma \int_0^{t_0} \|u_\tau(\tau)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 d\tau.$$

By the definition of  $F(t)$  and  $F(t_0) = 0$ , we can see  $K(u(t_0)) = \|u_t(t_0)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 \geq 0$ , so (1.7) leads to

$$J(u(t_0)) \geq \frac{p-1}{2(p+1)} \|\nabla_{\mathbb{B}} u(t_0)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 \geq \frac{\lambda_1(p-1)}{2(p+1)} \|u(t_0)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2,$$

here we also used (1.8). Together above inequality with (5.4) we can obtain

$$(5.5) \quad I(0) \geq \frac{1}{2} \|u_t(t_0)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \frac{\lambda_1(p-1)}{2(p+1)} \|u(t_0)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \gamma \int_0^{t_0} \|u_\tau(\tau)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 d\tau.$$

By means of the Cauchy inequality we can see that

$$(5.6) \quad -2\sqrt{\frac{1}{2}} \sqrt{\frac{\lambda_1(p-1)}{2(p+1)}} \int_{\mathbb{B}} u_t(t_0) u(t_0) \frac{dx_1}{x_1} dx' \leq \frac{1}{2} \|u_t(t_0)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \frac{\lambda_1(p-1)}{2(p+1)} \|u(t_0)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2$$

and

$$-2\gamma \int_{\mathbb{B}} u_t(t) u(t) \frac{dx_1}{x_1} dx' \leq \|u_t(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \gamma^2 \|u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2.$$

It follows from above inequality and the definition of  $\varphi(t)$  that

$$\begin{aligned}
 \gamma \int_0^{t_0} \|u_\tau(\tau)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 d\tau &\geq -2\gamma^2 \int_0^{t_0} \int_{\mathbb{B}} u_\tau(\tau) u(\tau) \frac{dx_1}{x_1} dx' d\tau - \gamma^3 \int_0^{t_0} \|u(\tau)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 d\tau \\
 &= -\gamma^2 \int_0^{t_0} \frac{d}{d\tau} \|u(\tau)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 d\tau - \gamma^3 \int_0^{t_0} \|u(\tau)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 d\tau \\
 &= -\gamma^2 \|u(t_0)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \gamma^2 \|u_0\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 - \gamma^3 \int_0^{t_0} \|u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 d\tau \\
 &= -\gamma^2 \varphi(t_0) + \gamma^2 \|u_0\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2.
 \end{aligned}$$

Substituting above inequality and (5.6) into (5.5), there hold

$$(5.7) \quad I(0) \geq -2\sqrt{\frac{\lambda_1(p-1)}{4(p+1)}} \int_{\mathbb{B}} u_t(t_0) u(t_0) \frac{dx_1}{x_1} dx' - \gamma^2 \varphi(t_0) + \gamma^2 \|u_0\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2.$$

Furthermore, we can infer from (5.2) that

$$\varphi'(t_0) = 2 \int_{\mathbb{B}} u(t_0) u_t(t_0) \frac{dx_1}{x_1} dx' + \gamma \|u(t_0)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 \geq 2 \int_{\mathbb{B}} u(t_0) u_t(t_0) \frac{dx_1}{x_1} dx'.$$

Combing above inequality and (5.7), we reach

$$(5.8) \quad I(0) \geq -\sqrt{\frac{\lambda_1(p-1)}{4(p+1)}} \varphi'(t_0) - \gamma^2 \varphi(t_0) + \gamma^2 \|u_0\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2.$$

On the other hand, combining the assumption  $F(t) > 0$  for  $t \in (0, t_0)$ , the definition of  $F(t)$  and (5.3) we know that  $\varphi''(t) = -2F(t) < 0$  for all  $t \in (0, t_0)$ . Hence, by the continuity of  $\varphi(t)$  and  $\varphi'(t)$  we can see that there exists two constants  $\xi_1, \xi_2 \in (0, t_0)$  such that

$$\varphi(t_0) = \varphi(0) + \varphi'(0)t_0 + \varphi''(\xi_1)t_0^2 < \varphi(0) + \varphi'(0)t_0$$

and

$$\varphi'(t_0) = \varphi'(0) + \varphi''(\xi_2)t_0^2 < \varphi'(0).$$

Since  $I(0) > 0$ , by the assumption (5.1) and (5.2) we obtain  $\varphi'(0) < 0$ , thus, substituting the above two inequalities into (5.8) and calculating the values of  $\varphi'(0), \varphi(0)$  we get

$$\begin{aligned}
 I(0) &\geq -\sqrt{\frac{\lambda_1(p-1)}{4(p+1)}} \varphi'(0) - \gamma^2 \varphi(0) + \gamma^2 \|u_0\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 \\
 &= -\sqrt{\frac{\lambda_1(p-1)}{4(p+1)}} \left( 2 \int_{\mathbb{B}} u_0 u_1 \frac{dx_1}{x_1} dx' + \gamma \|u_0\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 \right),
 \end{aligned}$$

which contradicts (5.1), so our claim is true, i.e.,  $K(u(t)) > \|u_t(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 \geq 0$  for all  $t \in (0, T)$ .

Since  $K(u(t)) \geq 0$  for all  $t \in [0, T]$ , so we can infer from (1.7) that

$$J(u(t)) \geq \frac{p-1}{2(p+1)} \|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2.$$

Together with (1.11), the definition of  $I(t)$  and above inequality we can see

$$\begin{aligned} I(0) &\geq I(t) + \gamma \int_0^t \|u_\tau(\tau)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 d\tau \\ &= \frac{1}{2} \|u_t(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \frac{1}{p+1} \int_0^t \|(\frac{d}{d\tau} g_\tau(x))^{\frac{1}{p+1}} u(\tau)\|^{p+1} d\tau + J(u(t)) + \gamma \int_0^t \|u_\tau(\tau)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 d\tau \\ &\geq \frac{1}{2} \|u_t(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \frac{p-1}{2(p+1)} \|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2. \end{aligned}$$

Namely, we establish the uniformly boundedness of  $u_t(t)$  in  $\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})$  and  $u(t)$  in  $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$  for  $t \in [0, T)$ , so we get  $T = +\infty$ , i.e., the weak solution of problem (1.1) exists globally.  $\square$

Before proceeding further, we introduce following lemma, which can be found in [21, 22] and will be used in our proof.

**Lemma 5.2.** *Suppose that for  $t \geq t_0$ ,  $t_0$  is a positive constant, a nonnegative, twice-differentiable function  $\phi(t)$  satisfies the inequality*

$$\phi''(t)\phi(t) - (\alpha + 1)(\phi'(t))^2 \geq 0,$$

where  $\alpha > 0$  is a constant. If  $\phi(t_0) > 0$  and  $\phi'(t_0) > 0$ , then  $\phi(t) \rightarrow +\infty$  as  $t \rightarrow t_1$ , here  $t_1$  satisfies  $t_1 \leq T_0 = \phi(t_0)/(\alpha\phi'(t_0)) + t_0$ .

**Theorem 5.3.** *Let  $u(t) = u(x, t)$  be the weak solution of problem (1.1) with  $I(0) > 0$ . If initial value satisfies:*

$$(5.9) \quad 2 \int_{\mathbb{B}} u_0 u_1(t) \frac{dx_1}{x_1} dx' + \gamma \|u_0\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 - \frac{2(p+1)}{\kappa} I(0) > 0,$$

where  $\kappa$  is a positive constant defined in (5.13), then  $u(t)$  blows up in finite time.

*Proof.* Arguing by contradiction, we assume that  $u(t)$  exists globally. We define a functional as follow:

$$H(t) := 2 \int_{\mathbb{B}} u(t) u_t(t) \frac{dx_1}{x_1} dx' + \gamma \|u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 - \frac{2(p+1)}{\kappa} I(0),$$

where  $\kappa$  is a positive constant will be give later. Comparing the definition of  $H(t)$  with (5.2) and using (5.3) we can see that

$$(5.10) \quad H'(t) = -2K(u(t)) + 2\|u_t(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2.$$

Combining (1.7), the definition of  $I(t)$  and (1.11), we obtain

$$\begin{aligned} &-2K(u(t)) \\ &= -2(p+1)J(u(t)) + (p-1) \left( \|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \|\sqrt{V(x)}u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 \right) \\ &\geq -2(p+1)I(t) + (p+1)\|u_t(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + 2 \int_0^t \|(\frac{d}{d\tau} g_\tau(x))^{\frac{1}{p+1}} u(\tau)\|^{p+1} d\tau + (p-1)\|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 \\ &\geq 2(p+1)\gamma \int_0^t \|u_\tau(\tau)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 d\tau - 2(p+1)I(0) + (p+1)\|u_t(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + (p-1)\|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2, \end{aligned}$$

substituting above inequality into (5.10) and using (1.8), there hold

$$\begin{aligned}
 (5.11) \quad & H'(t) \\
 & \geq (p+3)\|u_t(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + (p-1)\|\nabla_{\mathbb{B}} u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 - 2(p+1)I(0) + 2(p+1)\gamma \int_0^t \|u_\tau(\tau)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 d\tau \\
 & \geq (p+3)\|u_t(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \lambda_1(p-1)\|u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 - 2(p+1)I(0) + 2(p+1)\gamma \int_0^t \|u_\tau(\tau)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 d\tau \\
 & \geq (p+3)\|u_t(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \lambda_1(p-1)\eta\|u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \lambda_1(p-1)(1-\eta)\|u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 - 2(p+1)I(0),
 \end{aligned}$$

where  $\eta$  is a positive constant will be define below.

It follows from the Cauchy inequality that

$$2\sqrt{\lambda_1\eta(p-1)(p+3)} \int_{\mathbb{B}} u(t)u_t(t) \frac{dx_1}{x_1} \leq (p+3)\|u_t(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \lambda_1\eta(p-1)\|u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2,$$

which combines (5.11) leads to

$$(5.12) \quad H'(t) \geq 2\sqrt{\lambda_1\eta(p-1)(p+3)} \int_{\mathbb{B}} u(t)u_t(t) \frac{dx_1}{x_1} + \lambda_1(p-1)(1-\eta)\|u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 - 2(p+1)I(0).$$

Considering the following equation respect to  $\eta$ ,

$$\sqrt{\lambda_1\eta(p-1)(p+3)} = \frac{1}{\gamma}\lambda_1(p-1)(1-\eta).$$

By a straightforward computation we can turn above equation into following equivalent form:

$$\eta^2 - \left(2 + \frac{(p+3)\gamma^2}{\lambda_1(p-1)}\right)\eta + 1 = 0,$$

it is easy to see that it has two positive roots, then we define  $\eta$  be the one of roots of above equation. Let

$$(5.13) \quad \kappa := \sqrt{\lambda_1\eta(p-1)(p+3)} > 0.$$

So we can infer from (5.12) and the definition of  $H(t)$  that

$$H'(t) \geq \kappa \left( 2 \int_{\mathbb{B}} u(t)u_t(t) \frac{dx_1}{x_1} + \gamma\|u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 - \frac{2(p+1)}{\kappa}I(0) \right) = \kappa H(t),$$

which implies  $H(t) \geq H(0)e^{\kappa t}$  for all  $t \geq 0$ . Due to the assumption (5.9) and the definition of  $H(t)$  we know  $H(0) > 0$ , so we get

$$\lim_{t \rightarrow +\infty} H(t) = +\infty,$$

then

$$(5.14) \quad \lim_{t \rightarrow +\infty} \left( \int_{\mathbb{B}} u(t)u_t(t) \frac{dx_1}{x_1} + \|u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 - \frac{2(p+1)}{\kappa}I(0) \right) = +\infty,$$

which combines the Cauchy inequality implies that

$$(5.15) \quad \lim_{t \rightarrow +\infty} \left( \|u_t(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \|u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 - \frac{2(p+1)}{\kappa} I(0) \right) = +\infty.$$

In order to complete our proof, we define an other auxiliary functional by

$$\phi(t) := \|u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \gamma \int_0^t \|u(\tau)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 d\tau + \gamma(T-t)\|u_0\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2,$$

then we can see

$$(5.16) \quad \phi'(t) = 2 \int_{\mathbb{B}} u(t) u_t(t) \frac{dx_1}{x_1} + \gamma \|u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 - \gamma \|u_0\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2.$$

By (5.14) we know that there exists a sufficient large  $t_0 > 0$  such that

$$\phi'(t) > 0, \quad \forall t \in (t_0, +\infty).$$

On the other hand, for a fixed  $p > 1$ , it is easy to see that there exist  $\frac{4}{p+3} < \xi < 1, \epsilon > 0$  such that

$$(5.17) \quad \xi(p+3) > 4 + \epsilon, \quad p > 1 + \frac{\epsilon}{2}.$$

In fact, since  $\xi > \frac{4}{p+3}$ , so we can choose  $0 < \epsilon < \min\{2p-2, \xi(p+3)-4\}$ , which satisfies (5.17). Comparing (5.16) and the definition of  $H(t)$ , using further the second inequality in (5.11) we obtain

$$\begin{aligned} \phi''(t) &= H'(t) \\ &\geq (p+3)\|u_t(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \lambda_1(p-1)\|u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 - 2(p+1)I(0) + 2(p+1)\gamma \int_0^t \|u_\tau(\tau)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 d\tau. \end{aligned}$$

We infer from (5.15) that there exists a sufficient large  $t_1$  such that, for any  $t \in (t_1, +\infty)$

$$\begin{aligned} \phi''(t) &> (p+3)\|u_t(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \lambda_1(p-1)\|u(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + 2(p+1)\gamma \int_0^t \|u_\tau(\tau)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 d\tau \\ &\geq (p+3)\|u_t(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + 2(p+1)\gamma \int_0^t \|u_\tau(\tau)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 d\tau \\ &> \xi(p+3)\|u_t(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + 2(p+1)\gamma \int_0^t \|u_\tau(\tau)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 d\tau. \end{aligned}$$

Hence, for any  $t \in (t_1, +\infty)$ , by (5.17) we get

$$\begin{aligned} \phi''(t) &> (4+\epsilon)\|u_t(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + 2(p+1)\gamma \int_0^t \|u_t(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 \\ &> (4+\epsilon) \left( \|u_t(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \gamma \int_0^t \|u_t(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 \right). \end{aligned}$$

Furthermore, for any  $t \in (t_1, +\infty)$ , combining (5.16) and the definition of  $\phi(t)$  we have

$$[\phi'(t)]^2 < 4\phi(t) \left( \|u_t(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 + \gamma \int_0^t \|u_t(t)\|_{\mathcal{L}_2^{\frac{n}{2}}(\mathbb{B})}^2 \right).$$



Thus, taking  $T_0 = \max\{t_0, t_1\}$ , for all  $t \in (T_0, +\infty)$  we arrive at

$$\phi(t)\phi''(t) - \left(1 + \frac{\epsilon}{4}\right) [\phi'(t)]^2 > 0,$$

then we can infer from Lemma 5.2 that there exists a sufficiently large  $T > T_0$  such that

$$\lim_{t \rightarrow T^-} \phi(t) = +\infty,$$

which contradicts the assumption that  $u(t)$  is a global solution, so our proof is complete.  $\square$

## Acknowledgements

This work is partially supported by the NSFC (Grant Nos. 11571062 and 11771062), the Fundamental Research Funds for the Central Universities(Grant No. 106112016CDJXZ238826).

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