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Classification of meromorphic integrals for autonomous nonlinear ordinary differential equations with two dominant monomials

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A R T I C L E I N F O

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A B S T R A C T

The Nevanlinna theory is used to derive the general structure of transcendental meromorphic solutions for a wide class of autonomous nonlinear ordinary differential equations with two dominant monomials. An algebro-geometric method, which enables one to obtain these solutions explicitly, is described. New simply periodic solutions of the Lorenz system are obtained.

1. Introduction

Solutions of ordinary differential equations that are meromorphic in the complex plane are of great theoretical and practical importance. In many cases physically relevant solutions belong to the class of meromorphic functions. Powerful tools of the Nevanlinna theory can be used to study existence and properties of meromorphic solutions [3,4,6–8,14,15,17,18,20,23,25].

Let us consider an Nth-order autonomous algebraic ordinary differential equation

$$\sum_j \alpha_j w^{j_0} \left\{\frac{dw}{dz}\right\}^{j_1} \ldots \left\{\frac{d^N w}{dz^N}\right\}^{j_N} = 0, \tag{1.1}$$

where $j = (j_0, \ldots, j_N)$ is a multi-index. The number $j_0+j_1+\ldots+j_N$ is called the degree of the corresponding monomial. A monomial is said to be dominant if it has the highest degree among other monomials of the differential equation. A. Eremenko studied autonomous algebraic ordinary differential equation with one dominant monomial and finite number of admissible Laurent series with a pole at the origin. A. Eremenko

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proved that transcendental meromorphic solutions of such equations are either elliptic or rational in $\exp(az)$, $a \in \mathbb{C}$ [14]. Further, the general structure of these solutions was clarified in [10,11]. In addition a method, which can be used to find these solutions in explicit form, was given in articles [10,11], see also [12,13].

In this article we study algebraic ordinary differential equations possessing two dominant monomials forming the following balance

$$\lambda w^k \{w_z - \mu w\} = 0, \quad \lambda \neq 0, \quad \mu \neq 0, \quad k \in \mathbb{N}. \tag{1.2}$$

Our results are presented in the following theorems.

Theorem 1.1. *Transcendental meromorphic functions with finite number of poles that satisfy equation (1.1) with two dominant monomials given by (1.2) are entire and take the form*

$$w(z) = h_0 + h_1 \exp\left[\mu z\right], \tag{1.3}$$

where h_0 and h_1 are constants.

Theorem 1.2. *Suppose that there exists at most $M \in \mathbb{N}$ pairwise distinct Laurent series*

$$w^{(j)}(z) = \sum_{n=1}^{p_j} \frac{c_{-n}^{(j)}}{z^n} + \sum_{n=0}^{\infty} c_n^{(j)} z^n, \quad p_j \in \mathbb{N}, \quad j = 1, \ldots, M \tag{1.4}$$

satisfying equation (1.1) with two dominant monomials given by (1.2). Then transcendental meromorphic functions with infinite number of poles that solve this equation are either elliptic or simply periodic of the form

$$w(z) = b\left\{\sum_{j=1}^{M} \varepsilon_j \sum_{n=1}^{p_j} \frac{(-1)^{n-1} c_{-n}^{(j)}}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}\right\} \cot\left(b\{z - z_j\}\right) + h_0 + h_1 \exp\left[\mu z\right], \tag{1.5}$$

where z_1, ..., z_M are distinct poles lying in a period strip, $\varepsilon_j = 1$ if $w(z)$ involves poles characterized by the series $w^{(j)}(z)$ or $\varepsilon_j = 0$ otherwise, h_0 and h_1 are constant, and $2b = q\mu i$, $q \in \mathbb{Q}/\{0\}$, $q > 0$ whenever $h_1 \neq 0$.

If there exists infinite number of Laurent series of the form (1.4) satisfying equation (1.1) with two dominant monomials, then our results can be used to classify meromorphic simply periodic solutions with finite number of poles in a period strip. The following theorem is valid.

Theorem 1.3. *Meromorphic simply periodic functions with finite number of poles in a period strip that satisfy equation (1.1) with two dominant monomials given by (1.2) are of the form (1.5).*

Theorems 1.1 and 1.2 establish the general structure of transcendental meromorphic solutions. Another important problem is to find these solutions in explicit form. In this article we describe such a method. In fact we generalize the method of articles [10,11]. The case of elliptic solutions was considered in detail in [6,12,13].

This article is organized as follows. In section 2 we use the Nevanlinna theory to prove our results and in section 3 we describe a method, which allows one to construct solutions in question explicitly. Section 4 is devoted to an example: we study two third-order ordinary differential equations related to the Lorenz system and present the general structure of meromorphic simply periodic solutions with finite number of poles in a period strip. We find explicitly several families of solutions. These solutions of the Lorenz system seem to be new.

2. Proof of main results

In what follows we shall use some basic results of the Nevanlinna theory. By $T(r, f)$ we denote the Nevanlinna characteristic function of a meromorphic function $f(z)$. We recall that $T(r, f) = m(r, f) + N(r, f)$, where $m(r, f)$ is the proximity function and $N(r, f)$ is the integrated counting function. For more details on the Nevanlinna theory see, for example [18,20].

Further we shall need the following lemmas.

Lemma 2.1. *Suppose Laurent series $w^{(j)}(z)$, see (1.4), with uniquely determined coefficients satisfies an algebraic ordinary differential equation; then this equation admits at most one meromorphic solution having a pole $z = 0$ with Laurent series $w^{(j)}(z)$.*

This lemma can be easily proved using properties of Laurent series and uniqueness of analytic continuation.

Lemma 2.2 *(Clunie). Let $f(z)$ be a transcendental meromorphic function satisfying the following equation*

$$f^k P(z, f, f_z, \ldots) = Q(z, f, f_z, \ldots), \tag{2.1}$$

where P, Q are polynomials in $f(z)$ and its derivatives with rational coefficients. If the degree of Q is at most k, then

$$m(r, P(z, f, f_z, \ldots)) = O(\log\{rT(r, f)\}), \quad r \to \infty \tag{2.2}$$

possibly outside a set of finite linear measure.

The proof of this lemma is given in [4,18,20]. Recall that the order ϱ of a meromorphic function $f(z)$ is defined as

$$\varrho = \overline{\lim_{r\to\infty}} \frac{\log T(r, f)}{\log r}. \tag{2.3}$$

Let us take an autonomous algebraic ordinary differential equation (1.1) with two dominant monomials (1.2). We can present such an equation in the form

$$w^k\{w_z - \mu w\} = Q(w, w_z, \ldots), \quad \mu \neq 0, \quad k \in \mathbb{N}, \tag{2.4}$$

where Q is a polynomial of its arguments with degree at most k. While proving Theorems 1.1, 1.2 we suppose that all asymptotic relations are valid for sufficiently large values of r possibly outside an exceptional set as that arising in Lemma 2.2.

Proof of Theorem 1.1. Let $w(z)$ be a transcendental meromorphic solution of equation (2.4). In addition let us suppose that $w(z)$ possesses finite number of poles. Hence we can present $w(z)$ in the form

$$w(z) = W(z) + R_1(z), \tag{2.5}$$

where $W(z)$ is a transcendental entire function and $R_1(z)$ is a rational function. Applying Clunie Lemma 2.2 to equation (2.4) yields $m(r, w_z - \mu w) = O(\log\{rT(r, w)\})$. Since the function $w(z)$ has finite number of poles, we see that the integrated counting function is $N(r, w) = O(\log r)$. As a result we obtain $N(r, w_z - \mu w) = O(\log r)$. Further, we can estimate the Nevanlinna characteristic function: $T(r, w_z - \mu w) =$

$O(\log\{rT(r,w)\})$. It follows from this relation that $w(z)$ is of finite order and $T(r, w_z - \mu w) = O(\log r)$. Consequently the combination $w_z - \mu w$ is a rational function. Let us denote this function by $R_2(z)$ and consider the equation $w_z - \mu w = R_2(z)$. Substituting (2.5) into this equation yields $W_z - \mu W = S_1(z)$, where $S_1(z) = R_2 + \mu R_1 - R_{1,z}$. We claim that the function $S_1(z)$ is a polynomial in z. Indeed, assuming the contrary and performing the singularity analysis near a pole $z_0 \in \mathbb{C}$ of $S_1(z)$ in equation $W_z - \mu W = S_1(z)$, we obtain that z_0 is a pole or a branch point of $W(z)$. But $W(z)$ is entire. It is a contradiction.

The general solution of equation $W_z - \mu W = S_1(z)$ takes the form

$$W(z) = h_1 \exp[\mu z] + S_2(z). \tag{2.6}$$

In this expression $S_2(z)$ is a polynomial in z, h_1 is an arbitrary constant. This yields the general expression for transcendental meromorphic solutions with finite number of poles:

$$w(z) = h_1 \exp[\mu z] + R_3(z) \tag{2.7}$$

Here $R_3(z) = R_1(z) + S_2(z)$ is a rational function. If $h_1 = 0$, then $w(z)$ is not transcendental. Thus we set $h_1 \neq 0$. Substituting expression (2.7) into equation (2.4) and setting to zero the coefficient at $\exp[k\mu z]$, we get $R_{3,z} - \mu R_3 + A = 0$, where A is a constant. Any rational solution of such an equation is a constant. This completes the proof. $\square$

Now let us proceed to Theorem 1.2.

Proof of Theorem 1.2. Let $w(z)$ be a transcendental meromorphic function with infinite number of poles that solves equation (2.4). Since $w(z)$ possesses infinite number of poles, we see that there are distinct poles z_1 and z_2 such that the functions $w(z + z_1)$, $w(z + z_2)$ have coinciding Laurent series in a neighborhood of the origin. In addition these functions solve equation (2.4) since the latter is autonomous. From Lemma 2.1 it follows that $w(z + z_1) = w(z + z_2)$. Further we get $w(z) = w(z + z_2 - z_1)$. Consequently, $w(z)$ is periodic. A periodic meromorphic function is either elliptic or simply periodic. If $w(z)$ is elliptic, then the theorem is proved. Let us suppose that $w(z)$ is simply periodic. Arguing as above, we see that $w(z)$ cannot have more than M poles in a period strip.

The Laurent series in a neighborhood of the origin that satisfy equation (2.4) are given by (1.4). From equality

$$\frac{\pi}{T}\cot\left(\frac{\pi z}{T}\right) = \frac{1}{z} + \sum_{n\in\mathbb{Z},\, n\neq 0}\left[\frac{1}{z-nT} + \frac{1}{nT}\right] \tag{2.8}$$

with T being a principle period, we see that $w(z)$ can be presented in the form

$$\begin{gathered} w(z) = W(z) + R_1(z), \\ R_1(z) = b\left\{\sum_{j=1}^{M}\varepsilon_j\sum_{n=1}^{p_j}\frac{(-1)^{n-1}c_{-n}^{(j)}}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\right\}\cot\left(b\{z-z_j\}\right), \quad b = \frac{\pi}{T}, \end{gathered} \tag{2.9}$$

where $z_1, \ldots, z_M$ are distinct poles lying in a period strip, $\varepsilon_j = 1$ if $w(z)$ involves poles characterized by the series $w^{(j)}(z)$ or $\varepsilon_j = 0$ otherwise, and $W(z)$ is a periodic entire function.

Applying Clunie Lemma 2.2 to equation (2.4) yields $m(r, w_z - \mu w) = O(\log\{rT(r,w)\})$. Integrated counting function in this case is $N(r,w) = O(r)$ and we obtain $N(r, w_z - \mu w) = O(r)$. Thus we conclude that $w(z)$ is of finite order and $T(r, w_z - \mu w) = O(r)$.

As a result there exists $a \in \mathbb{C}$ such that the function $w_z - \mu w$ is rational in $\exp(az)$. Let us consider the following first-order ordinary differential equation $w_z - \mu w = R_2(z)$, where $R_2(z)$ is rational in $\exp(az)$. Substituting expression (2.9) into this equation, we find the ordinary differential equation for transcendental entire function $W(z)$: $W_z - \mu W = S(z)$, where $S(z) = R_2 - R_{1,z} + \mu R_1$. We claim that the function $S(z)$ does not have poles. Indeed, assuming the contrary and performing the singularity analysis near a pole $z_0 \in \mathbb{C}$ of $S(z)$ in equation $W_z - \mu W = S(z)$, we get that z_0 is a pole or a branch point of $W(z)$. But $W(z)$ is entire. It is a contradiction. Consequently, the function $S(z)$ reads as $S(z) = S_1(z) + S_2(z)$, where $S_1(z)$ is a polynomial in $\exp(az)$ and $S_2(z)$ is a polynomial in $\exp(-az)$. The general periodic solution of equation $W_z - \mu W = S_1(z) + S_2(z)$ is

$$W(z) = h_1 \exp[\mu z] + Q_1(z) + Q_2(z), \tag{2.10}$$

where h_1 is an arbitrary constant and $Q_1(z)$, $Q_2(z)$ are polynomials in $\exp(az)$, $\exp(-az)$ accordingly. Substituting relation (2.10) into expression (2.9), we obtain

$$\begin{gathered} w(z) = b\left\{\sum_{j=1}^{M} \varepsilon_j \sum_{n=1}^{p_j} \frac{(-1)^{n-1} c_{-n}^{(j)}}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}\right\} \cot\left(b\{z - z_j\}\right) + h_1 \exp[\mu z] \\ +Q_1(z) + Q_2(z). \end{gathered} \tag{2.11}$$

Let us note that the parameters a, b, and μ are not independent because function (2.11) is meromorphic and simply periodic. There exists a principle period τ of the function $w(z)$ and all these parameters are expressible via τ.

Further let us consider the following asymptotic representation

$$\begin{gathered} w(z) = \alpha_1 \exp[A_1 z] + \alpha_0 \exp[A_0 z] + (\text{lower order terms}), \\ \operatorname{Re}\{A_1 z\} > \operatorname{Re}\{A_0 z\}, \quad z \to \infty, \end{gathered} \tag{2.12}$$

where $\alpha_1 \neq 0$, α_0, A_1, A_0 are constants and z tends to infinity along such a pass that $\operatorname{Re}\{A_1 z\} > 0$. Substituting this expression into equation (2.4) and setting to zero coefficients at $\exp[A_1(k+1)z]$, $\exp[(A_1 k + A_2)z]$ yields $A_1 = \mu$, $A_0 = 0$.

Combining expressions (2.11) and (2.12), we obtain

$$w(z) = b\left\{\sum_{j=1}^{M} \varepsilon_j \sum_{n=1}^{p_j} \frac{(-1)^{n-1} c_{-n}^{(j)}}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}\right\} \cot\left(b\{z - z_j\}\right) + h_1 \exp[\mu z] + h_0. \tag{2.13}$$

Concluding the proof it remains to mention that the functions $\exp[\mu z]$, $\cot[b\{z - z_j\}]$ should have the same principal period. This gives $2b = q\mu i$, $q \in \mathbb{Q}/\{0\}$, $q > 0$ whenever $h_1 \neq 0$. □

If equation (2.4) admits Laurent series in a neighborhood of the origin such that one or several coefficients are arbitrary, then there may exist meromorphic solutions with infinite number of poles that have more complicated structure. However if we require that the meromorphic solution $w(z)$ is simply periodic with finite number of poles in a period strip, then as a consequence of Theorem 1.2 we obtain Theorem 1.3.

3. Method applied

To begin this section let us mention that solutions of Theorem 1.1 can be obtained explicitly by substituting relation (1.3) into the original equation and setting to zero coefficients at $\exp[m\mu z]$, $m = 0, \dots, k$. If this system of algebraic equations is inconsistent, then equation (2.4) does not have transcendental meromorphic solutions with finite number of poles.

A method of finding solutions of Theorem 1.2 in explicit form is based on the fact that the Laurent series in a neighborhood of poles for solutions (1.5) and the corresponding Laurent series satisfying the original equation should coincide. The case of elliptic solutions and simply periodic with $h_1 = 0$ was studied in detail in articles [6,10–13]. In what follows we consider the case of simply periodic solutions (1.5) with $h_1 \neq 0$. The following lemma is valid.

Lemma 3.1. *Let z tend to infinity along such a pass that $Re\{\mu z\} > 0$, then solutions (1.5) can be presented asymptotically as follows*

$$w(z) = h_1 \exp[\mu z] + h_0 + \frac{q\mu}{2} \sum_{j=1}^{M} \varepsilon_j c_{-1}^{(j)} + o(1),\ Re\{\mu z\} > 0,\ z \to \infty. \tag{3.1}$$

Proof. We begin the proof by substituting $2b = q\mu i$, $q \in \mathbb{Q}/\{0\}$, $q > 0$ into the relation

$$\cot[b\{z - z_j\}] = \frac{\{1 + \exp[-2ib(z - z_j)]\}\, i}{1 - \exp[-2ib(z - z_j)]} \tag{3.2}$$

The result is

$$\cot[b\{z - z_j\}] = \frac{\{1 + \exp[q\mu(z - z_j)]\}\, i}{1 - \exp[q\mu(z - z_j)]}. \tag{3.3}$$

Calculating the limit $z \to \infty$, $\mathrm{Re}\{\mu z\} > 0$ yields

$$\cot[b\{z - z_j\}] = -i + o(1), \quad \mathrm{Re}\{\mu z\} > 0, \quad z \to \infty. \tag{3.4}$$

Further, using relations

$$\begin{aligned} &\frac{d}{dz}\cot[b\{z - z_j\}] = -b\left\{1 + \cot^2[b\{z - z_j\}]\right\}, \\ &\frac{d^2}{dz^2}\cot[b\{z - z_j\}] = -2b\cot[b\{z - z_j\}]\,\frac{d}{dz}\cot[b\{z - z_j\}], \\ &\qquad\qquad \dots \end{aligned} \tag{3.5}$$

we obtain by induction the following expressions

$$\frac{d^n}{dz^n}\cot[b\{z - z_j\}] = o(1), \quad \mathrm{Re}\{\mu z\} > 0, \quad z \to \infty, \quad n \in \mathbb{N}. \tag{3.6}$$

We complete the proof by substituting these asymptotic expressions into (1.5). Note that $a_{1,2} = \pm i$ are Picard exceptional values of the meromorphic function $\cot bz$. $\square$

Suppose we wish to find solutions (1.5) with $h_1 \neq 0$ of equation (2.4). Our algorithm can be subdivided into several steps.

Step 1. Perform local singularity analysis for solutions of equation (2.4). Construct all the Laurent series of the form (1.4).

Step 2. Write down general expressions (1.5).

Step 3. For the solutions of step 2 find the Laurent series in a neighborhood of the poles.

Step 4. Substitute asymptotic representation

$$w(z) = h_1 \exp[\mu z] + \alpha_0 + o(1), \quad \mathrm{Re}\{\mu z\} > 0, \quad z \to \infty, \tag{3.7}$$

into the original equation and find the constant α_0 by setting to zero the coefficient at $\exp[k\mu z]$. From Lemma 3.1 it follows that the following relation is valid

$$\alpha_0 = h_0 + \frac{q\mu}{2} \sum_{j=1}^{M} \varepsilon_j c_{-1}^{(j)}. \tag{3.8}$$

Step 5. Substitute all those Laurent series found at step 3 that are captured by a supposed solution into the original equation and set to zero coefficients at negative and zero powers of the expression $\{z - z_j\}$. At this step it is convenient to introduce notation

$$A_{j,l} = b\cot(b\{z_j - z_l\}), \quad j > l; \quad B_j = \exp[\mu z_j], \quad 1 \le j, l \le M. \tag{3.9}$$

Step 6. Solve the algebraic system obtained at step 5 and equation (3.8). It is necessary to take into account that $\{A_{j,l}, B_j\}$ are not independent and $b = (q\mu i)/2$.

Remark 1. Without loss of generality it can be assumed that $z_1 = 0$. Under such an assumption the addition formula

$$\cot(s - t) = \frac{\cot s \cot t + 1}{\cot t - \cot s} \tag{3.10}$$

allows us to rewrite solutions (1.5) as

$$\begin{gathered} w(z) = h_1 \exp[\mu z] + h_0 + b\varepsilon_1 \left\{ \sum_{n=1}^{p_1} \frac{(-1)^{n-1} c_{-n}^{(j)}}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \right\} \cot(bz) \\ + b \left\{ \sum_{j=2}^{M} \varepsilon_j \sum_{n=1}^{p_j} \frac{(-1)^{n-1} c_{-n}^{(j)}}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \right\} \frac{\{A_{j,1}\cot(bz) + b\}}{\{A_{j,1} - b\cot(bz)\}}, \quad b = \frac{q\mu i}{2}. \end{gathered} \tag{3.11}$$

Remark 2. As soon as the simply meromorphic solution $w(z)$ is found one should recall that in fact there exists the family of solutions $w(z - z_0)$ with arbitrary z_0.

In the next section we shall consider an example.

4. Lorenz model

The famous Lorenz model is given by the following system of polynomial ordinary differential equations [22]

$$\begin{cases} x_t = \sigma(y - x), \\ y_t = rx - y - zx, \\ z_t = xy - \beta z. \end{cases} \tag{4.1}$$

In this article we consider the Lorenz system in the framework of analytic theory of differential equations. The time t and the parameters (β, σ, r) are supposed to be complex variables. In their turn the functions $x(t)$, $y(t)$, $z(t)$ are supposed to be complex-valued.

The Lorenz system becomes linear if we set $\sigma = 0$. The general solutions of the Lorenz system with $\sigma \neq 0$ are known if the parameters (β, σ, r) take the following values [24]:

$$(\beta,\sigma,r) = \left(0,\frac{1}{3},r\right), \quad (\beta,\sigma,r) = \left(1,\frac{1}{2},0\right), \quad (\beta,\sigma,r) = \left(2,1,\frac{1}{9}\right). \tag{4.2}$$

It is a remarkable fact that in all these cases the Lorenz system passes the Painlevé test. The general solutions of the Lorenz system with the parameters given by (4.2) belong to the class of simply periodic meromorphic functions. Further, a natural question arises: can there exist periodic meromorphic solutions at other values of the parameters? Our aim is to obtain new families of exact simply periodic meromorphic solutions.

The Lorenz system possesses Darboux polynomials at certain values of the parameters. The Darboux polynomials produce time-dependent first integrals. The final classification of Darboux polynomials was obtained by Llibre and Zhang [21].

In what follows we suppose that the parameters (β, σ, r) do not take values that make the Lorenz system integrable. In particular, we set $\sigma \neq 0$. Solving the first and the second equations of the system with respect to $y(t)$ and $z(t)$ yields

$$y(t) = \frac{x_t}{\sigma} + x, \quad z(t) = -\frac{x_{tt}}{\sigma x} - \frac{(\sigma+1)x_t}{\sigma x} + r - 1. \tag{4.3}$$

If $x(t)$ is a simply periodic meromorphic function, then so do $y(t)$ and $z(t)$. Substituting these relations into the third equation of the system, we get the following third-order ordinary differential equation

$$\begin{gathered} x x_{ttt} - \{x_t - (1+\beta+\sigma)x\}x_{tt} - (\sigma+1)x_t^2 + x\{x^2 + \beta(\sigma+1)\}x_t \\ +\sigma x^4 + \beta\sigma(1-r)x^2 = 0 \end{gathered} \tag{4.4}$$

Let us note that introducing the new function $w(t) = x^2(t)$ we obtain another algebraic third-order ordinary differential equation

$$\begin{gathered} w^2 w_{ttt} - w\{2w_t - (1+\beta+\sigma)w\}w_{tt} + w_t^3 - \left(1+\sigma+\frac{\beta}{2}\right) w w_t^2 \\ +w^2\{w + \beta(\sigma+1)\}w_t + 2\sigma w^4 + 2\beta\sigma(1-r)w^3 = 0. \end{gathered} \tag{4.5}$$

Our aim is to investigate the structure of simply periodic meromorphic functions with finite number of poles in a period strip that satisfy equation (4.4). It turns out that it is convenient to use equation (4.5) instead of equation (4.4) since the former possesses only one family of Laurent series in a neighborhood of a pole.

We use the Painlevé methods to obtain Laurent series in a neighborhood of a pole that satisfy equations (4.4), (4.5). For more details on the Painlevé methods, related algorithms and examples see [1,2,5,9,16]. These equations are autonomous, consequently, without loss of generality we shall construct the Laurent series in a neighborhood of the origin. Let us begin with equation (4.5). The dominant behavior and the Fuchs indices are the following

$$w(t) = -\frac{4}{t^2}; \quad j = -1, \quad 2, \quad 4. \tag{4.6}$$

The Laurent series of the form

$$w(t) = -\frac{4}{t^2} + \frac{4(1-3\sigma+2\beta)}{3t} + \sum_{k=2}^{\infty} c_k t^{k-2}, \quad 0 < |t| < \delta_1 \tag{4.7}$$

exists provided that one of the following conditions is valid: $\beta = 1 - 3\sigma$, $\beta = 2\sigma$. This series contains two arbitrary coefficients c_2 and c_4 in integrable cases. In other cases the coefficient c_2 is no longer arbitrary. Consequently we see that equation (4.5) necessarily possess meromorphic solutions with at least one pole only under one of the following restrictions: $\beta = 1 - 3\sigma$, $\beta = 2\sigma$.

Note that another dominant balance $w^2 w_{ttt} - 2ww_t w_{tt} + w_t^3 = 0$ gives the asymptotic behavior of the form $w(t) = a_0$ or $w(t) = a_0(t - t_0)^2$ with an arbitrary coefficient a_0. These solutions cannot generate Laurent series in a neighborhood of a pole t_0.

Similarly, we obtain two families of Laurent series in a neighborhood of the origin that satisfy equation (4.4) with $\beta = 1 - 3\sigma$ or $\beta = 2\sigma$. They are the following

$$x^{(1,2)}(t) = \frac{\pm 2i}{t} + \sum_{k=1}^{\infty} c_k^{(1,2)} t^{k-1}, \quad 0 < |t| < \delta_2^{(1,2)}. \tag{4.8}$$

The coefficients $c_4^{(1,2)}$ are arbitrary, while the coefficients $c_2^{(1,2)}$ are arbitrary only in integrable cases.

Again the dominant balance $xx_{ttt} - x_t x_{tt} = 0$ does not give solutions that generate Laurent series in a neighborhood of poles lying in $\mathbb{C}$.

Equations (4.4), (4.5) possess two dominant monomials forming the following ordinary differential equations

$$\begin{aligned} &x^3(x_t + \sigma x) = 0; \\ &w^3(w_t + 2\sigma w) = 0. \end{aligned} \tag{4.9}$$

Using results of sections 1 and 3, we get the following theorems.

Theorem 4.1. *Transcendental meromorphic solutions of equation* (4.5) *possessing finite number of poles are of the form*

$$w(t) = h_1 \exp(-2\sigma t), \quad \beta r = 0, \tag{4.10}$$

where h_1 is an arbitrary constant.

Theorem 4.2. *Simply periodic non-entire meromorphic solutions of equation* (4.5) *possessing finite number of poles in a period strip are of the form*

$$\begin{aligned} w(t) = -4b \sum_{m=1}^{M} &\left\{ b \cot^2\left(b\{t - t_m\}\right) - \frac{(1 - 3\sigma + 2\beta)}{3} \cot\left(b\{t - t_m\}\right) \right\} \\ &+ h_0 - 4b^2 M + h_1 \exp(-2\sigma t), \quad M \in \mathbb{N}, \end{aligned} \tag{4.11}$$

where $t_1, \ldots, t_M$ are distinct poles lying in a period strip and $b = -i\sigma q$, $q \in \mathbb{Q}/\{0\}$, $q > 0$ whenever $h_1 \neq 0$.

Theorem 4.3. *Transcendental meromorphic solutions of equation* (4.4) *possessing finite number of poles are of the form*

$$w(t) = p_1 \exp(-\sigma t), \quad \beta r = 0, \tag{4.12}$$

where p_1 is an arbitrary constant.

Theorem 4.4. *Simply periodic non-entire meromorphic solutions of equation (4.4) possessing finite number of poles in a period strip are of the form*

$$\begin{gathered} x(t)=2ib\left\{\sum_{m=1}^{M_1}\cot\left(b\left\{t-t_m^{(1)}\right\}\right)-\sum_{m=1}^{M_2}\cot\left(b\left\{t-t_m^{(2)}\right\}\right)\right\} \\ +p_0+p_1\exp(-\sigma t), \quad M_1, M_2\in\mathbb{N}_0, \quad M_1+M_2>0, \end{gathered} \tag{4.13}$$

where $t_1^{(j)}, \ldots, t_{M_j}^{(j)}$, $j=1, 2$ *are distinct poles lying in a period strip and* $2b=-i\sigma s$, $s\in\mathbb{Q}/\{0\}$, $s>0$ *whenever* $p_1\neq 0$.

Let us construct simply periodic solutions (4.11) with $M=1$ in explicit form. We shall use the method of section 3. The case $h_1=0$ was studied in [19]. Consequently, we set $h_1\neq 0$. Relation (3.8) takes the form

$$h_0-\frac{4(1-3\sigma+2\beta b)\sigma q}{3}=\beta r. \tag{4.14}$$

The Laurent series expansion of function (4.11) with $M=1$ and $t_1=0$ in a neighborhood of the origin is the following

$$\begin{gathered} w(t)=-\frac{4}{t^2}+\frac{c_1}{t}+h_0+h_1+\frac{8b^2}{3}-\left(2h_1\sigma+\frac{c_1b^2}{3}\right)t+\left(2h_1\sigma^2-\frac{4b^4}{15}\right)t^2 \\ -\left(\frac{4h_1\sigma^3}{3}+\frac{c_1b^4}{45}\right)t^3+\left(\frac{2h_1\sigma^4}{3}-\frac{8b^6}{189}\right)t^4+\ldots, \end{gathered} \tag{4.15}$$

where we use notation

$$c_1=\frac{4(1-3\sigma+2\beta)}{3}. \tag{4.16}$$

Substituting series (4.15) into equation (4.5) and setting to zero the coefficients at negative and zero powers of t, we obtain the system of eight algebraic equations. One of these equation gives: $\beta=1-3\sigma$ or $\beta=2\sigma$. Further, we combine this system with relation (4.14). Solving obtained equations, we find two simply periodic solutions of the form (4.11). These solutions exist under certain conditions on the parameters of the original equation. Our results are the following

$$\begin{gathered} (I):\ w(t)=-\frac{4}{121}\left(\coth^2\left\{\frac{t}{11}\right\}-4\coth\left\{\frac{t}{11}\right\}+3+4\exp\left[-\frac{2t}{11}\right]\right) \\ (II):\ w(t)=-\frac{4}{49}\left(\coth^2\left\{\frac{t}{7}\right\}-4\coth\left\{\frac{t}{7}\right\}+3+4\exp\left[-\frac{2t}{7}\right]\right) \end{gathered} \tag{4.17}$$

The first solution corresponds to the case $\beta=2\sigma$, the second solution solves the original equation whenever $\beta=1-3\sigma$. For all these solutions the parameter q equals 1. Other parameters are given by

$$\begin{gathered} (I):\quad (\beta,\sigma,r)=\frac{1}{11}(2,1,-16); \\ (II):\quad (\beta,\sigma,r)=\frac{1}{7}(4,1,-8). \end{gathered} \tag{4.18}$$

Finally, let us recall the relation $w(t) = x^2(t)$ and find solutions of equation (4.4). The result is

$$
\begin{aligned}
(I): \, x(t) &= \pm\frac{i}{11}\left(\coth\left\{\frac{t}{22}\right\} - \coth^{-1}\left\{\frac{t}{22}\right\} - 4\exp\left[-\frac{t}{11}\right]\right)\\
(II): \, x(t) &= \pm\frac{i}{7}\left(\coth\left\{\frac{t}{14}\right\} - \coth^{-1}\left\{\frac{t}{14}\right\} - 4\exp\left[-\frac{t}{7}\right]\right)
\end{aligned}
\tag{4.19}
$$

Simply periodic solutions (4.19) possess two distinct poles in a period strip. They can be presented in the form

$$
\begin{aligned}
(I): \, x(t) &= \pm\frac{i}{11}\left(\coth\left\{\frac{t}{22}\right\} - \coth\left\{\frac{t}{22} - \frac{\pi i}{2}\right\} - 4\exp\left[-\frac{t}{11}\right]\right)\\
(II): \, x(t) &= \pm\frac{i}{7}\left(\coth\left\{\frac{t}{14}\right\} - \coth\left\{\frac{t}{14} - \frac{\pi i}{2}\right\} - 4\exp\left[-\frac{t}{7}\right]\right)
\end{aligned}
\tag{4.20}
$$

Recall that we omit the arbitrary constant t_0 resulting from the invariance of equations (4.4), (4.5) under the substitution $t \mapsto t - t_0$. Let us also mention that the Lorenz system has no Darboux polynomials in the case $\beta = 1 - 3\sigma$ [21]. Solutions (4.20) seem to be new.

5. Conclusion

In this article we have found the general structure of transcendental meromorphic solutions for a wide class of autonomous nonlinear ordinary differential equations with two dominant monomials. We have described an algebro-geometric method, which can be used to obtain these solutions in explicit form. As an example we have studied third-order ordinary differential equations (4.4) and (4.5) related to the Lorenz model. We have derived new simply periodic meromorphic solutions of these equations. Let us note that these solutions can not be found with the help of the tanh-function method and related methods.

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References

[1] A.D. Bruno, Power Geometry in Algebraic and Differential Equations, Elsevier Science (North-Holland), 2000.
[2] A.D. Bruno, Asymptotic behaviour and expansions of solutions of an ordinary differential equation, Russian Math. Surveys 59 (3) (2004) 429–481.
[3] Y.M. Chiang, R. Halburd, On the meromorphic solutions of an equation of Hayman, J. Math. Anal. Appl. 281 (2003) 663–677.
[4] J. Clunie, On integral and meromorphic functions, J. Lond. Math. Soc. 37 (1962) 17–27.
[5] R. Conte, The Painlevé approach to nonlinear ordinary differential equations, in: R. Conte (Ed.), The Painlevé Property One Century Later, in: CRM Series in Mathematical Physics, Springer-Verlag, New York, 1999, pp. 77–180.
[6] R. Conte, M. Musette, Elliptic general analytic solutions, Stud. Appl. Math. 123 (2009) 63–81.
[7] R. Conte, T.-W. Ng, Meromorphic solutions of a third order nonlinear differential equation, J. Math. Phys. 51 (2010) 033518.
[8] R. Conte, T.-W. Ng, K.K. Wong, Exact meromorphic stationary solutions of the real cubic Swift–Hohenberg equation, Stud. Appl. Math. 129 (1) (2012) 117–131.
[9] M.V. Demina, N.A. Kudryashov, Power and non-power expansions of the solutions for the fourth-order analogue to the second Painlevé equation, Chaos Solitons Fractals 32 (1) (2007) 124–144.
[10] M.V. Demina, N.A. Kudryashov, From Laurent series to exact meromorphic solutions: the Kawahara equation, Phys. Lett. A 374 (2010) 4023–4029.
[11] M.V. Demina, N.A. Kudryashov, Explicit expressions for meromorphic solutions of autonomous nonlinear ordinary differential equations, Commun. Nonlinear Sci. Numer. Simul. 16 (2011) 1127–1134.

[12] M.V. Demina, N.A. Kudryashov, On elliptic solutions of nonlinear ordinary differential equations, Appl. Math. Comput. 217 (23) (2011) 9849–9853.
[13] M.V. Demina, N.A. Kudryashov, Elliptic solutions in the Hénon–Heiles model, Commun. Nonlinear Sci. Numer. Simul. 19 (3) (2014) 471–482.
[14] A. Eremenko, Meromorphic traveling wave solutions of the Kuramoto–Sivashinsky equation, J. Math. Phys. Anal. Geom. 2 (3) (2006) 278–286.
[15] A.E. Eremenko, L. Liao, T.W. Ng, Meromorphic solutions of higher order Briot–Bouquet differential equations, Math. Proc. Cambridge Philos. Soc. 146 (1) (2009) 197–206.
[16] A. Goriely, Integrability and Nonintegrability of Dynamical Systems, World Scientific, 2001.
[17] R. Halburd, J. Wang, All admissible meromorphic solutions of Hayman's equation, Int. Math. Res. Not. IMRN 18 (2015) 8890–8902.
[18] W. Hayman, Nevanlinna Theory and Complex Differential Equations, Clarendon Press, Oxford, 1964.
[19] N.A. Kudryashov, Analytical solutions of the Lorenz system, Regul. Chaotic Dyn. 20 (2015) 123–133.
[20] I. Laine, Meromorphic Functions, de Gruyter, Berlin, New York, 1993.
[21] J. Llibre, X. Zhang, Invariant algebraic surfaces of the Lorenz system, J. Math. Phys. 43 (2002) 1622–1645.
[22] E.N. Lorenz, Deterministic nonperiodic flow, J. Atmos. Sci. 20 (1963) 130–141.
[23] R. Nevanlinna, Analytic Functions, Springer, Berlin, 1970.
[24] M. Tabor, J. Weiss, Analytic structure of the Lorenz system, Phys. Rev. A 24 (4) (1981) 2157–2167.
[25] Y. Zhang, Z. Gao, J. Zhang, All admissible meromorphic solutions of certain type of second degree second order algebraic ODEs, J. Math. Anal. Appl. 452 (2) (2017) 1182–1193.