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Compact almost automorphic weak solutions for some monotone differential inclusions: Applications to parabolic and hyperbolic equations

Brahim Es-sebbar [a], Khalil Ezzinbi [b], Samir Fatajou [c], Mohamed Ziat [d,*]

[a] *Département de Mathématiques, Faculté des Sciences et Techniques Guéliz, Université Cadi Ayyad, B.P. 549, Marrakesh, Morocco*
[b] *Département de Mathématiques, Faculté des Sciences Semlalia, Université Cadi Ayyad, B.P. 2390, Marrakesh, Morocco*
[c] *École Nationale des Sciences Appliquées de Safi, Université Cadi Ayyad, Route Sidi Bouzid, B.P. 63, Safi, Morocco*
[d] *Laboratoire de Mathématiques Appliquées et Calcul Scientifique, Faculté des Sciences et Techniques, Université Sultan Moulay Slimane, B.P. 523, Béni Mellal, Morocco*

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A B S T R A C T

We study the existence of compact almost automorphic weak solutions for the differential inclusion $u'(t) + \mathcal{A}u(t) \ni f(t)$ for $t \in \mathbb{R}$, where $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ is *maximal monotone* and the forcing term f is compact almost automorphic. We prove that the existence of a uniformly continuous weak solution on $\mathbb{R}^+$ having a relatively compact range over $\mathbb{R}^+$ implies the existence of a compact almost automorphic weak solution. For that goal, we use Amerio's principle. We prove also the existence, uniqueness, and global attractivity of a compact almost automorphic weak solution where $\mathcal{A}$ is *strongly maximal monotone*. For illustration, some applications are provided for parabolic and hyperbolic equations.

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1. Introduction

Let $\mathcal{H}$ be a real Hilbert space and $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ be a multivalued operator with domain $D(\mathcal{A})$. We consider the following differential inclusions:

$$u'(t) + \mathcal{A}u(t) \ni f(t) \quad \text{for } t \in \mathbb{R}, \tag{1.1}$$

$$u'(t) + \mathcal{A}u(t) \ni g(t, u(t)) \quad \text{for } t \in \mathbb{R}, \tag{1.2}$$

* Corresponding author.
E-mail address: ziat.maths@gmail.com (M. Ziat).

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where $f : \mathbb{R} \longrightarrow \mathcal{H}$ and $g : \mathbb{R} \times \mathcal{H} \longrightarrow \mathcal{H}$ are continuous functions. Many studies have been devoted to the existence of periodic and almost periodic solutions for the differential inclusion (1.1) when the operator $\mathcal{A}$ is maximal monotone on $\mathcal{H}$ and the forcing term f is periodic or almost periodic. Brézis [13, Theorem 3.4, p. 65] proved that for any $f \in L^1(a, b; \mathcal{H})$ and $u_0 \in \overline{D(\mathcal{A})}$ there exists a unique weak solution of the following differential inclusion:

$$\begin{cases} u'(t) + \mathcal{A}u(t) \ni f(t) \quad \text{for } t \in [a, b], \\ u(a) = u_0. \end{cases} \tag{1.3}$$

Brézis [13, Theorem 3.15, p. 95] showed that if $\mathcal{A}$ is maximal monotone, then for each $f \in L^1(0, T; \mathcal{H})$ the differential inclusion

$$\begin{cases} u'(t) + \mathcal{A}u(t) \ni f(t), \\ u(0) = u(T) \end{cases} \tag{1.4}$$

has at least a weak solution. Baillon and Haraux [5] studied the following differential inclusion:

$$u'(t) + \partial\phi(u(t)) \ni f(t) \quad \text{for } t \in [0, +\infty), \tag{1.5}$$

where ϕ is a proper, convex, and lower semicontinuous function and $f \in L^2([0, +\infty), \mathcal{H})$ is T-periodic. They proved that if a T-periodic solution of (1.5) exists on $\mathbb{R}$, then for each solution u of (1.5) on $\mathbb{R}^+$ there exists a periodic strong solution w of (1.5) on $\mathbb{R}$ such that

$$u(t) \rightharpoonup w(t) \quad \text{as } t \longrightarrow +\infty.$$

Haraux [19] proved that if the forcing term $f : \mathbb{R} \longrightarrow \mathcal{H}$ is S^2-almost periodic, then each weak solution of (1.5) on $\mathbb{R}^+$ is asymptotic to an almost periodic weak solution of (1.5) on $\mathbb{R}$. Haraux [21] also proved that if (1.1) has a uniformly continuous weak solution on $\mathbb{R}^+$ and its range over $\mathbb{R}^+$ is relatively compact, then it has an almost periodic weak solution on $\mathbb{R}$ when f is almost periodic [21, Theorem 1, p. 295]. Furthermore, Haraux [23] proved that if the forcing term $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ is S^1-almost periodic, then all bounded solutions on $\mathbb{R}$ of (1.1) are almost periodic.

Bochner [9–11] introduced the concept of almost automorphy as a generalization of almost periodicity [10,12]. This concept was then investigated in depth by Veech [26,27] and many other authors. Almost automorphy is a weak version of almost periodicity, so many results and methods in the theory of almost periodicity are complicated in the almost automorphic framework.

The aim of this work is to study the existence of compact almost automorphic weak solutions for (1.1) and (1.2). If $\mathcal{A}$ is maximal monotone and f is compact almost automorphic, we prove that if (1.1) has a uniformly continuous weak solution on $\mathbb{R}^+$ having a relatively compact range over $\mathbb{R}^+$, then it has at least a compact almost automorphic weak solution on $\mathbb{R}$. Our main result is proved by use of the minmax principle due to Amerio [1]. As an application, we study the following partial differential equation:

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}u(t,x) - \Delta u(t,x) + \beta\left(\dfrac{\partial}{\partial t}u(t,x)\right) \ni f(t,x) \quad \text{for } (t,x) \in \mathbb{R} \times \Omega, \\ u(t,x) = 0 \quad \text{for } (t,x) \in \mathbb{R} \times \partial\Omega, \end{cases} \tag{1.6}$$

where Ω is a bounded open set in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ such that $\dim(\Omega) \geq 2$, β is a strongly monotone graph in $\mathbb{R} \times \mathbb{R}$, and f is a compact almost automorphic function in $L^2(\Omega)$.

If $\mathcal{A}$ is strongly maximal monotone and f is compact almost automorphic, we prove that (1.1) has a unique bounded weak solution that is compact almost automorphic and globally attractive. Moreover, we use the contraction principle to prove the existence and uniqueness of compact almost automorphic weak solutions for (1.2) where g is compact almost automorphic in t and Lipschitzian with respect to the second argument.

This work is organized as follows. In Section 2, we recall some definitions and basic results that we need to prove our results. Section 3 is devoted to the existence of compact almost automorphic weak solutions for the differential inclusion (1.1) where $\mathcal{A}$ is maximal monotone. In Section 4, we study the existence and uniqueness of compact almost automorphic weak solutions of the differential inclusion (1.2) where $\mathcal{A}$ is strongly maximal monotone. Finally, in Section 5 we give some applications to illustrate the main results of this work.

2. Monotone operators and almost automorphic functions

In this section we recall some basic results for maximal monotone operators and almost automorphic functions.

2.1. Maximal monotone operators

Let $\mathcal{H}$ be a real Hilbert space equipped with its norm $|\cdot|$ arising from its inner product $(\cdot,\cdot)_{\mathcal{H}}$.

Definition 2.1. Let $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ be a multivalued operator. Its domain is defined by

$$D(\mathcal{A}) = \{x \in \mathcal{H} : \mathcal{A}x \text{ is nonempty in } \mathcal{H}\}.$$

(i) The range of $\mathcal{A}$ is defined by

$$R(\mathcal{A}) = \bigcup_{x\in\mathcal{H}} \mathcal{A}x.$$

(ii) The graph of $\mathcal{A}$ is defined by

$$G(\mathcal{A}) = \{(x, y) \in \mathcal{H}^2 : y \in \mathcal{A}x\}.$$

(iii) $\mathcal{A}$ is *monotone* if

$$(\mathcal{A}x - \mathcal{A}y, x - y)_{\mathcal{H}} \ \geq 0 \quad \text{for all } x, y \in D(\mathcal{A}), \tag{2.1}$$

which means that for each $x_1 \in D(A)$ and $x_2 \in D(A)$, one has

$$(y_1 - y_2, x_1 - x_2)_{\mathcal{H}} \ \geq 0 \quad \text{for all } y_1 \in \mathcal{A}x_1 \text{ and } y_2 \in \mathcal{A}x_2. \tag{2.2}$$

(iv) $\mathcal{A}$ is *maximal monotone* if it is monotone and $R(I + \mathcal{A}) = \mathcal{H}$, where I is the identity operator on $\mathcal{H}$.

(v) $\mathcal{A}$ is α*-strongly maximal monotone* ($\alpha > 0$) if it is maximal monotone and

$$(\mathcal{A}x - \mathcal{A}y, x - y)_{\mathcal{H}} \ \geq \alpha|x - y|^2 \quad \text{for all } x, y \in D(\mathcal{A}). \tag{2.3}$$

Remark 2.2. Let $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ be a multivalued operator and $\alpha > 0$. Then

$$\mathcal{A} \text{ is maximal monotone if and only if } \mathcal{A} + \alpha I \text{ is } \alpha\text{-strongly maximal monotone.}$$

Theorem 2.3. [13, p. 27] *If $\mathcal{A} : \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ is maximal monotone, then $\overline{D(\mathcal{A})}$ is convex.*

Consider the following differential inclusion:

$$\begin{cases} u'(t) + \mathcal{A}u(t) \ni f(t) & \text{for } t \in [a,b], \\ u(a) = u_0 \in \mathcal{H}. \end{cases} \tag{2.4}$$

Definition 2.4. [13] Let $f \in L^1(a,b;\mathcal{H})$. A continuous function $u : [a,b] \longrightarrow \mathcal{H}$ is a strong solution of the differential inclusion (2.4) if u is absolutely continuous on each compact of $]a,b[$, $u(t) \in D(\mathcal{A})$ almost everywhere on $[a,b]$ and (2.4) is satisfied almost everywhere on $[a,b]$.

Definition 2.5. [13] A function u is a weak solution of (2.4) if there exist $f_n \in L^1(a,b;\mathcal{H})$, $u_n \in C(a,b;\mathcal{H})$ such that u_n is a strong solution of

$$u_n'(t) + \mathcal{A}u_n(t) \ni f_n(t)$$

on $[a,b]$, $f_n \longrightarrow f$ in $L^1(a,b;\mathcal{H})$ and $u_n \longrightarrow u$ in $C(a,b;\mathcal{H})$.

Definition 2.6. Let $f \in L^1_{loc}(\mathbb{R},\mathcal{H})$. u is a weak solution of (2.4) on $\mathbb{R}$ if it is a weak solution of (2.4) on every compact interval of $\mathbb{R}$.

The following results regarding the existence and estimations of weak solutions are needed in this work.

Theorem 2.7. [21, Theorem 36, p. 76]. *Assume that $\mathcal{A}$ is maximal monotone and $f \in L^1(a,b;\mathcal{H})$. If $u_0 \in \overline{D(\mathcal{A})}$, then there exists a unique weak solution of (2.4). Moreover, if u and v are two weak solutions of $u'(t) + \mathcal{A}u(t) \ni f(t)$ and $v'(t) + \mathcal{A}v(t) \ni g(t)$, respectively, then,*

$$|u(t) - v(t)| \le |u(s) - v(s)| + \int_s^t |f(\sigma) - g(\sigma)| \mathrm{d}\sigma \quad \text{for } a \le s \le t \le b. \tag{2.5}$$

Theorem 2.8. [28] *Assume that $\mathcal{A}$ is α-strongly maximal monotone. Let I be an interval of $\mathbb{R}$ and $\widetilde{f}, \widehat{f} \in L^1_{loc}(I;\mathcal{H})$. If $\widetilde{u}$ and $\widehat{u}$ are weak solutions on I of $\widetilde{u}'(t) + \mathcal{A}\widetilde{u}(t) \ni \widetilde{f}(t)$ and $\widehat{u}'(t) + \mathcal{A}\widehat{u}(t) \ni \widehat{f}(t)$, respectively, then for any s and t in I, $s \le t$, we have*

$$|\widetilde{u}(t) - \widehat{u}(t)| \le e^{-\alpha(t-s)}|\widetilde{u}(s) - \widehat{u}(s)| + \int_s^t e^{-\alpha(t-\sigma)}|\widetilde{u}(\sigma) - \widehat{f}(\sigma)| \mathrm{d}\sigma. \tag{2.6}$$

2.2. Almost automorphic functions

Let $(X, |\cdot|_X)$ and $(Y, |\cdot|_Y)$ be Banach spaces and $BC(\mathbb{R}, X)$ be the space of bounded continuous functions from $\mathbb{R}$ to X equipped with the supremum norm.

Definition 2.9. [14,18] A continuous function $f : \mathbb{R} \longrightarrow X$ is almost periodic if for every $\varepsilon > 0$ there exists a positive number l such that every interval of length l contains a number τ such that

$$\sup_{t\in\mathbb{R}} |f(t+\tau) - f(t)|_X < \varepsilon.$$

We denote by $AP(\mathbb{R}, X)$ the space of almost periodic X-valued functions.

Theorem 2.10. [18] *Every almost periodic function is uniformly continuous.*

Bochner [11] introduced the concept of almost automorphy, which is more general than almost periodicity.

Definition 2.11. [11,24,25] A continuous function $f : \mathbb{R} \longrightarrow X$ is almost automorphic if for every sequence of real numbers $(t'_n)_n$ there exist a subsequence $(t_n)_n$ and a function g such that for each $t \in \mathbb{R}$

$$f(t+t_n) \longrightarrow g(t) \quad \text{as } n \longrightarrow +\infty \tag{2.7}$$

and

$$g(t-t_n) \longrightarrow f(t) \quad \text{as } n \longrightarrow +\infty. \tag{2.8}$$

We denote by $AA(\mathbb{R}, X)$ the space of all almost automorphic X-valued functions.

Definition 2.12. [15] A continuous function $f : \mathbb{R} \longrightarrow X$ is compact almost automorphic if for each $(t'_n)_n \subseteq \mathbb{R}$ there exist a subsequence $(t_n)_n$ and a function g such that

$$\lim_{n\to+\infty} f(t+t_n) = g(t) \quad \text{and} \quad \lim_{n\to+\infty} g(t-t_n) = f(t) \tag{2.9}$$

uniformly on any compact subset of $\mathbb{R}$.
We denote by $AA_c(\mathbb{R}, X)$ the space of all such functions.

Remark 2.13. (1) Each almost automorphic function $f : \mathbb{R} \longrightarrow X$ has a relatively compact range; hence, it is bounded.
(2) Since the convergence in Definition 2.11 is a pointwise convergence, the function g is only measurable and not necessarily continuous. The function g in Definition 2.12 is continuous.

Theorem 2.14. [16] *A function f is compact almost automorphic if and only if it is almost automorphic and uniformly continuous.*

Example 2.15. [6, Example 3.1] Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be such that

$$f(t) = \sin\left(\frac{1}{2+\cos(t)+\cos(\sqrt{2}t)}\right) \quad \text{for } t \in \mathbb{R}.$$

f is almost automorphic, but it is not uniformly continuous on $\mathbb{R}$; hence, it is not almost periodic.

Example 2.16. [17,27] Let θ be an irrational real number. Then for all $n \in \mathbb{Z}$, $\cos(2\pi n\theta) \neq 0$. Let $(h_n)_n$ be the sequence defined by

$$h_n = \operatorname{sgn}\cos(2\pi n\theta) = \begin{cases} +1 \text{ if } \cos(2\pi n\theta) > 0, \\ -1 \text{ if } \cos(2\pi n\theta) < 0. \end{cases} \tag{2.10}$$

Let f be given by

$$f(t) = h_n + (t-n)(h_{n+1} - h_n) \quad \text{for } t \in [n, n+1].$$

Then f is compact almost automorphic, but it is not almost periodic.

Remark 2.17. We have that

$$AP(\mathbb{R}, X) \subsetneq AA_c(\mathbb{R}, X) \subsetneq AA(\mathbb{R}, X) \subsetneq BC(\mathbb{R}, X). \tag{2.11}$$

Theorem 2.18. [15] *$AA_c(\mathbb{R}, X)$ endowed with the supremum norm is a Banach space.*

Definition 2.19. [15] A continuous function $f : \mathbb{R} \times X \longrightarrow Y$ is compact almost automorphic in $t \in \mathbb{R}$ if for every $(t'_n)_n \subseteq \mathbb{R}$ there exist a subsequence $(t_n)_n$ and a function g such that

$$f(t + t_n, x) \longrightarrow g(t, x) \quad \text{as } n \longrightarrow +\infty \tag{2.12}$$

and

$$g(t - t_n, x) \longrightarrow f(t, x) \quad \text{as } n \longrightarrow +\infty \tag{2.13}$$

uniformly on any compact set in $\mathbb{R}$ and for any $x \in X$.
The space of such functions is denoted by $AA_c(\mathbb{R} \times X, Y)$.

Theorem 2.20. [15] *Let $f \in AA_c(\mathbb{R} \times X, Y)$ be Lipschitzian with respect to the second argument. If $x \in AA_c(\mathbb{R}, X)$, then the composition function $t \longmapsto f(t, x(t))$ belongs to $AA_c(\mathbb{R}, Y)$.*

3. Compact almost automorphic weak solutions of (1.1) where $\mathcal{A}$ is maximal monotone

In the sequel, we prove the existence of compact almost automorphic weak solutions of (1.1) where the operator $\mathcal{A}$ is maximal monotone.

Theorem 3.1. *Suppose that f is compact almost automorphic and $\mathcal{A}$ is maximal monotone. If (1.1) has a uniformly continuous weak solution on $\mathbb{R}^+$ having a relatively compact range over $\mathbb{R}^+$, then (1.1) has at least a compact almost automorphic weak solution.*

For the proof of Theorem 3.1, we need the following lemmas.

Lemma 3.2. [13] *Let $f,\ f_n \in L^1([a, b]; \mathcal{H})$. Assume that x_n is a weak solution of $x'_n(t) + \mathcal{A}x_n(t) \ni f_n(t)$ on $[a, b]$. If $x_n \longrightarrow x$ uniformly on $[a, b]$ and $f_n \longrightarrow f$ in $L^1([a, b]; \mathcal{H})$, then x is a weak solution of (1.1) on $[a, b]$.*

Lemma 3.3. *Let $F \in L^1_{loc}(\mathbb{R}, \mathcal{H})$ and x be a weak solution on $\mathbb{R}$ of the following differential inclusion:*

$$x'(t) + \mathcal{A}x(t) \ni F(t). \tag{3.1}$$

Assume that x is uniformly continuous on $\mathbb{R}$ and there exists a compact set K of $\mathcal{H}$ such that

$$x(t) \in K \quad \text{for all } t \in \mathbb{R}. \tag{3.2}$$

If there exist a sequence $(t_n)_n \subseteq \mathbb{R}$ and a function $G : \mathbb{R} \longrightarrow \mathcal{H}$ such that

$$F(t + t_n) \longrightarrow G(t) \quad \text{in } L^1_{loc}(\mathbb{R}, \mathcal{H}) \text{ as } n \longrightarrow +\infty, \tag{3.3}$$

then there exists a subsequence of $(t_n)_n$ denoted by $(s_n)_n$ such that

$$x(t + s_n) \longrightarrow y(t) \quad \text{as } n \longrightarrow +\infty \tag{3.4}$$

uniformly on any compact subset of $\mathbb{R}$, where y is a weak solution on $\mathbb{R}$ of the following differential inclusion:

$$y'(t) + \mathcal{A}y(t) \ni G(t). \tag{3.5}$$

Furthermore, y is uniformly continuous on $\mathbb{R}$ and $y(t) \in K$ for all $t \in \mathbb{R}$.

Proof. For each $n \in \mathbb{N}$, we define x_n and F_n on $\mathbb{R}$ by $x_n(t) = x(t+t_n)$ and $F_n(t) = F(t+t_n)$. By (3.2), $(x_n)_n$ satisfies $x_n(t) \in K$ for each $t \in \mathbb{R}$ and $n \in \mathbb{N}$. Consequently $\{x_n(t) : n \in \mathbb{N}\}$ is a relatively compact set in $\mathcal{H}$ for each $t \in \mathbb{R}$. Since x is uniformly continuous on $\mathbb{R}$, the sequence $(x_n)_n$ is uniformly equicontinuous on $\mathbb{R}$. By the Arzelà-Ascoli theorem, $\{x_n : n \in \mathbb{N}\}$ is a relatively compact subset of $BC(\mathbb{R}, \mathcal{H})$ endowed with the topology of compact convergence. From the sequence $(t_n)_n$, we can extract a subsequence $(s_n)_n$ such that there exists $y \in BC(\mathbb{R}, \mathcal{H})$ such that $(x_n)_n$ converges to y uniformly on each compact subset of $\mathbb{R}$ and hence (3.4) holds. Furthermore, since x_n is a weak solution of (1.1) with F_n and F_n converges to G in $L^1_{loc}(\mathbb{R}, \mathcal{H})$, the use of Lemma 3.2 allows us to conclude that y is a weak solution of (3.5); moreover, $y(t) \in K$ for all $t \in \mathbb{R}$. Therefore, y is uniformly continuous on $\mathbb{R}$ because it is a limit of the sequence $(x_n)_n$, which is uniformly equicontinuous on $\mathbb{R}$. $\square$

Lemma 3.4. *Suppose that f is compact almost automorphic. If (1.1) has a uniformly continuous weak solution u_0 on $\mathbb{R}^+$ having a relatively compact range over $\mathbb{R}^+$, then it has a uniformly continuous weak solution u^* on $\mathbb{R}$; moreover, its range over $\mathbb{R}$ is relatively compact.*

Proof. Let $(t_n)_n \subseteq \mathbb{R}$ be such that

$$\lim_{n \to +\infty} t_n = +\infty. \tag{3.6}$$

If $t \in [-1, 1]$, then for sufficiently large n the sequence of functions $u_n : t \longmapsto u_0(t + t_n)$ is well defined and uniformly equicontinuous. By the Arzelà-Ascoli theorem, there exist a function v and a subsequence $(t^1_n)_n \subset (t_n)_n$ such that

$$u_0(t + t^1_n) \longrightarrow v(t) \quad \text{as } n \longrightarrow +\infty$$

uniformly on $[-1, 1]$. Using the same argument, we deduce that for each $p \in \mathbb{N}^*$ there exists a subsequence $(t^p_n)_n \subset (t^{p-1}_n)_n \subset \cdots \subset (t_n)_n$ such that

$$u_0(t + t^p_n) \longrightarrow v(t) \quad \text{as } n \longrightarrow +\infty$$

uniformly on $[-p, p]$. Let $(t'_n)_n := (t^n_n)_n$ be the Cantor's diagonal sequence. Then

$$u_0(t + t'_n) \longrightarrow v(t) \quad \text{as } n \longrightarrow +\infty \tag{3.7}$$

uniformly on any compact subset of $\mathbb{R}$. Since f is compact almost automorphic, there exist a continuous function g and a subsequence $(t''_n)_n \subset (t'_n)_n$ such that

$$f(t + t''_n) \longrightarrow g(t) \quad \text{as } n \longrightarrow +\infty, \tag{3.8}$$

$$g(t - t''_n) \longrightarrow f(t) \quad \text{as } n \longrightarrow +\infty \tag{3.9}$$

uniformly on any compact subset of $\mathbb{R}$. Moreover, using (3.7) and (3.8), we find that v is a weak solution on $\mathbb{R}$ of (1.1) with g. Furthermore, v is also equicontinuous. By applying the above argument to the returning sequence $(-t''_n)_n$, we obtain a subsequence $(t'''_n)_n \subset (t''_n)_n$ and a function u^* such that

$$v(t + t'''_n) \longrightarrow u^*(t) \quad \text{as } n \longrightarrow +\infty \tag{3.10}$$

uniformly on any compact subset of $\mathbb{R}$. By (3.9), (3.10), and Lemma 3.2, we deduce that u^* is a weak solution on $\mathbb{R}$ of (1.1). Furthermore, the function u^* is uniformly continuous on $\mathbb{R}$ and its range is contained in the closure of the range of u_0; hence, it is relatively compact. $\square$

The next lemma plays a crucial role in this work.

Lemma 3.5. [21, Lemma 30, p. 220] *Let S be a contraction defined on a closed convex subset of $\mathcal{H}$ and $x, y \in \mathcal{H}$ be such that $|Sx - Sy| = |x - y|$. Then*

$$S\left(\frac{x+y}{2}\right) = \frac{S(x) + S(y)}{2}.$$

Proof of Theorem 3.1. We use Amerio's principle. Let $K = \overline{\mathrm{Co}}(u^*(\mathbb{R}))$ be the closed convex hull of $u^*(\mathbb{R})$ in $\mathcal{H}$, where u^* is given in Lemma 3.4. Let Λ and Γ be the sets defined by

$$\Lambda = \left\{u \in C(\mathbb{R}, \mathcal{H}) : u(\mathbb{R}) \subset K \text{ and } \sup_{t\in\mathbb{R}} |u(t+\sigma) - u(t)| \leq \sup_{t\in\mathbb{R}} |u^*(t+\sigma) - u^*(t)| \text{ for all } \sigma \in \mathbb{R}\right\},$$

$$\Gamma = \left\{u \in \Lambda : \quad u \text{ is a weak solution of the differential inclusion (1.1) on } \mathbb{R}\right\}.$$

We define the operator $J : \Lambda \longrightarrow \mathbb{R}^+$ by

$$J(u) = \sup_{t\in\mathbb{R}} |u(t)| \quad \text{for } u \in \Lambda.$$

We say that $\widetilde{u}$ is a *minimal weak solution* of (1.1) if

$$\widetilde{u} \in \Gamma \quad \text{and} \quad J(\widetilde{u}) = \inf_{u\in\Gamma} J(u).$$

We divide the proof into three steps:

Step 1. We claim that (1.1) has a minimal weak solution $\widehat{u}$ on $\mathbb{R}$. Let

$$\delta = \inf_{u\in\Gamma} J(u). \tag{3.11}$$

Then, by Lemma 3.4, Γ is nonempty since $u^* \in \Gamma$. Hence, δ exists in $\mathbb{R}$. Consequently, there exists a sequence $(u_n)_n$ in Γ such that

$$\lim_{n\to+\infty} J(u_n) = \delta. \tag{3.12}$$

By the definition of Γ, for each $t \in \mathbb{R}$, $\{u_n(t) : n \in \mathbb{N}\}$ is a subset of the compact K and $(u_n)_n$ is uniformly equicontinuous on $\mathbb{R}$. Using the Arzelà-Ascoli theorem, we assert that $\{u_n : n \in \mathbb{N}\}$ is a relatively compact subset of $BC(\mathbb{R}, \mathcal{H})$ endowed with the topology of compact convergence. Thus, there exists a subsequence of $(u_n)_n$ denoted also by $(u_n)_n$ such that

$$u_n(t) \longrightarrow \widehat{u}(t) \quad \text{as } n \longrightarrow +\infty \tag{3.13}$$

uniformly on any compact subset of $\mathbb{R}$. Since $u_n'(t) + \mathcal{A}u_n(t) \ni f(t)$ in the sense of weak solutions, the use of (3.13) together with Lemma 3.2 implies that $\widehat{u}$ is a weak solution on $\mathbb{R}$ of (1.1) and $\widehat{u} \in \Lambda$; consequently, $\widehat{u} \in \Gamma$. Hence, we obtain

$$\delta \leq J(\widehat{u}). \tag{3.14}$$

We note that J is lower semicontinuous with respect to the topology of compact convergence; namely, if $\lim\limits_{n\to\infty} x_n = x$ uniformly on compact subsets of $\mathbb{R}$, then $J(x) \le \liminf_{n\to\infty} J(x_n)$. By (3.13), we get

$$J(\widehat{u}) \le \liminf_{n\to+\infty} J(u_n). \tag{3.15}$$

By (3.12), (3.14), and (3.15), we deduce that

$$J(\widehat{u}) = \delta = \inf_{u\in\Gamma} J(u). \tag{3.16}$$

Step 2. We claim that the minimal weak solution $\widehat{u}$ is unique. Let $u, v \in \Gamma$ be such that

$$J(u) = J(v) = \delta. \tag{3.17}$$

Let $(t_n)_n \subseteq \mathbb{R}$ be such that

$$\lim_{n\to+\infty} t_n = -\infty. \tag{3.18}$$

From the compact almost automorphy of f, there exist a continuous function g and a subsequence of $(t_n)_n$ denoted also by $(t_n)_n$ such that

$$\begin{aligned} f(t+t_n) &\longrightarrow g(t) \quad \text{as } n \longrightarrow +\infty, \\ g(t-t_n) &\longrightarrow f(t) \quad \text{as } n \longrightarrow +\infty \end{aligned}$$

uniformly on any compact subset of $\mathbb{R}$. Now let us prove that

$$u(t+t_n) \longrightarrow u_1(t) \quad \text{as } n \longrightarrow +\infty, \tag{3.19}$$

$$u_1(t-t_n) \longrightarrow u_2(t) \quad \text{as } n \longrightarrow +\infty, \tag{3.20}$$

$$v(t+t_n) \longrightarrow v_1(t) \quad \text{as } n \longrightarrow +\infty, \tag{3.21}$$

$$v_1(t-t_n) \longrightarrow v_2(t) \quad \text{as } n \longrightarrow +\infty \tag{3.22}$$

uniformly on any compact subset of $\mathbb{R}$, where u_2 and v_2 are two minimal weak solutions on $\mathbb{R}$ of (1.1). Since $u \in \Gamma$, it is uniformly continuous on $\mathbb{R}$ and $u(\mathbb{R}) \subset K$. Applying Lemma 3.3 to $x = u$, $F = f$, and the sequence $(t_n)_n$, we obtain (3.19), where u_1 is a weak solution on $\mathbb{R}$ of the following differential inclusion:

$$u_1'(t) + \mathcal{A}u_1(t) \ni g(t).$$

Moreover, u_1 is uniformly continuous on $\mathbb{R}$ and $u_1(\mathbb{R}) \subset K$, which implies that $u_1 \in \Lambda$. Applying again Lemma 3.3 to $x = u_1$, $F = g$, and the returning sequence $(-t_n)_n$, we obtain (3.20), where u_2 is a weak solution on $\mathbb{R}$ of (1.1) with $u_2 \in \Gamma$. It follows from (3.19) and (3.20) that

$$J(u_2) \le J(u_1) \le J(u). \tag{3.23}$$

Using (3.17), we obtain $J(u_2) = \delta$, and consequently u_2 is a weak minimal solution on $\mathbb{R}$ of (1.1). Applying the same argument to v, we obtain (3.21) and (3.22), where v_2 is a weak minimal solution on $\mathbb{R}$ of (1.1). Since $u'(t) + \mathcal{A}u(t) \ni f(t)$ and $v'(t) + \mathcal{A}v(t) \ni f(t)$ in the sense of weak solutions and the operator $\mathcal{A}$ is monotone, we find by using inequality (2.5) that the function $t \longmapsto |u(t) - v(t)|$ is nonincreasing. By (3.18), we obtain

$$\lim_{n\to+\infty}|u(t+t_n)-v(t+t_n)|=\sup_{\sigma\in\mathbb{R}}|u(\sigma)-v(\sigma)|. \tag{3.24}$$

It follows from (3.19)–(3.22) that for each $t\in\mathbb{R}$

$$\begin{aligned}\lim_{m\to+\infty}\lim_{n\to+\infty}|u(t+t_n-t_m)-v(t+t_n-t_m)|&=\lim_{m\to+\infty}|u_1(t-t_m)-v_1(t-t_m)|\\&=|u_2(t)-v_2(t)|.\end{aligned} \tag{3.25}$$

Combining (3.24) and (3.25), we find that for each $t\in\mathbb{R}$

$$|u_2(t)-v_2(t)|=\sup_{\sigma\in\mathbb{R}}|u(\sigma)-v(\sigma)|=c. \tag{3.26}$$

Consequently, we have

$$|u_2(t)-v_2(t)|=|u_2(0)-v_2(0)|\quad\text{for } t\in\mathbb{R}. \tag{3.27}$$

Let $S_t:\overline{D(\mathcal{A})}\longrightarrow\overline{D(\mathcal{A})}$ be the operator defined for each $x_0\in\overline{D(\mathcal{A})}$ by

$$S_tx_0=x(t),$$

where x is the unique weak solution on $\mathbb{R}$ of (1.1) with initial data $x(0)=x_0$. Taking $f=g$ in (2.5), we deduce that the operator S_t is contractive on the closed convex set $\overline{D(\mathcal{A})}$. It follows from (3.27) that

$$|S_tu_2(0)-S_tv_2(0)|=|u_2(0)-v_2(0)|.$$

Using Lemma 3.5, we obtain

$$S_t\left(\frac{u_2(0)+v_2(0)}{2}\right)=\frac{1}{2}\big(\mathbb{S}_tu_2(0)+S_tv_2(0)\big)=\frac{u_2(t)+v_2(t)}{2}.$$

We conclude that $\dfrac{u_2+v_2}{2}$ is also a weak solution on $\mathbb{R}$ of (1.1). Since $u_2(\mathbb{R})\subset K$, $v_2(\mathbb{R})\subset K$, and K is convex, $\left(\dfrac{u_2+v_2}{2}\right)(\mathbb{R})\subset K$ and $\dfrac{u_2+v_2}{2}\in\Gamma$. Hence,

$$\delta=\inf_{u\in\Gamma}J(u)\le J\left(\frac{1}{2}u_2+\frac{1}{2}v_2\right)=\sup_{t\in\mathbb{R}}\left|\frac{1}{2}u_2(t)+\frac{1}{2}v_2(t)\right|. \tag{3.28}$$

By the parallelogram law, we get

$$\sup_{t\in\mathbb{R}}\left|\frac{1}{2}u_2(t)+\frac{1}{2}v_2(t)\right|^2+\frac{1}{4}c^2\le\frac{1}{2}\sup_{t\in\mathbb{R}}|u_2(t)|^2+\frac{1}{2}\sup_{t\in\mathbb{R}}|v_2(t)|^2. \tag{3.29}$$

By (3.28) and (3.29), we obtain $\delta^2+\dfrac{1}{4}c^2\le\dfrac{1}{2}\delta^2+\dfrac{1}{2}\delta^2$. Hence, $|u_2(t)-v_2(t)|=c\le 0$ for all $t\in\mathbb{R}$; consequently, $u_2=v_2$, which implies by (3.26) that $u=v$.

Step 3. We claim that the unique minimal weak solution $\widehat{u}$ is compact almost automorphic. Let $(t'_n)_n\subseteq\mathbb{R}$. We have to prove that there exist a subsequence $(t_n)_n$ of $(t'_n)_n$ and a continuous function ν such that

$$\widehat{u}(t+t_n)\longrightarrow\nu(t)\quad\text{as } n\longrightarrow+\infty, \tag{3.30}$$

$$\nu(t-t_n)\longrightarrow\widehat{u}(t)\quad\text{as } n\longrightarrow+\infty \tag{3.31}$$

uniformly on any compact subset of $\mathbb{R}$. From the compact almost automorphy of f, there exists a subsequence $(t_n)_n \subset (t'_n)_n$ such that

$$f(t+t_n) \longrightarrow g(t) \quad \text{as } n \longrightarrow +\infty,$$
$$g(t-t_n) \longrightarrow f(t) \quad \text{as } n \longrightarrow +\infty$$

uniformly on any compact subset of $\mathbb{R}$. Since $\widehat{u} \in \Gamma$, it is uniformly continuous on $\mathbb{R}$, and $\widehat{u}(t) \in K$ for all $t \in \mathbb{R}$. Applying Lemma 3.3 to $x = \widehat{u}$, $F = f$, and the sequence $(t_n)_n$, we obtain (3.30), where ν is a weak solution on $\mathbb{R}$ of the following differential inclusion:

$$\nu'(t) + \mathcal{A}\nu(t) \ni g(t).$$

Furthermore, ν is uniformly continuous on $\mathbb{R}$ and $\nu(t) \in K$ for all $t \in \mathbb{R}$ since $\nu \in \Lambda$. Using (3.30), we obtain

$$J(\nu) \le J(\widehat{u}). \tag{3.32}$$

Applying Lemma 3.3 to $x = \nu$, $F = g$, and the returning sequence $(-t_n)_n$, we find for a subsequence that

$$\nu(t-t_n) \longrightarrow \omega(t) \quad \text{as } n \longrightarrow +\infty \tag{3.33}$$

uniformly on any compact subset of $\mathbb{R}$, where $\omega \in \Gamma$. From (3.33) we obtain

$$J(\omega) \le J(\nu). \tag{3.34}$$

By (3.32) and (3.34), we get

$$J(\omega) \le J(\widehat{u}) = \inf_{u \in \Gamma} J(u).$$

Consequently,

$$J(\omega) = J(\widehat{u}) = \inf_{u \in \Gamma} J(u).$$

By the uniqueness of the minimal weak solution of (1.1) (from steps 1 and 2), we deduce that $\omega = \widehat{u}$, (3.31) holds, and $\widehat{u}$ is compact almost automorphic. $\square$

4. Compact almost automorphic weak solutions of (1.1) and (1.2) where $\mathcal{A}$ is strongly maximal monotone

In the sequel, we prove the existence and uniqueness of compact almost automorphic weak solutions for the differential inclusions (1.1) and (1.2) where the operator $\mathcal{A}$ is strongly maximal monotone.

Theorem 4.1. *Assume that $\mathcal{A}$ is α-strongly maximal monotone ($\alpha > 0$) with $0 \in \mathcal{A}0$ and $f \in AA_c(\mathbb{R}, \mathcal{H})$. Then (1.1) has a unique compact almost automorphic weak solution u_f that is globally attractive.*

Proof. The proof is divided in five steps:
Step 1. We claim that the differential inclusion (1.1) has a bounded weak solution u_f on $\mathbb{R}$. Let $n \in \mathbb{N}$ and consider the following problem:

$$\begin{cases} u'(t) + \mathcal{A}u(t) \ni f(t), \\ u(-n) = 0. \end{cases} \tag{4.1}$$

Then (4.1) has a unique weak solution u_n on $[-n, \infty)$. Since $\mathcal{A}0 \ni 0$, it follows by Theorem 2.8, with $\widetilde{u} = u_n$, $\widetilde{f} = f$, $\widehat{u} = 0$, and $\widehat{f} = 0$, that

$$|u_n(t)| \leq \int_{-n}^{t} e^{-\alpha(t-\sigma)}|f(\sigma)|\mathrm{d}\sigma \quad \text{for } t \in [-n, \infty).$$

The compact almost automorphy of the function f implies its boundedness. Let $M_f = \sup_{t\in\mathbb{R}} |f(t)|$. Then the last inequality gives

$$|u_n(t)| \leq \frac{M_f}{\alpha}(1 - e^{-\alpha(t+n)}) \quad \text{for } t \in [-n, +\infty[;$$

consequently,

$$|u_n(t)| \leq \frac{M_f}{\alpha} \quad \text{for } t \in [-n, +\infty[. \tag{4.2}$$

Let $I = [a, b]$ and n and m be such that $-n \leq -m \leq a$. Using Theorem 2.8 for $\widetilde{u} = u_n$, $\widetilde{f} = f$, $\widehat{u} = u_m$, and $\widehat{f} = f$, we get

$$|u_n(t) - u_m(t)| \leq e^{-\alpha(t+m)}|u_n(-m) - u_m(-m)| = e^{-\alpha(t+m)}|u_n(-m)| \quad \text{for } t \in I. \tag{4.3}$$

Inequalities (4.2) and (4.3) imply that

$$|u_n(t) - u_m(t)| \leq \frac{M_f}{\alpha} e^{-\alpha(a+m)} \quad \text{for } t \in I,$$

which implies that $(u_n)_n$ is a Cauchy sequence in $C(I, \mathcal{H})$ and hence it converges to u_f in $C(I, \mathcal{H})$. By Lemma 3.2, the function u_f is a weak solution of (1.1) on I. Since I is arbitrary, by (4.2), u_f is a bounded weak solution on $\mathbb{R}$ of (1.1).

Step 2. We claim that u_f is unique. Suppose that v is another bounded weak solution on $\mathbb{R}$ of (1.1). By Theorem 2.8, we have for $t, \sigma \in \mathbb{R}$, $t \geq \sigma$,

$$|u_f(t) - v(t)| \leq e^{-\alpha(t-\sigma)}|u_f(\sigma) - v(\sigma)|. \tag{4.4}$$

Since u_f and v are bounded, by letting $\sigma \longrightarrow -\infty$ in (4.4), we obtain $u_f = v$. The global attractivity also follows from (4.4) by our taking $\sigma = 0$ and letting $t \longrightarrow +\infty$.

Step 3. We claim that the range of u_f is relatively compact. Let $(t'_n)_n \subseteq \mathbb{R}$. From the almost automorphy of f, there exists a subsequence $(t_n)_n \subset (t'_n)_n$ such that for each $t \in \mathbb{R}$

$$|f(t + t_p) - f(t + t_q)| \longrightarrow 0$$

as $p, q \longrightarrow \infty$. Let

$$\begin{cases} u_n(t) := u_f(t + t_n) & \text{for } t \in \mathbb{R}, \\ f_n(t) := f(t + t_n) & \text{for } t \in \mathbb{R}. \end{cases}$$

We claim that $(u_n(t))_n$ is a Cauchy sequence in $\mathcal{H}$ for each $t \in \mathbb{R}$. u_p and u_q are weak solutions of the following differential inclusions:

$$\begin{cases} u_p'(t) + \mathcal{A}u_p(t) \ni f_p(t) & \text{for } t \in \mathbb{R}, \\ u_q'(t) + \mathcal{A}u_q(t) \ni f_q(t) & \text{for } t \in \mathbb{R}. \end{cases} \tag{4.5}$$

Applying Theorem 2.8 to (4.5), we find that for $t \geq \sigma$

$$|u_p(t) - u_q(t)| \leq e^{-\alpha(t-\sigma)}|u_p(\sigma) - u_q(\sigma)| + \int_\sigma^t e^{-\alpha(t-s)}|f_p(s) - f_q(s)|\mathrm{d}s.$$

Using the boundedness of u_f and letting $\sigma \longrightarrow -\infty$, we deduce that for each $t \in \mathbb{R}$

$$\begin{aligned} |u_p(t) - u_q(t)| &\leq \int_{-\infty}^t e^{-\alpha(t-s)}|f_p(s) - f_q(s)|\mathrm{d}s \\ &= \int_{-\infty}^t e^{-\alpha(t-s)}|f(s+t_p) - f(s+t_q)|\mathrm{d}s. \end{aligned}$$

Using Lebesgue's dominated convergence theorem, we conclude that $(u_n(t))_n$ is a Cauchy sequence in $\mathcal{H}$ for each $t \in \mathbb{R}$. Therefore, u_f has a relatively compact range.

Step 4. We claim that the solution u_f is uniformly continuous on $\mathbb{R}$. By Theorem 2.8 with $\widetilde{u} = u_f(\cdot + h)$, $\widehat{u} = u_f(\cdot)$, $\widetilde{f} = f(\cdot + h)$, and $\widehat{f} = f(\cdot)$, we find that for $t \geq \sigma$

$$|u_f(t+h) - u_f(t)| \leq e^{-\alpha(t-\sigma)}|u_f(\sigma+h) - u_f(\sigma)| + \int_\sigma^t e^{-\alpha(t-s)}|f(s+h) - f(s)|\mathrm{d}s.$$

Since u_f is bounded, we find by letting $\sigma \longrightarrow -\infty$ that for each $t \in \mathbb{R}$

$$\begin{aligned} |u_f(t+h) - u_f(t)| &\leq \int_{-\infty}^t e^{-\alpha(t-s)}|f(s+h) - f(s)|\mathrm{d}s \\ &\leq \frac{1}{\alpha} \sup_{t\in\mathbb{R}} |f(t+h) - f(t)|, \end{aligned}$$

which implies that

$$\sup_{t\in\mathbb{R}} |u_f(t+h) - u_f(t)| \leq \frac{1}{\alpha} \sup_{t\in\mathbb{R}} |f(t+h) - f(t)|.$$

By Theorem 2.14, we find that u_f is uniformly continuous on $\mathbb{R}$.

Step 5. We claim that u_f is compact almost automorphic. Let $(t_n)_n \subseteq \mathbb{R}$. Since f is compact almost automorphic, there exist a subsequence $(t_n')_n \subset (t_n)_n$ and a continuous function $g : \mathbb{R} \longrightarrow \mathcal{H}$ such that

$$\begin{aligned} |f(t+t_n') - g(t)| &\longrightarrow 0 \quad \text{as } n \longrightarrow +\infty, \\ |g(t-t_n') - f(t)| &\longrightarrow 0 \quad \text{as } n \longrightarrow +\infty \end{aligned}$$

uniformly on any compact subset of $\mathbb{R}$. By Lemma 3.3, one can extract another subsequence $(t_n'')_n \subset (t_n')_n \subset (t_n)_n$ such that

$$u_f(t+t_n'') \longrightarrow y(t) \quad \text{as } n \longrightarrow +\infty \tag{4.6}$$

uniformly on any compact subset of $\mathbb{R}$, where y is a weak solution on $\mathbb{R}$ of the following differential inclusion:

$$y'(t) + \mathcal{A}y(t) \ni g(t).$$

Since y is also uniformly continuous and its range is relatively compact, by applying the same procedure to the function y using the returning sequence $(-t''_n)_n$, we have another subsequence $(t'''_n)_n \subset (t''_n)_n \subset (t'_n)_n \subset (t_n)_n$ such that

$$y(t - t'''_n) \longrightarrow z(t) \quad \text{as } n \longrightarrow +\infty \tag{4.7}$$

uniformly on any compact subset of $\mathbb{R}$, where z is a bounded weak solution of (1.1) on $\mathbb{R}$. From the uniqueness of the bounded weak solution (step 2), we conclude that $z = u_f$. Thus, it follows from (4.6) and (4.7) that u_f is compact almost automorphic. $\square$

Theorem 4.2. *Assume that $\mathcal{A}$ is α-strongly maximal monotone ($\alpha > 0$) with $0 \in \mathcal{A}0$ and $g : \mathbb{R} \times \mathcal{H} \longrightarrow \mathcal{H}$ is compact almost automorphic in t and Lipschitzian with respect to the second argument. Then (1.2) has a unique compact almost automorphic weak solution provided that $Lip(g) < \alpha$, where $Lip(g)$ is the Lipschitz constant of g.*

Proof. Let $v : \mathbb{R} \longrightarrow \mathcal{H}$ be a compact almost automorphic function. Consider the following differential inclusion:

$$u'(t) + \mathcal{A}u(t) \ni g(t, v(t)) \quad \text{for } t \in \mathbb{R}. \tag{4.8}$$

By Theorem 2.20, the function $t \longmapsto g(t, v(t))$ is compact almost automorphic. It follows from Theorem 4.1 that the differential inclusion (4.8) has a unique compact almost automorphic weak solution u_v. Let T be defined by

$$\begin{aligned} T : AA_c(\mathbb{R}, \mathcal{H}) &\longrightarrow AA_c(\mathbb{R}, \mathcal{H}), \\ v &\longmapsto u_v. \end{aligned}$$

Then T is well defined. Let $v, w \in AA_c(\mathbb{R}, \mathcal{H})$. Applying Theorem 2.8 to $\widetilde{u} = u_v$, $\widetilde{f} = g(\cdot, v(\cdot))$, $\widehat{u} = u_w$, and $\widehat{f} = g(\cdot, w(\cdot))$, we obtain

$$|Tv(t) - Tw(t)| \leq e^{-\alpha(t-\sigma)}|u_v(\sigma) - u_w(\sigma)| + \int_\sigma^t e^{-\alpha(t-s)}|g(s, v(s)) - g(s, w(s))|\mathrm{d}s \quad \text{for } t \geq \sigma.$$

Letting $\sigma \longrightarrow -\infty$, we find that for each $t \in \mathbb{R}$

$$\begin{aligned} |Tv(t) - Tw(t)| &\leq \int_{-\infty}^t e^{-\alpha(t-s)}|g(s, v(s)) - g(s, w(s))|\mathrm{d}s \\ &\leq \frac{Lip(g)}{\alpha}|v - w|_\infty. \end{aligned}$$

This means that T is a strict contraction. We deduce that the operator T has a unique fixed point that is the unique compact almost automorphic weak solution of (1.2). $\square$

5. Hyperbolic and parabolic equations

5.1. A dissipative hyperbolic system

We give an existence theorem of compact almost automorphic weak solutions for the following dissipative nonlinear wave inclusion:

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}u(t,x) - \Delta u(t,x) + \beta\left(\dfrac{\partial}{\partial t}u(t,x)\right) \ni \theta(t,x) \quad \text{for } (t,x) \in \mathbb{R}\times\Omega, \\ u(t,x) = 0 \quad \text{for } (t,x) \in \mathbb{R}\times\partial\Omega, \end{cases} \tag{5.1}$$

where Δ is the Laplacian operator. We assume that
(A1) Ω is a bounded open set in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ such that $\dim(\Omega) \geq 2$.
(A2) β is a strongly maximal monotone graph in $\mathbb{R}\times\mathbb{R}$ with $0 \in \beta 0$ such that

$$|\beta^0 w| \leq C_1|w|^k + C_2, \quad \text{with} \quad 0 \leq k \leq \frac{N+2}{N-2}, \tag{5.2}$$

where $\beta^0 w = \mathrm{Proj}_{\beta(w)}(0)$.
(A3) $\theta : \mathbb{R}\times\overline{\Omega} \longrightarrow \mathbb{R}$ satisfies $\dfrac{\partial\theta}{\partial t} \in S^2(\mathbb{R}, L^2(\Omega))$, where

$$S^2(\mathbb{R}, L^2(\Omega)) = \left\{ h \in L^2_{loc}(\mathbb{R}, L^2(\Omega)) : \sup_{t\in\mathbb{R}} \int_t^{t+1} |h(s)|^2_{L^2(\Omega)} \mathrm{d}s < +\infty \right\},$$

and the function $t \longmapsto \theta(t,\cdot)$ is in $AA_c(\mathbb{R}, L^2(\Omega))$. That is, for any $(t'_n)_n \subseteq \mathbb{R}$, there exist a subsequence $(t_n)_n$ and a continuous function $\tilde{\theta} : \mathbb{R}\times\overline{\Omega} \longrightarrow \mathbb{R}$ such that

$$\int_\Omega |\theta(t+t_n,\omega) - \tilde{\theta}(t,\omega)|^2 \mathrm{d}\omega \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty, \tag{5.3}$$

$$\int_\Omega |\tilde{\theta}(t-t_n,\omega) - \theta(t,\omega)|^2 \mathrm{d}\omega \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty \tag{5.4}$$

uniformly on any compact subset of $\mathbb{R}$.

Let $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$ be the Hilbert space endowed with the norm

$$|(\phi_1,\phi_2)|_{\mathcal{H}} = \left(\int_\Omega (|\nabla\phi_1(s)|^2 + |\phi_1(s)|^2 + |\phi_2(s)|^2)\mathrm{d}s \right)^{\frac{1}{2}},$$

and let B be the canonical extension of β to $L^2(\Omega)$ taken from [21, p. 53]

$$(u,v) \in G(B) \quad \text{if and only if} \quad (u(x),v(x)) \in G(\beta) \ \text{ for almost all } x \in \Omega. \tag{5.5}$$

Let

$$\begin{cases} D(\mathcal{L}) = H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega) \\ \mathcal{L} = \begin{pmatrix} 0 & -I \\ -\Delta & 0 \end{pmatrix}, \end{cases}$$

$$\begin{cases} D(\mathcal{B}) = H_0^1(\Omega) \times D(B) \\ \mathcal{B} = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}. \end{cases}$$

Lemma 5.1. [21, p. 93] *$D(\mathcal{A}) = D(\mathcal{L}) \cap D(\mathcal{B})$, $\mathcal{A} = \mathcal{L} + \mathcal{B}$ is maximal monotone on $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$.*

Let $f : \mathbb{R} \longrightarrow \mathcal{H}$ be the function defined by

$$f(t)(\omega) = \begin{pmatrix} 0 \\ \theta(t,\omega) \end{pmatrix} \quad \text{for } t \in \mathbb{R} \text{ and } \omega \in \Omega. \tag{5.6}$$

Then, by assumption (A3), $f \in AA_c(\mathbb{R}, \mathcal{H})$. If we take $U = \begin{pmatrix} u \\ \frac{\partial u}{\partial t} \end{pmatrix}$, then (5.1) takes the following abstract form:

$$U'(t) + \mathcal{A}U(t) \ni f(t) \quad \text{for } t \in \mathbb{R}. \tag{5.7}$$

Lemma 5.2. [8, Theorem 2.1] *Let $U(t) = \begin{pmatrix} u \\ \frac{\partial u}{\partial t} \end{pmatrix}$ be a solution that starts at $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in D(\mathcal{A})$. Then*

$$\frac{\partial^2 u}{\partial t^2} \in L^\infty(\mathbb{R}^+, L^2(\Omega)) \quad \text{and} \quad \frac{\partial u}{\partial t} \in L^\infty(\mathbb{R}^+, H_0^1(\Omega)).$$

Lemma 5.3. [8] *Let $U(t) = \begin{pmatrix} u \\ \frac{\partial u}{\partial t} \end{pmatrix}$ be a solution that starts at $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in D(\mathcal{A})$. Then $U(t)$ has a relatively compact range in the energy space $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$.*

As a consequence, we have the following result.

Theorem 5.4. (5.1) *has at least a weak solution in $AA_c(\mathbb{R}, H_0^1(\Omega) \times L^2(\Omega))$.*

Proof. By Lemmas 5.2 and 5.3, any trajectory $(U(t))_{t\geq 0}$ that starts at $U_0 \in D(\mathcal{A})$ is uniformly continuous on $\mathbb{R}^+$ in $H_0^1(\Omega) \times L^2(\Omega)$ and its range over $\mathbb{R}^+$ is relatively compact. In view of Theorem 3.1, (5.7) has at least a compact almost automorphic weak solution. □

Remark 5.5. (5.1) was considered in the periodic case and the almost periodic case in [2–4,7,8,20–22].

5.2. A dissipative parabolic system

Consider the following system:

$$\begin{cases} \dfrac{\partial}{\partial t} w(t,x) - \Delta w(t,x) + \beta\left(w(t,x)\right) + \alpha w(t,x) \ni \gamma(w(t,x))) + h(t,x) \quad \text{for } (t,x) \in \mathbb{R} \times \Omega, \\ w(t,x) = 0 \quad \text{for } (t,x) \in \mathbb{R} \times \partial\Omega, \end{cases} \tag{5.8}$$

where $\alpha > 0$. Assume that
(B1) Ω is a smooth subset of $\mathbb{R}^N$ with a regular boundary $\partial\Omega$.
(B2) β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta 0$.
(B3) $\gamma : \mathbb{R} \longrightarrow \mathbb{R}$ is a Lipschitzian function such that $\gamma(0) = 0$. Let L_γ be its Lipschitz constant.
(B4) The function $t \longmapsto h(t, \cdot)$ belongs to $AA_c(\mathbb{R}, L^2(\Omega))$.

Let B be the canonical extension of β to $L^2(\Omega)$ defined by

$$(u,v) \in G(B) \quad \text{if and only if} \quad (u(x),v(x)) \in G(\beta) \;\; \text{for almost all}\, x \in \Omega. \tag{5.9}$$

Let A_1 be defined in $L^2(\Omega)$ by

$$\begin{cases} D(A_1) = \left\{u \in H^2(\Omega) \cap H_0^1(\Omega); \beta\,(u) \in L^2(\Omega)\right\}, \\ A_1 u = -\Delta u + Bu. \end{cases}$$

It is known from [21, p. 88] that A_1 is maximal monotone. Hence, by Remark 2.2 the operator

$$\begin{cases} D(\mathcal{A}) = D(A_1), \\ \mathcal{A}u = A_1 u + \alpha u \end{cases}$$

is α-strongly maximal monotone. Using the fact that $0 \in \beta 0$, we get $0 \in \mathcal{A}0$. Take $\mathcal{H} = L^2(\Omega)$.
We consider the function $f : \mathcal{H} \longrightarrow \mathcal{H}$ defined by

$$f(x)(\omega) = \gamma(x(\omega)) \quad \text{for } x \in \mathcal{H} \text{ and } \omega \in \Omega.$$

By assumption (B3), one sees that f is well defined. Using assumption (B3), we obtain that f is Lipschitzian with a Lipschitz constant $L_f = L_\gamma$. Furthermore, $f \in C(\mathcal{H},\mathcal{H})$.
Let $H : \mathbb{R} \longrightarrow \mathcal{H}$ be defined by

$$H(t)(\omega) = h(t,\omega) \quad \text{for } t \in \mathbb{R} \text{ and } \omega \in \Omega.$$

Assumption (B4) implies that $H \in AA_c(\mathbb{R},\mathcal{H})$.
Let $g : \mathbb{R} \times \mathcal{H} \longrightarrow \mathcal{H}$ be defined by

$$g(t,x) = f(x) + H(t) \quad \text{for } t \in \mathbb{R} \text{ and } x \in \mathcal{H}.$$

We deduce that $g \in AA_c(\mathbb{R} \times \mathcal{H},\mathcal{H})$ and g is Lipschitzian with respect to the second argument with a Lipschitz constant $L_g = L_\gamma$.
If we take $u(\cdot)(x) = w(\cdot,x)$, then (5.8) takes the following abstract form:

$$u'(t) + \mathcal{A}u(t) \ni g(t,u(t)) \quad \text{for } t \in \mathbb{R} \tag{5.10}$$

in the Hilbert space $\mathcal{H}$. If we suppose that $L_\gamma < \alpha$, then all the assumptions in Theorem 4.2 are fulfilled. Consequently, we get the following result.

Theorem 5.6. *The system (5.8) has a unique compact almost automorphic weak solution provided that $L_\gamma < \alpha$.*

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