



# The limit theorems for a previous $k$ -sum dependent model

Deepak Singh<sup>a</sup>, Somesh Kumar<sup>a,\*</sup>, P. Vellaisamy<sup>b</sup>

<sup>a</sup> Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur, 721302, India

<sup>b</sup> Department of Mathematics, Indian Institute of Technology Bombay, Powai Mumbai, 400076, India



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## ABSTRACT

The main goal of this paper is to establish limit theorems for the sums of dependent Bernoulli random variables. Here each successive random variable depends on previous few random variables. A previous  $k$ -sum dependent model is considered. This model is a combination of the previous all sum and the previous  $k$ -sum dependent models. The strong law of large numbers, the central limit theorem and the law of iterated logarithm for the sums of random variables following this model are established. A new approach using martingale differences is developed to prove these results.

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## 1. Introduction

The classical binomial distribution has been extended extensively by many authors. When Bernoulli variables are identical and satisfy the Markov chain model, then the distribution of the sum of  $n$  Bernoulli variables is known as the Markov binomial distribution (MBD) (see, Edwards [5]). Wang [16] showed that the limiting distribution of MBD is a compound Poisson distribution. The closed form of the exact distribution and the moments were derived by Bhat et al. [2] using combinatorial arguments. Hille et al. [9] have considered some Markov chain operators and established an exponential rate of convergence and the central limit theorem on these.

If any one or both of the assumptions of independence and identical Bernoulli variables do not hold, then the distribution of the sum of  $n$  Bernoulli random variables is called a Generalized Binomial Distribution (GBD) (see Drezner and Furnum [4] and references therein). The results of Drezner and Furnum [4] were established in a simpler way by Vellaisamy [13] using an alternative approach. He also derived an upper bound for the Poisson approximation to the GBD. Vellaisamy and Punnen [14] have explored more about the nature of the previous all sum dependent model and the binomial model.

\* Corresponding author.

E-mail addresses: [deepak.singh@maths.iitkgp.ac.in](mailto:deepak.singh@maths.iitkgp.ac.in) (D. Singh), [smsh@iitkgp.ac.in](mailto:smsh@iitkgp.ac.in) (S. Kumar), [pv@math.iitb.ac.in](mailto:pv@math.iitb.ac.in) (P. Vellaisamy).

Vellaisamy and Sankar [15] introduced a dependent model based on previous few observations to design sampling plans for dependent production processes. This model includes as special cases the classical *i.i.d.* model, an independent and non-identical model, the Markov dependent model (Bhat et al. [2], Antzoulakos and Philippou [1]) and a previous all sum dependent model (Drezner and Furnum [4], Vellaisamy [13]). We describe below a more general model. This is a combination of the previous all sum and the previous  $k$ -sum models.

Let  $\{X_n, n \geq 1\}$  be a sequence of Bernoulli random variables and let  $S_n = \sum_{j=1}^n X_j$  for  $n \geq 1$ . The important departure in this formulation of the model from that of a previous  $k$ -sum model is that here for a fixed value of  $k$  ( $\geq 1$ ), the conditional probability of success in the  $i$ th trial depends on the number of successes in all the previous trials if  $i \leq k+1$  and on those of immediate previous  $k$  trials if  $i \geq k+2$ , that is,

$$\begin{aligned} P(X_1 = 1) &= p \quad \text{and} \\ P(X_i = 1 | \mathcal{F}_{i-1}) &= (1 - \theta)p + \frac{\theta}{i-1} S_{i-1} \quad \text{for } 2 \leq i \leq k+1 \\ P(X_i = 1 | \mathcal{F}_{i-1}) &= (1 - \theta)p + \frac{\theta}{k} S_{i,k} \quad \text{for } k+2 \leq i \leq n \end{aligned} \quad (1)$$

where  $0 \leq \theta < 1$ ,  $0 < p < 1$ ,  $\mathcal{F}_i$  denotes the  $\sigma$ -field generated by random variables  $X_1, \dots, X_i$ ,  $S_{k+1,k} = S_k$  and  $S_{i,k} = S_{i-1} - S_{i-k-1}$ .

The conditional model as defined in (1) arises frequently in real life situations. Consider a production process, where the conformity of an item, say  $X_i$ , may depend on all the previous conforming items if few items are produced. However, after a while, it will depend on the number of conforming items in previous  $k$  trials. This model is new and has not been discussed in the literature.

Limit theorems for sums of correlated Bernoulli random variables have been studied by some authors over the last twenty years. Heyde [7] established the weak law of large numbers, the central limit theorem, the short range and long range dependence phenomena for the previous all sum dependent model proposed by [4] and [13]. These results were extended to a more general dependent model by James et al. [10]. Mourrat [11] has considered the martingale sequence and explored the rate of convergence in the central limit theorem.

There has been an active interest on the models with short-range and long-range dependence during the last two decades (see [3], [8]). The testing of short-range/long-range dependence under some specified models has practical implications in econometrics, atmospheric and agricultural sciences.

In the classical martingale limit theory, we need to define a sequence  $\{a_n\}$  such that  $\{(S_n - \mathbb{E}(S_n))/a_n\}$  is a martingale sequence. But, this construction seems to fail for more general dependent processes. We refer to Hall and Heyde [6] for more details on this.

It turns out that it is quite difficult to construct a sequence  $\{a_n\}$  in the model defined in (1) so that the sequence  $\{(S_n - \mathbb{E}(S_n))/a_n\}$  is a martingale. Hence we propose a somewhat different approach for establishing the strong limit theorems for the sums of correlated random variables satisfying the Model (1).

This paper is organized as follows. Some preliminary results are discussed in Section 2. The moment structure, the strong law of large numbers (SLLN), the central limit theorem (CLT) and the law of iterated logarithm (LIL) for the sums of  $n$  random variables are established in Section 3. Some concluding remarks are given in Section 4. Proofs are given in Section 5.

## 2. Some preliminary results and notations

Let  $(\Omega, \Sigma, P)$  be a probability space where  $\Omega$  is a set,  $\Sigma$  a  $\sigma$ -field of subsets of  $\Omega$  and  $P$  a probability measure defined on  $\Sigma$ . Let  $\mathbb{N}$  be the set of natural numbers and  $\{\mathcal{G}_n, n \in \mathbb{N}\}$  be the  $\sigma$ -field generated by the random variables  $\{X_1, \dots, X_n\}$  such that  $\mathcal{G}_n \subseteq \mathcal{G}_{n+1}$  for each  $n \in \mathbb{N}$ . Let  $\{X_n, \mathcal{G}_n, n \in \mathbb{N}\}$  be a martingale sequence and let

$$Y_n = X_n - \mathbb{E}(X_n | \mathcal{G}_{n-1})$$

with  $\mathbb{E}|Y_n| < \infty$ . Then  $\{Y_n, \mathcal{G}_n, n \geq 1\}$  is called a martingale difference sequence. The following lemma is a special case of Theorem 2.17 of Hall and Heyde [6] for  $p = 2$ .

**Lemma 2.1.** *Let  $\{Y_n, \mathcal{G}_n, n \geq 1\}$  be a sequence of bounded martingale differences. If  $\sum_{i=1}^{\infty} \mathbb{E}(Y_i^2 | \mathcal{G}_{i-1}) < \infty$  almost surely, then  $\sum_{i=1}^n Y_i$  converges almost surely.*

Next result is a special case of Corollary 3.1 of Hall and Heyde [6] for  $X_{ni} = Y_i/\sqrt{n}$ .

**Lemma 2.2.** *Let  $\{Y_n, \mathcal{G}_n, n \geq 1\}$  be a sequence of martingale differences. Assume that  $\frac{1}{n} \sum_{i=1}^n \mathbb{E}(Y_i^2 | \mathcal{G}_{i-1}) \xrightarrow{p} \sigma^2$ , then  $\frac{\sum_{i=1}^n Y_i}{\sqrt{n}} \xrightarrow{d} N(0, \sigma^2)$ .*

The following Lemma will be used in proving Theorem 3.2.

**Lemma 2.3.** *Let  $f, g$  and  $h$  be three real valued functions defined on a common domain  $D$  such that*

$$g(x) \leq f(x) \leq h(x) \quad \text{for all } x \in D.$$

*If  $\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L$  and if  $\lim_{x \rightarrow a} f(x) = 0$ , then  $L = 0$ .*

### 3. Main results

In this section, we study in detail the moment structure and the limiting behavior of the sums of  $n$  dependent Bernoulli random variables as defined in (1). In the following lemmas, the mean and the variance of  $S_n$  are derived.

**Lemma 3.1.** *Under Model (1),  $\mathbb{E}(S_n) = np$ ,  $n \geq 1$ . Hence,  $\{X_n, n \geq 1\}$  are identically distributed Bernoulli random variables with  $P(X_n = 1) = p$ .*

**Corollary 3.1.** *Under Model (1),  $\mathbb{E}(S_{i,k}) = k p$ .*

**Proof.** The proof follows using the relation  $S_{i,k} = S_{i-1} - S_{i-k-1}$  in Lemma 3.1.  $\square$

Next, we derive the variance of  $S_n$ . It is observed that the covariance structure between Bernoulli random variables is complicated and it is difficult to obtain a closed form of the variance of  $S_n$ .

**Lemma 3.2.** *Let  $q = (1 - p)$ . Under Model (1), we have*

(i) *For  $2 \leq i \leq k + 1$*

$$\mathbb{V}(S_i) = \begin{cases} pq \left\{ \frac{i - (B(i, 2\theta))^{-1}}{(1 - 2\theta)} \right\}, & \theta \neq \frac{1}{2} \\ ipq \sum_{j=1}^i \frac{1}{j}, & \theta = \frac{1}{2} \end{cases}.$$

(ii) For  $k + 2 \leq i \leq n$

$$\begin{aligned} \mathbb{V}(S_i) &= pq + \left(1 + \frac{2\theta}{k}\right) \mathbb{V}(S_{i-1}) - \frac{2\theta}{k} \mathbb{V}(S_{i-k-1}) \\ &\quad - \frac{2\theta}{k} \sum_{m=i-k}^{i-1} \sum_{n=1}^{i-k-1} \text{Cov}(X_m, X_n), \end{aligned}$$

where  $\text{Cov}(X_m, X_n)$  denotes the covariance between  $X_m$  and  $X_n$  and  $B(a, b)$  denotes the standard beta function given by

$$B(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx, \quad a > 0, b > 0.$$

Next, we consider the sequence  $\{S_{i,k}, i \geq k+1\}$  and prove that the variance of  $S_{i,k}$  is the same for all  $i \geq k+1$ . Further we observe that this expression is useful in proving the central limit theorem.

**Corollary 3.2.** Under Model (1), the variance of  $S_{i,k}$  is given by

$$\mathbb{V}(S_{i,k}) = p q \left\{ 1 + \sum_{m=1}^{k-1} \prod_{j=0}^{m-1} \left( 1 + \frac{2\theta}{k-j-1} \right) \right\}.$$

Bernstein has given a result on the weak law of large numbers (see Spanos [12], page 474) which uses a condition on the covariance between random variable  $X_i$  and  $X_j$ . Let  $\{X_n, n \geq 1\}$  be a sequence of random variables with  $\mathbb{E}|X_i|^2 < \infty$  and  $\text{Cov}(X_i, X_j) \rightarrow 0$  as  $|i-j| \rightarrow \infty$  (that is,  $X_n$ 's are asymptotically uncorrelated in the sense that the correlation between  $X_i$  and  $X_j$  tends to zero as the distance between indices  $i$  and  $j$  becomes large). Then the weak law of large numbers (WLLN) is shown to hold on the sequence  $\{X_n\}$ . In the following theorem, we prove that the above condition holds for the model defined in (1). This result is later used in proofs of the strong law of large numbers and the central limit theorem.

**Theorem 3.1.** Under Model (1), the covariance is given as

$$\text{Cov}(X_n, X_j) \rightarrow 0 \quad \text{as } |n-j| \rightarrow \infty. \quad (2)$$

### 3.1. Limit theorems for the dependent model

In this section, we establish asymptotic results such as the strong law of large numbers, the central limit theorem and the law of iterated logarithm for the sums of Bernoulli random variables following the dependent model as defined in (1). For the model of Heyde [7], the construction of the martingale difference sequences is relatively straightforward. However, for our model that approach does not seem to work. We construct the martingale difference sequence by a direct decomposition of the sequence  $\{X_n\}$  in two parts.

Let  $\{L_n, \mathcal{F}_n, n \geq 1\}$  be a martingale difference sequence such that  $L_1 = X_1 - p$  and

$$L_i = \begin{cases} \left( X_i - (1-\theta)p - \frac{\theta}{i-1} S_{i-1} \right), & 2 \leq i \leq k+1 \\ \left( X_i - (1-\theta)p - \frac{\theta}{k} S_{i,k} \right), & k+2 \leq i \leq n \end{cases}, \quad (3)$$

where  $S_{i,k} = S_{i-1} - S_{i-k-1}$ . The strong law of large numbers is proved in the following theorem.

**Theorem 3.2.** *Under Model (1), we have the SLLN as*

$$\frac{S_n}{n} \rightarrow p \quad \text{almost surely.}$$

We also establish the strong law of large numbers for the sequence  $\{S_{i,k}\}$ . This will be helpful for proving the central limit theorem for the sequence  $\{S_n\}$ . The result is stated below.

**Theorem 3.3.** *Under Model (1), we have*

$$\frac{1}{n} \sum_{i=k+1}^n S_{i,k} \rightarrow k p \quad \text{almost surely.}$$

The next theorem gives the weak law of large numbers for another sequence of correlated random variables. This result is also used in the proving the central limit theorem for the sequence  $\{S_n\}$ .

**Theorem 3.4.** *Let  $Y_i = X_i S_{i,k}$ ,  $i \geq k+1$ . Under Model (1), we get*

$$\frac{1}{n} \sum_{i=k+1}^n Y_i \xrightarrow{P} \mu,$$

where  $\mu = k p^2 + \frac{\theta}{k} \mathbb{V}(S_{k+1,k})$ .

In the following theorems, the central limit theorem and the law of iterated logarithm are established.

**Theorem 3.5.** *Under Model (1), the CLT is given as*

$$\frac{S_n - np}{\sqrt{n}} \xrightarrow{d} N\left(0, \frac{\sigma^2}{(1-\theta)^2}\right), \quad (4)$$

where  $\sigma^2 = p(1-p) - \frac{\theta^2}{k^2} \mathbb{V}(S_{k+1,k})$ .

**Theorem 3.6.** *Under Model (1), the LIL is given as*

$$\limsup_{n \rightarrow \infty} \frac{S_n - np}{\sqrt{2n \log \log n}} = \frac{\sigma}{1-\theta} \quad \text{a.s.} \quad (5)$$

and

$$\liminf_{n \rightarrow \infty} \frac{S_n - np}{\sqrt{2n \log \log n}} = -\frac{\sigma}{1-\theta} \quad \text{a.s.} \quad (6)$$

In the following corollaries, we consider a special case of the Model (1).

**Corollary 3.3.** *For  $k = 1$ , the Model (1) reduces to the Markov dependent model and the central limit theorem is*

$$\frac{S_n - np}{\sqrt{n}} \xrightarrow{d} N\left(0, \frac{p(1-p)(1+\theta)}{(1-\theta)}\right).$$

**Corollary 3.4.** For  $k = 1$ , the law of iterated logarithm is

$$\limsup_{n \rightarrow \infty} \frac{S_n - np}{\sqrt{2n \log \log n}} = \sqrt{\frac{p(1-p)(1+\theta)}{(1-\theta)}} \quad a.s.$$

and

$$\liminf_{n \rightarrow \infty} \frac{S_n - np}{\sqrt{2n \log \log n}} = -\sqrt{\frac{p(1-p)(1+\theta)}{(1-\theta)}} \quad a.s.$$

The sample Allen variance (SAV) is based on an empirical observation of a non-stationary process (see, [7] and [8]). The SAV can be used to determine whether a process is short range or long range dependent. If  $W_n$  is a zero mean process then the sample Allen variance is  $SAV = (\sum_{i=1}^n W_i)^2 / \sum_{i=1}^n W_i^2$ . The process is said to be short range or long range dependent according to SAV converges or diverges as  $n \rightarrow \infty$ . With this concept, we have the following remark.

**Remark 3.1.** Under Model (1),  $SAV = \frac{\sigma^2 \chi_1^2}{p(1-p)(1-\theta)^2} < \infty$ . Hence, the process is short range dependent.

**Remark 3.2.** For the previous all sum dependent model considered by [7], the central limit theorem for  $S_n$  holds when dependent parameter  $\theta \in (0, 0.5]$ . Thus, the range of a dependent parameter  $\theta$  in Theorem 3.5, will be decreased when the value of  $k$  increases.

#### 4. Conclusion

A new dependent Bernoulli model is analyzed. The model generalizes some previously known models in the literature. Specifically, the new model is an extension of a previous all sum model. The strong law of large numbers, the central limit theorem and the law of iterated logarithm are established for this model.

#### 5. Proofs

**Proof of Lemma 3.1.** The initial condition  $P(X_1 = 1) = p$  gives  $\mathbb{E}(S_1) = p$ . Consider next the case  $2 \leq i \leq k+1$  so that

$$P(X_i = 1 | \mathcal{F}_{i-1}) = (1-\theta)p + \frac{\theta}{i-1} S_{i-1}. \quad (7)$$

Assume now  $\mathbb{E}(S_{i-1}) = (i-1)p$ . Then

$$\begin{aligned} \mathbb{E}(S_i) &= \mathbb{E}(S_{i-1} + X_i) \\ &= (i-1)p + \mathbb{E}\{\mathbb{E}(X_i | \mathcal{F}_{i-1})\} \\ &= (i-1)p + \mathbb{E}\left((1-\theta)p + \frac{\theta}{i-1} S_{i-1}\right) \\ &= ip \quad \text{for } 1 \leq i \leq k+1. \end{aligned}$$

For  $k+2 \leq i \leq n$ , we have  $P(X_i = 1 | \mathcal{F}_{i-1}) = (1-\theta)p + \frac{\theta}{k} S_{i,k}$ . Now

$$\mathbb{E}(S_{k+2}) = \mathbb{E}(S_{k+1}) + \mathbb{E}\{\mathbb{E}(X_{k+2} | \mathcal{F}_{k+1})\}$$

$$\begin{aligned}
&= (k+1)p + (1-\theta)p + \frac{\theta}{k} \mathbb{E}(S_{k+1} - S_1) \\
&= (k+2)p.
\end{aligned}$$

Assume next  $\mathbb{E}(S_{k+r}) = (k+r)p$ , for all  $r \geq 2$ . Then

$$\begin{aligned}
\mathbb{E}(S_{k+r+1}) &= \mathbb{E}(S_{k+r} + X_{k+r+1}) \\
&= \mathbb{E}(S_{k+r}) + \mathbb{E}\{\mathbb{E}(X_{k+r+1} | \mathcal{F}_{k+r})\} \\
&= \mathbb{E}(S_{k+r}) + (1-\theta)p + \frac{\theta}{k} \mathbb{E}(S_{k+r} - S_r) \\
&= (k+r+1)p.
\end{aligned}$$

Using the principle of mathematical induction, this proves that  $\mathbb{E}(S_n) = np$ ,  $\forall n \geq 1$ . Thus  $\{X_n, n \geq 1\}$  is a sequence of identically distributed Bernoulli random variables.  $\square$

**Proof of Lemma 3.2.** From Vellaisamy [13], we have for  $i \leq k+1$

$$\mathbb{V}(S_i) = \begin{cases} pq \left\{ \frac{i - (B(i, 2\theta))^{-1}}{(1-2\theta)} \right\}, & \theta \neq \frac{1}{2} \\ ipq \sum_{j=1}^i \frac{1}{j}, & \theta = \frac{1}{2} \end{cases}. \quad (8)$$

Next, we consider the case for  $i \geq k+2$ . The covariance is defined as

$$Cov(X_i, S_{i-1}) = \mathbb{E}(X_i S_{i-1}) - \mathbb{E}(X_i) \mathbb{E}(S_{i-1}). \quad (9)$$

First we take

$$\mathbb{E}(X_i S_{i-1}) = \mathbb{E}\{S_{i-1} \mathbb{E}(X_i | \mathcal{F}_{i-1})\}.$$

Using the model defined in (1), we can write

$$\mathbb{E}(X_i S_{i-1}) = (1-\theta)p \mathbb{E}(S_{i-1}) + \frac{\theta}{k} \mathbb{E}(S_{i,k} S_{i-1}).$$

Consequently Equation (9) yields

$$\begin{aligned}
Cov(X_i, S_{i-1}) &= \frac{\theta}{k} Cov(S_{i,k}, S_{i-1}) \\
&= \frac{\theta}{k} Cov\{(S_{i-1} - S_{i-k-1}), S_{i-1}\}. \quad (10)
\end{aligned}$$

Some further simplification leads to

$$\begin{aligned}
Cov(X_i, S_{i-1}) &= \frac{\theta}{k} \mathbb{V}(S_{i-1}) - \frac{\theta}{k} Cov(S_{i-1}, S_{i-k-1}) \\
&= \frac{\theta}{k} \mathbb{V}(S_{i-1}) - \frac{\theta}{k} \sum_{j=i-k}^{i-1} Cov(X_j, S_{i-k-1}) - \frac{\theta}{k} \mathbb{V}(S_{i-k-1}).
\end{aligned}$$

Thus the variance is

$$\begin{aligned}\mathbb{V}(S_i) &= p(1-p) + \mathbb{V}(S_{i-1}) + 2\text{Cov}(X_i, S_{i-1}) \\ &= p q + \left(1 + \frac{2\theta}{k}\right) \mathbb{V}(S_{i-1}) - \frac{2\theta}{k} \mathbb{V}(S_{i-k-1}) - \frac{2\theta}{k} \sum_{j=i-k}^{i-1} \text{Cov}(X_j, S_{i-k-1}).\end{aligned}$$

This completes the proof.  $\square$

**Proof of Corollary 3.2.** The covariance between  $X_i$  and  $S_{i,k}$  for  $i \geq k+1$  is

$$\begin{aligned}\text{Cov}(X_i, S_{i,k}) &= \mathbb{E}(X_i S_{i,k}) - \mathbb{E}(X_i) \mathbb{E}(S_{i,k}) \\ &= \mathbb{E}(S_{i,k} \mathbb{E}(X_i | \mathcal{F}_{i-1})) - \mathbb{E}(X_i) \mathbb{E}(S_{i,k}).\end{aligned}$$

Using the Model (1), we get

$$\text{Cov}(X_i, S_{i,k}) = \frac{\theta}{k} \mathbb{V}(S_{i,k}). \quad (11)$$

Next, we consider the variance of  $S_{i+1,k+1}$

$$\begin{aligned}\mathbb{V}(S_{i+1,k+1}) &= \mathbb{V}(X_i + S_{i,k}) \\ &= \mathbb{V}(X_i) + \mathbb{V}(S_{i,k}) + 2 \text{Cov}(X_i, S_{i,k}).\end{aligned}$$

Substituting the covariance of  $X_i$  and  $S_{i,k}$  from Equation (11) in the above relation, we get

$$\mathbb{V}(S_{i+1,k+1}) = p q + \left(1 + \frac{2\theta}{k}\right) \mathbb{V}(S_{i,k}),$$

where  $q = 1 - p$ . This leads to

$$\begin{aligned}\mathbb{V}(S_{i+1,k+1}) &= p q + p q \left(1 + \frac{2\theta}{k}\right) + p q \left(1 + \frac{2\theta}{k}\right) \left(1 + \frac{2\theta}{k-1}\right) + \dots \\ &\quad + \left\{ \left(1 + \frac{2\theta}{k}\right) \left(1 + \frac{2\theta}{k-1}\right) \dots \left(1 + \frac{2\theta}{1}\right) \right\} \mathbb{V}(S_{i-k+1,1}).\end{aligned}$$

Hence

$$\mathbb{V}(S_{i+1,k+1}) = p q \left\{ 1 + \sum_{m=1}^k \prod_{j=0}^{m-1} \left(1 + \frac{2\theta}{k-j}\right) \right\}.$$

Finally, replacing  $i$  by  $i-1$  and  $k$  by  $k-1$  in the above relation yields

$$\mathbb{V}(S_{i,k}) = p q \left\{ 1 + \sum_{m=1}^{k-1} \prod_{j=0}^{m-1} \left(1 + \frac{2\theta}{k-j-1}\right) \right\}.$$

This completes the proof.  $\square$



**Proof of Theorem 3.1.** Consider first the case for  $j = 1$  and  $n \geq 1$ . Since  $X_i$ 's are Bernoulli random variables, initially, we have  $\mathbb{V}(X_1) = p(1-p) \leq \frac{1}{4}$ ,  $0 < p < 1$ . Next, the covariance between  $X_2$  and  $X_1$  is

$$\begin{aligned} \text{Cov}(X_2, X_1) &= \mathbb{E}(X_2 X_1) - \mathbb{E}(X_2)\mathbb{E}(X_1) \\ &= \mathbb{E}(X_1 \mathbb{E}(X_2 | \mathcal{F}_1)) - \mathbb{E}(X_2)\mathbb{E}(X_1). \end{aligned}$$

Using the conditional model defined in (1), we get

$$|\text{Cov}(X_2, X_1)| = \theta |\text{Cov}(X_1, X_1)| \leq C \theta,$$

where  $0 < C \leq \frac{1}{4}$ . Consequently, for  $2 \leq i \leq k+1$

$$\text{Cov}(X_i, X_1) = \frac{\theta}{i-1} \sum_{j=1}^{i-1} \text{Cov}(X_j, X_1).$$

Using Cauchy-Schwarz inequality, it is seen that  $|\text{Cov}(X_j, X_1)| \leq p(1-p)$  and so applying triangle inequality, we get

$$|\text{Cov}(X_i, X_1)| \leq C \theta, \quad (12)$$

where  $0 < C < \frac{1}{4}$  is a constant. Next, we consider the case  $i \geq k+2$ .

$$\begin{aligned} \text{Cov}(X_{k+2}, X_1) &= \mathbb{E}(X_{k+2} X_1) - \mathbb{E}(X_{k+2})\mathbb{E}(X_1) \\ &= \mathbb{E}(X_1 \mathbb{E}(X_{k+2} | \mathcal{F}_{k+1})) - \mathbb{E}(X_{k+2})\mathbb{E}(X_1). \end{aligned}$$

Using the expression from Model (1), we get

$$\text{Cov}(X_{k+2}, X_1) = \frac{\theta}{k} \sum_{j=2}^{k+1} \text{Cov}(X_j, X_1)$$

Replacing with upper bounds, yields

$$|\text{Cov}(X_{k+2}, X_1)| \leq C \theta^2. \quad (13)$$

Further we observe that the Equation (13) will hold up to  $(2k+2)$  number of trials. The next power of  $\theta$  will be increased for the trials  $(2k+3)$ ,  $(3k+4)$  and so on. Thus

$$|\text{Cov}(X_n, X_1)| \leq C \theta \left\{ \frac{(n-1)}{(k+1)} + 1 \right\}. \quad (14)$$

Since  $0 \leq \theta < 1$ ,  $|\text{Cov}(X_n, X_1)| \rightarrow 0$  as  $n \rightarrow \infty$ . The above expression is generalized for all  $j \geq 1$ ,  $k \geq 1$  as

$$\begin{aligned} \text{Cov}(X_{k+j}, X_j) &= \mathbb{E}(X_{k+j} X_j) - \mathbb{E}(X_{k+j})\mathbb{E}(X_j) \\ &= \mathbb{E}(X_j \mathbb{E}(X_{k+j} | \mathcal{F}_{k+j-1})) - \mathbb{E}(X_{k+j})\mathbb{E}(X_j). \end{aligned}$$

Applying Model (1), we get

$$\text{Cov}(X_{k+j}, X_j) = \frac{\theta}{k} \sum_{l=j}^{k+j-1} \text{Cov}(X_l, X_j).$$

Again we replace the covariance between  $X_l$  and  $X_j$  with an upper bound

$$|\text{Cov}(X_{k+j}, X_j)| \leq B \theta, \quad (15)$$

where  $0 < B < \frac{1}{4}$  is a constant. Consider next

$$\text{Cov}(X_{k+j+1}, X_j) = \frac{\theta}{k} \sum_{l=j+1}^{k+j} \text{Cov}(X_l, X_j).$$

With the help of triangle inequality and Equation (15), we have

$$|\text{Cov}(X_{k+j+1}, X_j)| \leq B \theta^2. \quad (16)$$

Note that the next increment on the power of  $\theta$  will occur for the trials  $(2k+j+2)$ ,  $(3k+j+3)$  and so on. Therefore, for all  $j \geq 1$ , we get

$$|\text{Cov}(X_n, X_j)| \leq B \theta \left\{ \frac{(n-j)}{(k+1)} + 1 \right\}, \quad (17)$$

where  $B$  is a constant and  $n \rightarrow \infty$ . Similarly, we fix the value of  $n$  and increase  $j$  to infinity, then we get

$$|\text{Cov}(X_n, X_j)| \leq B \theta \left\{ \frac{(j-n)}{(k+1)} + 1 \right\}. \quad (18)$$

Combining the Equation (17) and (18), we have

$$|\text{Cov}(X_n, X_j)| \leq B \theta \left\{ \frac{|n-j|}{(k+1)} + 1 \right\}.$$

Hence  $\text{Cov}(X_n, X_j) \rightarrow 0$  as  $|n-j| \rightarrow \infty$ . This completes the proof of the theorem.  $\square$

**Proof of Theorem 3.2.** Let  $\mathcal{F}_0$  be a trivial  $\sigma$ -field. Since  $|L_n| \leq 3$ ,  $\{L_n, \mathcal{F}_{n-1}, n \geq 1\}$  is a sequence of bounded martingale differences. Also define the sum of martingale differences as  $D_n = \sum_{i=1}^n L_i$ . Next, we decompose the sum as

$$D_n = \sum_{i=1}^{k+1} L_i + \sum_{i=k+2}^n L_i.$$

Using the expression for the martingale difference sequence  $\{L_i\}$  defined in (3) and some simplification yield

$$D_n = \left( S_n - np + \theta(n-1)p - \theta \sum_{i=2}^{k+1} \frac{S_{i-1}}{i-1} - \frac{\theta}{k} \sum_{i=k+2}^n S_{i,k} \right).$$

We obtain some upper and lower bounds of  $D_n$  as below.

$$\sum_{i=2}^{k+1} \frac{S_{i-1}}{k} \leq \sum_{i=2}^{k+1} \frac{S_{i-1}}{i-1} \leq \sum_{i=2}^{k+1} S_{i-1}.$$

Using above inequalities and carrying out some computations, we get the upper and lower bounds of  $D_n$  as

$$A_n \leq D_n \leq B_n, \quad (19)$$

where

$$A_n = \left\{ S_n - np + \theta(n-1)p - \frac{\theta}{k} \left( k \sum_{i=2}^{k+1} S_{i-1} + \sum_{i=k+2}^n S_{i,k} \right) \right\}$$

and

$$B_n = \left\{ S_n - np + \theta(n-1)p - \frac{\theta}{k} \left( \sum_{i=2}^{k+1} S_{i-1} + \sum_{i=k+2}^n S_{i,k} \right) \right\}.$$

Using  $S_{i,k} = S_{i-1} - S_{i-k-1}$  in the expressions on the right hand terms of  $A_n$  and  $B_n$ , we get

$$\left( \sum_{i=2}^{k+1} S_{i-1} + \sum_{i=k+2}^n S_{i,k} \right) = \sum_{i=n-k+1}^n S_{i-1} = kS_n - \sum_{i=1}^k \sum_{j=n-i+1}^n X_j.$$

The bounds reduce to

$$A_n = \left\{ (1-\theta)(S_n - np) - \theta p + \frac{\theta}{k} \sum_{i=1}^k \sum_{j=n-i+1}^n X_j - \frac{\theta(k-1)}{k} \sum_{i=2}^{k+1} S_{i-1} \right\}$$

and

$$B_n = \left\{ (1-\theta)(S_n - np) - \theta p + \frac{\theta}{k} \sum_{i=1}^k \sum_{j=n-i+1}^n X_j \right\}.$$

Consequently, Equation (19) yields

$$\{(1-\theta)(S_n - np) - O(k^2)\} \leq D_n \leq \{(1-\theta)(S_n - np) + O(k)\}. \quad (20)$$

Let  $\mathcal{L}_n = \frac{L_n}{n}$ . Then  $\{\mathcal{L}_n, \mathcal{F}_{n-1}, n \geq 1\}$  is again a sequence of bounded martingale differences. Therefore

$$\sum_{i=1}^{\infty} \mathbb{E}(\mathcal{L}_i^2 | \mathcal{F}_{i-1}) \leq \sum_{i=1}^{\infty} \frac{9}{i^2} < \infty.$$

Therefore Lemma 2.1 yields that  $\sum_{i=1}^n \mathcal{L}_i$  converges almost surely. Hence, the Kronecker's lemma leads to that  $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n L_i = \lim_{n \rightarrow \infty} \frac{D_n}{n} \rightarrow 0$  almost surely. Now using Lemma 2.3 in the Equation (20), we get

$$\frac{S_n}{n} \rightarrow p \text{ a.s.} \quad (21)$$

This completes the proof.  $\square$

**Proof of Theorem 3.3.** Consider a martingale difference sequence defined as

$$B_i = \left( X_i - (1 - \theta)p - \frac{\theta}{k} S_{i,k} \right) \quad \text{for } i \geq k + 1.$$

Since  $|B_i| \leq 3$ , then  $\{B_i, \mathcal{F}_{i-1}, i \geq k + 1\}$  is a sequence of bounded martingale differences. Next we define  $\mathcal{B}_n = \frac{B_n}{n}$ , then  $\{\mathcal{B}_n, \mathcal{F}_{n-1}\}$  is again a sequence of bounded martingale differences. Therefore

$$\sum_{i=k+1}^{\infty} \mathbb{E}(\mathcal{B}_i^2 | \mathcal{F}_{i-1}) \leq \sum_{i=k+1}^{\infty} \frac{9}{i^2} < \infty.$$

Lemma 2.1 then gives that  $\sum_{i=k+1}^n \mathcal{B}_i$  converges almost surely. Consequently, Kronecker's lemma yields that  $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=k+1}^n B_i \rightarrow 0$  almost surely.

Next, we consider

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=k+1}^n B_i = \left( \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=k+1}^n X_i - (1 - \theta)p - \lim_{n \rightarrow \infty} \frac{\theta}{k} \frac{1}{n} \sum_{i=k+1}^n S_{i,k} \right).$$

Using the strong law of large numbers for the sequence  $\{X_n\}$  as proved in Theorem 3.2, we get

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=k+1}^n B_i = \left( p - (1 - \theta)p - \lim_{n \rightarrow \infty} \frac{\theta}{k} \frac{1}{n} \sum_{i=k+1}^n S_{i,k} \right).$$

Since  $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=k+1}^n B_i \rightarrow 0$ , this implies that

$$\frac{1}{n} \sum_{i=k+1}^n S_{i,k} \rightarrow kp \quad \text{almost surely.} \quad (22)$$

This completes the proof.  $\square$

**Proof of Theorem 3.4.** Consider  $Y_i = X_i S_{i,k}$ . Then

$$\mathbb{E}(Y_i) = \mathbb{E}\{S_{i,k} \mathbb{E}(X_i | \mathcal{F}_{i-1})\}.$$

Using the Model (1), we can write

$$\begin{aligned} \mathbb{E}(Y_i) &= (1 - \theta)p \mathbb{E}(S_{i,k}) + \frac{\theta}{k} \mathbb{E}(S_{i,k})^2 \\ &= kp^2 + \frac{\theta}{k} \mathbb{V}(S_{i,k}). \end{aligned} \quad (23)$$

In the Corollary 3.2, we have proved that for every  $i \geq k + 1$ , the variance of  $S_{i,k}$  does not depend on  $i$  and will depend only on  $k$ . Hence, each  $Y_i$  has a constant mean value which is free from  $i$ . Further we observe that

$$\text{Cov}(Y_n, Y_m) = \text{Cov}(X_n S_{n,k}, X_m S_{m,k}) \leq \text{Cov}(S_{n,k}, S_{m,k}),$$

for a fixed value of  $m \geq k + 1$ . This gives

$$\text{Cov}(Y_n, Y_m) \leq \sum_{i=n-k}^{n-1} \sum_{j=m-k}^{m-1} \text{Cov}(X_i, X_j). \quad (24)$$

In Theorem 3.1, we have shown that the covariance between  $X_i$  and  $X_j$  will tend to zero as the distance between the indices  $i$  and  $j$  increases. That is, the random variables  $X_i$  and  $X_j$  become asymptotically uncorrelated. Therefore  $\text{Cov}(Y_n, Y_m) \rightarrow 0$  as  $n \rightarrow \infty$  and  $m \geq k + 1$ . Further, let  $S_n^* = \sum_{i=k+1}^n Y_i$ . Then

$$\mathbb{V}\left(\frac{S_n^*}{n}\right) = \frac{1}{n^2} \sum_{i=k+1}^n \mathbb{V}(Y_i) + \frac{1}{n^2} \sum_{i=k+1}^n \sum_{j=k+1}^n \text{Cov}(Y_i, Y_j). \quad (25)$$

So there exists a finite  $N$  such that for every  $\epsilon > 0$  and  $n \geq N$ , we have  $|\text{Cov}(Y_n, Y_m)| \leq \epsilon$ . Moreover,  $\mathbb{E}(Y_i^2) \leq k^2$  implies that  $\mathbb{V}(Y_i) \leq \beta$  for some constant  $\beta$ . Equation (25) then yields that

$$\left| \mathbb{V}\left(\frac{S_n^*}{n}\right) \right| \leq \frac{\beta}{n} + \frac{\epsilon}{2} \leq \delta$$

for every  $\delta > 0$ . Thus  $\mathbb{V}\left(\frac{S_n^*}{n}\right) \rightarrow 0$  as  $n \rightarrow \infty$ . Hence

$$\lim_{n \rightarrow \infty} P\left(\left|\frac{S_n^*}{n} - \mu\right| > \beta\right) \leq \lim_{n \rightarrow \infty} \frac{\mathbb{V}(S_n^*)}{n^2 \beta^2} \rightarrow 0.$$

This completes the proof.  $\square$

**Proof of Theorem 3.5.** Consider again the sum of martingale differences as  $D_n = \sum_{i=1}^n L_i$ . Now we decompose the sum of conditional mean of the square of martingale difference sequence  $\{L_i\}$  as

$$\sum_{i=1}^n \mathbb{E}(L_i^2 | \mathcal{F}_{i-1}) = \sum_{i=1}^{k+1} \mathbb{E}(L_i^2 | \mathcal{F}_{i-1}) + \sum_{i=k+2}^n \mathbb{E}(L_i^2 | \mathcal{F}_{i-1}). \quad (26)$$

Using the Model (1) and some simplification, we get

$$\begin{aligned} \sum_{i=1}^{k+1} \mathbb{E}(L_i^2 | \mathcal{F}_{i-1}) &= p(1-p) + \sum_{i=2}^{k+1} \left[ \left\{ (1-\theta)p + \frac{\theta}{i-1} S_{i-1} \right\} \right. \\ &\quad \left. - \sum_{i=2}^{k+1} \left[ \left\{ (1-\theta)p + \frac{\theta}{i-1} S_{i-1} \right\}^2 \right] \right], \end{aligned}$$

carrying out similar calculations for the other part of the sum, we obtain

$$\sum_{i=k+2}^n \mathbb{E}(L_i^2 | \mathcal{F}_{i-1}) = \sum_{i=k+2}^n \left[ \left\{ (1-\theta)p + \frac{\theta}{k} S_{i,k} \right\} - \left\{ (1-\theta)p + \frac{\theta}{k} S_{i,k} \right\}^2 \right].$$

Substituting the above expressions in the Equation (26), we obtain

$$\sum_{i=1}^n \mathbb{E}(L_i^2 | \mathcal{F}_{i-1}) = M_n + N \sum_{i=k+2}^n S_{i,k} - \frac{\theta^2}{k^2} \sum_{i=k+2}^n (S_{i,k})^2 + O(k),$$

where  $M_n = \{p(1-p) + n(1-\theta)p - n(1-\theta)^2p^2\}$  and  $N = \frac{\theta}{k} \{1 - 2(1-\theta)p\}$ . Hence for large values of  $n$ , we can write

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E}(L_i^2 | \mathcal{F}_{i-1}) = \underbrace{M + N \left( \frac{1}{n} \sum_{i=k+2}^n S_{i,k} \right)}_{(I)} - \underbrace{\frac{\theta^2}{k^2} \left( \frac{1}{n} \sum_{i=k+2}^n (S_{i,k})^2 \right)}_{(II)} + o(1), \quad (27)$$

where  $M = (1-\theta)p - \{(1-\theta)p\}^2$  as  $n \rightarrow \infty$ . Now, we simplify the expressions in Part (I) and Part (II). In the Theorem 3.3, the strong law of large numbers for the sequence  $\{S_{i,k}\}$  was established. Using this in the Part (I) of the Equation (27) can be written as

$$\begin{aligned} M + N \lim_{n \rightarrow \infty} \left( \frac{1}{n} \sum_{i=k+2}^n S_{i,k} \right) &= M + N(kp) \\ &= p(1-p) + (\theta p)^2. \end{aligned} \quad (28)$$

For the Part (II) of the Equation (27), we again use the definition of martingale difference sequence. Define a sequence  $\{W_i\}$ ,  $i \geq k+2$  as

$$W_i = S_{i,k} \left( X_i - (1-\theta)p - \frac{\theta}{k} S_{i,k} \right).$$

This gives  $\mathbb{E}|W_i| < \infty$  and  $\mathbb{E}(W_i | \mathcal{F}_{i-1}) = 0$ . The sequence  $\{W_i\}$  satisfies all the conditions of a martingale difference. Since  $W_i$  is bounded for all  $i \geq k+2$ , with the same arguments as given in the proof of Theorem 3.3, it can be proved easily that  $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=k+2}^n W_i \rightarrow 0$ . Thus

$$\left\{ \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=k+2}^n X_i S_{i,k} - (1-\theta)p \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=k+2}^n S_{i,k} - \frac{\theta}{k} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=k+2}^n (S_{i,k})^2 \right\} \rightarrow 0.$$

Using Theorem 3.3 and Theorem 3.4, we get

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=k+2}^n (S_{i,k})^2 \rightarrow (kp)^2 + \mathbb{V}(S_{k+1,k}). \quad (29)$$

Using the Equations (28) and (29) in the Equation (27), we get

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E}(L_i^2 | \mathcal{F}_{i-1}) \xrightarrow{P} p(1-p) - \frac{\theta^2}{k^2} \mathbb{V}(S_{k+1,k}) \quad \text{as } n \rightarrow \infty.$$

Next, we observe that the sequence  $\{L_n, n \geq 1\}$  is bounded. Hence, the conditional Lindeberg condition is also satisfied, and we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left( L_i^2 I \left( \left| \frac{L_i}{\sqrt{n}} \right| \geq \epsilon \right) | \mathcal{F}_{i-1} \right) \xrightarrow{P} 0 \quad (30)$$

for all  $\epsilon > 0$ . Using Lemma 2.2, we get

$$\frac{\sum_{i=1}^n L_i}{\sqrt{n}} = \frac{D_n}{\sqrt{n}} \xrightarrow{d} N(0, \sigma^2),$$

where  $\sigma^2 = p(1-p) - \frac{\theta^2}{k^2} \mathbb{V}(S_{k+1,k})$ . Now the upper and lower bounds of  $D_n$  obtained in (19) lead to

$$\frac{S_n - np}{\sqrt{n}} \xrightarrow{d} N\left(0, \frac{\sigma^2}{(1-\theta)^2}\right).$$

This completes the proof.  $\square$

**Proof of Theorem 3.6.** Note that all the conditions of Lemma 3.5 of [10] are satisfied. Thus, the law of the iterated logarithm is

$$\limsup_{n \rightarrow \infty} \frac{D_n}{\sqrt{2n \log \log n}} = \sigma,$$

and

$$\liminf_{n \rightarrow \infty} \frac{D_n}{\sqrt{2n \log \log n}} = -\sigma,$$

where  $\sigma^2 = p(1-p) - \frac{\theta^2}{k^2} \mathbb{V}(S_{k+1,k})$ . Using the upper and lower bounds of  $D_n$  as in (19) completes the proof.  $\square$

**Proof of Corollary 3.3.** The proof is a special case of Theorem 3.5 when  $k = 1$ .  $\square$

**Proof of Corollary 3.3.** The proof is a special case of Theorem 3.6 when  $k = 1$ .  $\square$

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