J. Math. Anal. Appl. 488 (2020) 124064

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

L^∞ bounds for Maxwell-gauged equations in $\mathbb{R}^{1+1}$ and their applications

Hyungjin Huh

Department of Mathematics, Chung-Ang University, Seoul, 06974, Republic of Korea

A R T I C L E   I N F O

Article history:
Received 8 January 2020
Available online 23 March 2020
Submitted by C. Clason

Keywords:
Maxwell-gauged $O(3)$ sigma model
Maxwell–Klein–Gordon
L^∞ bound
Growth-in-time of H^2 norm

A B S T R A C T

By making use of local energy conservation, we estimate the L^∞ norm of the first derivative of the solution to the Maxwell-gauged $O(3)$ sigma model in $\mathbb{R}^{1+1}$. As an application of the bound, we obtain the growth of the H^2 norm of the solution. Quadratic polynomial growth is obtained by improving exponential growth. A similar idea can be applied to the Maxwell–Klein–Gordon equations in $\mathbb{R}^{1+1}$.

1. Introduction

We are interested in the L^∞ norm of the first derivative of the solution to the Maxwell-gauged $O(3)$ sigma model [14] in $\mathbb{R}^{1+1}$.

$$D_0D_0\phi - D_1D_1\phi - (1-\phi_3)n + \phi\left(|D_0\phi|^2 - |D_1\phi|^2 + \phi_3(1-\phi_3)\right) = 0, \tag{1.1}$$
$$\partial_0 F_{10} = \langle n\times\phi, D_1\phi\rangle, \tag{1.2}$$
$$\partial_1 F_{10} = \langle n\times\phi, D_0\phi\rangle, \tag{1.3}$$

with initial data $\phi(x,0)=\phi_0$, $\partial_t\phi(x,0)=\phi_1$, $A_\mu(x,0)=a_\mu$, and $\partial_t A_\mu(x,0)=b_\mu$. Here, $\phi=(\phi_1,\,\phi_2,\,\phi_3)$ is a three-component real scalar field with unit norm $\phi_1^2+\phi_2^2+\phi_3^2=1$, and A_0 and A_1 are real number fields. The space–time derivatives are denoted by $\partial_0=\partial_t$, $\partial_1=\partial_x$. The covariant derivative is defined as

$$D_\mu\phi = \partial_\mu\phi + A_\mu(n\times\phi),$$

where $n=(0,0,1)$ and $F_{10}=\partial_1A_0-\partial_0A_1$. The usual inner product and cross-product on $\mathbb{R}^3$ are given by

$$\langle a,b\rangle = a_1b_1+a_2b_2+a_3b_3,$$

E-mail address: huh@cau.ac.kr.

https://doi.org/10.1016/j.jmaa.2020.124064
0022-247X/

$$a \times b = (a_2 b_3 - a_3 b_2,\, a_3 b_1 - a_1 b_3,\, a_1 b_2 - a_2 b_1).$$

Moreover, $|a|^2 = \langle a, a\rangle$.

The system of equations (1.1)–(1.3) is invariant under the following gauge transformations:

$$\phi = (z, \phi_3) \to (ze^{i\chi},\, \phi_3), \qquad A_\mu \to A_\mu - \partial_\mu \chi,$$

where χ is a real-valued smooth function on $\mathbb{R}^{1+1}$, and the notation $z = \phi_1 + i\phi_2$ is used. Therefore, a solution of the system (1.1)–(1.3) is formed by a class of gauge equivalent pairs $(\phi,\, A_\mu)$. The conserved energy for (1.1)–(1.3) is

$$E(t) = \int_{\mathbb{R}} |D_0\phi|^2 + |D_1\phi|^2 + F_{10}^2 + (1-\phi_3)^2 \, dx = E(0). \tag{1.4}$$

The initial value problems of the sigma model and Maxwell-gauged sigma model in $\mathbb{R}^{1+1}$ have been studied in [9] and [8], respectively. We refer to [11,15,17,18] for the Cauchy problem of the sigma model without gauge fields. The existence of a solution to the self-dual equations for the Maxwell-gauged $O(3)$ sigma model has been studied in [6,7].

The existence of a global solution to (1.1)–(1.3) was proven in [8] under the Lorenz gauge condition $\partial_0 A_0 - \partial_1 A_1 = 0$. For initial data

$$\phi_0 \in H^2(\mathbb{R}), \quad \phi_1 \in H^1(\mathbb{R}), \quad a_\mu \in H^2(\mathbb{R}), \quad b_\mu \in H^1(\mathbb{R}) \tag{1.5}$$

satisfying $\langle \phi_0,\, \phi_1\rangle = 0$, the initial value problem for (1.1)–(1.3) has a unique, global-in-time solution that belongs to

$$\begin{aligned} \phi &\in C([0,\infty),\, H^2(\mathbb{R})) \cap C^1([0,\infty),\, H^1(\mathbb{R})), \\ A_\mu &\in C([0,\infty),\, H^2(\mathbb{R})) \cap C^1([0,\infty),\, H^1(\mathbb{R})). \end{aligned}$$

By making use of local energy conservation, we estimate L^∞ bounds for F_{10} and $D_\alpha\phi$, which are independent of the gauge condition.

Theorem 1.1. *For the global solution of* (1.1)–(1.3) *with initial data* (1.5)*, we have*

$$\begin{aligned} \|F_{10}(\cdot,\, t)\|_{L^\infty(\mathbb{R})} &\le C + \sqrt{2}E^{\frac{3}{4}}(0)t^{\frac{1}{4}}, \\ \|D_\alpha\phi(\cdot,\, t)\|_{L^\infty(\mathbb{R})} &\le C + \sqrt{2}E^{\frac{3}{4}}(0)t^{\frac{1}{4}} + E^{\frac{1}{2}}(0)t^{\frac{1}{2}}, \end{aligned} \tag{1.6}$$

where C depends only on $\|F_{10}(\cdot, 0)\|_{L^\infty(\mathbb{R})}$ and $\|D_\alpha\phi(\cdot, 0)\|_{L^\infty(\mathbb{R})}$.

Several authors [1,2,4,5,12,16,19,20] have studied the growth of the Sobolev norms of the solutions to the Hamiltonian PDE. We refer to [2,4,5,12] for Schrödinger equations and [19] for the Dirac-Klein–Gordon system. The logarithmic Sobolev inequality was used in [2] to obtain an exponential bound for the H^2 norm. The so-called *upside-down I* method was used in [3,4,19].

We consider here the growth of the H^2 norm of the solution to (1.1)–(1.3). The exponential growth in [8] was obtained

$$\|D_\mu D_\nu \phi(\cdot, t)\|_{L^2(\mathbb{R})}^2 \lesssim e^{c(E(0)+1)t}.$$

By applying Sobolev's inequalities

$$\|D\phi\|^3_{L^6(\mathbb{R})} \lesssim \|D\phi\|^2_{L^2(\mathbb{R})}\|DD\phi\|_{L^2(\mathbb{R})} \quad \text{and} \quad \|D\phi\|^2_{L^\infty(\mathbb{R})} \lesssim \|D\phi\|_{L^2(\mathbb{R})}\|DD\phi\|_{L^2(\mathbb{R})}.$$

Instead of using Sobolev's inequalities, we make use of the L^∞ bounds in Theorem 1.1 to improve growth as follows:

Theorem 1.2. *For the global solution of (1.1)–(1.3) with initial data (1.5), we have*

$$\|D_\mu D_\nu \phi(\cdot, t)\|_{L^2(\mathbb{R})} \lesssim \|D_\mu D_\nu \phi(\cdot, 0)\|_{L^2(\mathbb{R})} + (1 + E(0))^2(1 + t)^2.$$

Remark. *The proofs of Theorems 1.1 and 1.2 can be applied to the solution of the $O(3)$ sigma model without gauge fields:*

$$\partial_t\partial_t\phi - \partial_x\partial_x\phi - (1 - \phi_3)n + \phi\left(|\partial_t\phi|^2 - |\partial_x\phi|^2 + \phi_3(1 - \phi_3)\right) = 0. \tag{1.7}$$

Then, the obtained bound for the H^2 norm is new even for (1.7). The usual argument using Sobolev and Gronwall's inequalities leads us to the exponential bound.

We also consider the following Maxwell–Klein–Gordon equations in $\mathbb{R}^{1+1}$:

$$\begin{aligned} D_0D_0\psi - D_1D_1\psi + \psi &= 0,\\ \partial_0 F_{10} &= \mathrm{Im}(\bar{\psi}D_1\psi),\\ \partial_1 F_{10} &= \mathrm{Im}(\bar{\psi}D_0\psi), \end{aligned} \tag{1.8}$$

where ψ is a complex-valued function, and the covariant derivative is defined as

$$D_\mu\psi = \partial_\mu\psi + iA_\mu\psi,$$

where $i = \sqrt{-1}$. The conserved energy of the system (1.8) is

$$H(t) = \int\limits_{\mathbb{R}} |D_0\psi|^2 + |D_1\psi|^2 + |\psi|^2 + F_{10}^2\,dx = H(0).$$

For the initial data belonging to $H^2(\mathbb{R})$, it is easy to prove, under the Lorenz gauge condition $\partial_0 A_0 - \partial_1 A_1 = 0$, the existence of global solutions of (1.8) with

$$\begin{aligned} \psi &\in C([0,\infty),\, H^2(\mathbb{R})) \cap C^1([0,\infty),\, H^1(\mathbb{R})),\\ A_\mu &\in C([0,\infty),\, H^2(\mathbb{R})) \cap C^1([0,\infty),\, H^1(\mathbb{R})). \end{aligned}$$

We refer to [10,13] for a proof of the existence of a global solution of the Maxwell–Klein–Gordon equations. The next result is concerned with the L^∞ bounds of $\|F_{01}(\cdot,t)\|_{L^\infty(\mathbb{R})}$ and $\|D_\alpha\psi(\cdot,t)\|_{L^\infty(\mathbb{R})}$.

Theorem 1.3. *For the global solution of (1.8) with H^2 as initial data, we have*

$$\begin{aligned} \|F_{10}(\cdot,\, t)\|_{L^\infty(\mathbb{R})} &\le C + H(0),\\ \|D_\alpha\psi(\cdot,\, t)\|_{L^\infty(\mathbb{R})} &\le C + H(0) + H^{\frac{1}{2}}(0)t^{\frac{1}{2}}, \end{aligned}$$

where C depends only on $\|F_{10}(\cdot,0)\|_{L^\infty}$ and $\|D_\alpha\psi(\cdot,0)\|_{L^\infty}$.

As an application of Theorem 1.3, we can obtain the H^2 bound for the solution of the Maxwell–Klein–Gordon equations (1.8) as follows:

$$\|D_\mu D_\nu \psi(\cdot, t)\|_{L^2(\mathbb{R})} \lesssim \|D_\mu D_\nu \psi(\cdot, 0)\|_{L^2(\mathbb{R})} + H^{\frac{1}{2}}(0)(H(0) + \|F(\cdot, 0)\|_{L^\infty(\mathbb{R})})\, t.$$

In Section 2, we obtain the L^∞ bounds (1.6) for the solution of the Maxwell-gauged $O(3)$ sigma model. Theorems 1.2 and 1.3 are proved in Sections 3 and 4, respectively. We use the summation convention, where we sum over repeated indices. We also use C to denote various constants and $A \lesssim B$ to denote an estimate of the form $A \le CB$.

2. Proof of Theorem 1.1

Recall the basic algebra associated with the covariant derivative:

$$\partial_\alpha \langle \phi, \psi \rangle = \langle D_\alpha \phi, \psi \rangle + \langle \phi, D_\alpha \psi \rangle, \tag{2.1}$$

$$D_0 D_1 \phi - D_1 D_0 \phi = F_{01}(n \times \phi), \tag{2.2}$$

where ϕ and ψ are three-component real scalar fields.

We first derive local energy conservation (2.4). Taking the inner product $D_0\phi$ with (1.1) and considering $\langle D_0\phi, \phi\rangle = 0$, we have

$$\begin{aligned} 0 &= \langle D_0 D_0 \phi, D_0 \phi \rangle - \langle D_1 D_1 \phi, D_0 \phi \rangle - (1 - \phi_3)\langle n, D_0 \phi \rangle \\ &= \partial_t \frac{1}{2}|D_0\phi|^2 - \partial_x \langle D_1\phi, D_0\phi\rangle + \langle D_1\phi, D_1 D_0 \phi\rangle + \partial_t \frac{1}{2}(1-\phi_3)^2, \end{aligned}$$

which becomes, by considering (2.2),

$$0 = \partial_t \frac{1}{2}|D_0\phi|^2 + \partial_t \frac{1}{2}|D_1\phi|^2 - \partial_x\langle D_1\phi, D_0\phi\rangle + F_{10}\langle n \times \phi, D_1\phi\rangle + \partial_t \frac{1}{2}(1-\phi_3)^2.$$

We also have, from (1.2),

$$0 = \partial_t \frac{1}{2} F_{10}^2 - F_{10}\langle n \times \phi, D_1 \phi\rangle.$$

Then, we obtain

$$\frac{\partial}{\partial t}\left(|D_0\phi|^2 + |D_1\phi|^2 + F_{10}^2 + (1-\phi_3)^2\right) - \frac{\partial}{\partial x} 2\langle D_1\phi, D_0\phi\rangle = 0. \tag{2.3}$$

We integrate (2.3) on the domain

$$D(x_0, t_0) = \{(x,t) |\, 0 < t < t_0,\ x_0 - t_0 + t < x < x_0 + t_0 - t\},$$

and apply Green's Theorem to obtain

$$\begin{aligned}
&\int_0^{t_0} \Big(|v_+|^2 + F_{10}^2 + (1-\phi_3)^2\Big)(x_0 - t_0 + s, s)\, ds \\
&+ \int_0^{t_0} \Big(|v_-|^2 + F_{10}^2 + (1-\phi_3)^2\Big)(x_0 + t_0 - s, s)\, ds \\
&= \int_{x_0-t_0}^{x_0+t_0} \Big(|D_0\phi|^2 + |D_1\phi|^2 + F_{10}^2 + (1-\phi_3)^2\Big)(s, 0)\, ds \\
&\le \int_{\mathbb{R}} \Big(|D_0\phi|^2 + |D_1\phi|^2 + F_{10}^2 + (1-\phi_3)^2\Big)(x, 0)\, dx = E(0),
\end{aligned} \tag{2.4}$$

where

$$v_- = D_0\phi - D_1\phi \quad \text{and} \quad v_+ = D_0\phi + D_1\phi.$$

We can obtain from (1.2) and (1.3):

$$(\partial_0 + \partial_1)F_{10} = \langle n \times \phi, v_+\rangle. \tag{2.5}$$

By integrating (2.5) along the characteristic and considering (2.4), we obtain

$$\begin{aligned}
|F_{10}(x,t)| &\le |F_{10}(x-t,0)| + \int_0^t |n \times \phi|\, |v_+|(x - t + s, s) ds \\
&\le |F_{10}(x-t,0)| + \sqrt{2}\left(\int_0^t |v_+|^2\right)^{\frac{1}{2}} \left(\int_0^t (1-\phi_3)^2\right)^{\frac{1}{4}} \left(\int_0^t 1\right)^{\frac{1}{4}} \\
&\le |F_{10}(x-t,0)| + \sqrt{2}E^{\frac{3}{4}}(0)t^{\frac{1}{4}},
\end{aligned}$$

where we use $|n \times \phi|^2 = 1 - \phi_3^2$ and $1 + \phi_3 \le 2$.

Using (2.2), (1.1) can be rewritten as

$$(D_0 + D_1)v_- = F_{10}(n \times \phi) + (1-\phi_3)n - \phi\Big(|D_0\phi|^2 - |D_1\phi|^2 + \phi_3(1-\phi_3)\Big).$$

Taking the inner product with v_-, we have

$$(\partial_0 + \partial_1)\frac{1}{2}|v_-|^2 = F_{10}\langle n \times \phi, v_-\rangle + (1-\phi_3)\langle n, v_-\rangle, \tag{2.6}$$

where we use $\langle \phi, v_-\rangle = 0$, which is from $|\phi|^2 = 1$ and (2.1). Considering $1 + \phi_3 \le 2$, (2.6) implies

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}|v_-(x+t,t)|^2 &\le |F_{10}|(\phi_1^2 + \phi_2^2)^{\frac{1}{2}}|v_-| + (1-\phi_3)|v_-| \\
&\le \sqrt{2}|F_{10}|(1-\phi_3)^{\frac{1}{2}}|v_-| + (1-\phi_3)|v_-|,
\end{aligned}$$

from which we obtain

$$\frac{d}{dt}|v_-(x+t,t)| \le \sqrt{2}|F_{10}|(1-\phi_3)^{\frac{1}{2}} + (1-\phi_3). \tag{2.7}$$

By integrating (2.7) along the characteristic and considering (2.4), we obtain

$$
\begin{aligned}
&|v_-(x,t)| \\
&\le |v_-(x-t,0)| + \sqrt{2}\left(\int_0^t F_{10}^2\right)^{\frac{1}{2}}\left(\int_0^t (1-\phi_3)^2\right)^{\frac{1}{4}}\left(\int_0^t 1\right)^{\frac{1}{4}} + \left(\int_0^t (1-\phi_3)^2\right)^{\frac{1}{2}}\left(\int_0^t 1\right)^{\frac{1}{2}} \\
&\le |v_-(x-t,0)| + \sqrt{2}E^{\frac{3}{4}}(0)t^{\frac{1}{4}} + E^{\frac{1}{2}}(0)t^{\frac{1}{2}}.
\end{aligned}
$$

Through a similar argument, we can obtain

$$
(\partial_0 - \partial_1)\frac{1}{2}|v_+|^2 + F_{10}\langle n\times\phi, v_+\rangle - (1-\phi_3)\langle n, v_+\rangle = 0,
$$

which leads us to

$$
|v_+(x,t)| \le |v_+(x+t,0)| + \sqrt{2}E^{\frac{3}{4}}(0)t^{\frac{1}{4}} + E^{\frac{1}{2}}(0)t^{\frac{1}{2}}.
$$

Then, we obtain

$$
\begin{aligned}
|D_\alpha\phi(x,\, t)| &\le \frac{1}{2}(|v_+| + |v_-|) \\
&\le \frac{1}{2}(|v_+(x+t,0)| + |v_-(x-t,0)|) + \sqrt{2}E^{\frac{3}{4}}(0)t^{\frac{1}{4}} + E^{\frac{1}{2}}(0)t^{\frac{1}{2}}.
\end{aligned}
$$

3. Proof of Theorem 1.2

We prove Theorem 1.2, which gives an improved H^2 bound for the solution to the Maxwell-gauged $O(3)$ sigma model (1.1)–(1.3).

We review the means of obtaining the H^2 bound provided in [8]. Using (2.1), (2.2), and integration by parts, we have

$$
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} |D_\mu D_\nu\phi|^2 = \int_{\mathbb{R}} \langle (D_0D_0 - D_1D_1)D_\nu\phi, D_0D_\nu\phi\rangle + F_{01}\langle n\times D_\nu\phi, D_1D_\nu\phi\rangle. \tag{3.1}
$$

Using D_ν on (1.1), we have

$$
\begin{aligned}
0 &= D_\nu\Big(D_0D_0\phi - D_0D_0\phi - (1-\phi_3)n + \phi\left(|D_0\phi|^2 - |D_1\phi|^2 + \phi_3(1-\phi_3)\right)\Big) \\
&= (D_0D_0 - D_1D_1)D_\nu\phi + \partial_0F_{\nu 0}(n\times\phi) + 2F_{\nu 0}(n\times D_0\phi) - \partial_1F_{\nu 1}(n\times\phi) - 2F_{\nu 1}(n\times D_1\phi) \\
&\quad + \partial_\nu\phi_3 n + D_\nu\phi\left(|D_0\phi|^2 - |D_1\phi|^2 + \phi_3(1-\phi_3)\right) + \partial_\nu\left(|D_0\phi|^2 - |D_1\phi|^2 + \phi_3(1-\phi_3)\right)\phi,
\end{aligned}
$$

where we use, as scalar function f and vector field ϕ,

$$
D_\mu(f\phi) = \partial_\mu f\phi + fD_\mu\phi.
$$

Considering $|\partial_\mu\phi_3| \le |D_\mu\phi|$ and

$$
\begin{aligned}
&\langle \partial_\nu\left(|D_0\phi|^2 - |D_1\phi|^2 + \phi_3(1-\phi_3)\right)\phi, D_0D_\nu\phi\rangle \\
&= -\partial_\nu\left(|D_0\phi|^2 - |D_1\phi|^2 + \phi_3(1-\phi_3)\right)\langle D_0\phi,\, D_\nu\phi\rangle,
\end{aligned}
$$

we have from (3.1):

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} |D_\mu D_\nu \phi|^2 \lesssim \int_{\mathbb{R}} |D\phi||DD\phi| + |F||D\phi||DD\phi| + |D\phi|^3|DD\phi|, \tag{3.2}$$

where $D\phi$ and F are used schematically. Then, we estimate the above integral in [8] as follows:

$$\begin{aligned}
&\int_{\mathbb{R}} |D\phi||DD\phi| + |F||D\phi||DD\phi| + |D\phi|^3|DD\phi| \\
&\lesssim \|D\phi\|_{L^2}\|DD\phi\|_{L^2} + \|F\|_{L^4}\|D\phi\|_{L^4}\|DD\phi\|_{L^2} + \|D\phi\|_{L^6}^3\|DD\phi\|_{L^2} \\
&\lesssim \|D\phi\|_{L^2}\|DD\phi\|_{L^2} + \|F\|_{L^2}^{\frac{1}{2}}\|\partial F\|_{L^2}^{\frac{1}{2}}\|D\phi\|_{L^2}^{\frac{1}{2}}\|DD\phi\|_{L^2}^{\frac{3}{2}} + \|D\phi\|_{L^2}^2\|DD\phi\|_{L^2}^2,
\end{aligned}$$

where the covariant Sobolev inequality $\|D\phi\|_{L^6(\mathbb{R})}^3 \lesssim \|D\phi\|_{L^2(\mathbb{R})}^2\|DD\phi\|_{L^2(\mathbb{R})}$ is used. Note that (1.2) and (1.3) imply $|\partial F| \le |D\phi|$. Taking into account (1.4), we obtain

$$\frac{d}{dt}\|D_\mu D_\nu \phi\|_{L^2(\mathbb{R})}^2 \lesssim (1+E(0))\|D_\mu D_\nu \phi\|_{L^2(\mathbb{R})}^2,$$

which leads us to an exponential bound $\|D_\mu D_\nu \phi\|_{L^2}^2 \lesssim e^{c(1+E(0))t}$.

Here, we can estimate (3.2) as follows:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|D_\mu D_\nu \phi\|_{L^2}^2 &\lesssim \|D\phi\|_{L^2}\|DD\phi\|_{L^2} + \|D\phi\|_{L^2}\|F\|_{L^\infty}\|DD\phi\|_{L^2} + \|D\phi\|_{L^\infty}^2\|D\phi\|_{L^2}\|DD\phi\|_{L^2} \\
&\lesssim E(0)^{\frac{1}{2}}(1+\|D\phi\|_{L^\infty}^2)\|DD\phi\|_{L^2},
\end{aligned}$$

from which we derive

$$\frac{d}{dt}\|DD\phi\|_{L^2} \lesssim E(0)^{\frac{1}{2}}(1+\|D\phi\|_{L^\infty}^2).$$

Applying Theorem 1.1, we arrive at

$$\|D_\mu D_\nu \phi(\cdot,t)\|_{L^2(\mathbb{R})} \lesssim \|D_\mu D_\nu \phi(\cdot,0)\|_{L^2(\mathbb{R})} + (1+E(0))^2(1+t)^2.$$

4. Proof of Theorem 1.3

We consider the Maxwell–Klein–Gordon equations in $\mathbb{R}^{1+1}$:

$$D_0D_0\psi - D_1D_1\psi + \psi = 0, \tag{4.1}$$

$$\partial_0 F_{10} = \mathrm{Im}(\bar{\psi}D_1\psi), \tag{4.2}$$

$$\partial_1 F_{10} = \mathrm{Im}(\bar{\psi}D_0\psi), \tag{4.3}$$

where $\psi \in \mathbb{C}$, and the covariant derivative is defined as $D_\mu\psi = \partial_\mu\psi + iA_\mu\psi$. We have the corresponding calculus for (2.1) and (2.2):

$$\begin{aligned}
\partial_\alpha(\phi\bar{\psi}) &= D_\alpha\phi\,\bar{\psi} + \phi\,\overline{D_\alpha\psi}, \\
D_0D_1\psi - D_1D_0\psi &= iF_{01}\psi,
\end{aligned} \tag{4.4}$$

where ϕ and ψ are complex-valued functions.

We derive the expression for the conservation of local energy. Multiplying (4.1) by $\overline{D_0\psi}$ and taking the real part, we have

$$\begin{aligned} 0 &= \partial_t(|D_0\psi|^2 + |\psi|^2) + 2\mathrm{Re}(\overline{D_1\psi} D_1 D_0\psi) - \partial_x 2\mathrm{Re}(\overline{D_1\psi} D_0\psi) \\ &= \partial_t(|D_0\psi|^2 + |\psi|^2) + 2\mathrm{Re}\big(\overline{D_1\psi}(D_0 D_1\psi - iF_{01}\psi)\big) - \partial_x 2\mathrm{Re}(\overline{D_1\psi} D_0\psi), \end{aligned} \tag{4.5}$$

where (4.4) is used. Multiplying (4.2) by $2F_{01}$, we have

$$\partial_t F_{01}^2 + 2\mathrm{Im}(\bar{\psi} D_1\psi)F_{01} = 0. \tag{4.6}$$

Adding (4.5) and (4.6), we get

$$\frac{\partial}{\partial t}\left(|D_0\psi|^2 + |D_1\psi|^2 + |\psi|^2 + F_{10}^2\right) - \frac{\partial}{\partial x} 2\mathrm{Re}(\overline{D_1}\psi, D_0\psi) = 0. \tag{4.7}$$

We integrate (4.7) on the domain $D(x_0, t_0)$ and apply Green's Theorem to obtain

$$\begin{aligned} &\int_0^{t_0} \Big(|\omega_+|^2 + |\psi|^2 + F_{10}^2\Big)(x_0 - t_0 + s, s)\, ds \\ &+ \int_0^{t_0} \Big(|\omega_-|^2 + |\psi|^2 + F_{10}^2\Big)(x_0 + t_0 - s, s)\, ds \\ &= \int_{x_0 - t_0}^{x_0 + t_0} \Big(|D_0\psi|^2 + |D_1\psi|^2 + |\psi|^2 + F_{10}^2 \Big)(s, 0)\, ds \\ &\le \int_{\mathbb{R}} \Big(|D_0\psi|^2 + |D_1\psi|^2 + |\psi|^2 + F_{10}^2 \Big)(x, 0)\, dx = H(0), \end{aligned} \tag{4.8}$$

where we denote

$$\omega_- = D_0\psi - D_1\psi \quad \text{ and } \quad \omega_+ = D_0\psi + D_1\psi.$$

From (4.7), we can derive the expression for energy conservation:

$$H(t) = \int_{\mathbb{R}} |D_0\psi|^2 + |D_1\psi|^2 + |\psi|^2 + F_{10}^2\, dx = H(0). \tag{4.9}$$

We now estimate L^∞ of F_{01} and $\omega_\pm$. From (4.1)–(4.3), we obtain

$$\begin{aligned} &(\partial_t + \partial_x)F_{01} = \mathrm{Im}(\bar{\psi}\omega_+), \\ &(D_0 + D_1)\omega_- + iF_{01}\psi + \psi = 0, \\ &(D_0 - D_1)\omega_+ - iF_{01}\psi + \psi = 0. \end{aligned} \tag{4.10}$$

Applying (4.8), F_{01} can be estimated as follows:

$$
\begin{aligned}
|F_{01}(x,t)| &\le |F_{01}(x-t,0)| + \int_0^t |\bar{\psi}\omega_+|(x-t+s,s)ds \\
&\le |F_{01}(x-t,0)| + \left(\int_0^t |\psi|^2 ds\right)^{\frac{1}{2}} \left(\int_0^t |\omega_+|^2 ds\right)^{\frac{1}{2}} \\
&\le |F_{01}(x-t,0)| + H(0).
\end{aligned}
\tag{4.11}
$$

Multiplying the second equation in (4.10) by $\bar{\omega}_-$ and taking the real part, we have

$$(\partial_t + \partial_x)\frac{1}{2}|\omega_-|^2 = -\mathrm{Im}(\bar{\psi}\omega_-)F_{01} - \mathrm{Re}(\bar{\psi}\omega_-).$$

Then, we have

$$
\begin{aligned}
|\omega_-(x,t)| &\le |\omega_-(x-t,0)| + \left(\int_0^t |\psi|^2 ds\right)^{\frac{1}{2}} \left(\int_0^t |F_{01}|^2 ds\right)^{\frac{1}{2}} + \left(\int_0^t |\psi|^2 ds\right)^{\frac{1}{2}} t^{\frac{1}{2}} \\
&\le |\omega_-(x-t,0)| + H(0) + H^{\frac{1}{2}}(0)t^{\frac{1}{2}}.
\end{aligned}
$$

Through a similar argument to that in Section 3, we can obtain the H^2 bound for the solution of the Maxwell–Klein–Gordon equations (4.1)–(4.3). We can derive

$$
\begin{aligned}
\|DD\psi(\cdot,t)\|_{L^2} &\lesssim C + \int_0^t \|\psi\partial F\|_{L^2} + \|FD\psi\|_{L^2}\, ds \\
&\lesssim C + \int_0^t \|\psi \mathrm{Im}(\bar{\psi}D\psi)\|_{L^2} + \|FD\psi\|_{L^2}\, ds \\
&\lesssim C + \int_0^t \|\psi\|_{L^\infty}^2 \|D\psi)\|_{L^2} + \|F\|_{L^\infty}\|D\psi\|_{L^2}\, ds,
\end{aligned}
$$

where C depends only on the H^2 norm of the initial data, and we use (4.2) and (4.3). By making use of $\|\psi\|_{L^\infty(\mathbb{R})}^2 \lesssim \|\psi\|_{L^2(\mathbb{R})}\|D\psi\|_{L^2(\mathbb{R})}$, and considering the energy conservation in (4.9) and (4.11), we finally have

$$
\begin{aligned}
\|DD\psi(\cdot,t)\|_{L^2} &\lesssim C + \int_0^t \|\psi\|_{L^\infty}^2 \|D\psi)\|_{L^2} + \|F\|_{L^\infty}\|D\psi\|_{L^2}\, ds \\
&\lesssim C + H^{\frac{1}{2}}(0)(H(0) + \|F(\cdot,0)\|_{L^\infty})\, t.
\end{aligned}
$$

Acknowledgment

This study was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF), funded by the Ministry of Education (2017R1D1A1B03028308).

References

[1] J. Bourgain, On the growth in time of higher Sobolev norms of smooth solutions of Hamiltonian PDE, Int. Math. Res. Not. (6) (1996) 277–304.
[2] H. Brézis, T. Gallouet, Nonlinear Schrödinger evolution equations, Nonlinear Anal. 4 (4) (1980) 677–681.
[3] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, Polynomial upper bounds for the orbital instability of the 1D cubic NLS below the energy norm, Discrete Contin. Dyn. Syst. 9 (1) (2003) 31–54.
[4] J. Colliander, S. Kwon, T. Oh, A remark on normal forms and the "upside-down" I-method for periodic NLS: growth of higher Sobolev norms, J. Anal. Math. 118 (1) (2012) 55–82.
[5] M. Guardia, V. Kaloshin, Growth of Sobolev norms in the cubic defocusing nonlinear Schrödinger equation, J. Eur. Math. Soc. 17 (1) (2015) 71–149.
[6] J. Han, H. Huh, Existence of solutions to the self-dual equations in the Maxwell-gauged O(3) sigma model, J. Math. Anal. Appl. 386 (1) (2012) 61–74.
[7] J. Han, K. Song, Condensate solutions of the self-dual O(3) Maxwell-Chern-Simons-Higgs equations with symmetric vacua, Calc. Var. Partial Differ. Equ. 58 (4) (2019) 135.
[8] H. Huh, Cauchy problems of the gauged sigma model, J. Math. Phys. 46 (5) (2005) 052303.
[9] M. Keel, T. Tao, Local and global well-posedness of wave maps on $\mathbb{R}^{1+1}$ for rough data, Int. Math. Res. Not. 21 (21) (1998) 1117–1156.
[10] S. Klainerman, M. Machedon, On the Maxwell–Klein–Gordon equation with finite energy, Duke Math. J. 74 (1) (1994) 19–44.
[11] S. Klainerman, Geometric and Fourier methods in nonlinear wave equations, https://web.math.princeton.edu/seri/homepage/papers/ucla1.pdf.
[12] A. Maspero, D. Robert, On time dependent Schrödinger equations: global well-posedness and growth of Sobolev norms, J. Funct. Anal. 273 (2) (2017) 721–781.
[13] V. Moncrief, Global existence of Maxwell-Klein-Gordon fields in (2+1)-dimensional spacetime, J. Math. Phys. 21 (8) (1980) 2291–2296.
[14] B.J. Schroers, Bogomol'nyi solitons in a gauged $O(3)$ sigma model, Phys. Lett. B 356 (2–3) (1995) 291–296.
[15] J. Shatah, M. Struwe, Geometric Wave Equations, Courant Lecture Notes in Mathematics, vol. 2, 1998.
[16] G. Staffilani, On the growth of high Sobolev norms of solutions for KdV and Schrödinger equations, Duke Math. J. 86 (1) (1997) 109–142.
[17] M. Struwe, Wave maps with and without symmetries, in: Evolution Equations, in: Clay Math. Proc., vol. 17, Amer. Math. Soc., Providence, RI, 2013, pp. 483–510.
[18] D. Tataru, The wave maps equation, https://math.berkeley.edu/~tataru/papers/wmams.pdf.
[19] A. Tesfahun, Growth-in-time of higher Sobolev norms of solutions to the 1D Dirac-Klein-Gordon system, J. Hyperbolic Differ. Equ. 16 (2) (2019) 313–332.
[20] J. Thirouin, Optimal bounds for the growth of Sobolev norms of solutions of a quadratic Szegö equation, Trans. Am. Math. Soc. 371 (5) (2019) 3673–3690.